DISPOSITION OF NON-NEGATIVE INTEGER NUMBERS AT THE NODES OF INTEGER LATTICE

V.N. BERESTOVSKII

ABSTRACT. The author considers for all non-negative integer numbers three natural dispositions at the nodes of Gaussian (square), Eisenstein (hexagonal) lattices, and non-negative quarter of Gaussian lattice in the complex plane. It is proved that for every such disposition, all prime numbers are visible from the origin.

UDK 511

1. Introduction

Felix Klein indicated in his book [1] a regular disposition of the set $\mathbb{Z}_+$ of all non-negative integer numbers at the nodes of Gaussian square lattice in the complex plane $\mathbb{C}$ by increasing order along a helix. This disposition can be understood also as a "continuous" bypass of the lattice without recurrence. Figure 1 below displays short segment of this helical disposition as well as some of results which we discuss in this paper. A colored version of this picture with long segment would be more impressive. This disposition is very interesting in some respect. I have found exact formulas for a mapping f, which indicates the place $x+iy = x(n)+iy(n)$ of a given non-negative integer number n, and for inverse mapping f^{-1}. It is proved to be that for this disposition, every prime number p is "visible" from the origin, i.e. the corresponding point $f(p)$ has coprime coordinates. Later on, Bumina has set similar results for a helical disposition $g(n)$ of the set $\mathbb{Z}_+$ at the nodes of Eisentstein (hexagonal) lattice in $\mathbb{C}$. I think that a chromatic illustration of these results for regular hexagonal helix must be even more impressive than for square helix. The obtained results were published in joint paper [2] without any detailed proof.

I give here a detailed proof of the mentioned results for these dispositions as well as essential simplifications of that pieces of proof which were given in the paper [2]. In particular, the statement that all prime numbers are visible from the origin, is a simple corollary of a new fact which I establish here. Namely, the complex plane $\mathbb{C}$ is naturally sudivided into four (six) sectors for square (hexagonal) helical disposition in such a way that a number n, disposed at any fixed sector, is presented as a polynomial function of the second degree (with zero constant term) on coordinates x, y. On the ground of this statement we give also explicit polynomial presentations for visible numbers.

I don't know whether is it possible to find a natural disposition of the set $\mathbb{Z}_+$ at the nodes of cubical lattice in Euclidean 3-space with similar properties including the property, concerning polynomials with zero constant terms.

I shall obtain similar results for a natural "continuous" "snakelike" disposition of non-negative integer numbers at the nodes of non-negative quarter of Gaussian lattice with interchanging only two polynomials when one goes along a sector from one layer to the next layer. The most simple "discontinuous" version of this disposition permits to preserve a polynomial in a sector. At the end of this paper we discuss an asymptotic ratio of quantities of prime and visible numbers for every of three disposition, mentioned above. We set also an unsolved combinatorial number-theoretic problem, connected with this.

2000 Mathematical Subject Classification: 11A25, 1A99, 11B83, 11D09, 11H06.

Key words and phrases: prime number, Gauss lattice, Eisenstein lattice

The author was supported by RFBR grant No. 02-01-00192-a. This paper was written while the author enjoyed the hospitality of Max-Planck-Institut für Mathematik, Bonn.

2. Square helix

Definition 1. *Gaussian lattice is the ring*

$$\Gamma_1 = \{z = x + iy \in \mathbb{C} : x, y \in \mathbb{Z}\}$$

with usual operations in $\mathbb{C}$.

Definition 2. *A disposition of all numbers from the set* $\mathbb{Z}_+$ *at the nodes of* Γ_1 *in increasing order along a helix, untwisting counterclockwise with the origin* $z = 0$, *we shall call a square helix (see* [1]*). In more detailes, we begin with* $z = 0$, *then fill in a layer, which is a regular quadrangle, after filling in the preceding smaller layer-quadrangle; the passage to the next layer is realized in the positive direction of the* x*-axis.*

196	**195**	194	**193**	192	**191**	190	189	188	**187**	186	**185**	184	**183**	182
197	144	**143**	142	141	140	**139**	138	**137**	136	135	134	**133**	132	**181**
198	**145**	100	**99**	98	**97**	96	95	94	**93**	92	**91**	90	**131**	180
199	146	**101**	64	**63**	62	**61**	60	**59**	58	**57**	56	**89**	130	**179**
200	147	102	**65**	36	**35**	34	33	32	**31**	30	**55**	88	129	178
201	148	**103**	66	**37**	16	**15**	14	**13**	12	**29**	54	**87**	128	**177**
202	**149**	104	**67**	38	**17**	4	**3**	**2**	**11**	28	**53**	86	**127**	176
203	150	105	68	39	18	**5**	0	**1**	10	27	52	85	126	175
204	**151**	106	**69**	40	**19**	6	**7**	8	**9**	26	**51**	84	**125**	174
205	152	**107**	70	**41**	20	**21**	22	**23**	24	**25**	50	**83**	124	**173**
206	153	108	**71**	42	**43**	44	45	46	**47**	48	**49**	82	123	172
207	154	**109**	72	**73**	74	**75**	76	**77**	78	**79**	80	**81**	122	**171**
208	**155**	110	**111**	112	**113**	114	115	116	**117**	118	**119**	120	**121**	170
209	156	**157**	158	159	160	**161**	162	**163**	164	165	166	**167**	168	**169**
210	**211**	212	**213**	214	**215**	216	217	218	**219**	220	**221**	222	**223**	224

Figure 1. Klein's square helix. Prime numbers are marked by Large boldface type, other odd visible numbers are marked by boldface type.

Proposition 1. *A disposition of the set* $\mathbb{Z}_+$ *along the square helix can be given by a formula*

$$f(n) = x(n) + iy(n), \tag{1}$$

where $x(n)$ *and* $y(n)$ *are defined by formulas*

$$x(n) = (-1)^{[\sqrt{n}]+1}([\frac{[\sqrt{n}]+1}{2}] - \max(0, n - [\sqrt{n}]^2 - [\sqrt{n}])), \tag{2}$$

$$y(n) = (-1)^{[\sqrt{n}]+1}([\frac{[\sqrt{n}]+1}{2}] + \min(0, n - [\sqrt{n}]^2 - [\sqrt{n}])), \tag{3}$$

where $[r]$ *denotes the integer part of a real number* r.

Proof. We remark at first that the k-th layer of the helix contains

$$(2k+1)^2 - (2k-1)^2 = 2(4k) = 8k$$

nodes. The k-th layers with $k = 0, 1, \dots, s$ contain alltogether

$$(2s+1)^2 - 1 = 2s(2(s+1)) = 4s(s+1)$$

natural numbers (besides 0). Hence

$$f((2s+1)^2) = (s+1) - si.$$

Let us walk along $(s+1)-$th layer.

After this, a first number n, which appears on the semidiagonal $y = x > 0$ will take a position $f(n) = (s+1) + (s+1)i$. Therefore this number is an even number

$$n = (2s+1)^2 + [(s+1) - (-s)] = (2s+1)^2 + (2s+1) = (2s+1)2(s+1) = 2(s+1)(2s+1).$$

We have

$$[\sqrt{n}] = 2s+1, n-[\sqrt{n}]^2-[\sqrt{n}] = (2s+1)^2+(2s+1)-(2s+1)^2-(2s+1) = 0.$$

Later on, on the semidiagonal $y = -x > 0$ will appear a number m with $f(m) = -(s+1)+(s+1)i$. Thus it must be

$$m = n+2(s+1) = (2s+1)^2+2(2s+1)+1 = (2(s+1))^2,$$

i.e. this number is a square of an even number.

Then, on the semidiagonal $y = x < 0$ we see a number l with $f(l) = -[(s+1)+(s+1)i]$. It means that

$$l = m+2(s+1) = (2(s+1))^2+2(s+1) = 2(s+1)(2(s+1)+1),$$

onemore, an even number. For this number, $[\sqrt{l}] = 2(s+1)$. Hence $l-[\sqrt{l}]^2-[\sqrt{l}] = 0$.

A number t with $f(t) = (s+1)-(s+1)i$ completes the $(s+1)$-th layer. It must be

$$t = l+2(s+1) = (2s+3)^2-1,$$

which is an even number.

After this excursion, one can try to find $f(n)$ for any number $n \in \mathbb{Z}_+$.

One defines at first a number $s = s(n)$ of a layer, containing a given number n. There are two possible cases: a)$[\sqrt{n}] = 2k$, b)$[\sqrt{n}] = 2k+1$. In the case a), $s(n) = k$. In the case b), $s(n) = k+1$. Therefore, it is possible to calculate the number s by the formula

$$s = [\frac{[\sqrt{n}]+1}{2}].$$

Let us follow further. In the case a), we calculate the number $t := n-[\sqrt{n}]^2-[\sqrt{n}]$. There are two possible subcases: $\alpha)t \le 0$, $\beta)$ $t > 0$.

Hence we have the following: in the subcase α), $f(n) = -s-(s+t)i$; in the subcase β), $f(n) = (-s+t)-si$.

By defining a disposition of the number n relatively to the semidiagonal $y = x > 0$, we find analogously that in the case α), $f(n) = s+si-ti = s+(s-t)i$, while in the case β), $f(n) = s+si+t = (s+t)+si$. Thus we see that in all cases the formulas from Proposition 1 are valid. ■

It's quite easily to get the following Corollary.

Corollary 1. *The formulas (2) and (3) of Proposition 1 are equivalent to the formulas*

$$y(n)-x(n) = (-1)^{[\sqrt{n}]+1}(n-[\sqrt{n}]^2-[\sqrt{n}]), \tag{4}$$

$$y(n)+x(n) = (-1)^{[\sqrt{n}]+1}(2[\frac{[\sqrt{n}]+1}{2}]-|n-[\sqrt{n}]^2-[\sqrt{n}]|). \tag{5}$$

Definition 3. *By disposing non-negative integer numbers along the square helix, we call a number n visible if the numbers x(n) and y(n) from Proposition 1 are coprime.*

The next proposition defines the inverse mapping f^{-1}.

Proposition 2. *A point $x+yi \in \Gamma_1$ is the image $f(n)$ of the number $n \in \mathbb{Z}_+$, where*

$$n = 4y^2-y-x, \quad \textit{if} \quad \max(|x|,|y|) = |y| = y,$$

$$n = 4y^2-3y+x, \quad \textit{if} \quad \max(|x|,|y|) = |y| = -y,$$

$$n = 4x^2-3x+y, \quad \textit{if} \quad \max(|x|,|y|) = |x| = x,$$

$$n = 4x^2-x-y, \quad \textit{if} \quad \max(|x|,|y|) = |x| = -x.$$

Proof. Evidently, the number s of a layer, in which lies a point $x+yi \in \Gamma_1$, is equal to the number $\max(|x|,|y|)$. We have the following four possible cases: 1)$s=|y|=y$; 2)$s=|y|=-y$; 3)$s=|x|=x$; 4)$s=|x|=-x$.

Using the observation of the s-th layer at the beginning of the proof of Proposition 1, one can easily make the following calculations in every case mentined above:

1) $n=(2s)^2-(x+s)=4y^2-y-x$,
2) $n=(2s+1)^2-1-(s-x)=4y^2-3y+x$,
3) $n=(2s-1)^2+y-(-s+1)=4x^2-3x+y$,
4) $n=(2s)^2+(s-y)=4x^2-x-y$.

The proposirtion is proved. ■

Theorem 1. *By disposing non-negative integer numbers along the square helix, all prime numbers are visible.*

Proof. It follows from Proposition 2 that a number $n=f^{-1}(x+yi)$ is divided by greatest common divisor of numbers x,y. Therefore, any prime number n is visible from the origin. ■

The next Corollary immediately follows from the proofs of Proposition 2 and Theorem 1.

Corollary 2. *The visible natural numbers for square helix disposition are exactly those which can be represented in one of the following form:*

$$n=4m^2\pm 3m\pm l,$$

$$n=4m^2\pm m\pm l,$$

where $m,l\in\mathbb{Z}_+$ are coprime numbers, $m\geq l$, and $m>0$.

3. Hexagonal helix

Definition 4. *Eisenstein lattice is the ring*

$$\Gamma_2=\{z=x+y(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)\in\mathbb{C}: x,y\in\mathbb{Z}\}$$

with usual operations in $\mathbb{C}$.

Definition 5. *A disposition of all numbers from the set $\mathbb{Z}_+$ at the nodes of Γ_2 in increasing order along a helix, untwisting counterclockwise, with the origin at $z=0$, we shall call a hexagonal helix. In more detailes, we begin with $z=0$, then fill in a layer, which is a regular hexagon, after filling in the preceding smaller layer-hexagon; the passage to the next layer is realized in the positive direction of the x-axis.*

Proposition 3. *Depending on k,s, where s and k are defined succussively by formulas*

$$s=[\frac{1+\sqrt{\frac{4}{3}(n-1)+1}}{2}]$$

and

$$k=[\frac{3s(s+1)-n}{s}],$$

the coordinates $(x,y)=(x(n),y(n))$ for the disposition $g(n)=x+y(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ of a number $n\in\mathbb{Z}_+$ along the hexagonal helix can be given by the following formulas

$$k=0.\quad x=n-3s^2-3s,\quad y=-s,$$

$$k=1.\quad x=-s,\quad y=3s^2+s-n,$$

$$k=2.\quad x=3s^2-n,\quad y=3s^2+s-n,$$

$$k=3.\quad x=3s^2-n,\quad y=s,$$

$$k=4.\quad x=s,\quad y=n-3s^2+2s,$$

$$k=5.\quad x=3s-3s^2+n,\quad y=n-3s^2+2s.$$

Proof. The layers of the disposition along hexagonal helix are regular hexagons. The l-th layer-hexagon contains $6l$ nodes from Γ_2 besides 0-th layer which contains one node. Therefore l-th layers with $l = 1, \dots, s$ include alltogether

$$\Sigma_{l=1}^{s} 6l = 6\frac{s(s+1)}{2} = 3s(s+1)$$

natural numbers. We see that a number n lies in s-th layer if and only if

$$3(s-1)s < n \leq 3s(s+1),$$

or, which is equivalent,

$$3(s-1)s \leq n-1 < 3s(s+1).$$

In turn, the last inequalities are equivalent to the inequalities

$$s^2 - s \leq \frac{n-1}{3} < s^2 + s \tag{6}$$

Evidently, subintervals $[s^2 - s, s^2 + s[, s \in \mathbb{N}_+$ constitute a partition of the set $\mathbb{R}_+$. Hence there is exactly one number $s \in \mathbb{N}$, satisfying the inequalities (6) for a given natural number n.

By taking the roots from the inequalities (6) and using the integer parts, we get

$$s - 1 \leq [\sqrt{\frac{n-1}{3}}] < s + 1,$$

or, which is equivalent,

$$s - 1 \leq [\sqrt{\frac{n-1}{3}}] \leq s.$$

Thus we have only two possibilities

a) $s = [\sqrt{\frac{n-1}{3}}]$ or

b) $s = [\sqrt{\frac{n-1}{3}}] + 1$.

The inequalities (6) imply that the case a) is satisfied if and only if the following inequalities

$$[\sqrt{\frac{n-1}{3}}]^2 \leq \frac{n-1}{3} < [\sqrt{\frac{n-1}{3}}]^2 + [\sqrt{\frac{n-1}{3}}] \tag{7}$$

are fulfilled. In all other cases we have the case b).

Let us check that one can calculate the number s as follows:

$$s = [\frac{1 + \sqrt{\frac{4}{3}(n-1) + 1}}{2}]. \tag{8}$$

Really, suppose that inequalities (7) are fulfilled. Then

$$\frac{1 + \sqrt{\frac{4}{3}(n-1) + 1}}{2} < \frac{1 + \sqrt{4[\sqrt{\frac{n-1}{3}}]^2 + 4[\sqrt{\frac{n-1}{3}}] + 1}}{2} =$$

$$\frac{1 + 2[\sqrt{\frac{n-1}{3}}] + 1}{2} = [\sqrt{\frac{n-1}{3}}] + 1.$$

Also taking into account the left inequality in (7), we see that the right side of the formula (8) gives the number $[\sqrt{\frac{n-1}{3}}]$, what is required.

Assume now that the right inequality in (7) is violated. Hence

$$\frac{1 + \sqrt{\frac{4}{3}(n-1) + 1}}{2} \geq \frac{1 + \sqrt{4[\sqrt{\frac{n-1}{3}}]^2 + 4[\sqrt{\frac{n-1}{3}}] + 1}}{2} = [\sqrt{\frac{n-1}{3}}] + 1.$$

Also one has

$$\frac{1+\sqrt{\frac{4}{3}(n-1)+1}}{2} < \frac{1+\sqrt{4([\sqrt{\frac{n-1}{3}}]+1)^2+1}}{2} < \frac{1+2([\sqrt{\frac{n-1}{3}}]+1)+1}{2} = [\sqrt{\frac{n-1}{3}}]+2,$$

because

$$\frac{n-1}{3} < ([\sqrt{\frac{n-1}{3}}]+1)^2.$$

The two last long inequalities imply that in the case under consideration, the right side of the formula (8) gives the number $[\sqrt{\frac{n-1}{3}}]+1$, what is required.

The lines $y=0, x=0, y=x$ divide a hexagon into six sectors. Enumerate them, beginning from lower, clockwise by the numbers $0,1,2,3,4,5$. It is not difficult to see that the number k of the sector, in which lies the number n, is calculated by the formula

$$k=[\frac{3s(s+1)-n}{s}],$$

if we suppose that under bypass clockwise, the last vertex of every side of bounding hexagon belongs to the next sector.

The vertices of the s-th layer, onemore under going around clockwise, if we begin with the node $g(3s(s+1))$, have the following coordinates realatively to the base $\{X=1, Y=-\frac{1}{2}+\frac{\sqrt{3}}{2}i\}$ of the lattice Γ_2:

$$g(3s(s+1))=(0,-s), g(3s^2+2s)=(-s,-s), g(3s^2+s)=(-s,0),$$
$$g(3s^2)=(0,s), g(3s^2-s)=(s,s), g(3s^2-2s)=(s,0).$$

Therefore, depending on $k, g(n)$ one has the following form:

$$k=0.\quad g(n)=(0,-s)+(3s(s+1)-n)(-1,0)=(n-3s^2-3s,-s),$$
$$k=1.\quad g(n)=(-s,-s)+(3s^2+2s-n)(0,1)=(-s,3s^2+s-n),$$
$$k=2.\quad g(n)=(-s,o)+(3s^2+s-n)(1,1)=(3s^2-n,3s^2+s-n),$$
$$k=3.\quad g(n)=(0,s)+(3s^2-n)(1,0)=(3s^2-n,s),$$
$$k=4.\quad g(n)=(s,s)+(3s^2-s-n)(0,-1)=(s,n-3s^2+2s),$$
$$k=5.\quad g(n)=(s,0)+(3s^2-2s-n)(-1,-1)=(3s-3s^2+n,n-3s^2+2s).$$

Proposition 3 is proved. ∎

Proposition 4. *A nonzero point $xX+yY\in\Gamma_2$ is the image $g(n)$ of the natural number n, where, depending on a number k of a sector,*

$$k=0.\quad n=3y^2-3y+x,\quad y<0,\quad y\le x\le 0;$$
$$k=1.\quad n=3x^2-x-y,\quad x<0,\quad x\le y\le 0;$$
$$k=2.\quad n=3(y-x)^2-x,\quad x\le 0\le y,\quad x<y;$$
$$k=3.\quad n=3y^2-x,\quad y\ge x\ge 0,\quad y>0;$$
$$k=4.\quad n=3x^2-2x+y,\quad x>0,\quad x\ge y\ge 0;$$
$$k=5.\quad n=3(x-y)^2-2(x-y)+y,\quad y\le 0\le x,\quad y<x.$$

Proof. Proposition 3 implies, depending on k, the equations below:

$$k=0.\quad s=-y,\quad n=x+3s^2+3s=3y^2-3y+x,$$
$$k=1.\quad s=-x,\quad n=3s^2+s-y=3x^2-x-y,$$
$$k=2.\quad s=y-x,\quad n=3s^2-x=3(y-x)^2-x,$$
$$k=3.\quad s=y,\quad n=3s^2-x=3y^2-x,$$
$$k=4.\quad s=x,\quad n=3s^2-2s+y=3x^2-2x+y,$$
$$k=5.\quad s=x-y,\quad n=3s^2-2s+y=3(x-y)^2-2(x-y)+y.$$

The inequalities for numbers x, y in corresponding sectors are easily verified. ∎

A visible number for the disposition along the hexagonal helix is defined analogously to the definition 3 according to the definitions 4 and 5.

Theorem 2. *By disposing non-negative integer numbers along the hexagonal helix, all prime numbers are visible.*

Proof. It follows from Proposition 4 that a number $n = g^{-1}(xX + yY)$ is divided by greatest common divisor of numbers x, y. Therefore, any prime number n is visible from the origin. ■

Corollary 3. *The visible natural numbers for hexagonal helix disposition are exactly those which admit a representation in one of the following form:*

$$n = 3m^2 \pm 2m \pm l,$$

$$n = 3m^2 \pm l,$$

where $m, l \in \mathbb{Z}_+$ *are coprime numbers,* $m \geq l$*, and* $m > 0$.

Proof. Following Proposition 4, we have for any natural number n one of the next presentations:

$$k = 0. \quad n = 3y^2 - 3y + x = 3y^2 + (x - y) - 2y = 3m^2 + 2m + l, \quad m \geq l \geq 0, \quad m > 0;$$

$$k = 1. \quad n = 3x^2 - x - y = 3x^2 - (y - x) - 2x = 3m^2 + 2m - l, \quad m \geq l \geq 0, \quad m > 0;$$

$$k = 2. \quad n = 3(y - x)^2 - x = 3m^2 + l, \quad m \geq l \geq 0, \quad m > 0;$$

$$k = 3. \quad n = 3y^2 - x = 3m^2 - l, \quad m \geq l \geq 0, \quad m > 0;$$

$$k = 4. \quad n = 3x^2 - 2x + y = 3m^2 - 2m + l, \quad m \geq l \geq 0, \quad m > 0;$$

$$k = 5. \quad n = 3(x - y)^2 - 2(x - y) + y = 3m^2 - 2m - l, \quad m \geq l \geq 0, \quad m > 0.$$

These presentations imply the corollary, if we remember that the visible natural numbers n are exactly those, for which the numbers x, y in $g(n) = xX + yY$ are coprime. ■

4. Snakelike disposition of non-negative integer numbers at the nodes of positive quadrant of Γ_1

Now we shall describe a version of the square helix disposition at the nodes of semigroup

$$\Gamma_{1,+} := \{x + yi : x, y \in \mathbb{Z}_+\}.$$

The k-th layer of this disposition consists of all nodes in $\Gamma_{1,+}$ such that $\max(x, y) = k \in \mathbb{Z}_+$. It has $2k + 1$ nodes. Therefore layers with numbers $k = 0, \ldots, s$ contain alltogether

$$\Sigma_{k=0}^{s}(2k + 1) = (s + 1)^2$$

nodes. The first (last) number in s-th layer is the number s^2 (respectively, $(s + 1)^2 - 1$). Thus in this layer will disposed numbers n, satisfying inequality

$$s^2 \leq n < (s + 1)^2.$$

Therefore we have the equation

$$s = [\sqrt{n}]. \tag{9}$$

In the middle of s-th layer (i.e. on the diagonal $x = y$) is situated the number $s(s + 1)$. So, the number

$$k = [\frac{(n - 1) - s^2}{s(s + 1)}] \tag{10}$$

indicates, in which half of this layer will be disposed a given number n. The initial number s^2 will take a position V at x-axis (y-axis), while the last number $(s + 1)^2 - 1$ will take a position at y-axis (x-axis), if s is odd (respectively, even). It is clear now the disposition, which we described right now, and why it is called snakelike.

More formally, it can be described as follows. Evidently,

$$V := s(\frac{1 - (-1)^s}{2}, \frac{1 + (-1)^s}{2}).$$

Now the disposition can be defined by a formula

$$h(n) = V + kA^k(V) + \frac{n-s^2}{s(s+1)}(1-k)A^{1-k}(V) - (\frac{n-s^2}{s(s+1)} - 1)kV, \tag{11}$$

where $\{\}$ and A denote the fractional part of a number and the linear permutation $A : (x, y) \to (y, x)$ respectively.

It is not difficult to prove the statements below.

Proposition 5. *The inverse mapping h^{-1} is defined as follows:*

$$h^{-1}(x,y) = x^2 + y, \quad \textit{if} \quad s = \max(x,y) = x \quad \textit{is} \quad \textit{odd},$$
$$h^{-1}(x,y) = y^2 + 2y - x, \quad \textit{if} \quad s = \max(x,y) = y \quad \textit{is} \quad \textit{odd},$$
$$h^{-1}(x,y) = x^2 + 2x - y, \quad \textit{if} \quad s = \max(x,y) = x \quad \textit{is} \quad \textit{even},$$
$$h^{-1}(x,y) = y^2 + x, \quad \textit{if} \quad s = \max(x,y) = y \quad \textit{is} \quad \textit{even}.$$

Theorem 3. *Under snakelike disposition of non-negative integer numbers, all prime numbers are visible.*

Corollary 4. *The visible natural numbers under snakelike disposition are exactly those which admit a presentation in one of the following form:*

$$n = m^2 + l,$$
$$n = m^2 + 2m - l,$$

where $m, l \in \mathbb{Z}_+$ are coprime numbers, $m \geq l$, and $m > 0$.

Theorem 4. *Any prime number n is represented in at least one form from every Corollary 2, 3, 4.*

We see that for snakelike disposition above, one needs to change (even only two) polynomials within a sector. Now we shal consider the most simple (although "discontinuous") version h_1 of this disposition, which permits to preserve a polynomial in a sector. For this we need to change only one detail above: any even layer (with number s also) begins with the number s^2 at x-axis, while the last number $(s+1)^2 - 1$ will take a position at y-axis.

Then we have the following more simple variant of Proposition 5.

Proposition 6. *The inverse mapping h_1^{-1} is defined as follows:*

$$h_1^{-1}(x,y) = x^2 + y, \quad \textit{if} \quad s = \max(x,y) = x,$$
$$h_1^{-1}(x,y) = y^2 + 2y - x, \quad \textit{if} \quad s = \max(x,y) = y.$$

Evidently, for this disposition we have exactly the same statements as in Theorem 3 and Corollary 4.

I want to remark that it is possible to find a natural disposition of the set $\mathbb{Z}_+$ at the nodes of cubical lattice $\mathbb{Z}^3$ in Euclidean 3-space E^3, using an observation that orthogonal projection in parallel to vector $w = (1, 1, 1)$ onto two-dimensional plane E^2 in E^3, orthogonal to w, maps the lattice $\mathbb{Z}^3$ onto regular hexagonal lattice in E^2, which can be identified with Γ_2 after some similarity of E^2. Then this projection coincides with linear map

$$(x, y, z) \to (x - z)X + (y - z)Y. \tag{12}$$

Under this, every face of some cube in E^3 projects onto some diamond, consisting of two regular triangles; each of these triangles is some sector of a solid regular hegxagon in Γ_2. On the ground of formula (12) and Proposition 4, one can define an inverse mapping F for some disposition of the set $\mathbb{Z}_+$ at the nodes of the lattice $\mathbb{Z}^3$. Moreover, the lattice $\mathbb{Z}^3$ is naturally subdivided into 24 additive subsemigroups such that for any of this semigroup, the mapping F is presented as a polynomial of the third degree on coordinates (x, y, z) in E^3. Unfortunately, only one half of these polynomials will have zero constant terms. Because of this, we shall not discussed this disposition in more detailes.

5. The ratio of quantities of primes and visible numbers

Thus for all dispositions, any line $y = \frac{m}{n}x$, $y = 0$, and $x = 0$ contains no more than two prime numbers; moreover, these numbers must be a nearest points to zero between integer points on this line. So, any such disposition gives us a kind of a sieve for search of prime numbers by excluding invisible numbers. Moreover, let us suppose for a moment that we found a place $(x(n), y(n))$ of a number n for some of these dispositions, and discovered that $g.c.d.(x(n), y(n)) = d > 1$. Then it follows from the proofs of Theorems 1,2 or 3 that n is divided by d. So, this gives us a method for a partial factoring of numbers. We can further refine this search for square and last two dispositions by removing the even numbers, using simple rule: these numbers are exactly those which take a position (x, y) such that $|x| + |y|$ is an even number. This rule is not valid for hexagonal helical disposition. One can use further the combination of these three sieves.

Unfortunately, we shall see in this section that in general case even this combination does not exlude all non-prime numbers.

Let us evaluate a quantity $\psi(n)$, $\phi(n)$, or $\theta(n)$ of visible numbers, which do not exceed a given number n, for dispositions, considered in Sections 2, 3, 4 respectively.

Remind for this some well-known facts from the theory of numbers (see, for example, [3]).

Definition 6. *Möbius function $\mu(n)$ is defined as follows*

$$\mu(n) := 0, \quad \text{if} \quad n \quad \text{is divided by} \quad m^2 \neq 1,$$

$$\mu(n) = (-1)^k, \quad \text{if} \quad n = p^1 \dots p^k \quad \text{where} \quad p^1, \dots, p^k \quad \text{are different prime numbers.}$$

Proposition 7. *For a given collection of rational fractions*

$$\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n},$$

let S be a number of uncancelled fractions, and S_d be a number of fractions, cancelleble by d. Then

$$S = \Sigma\mu(d)S_d. \tag{13}$$

Proposition 8. *Let $T(n)$ be the number of all noncancellable fractions $\frac{y}{x}$ such that $0 < x \leq n, \quad 0 < y \leq n, n \in \mathbb{N}$. Then*

$$T(n) \sim \frac{6n^2}{\pi^2}, \quad \text{when} \quad n \to +\infty.$$

Proof. Apply the equality (13).The number of fractions, cancelled by d, is equal to $S_d = [\frac{n}{d}]^2$. Therefore (see [3])

$$T(n) = \Sigma_{d=1}^n \mu(d)[\frac{n}{d}]^2 = n^2(\Sigma_{d=1}^n \frac{\mu(d)}{d^2} + O(\Sigma_{d=1}^n \frac{1}{nd}) = \frac{6n^2}{\pi^2} + O(n \log n).$$

■

Theorem 5. *The ratio of prime numbers to visible numbers for any of three dispositions, considered above, is one and the same up to equivalence (of infinitesimals), and*

$$\frac{p(n)}{\psi(n)} \sim \frac{p(n)}{\phi(n)} \sim \frac{p(n)}{\theta(n)} \sim \frac{\pi^2}{6 \log n},$$

when $n \to \infty$.

Proof. It is known that for the number $p(n)$ of the primes, which does not exceed a natural number n, one has the asymptotic formula $p(n) \sim \frac{n}{\log n}$.

For the square helix, the number of all primes, contained in the solid r-th square, is equal to the number

$$p((2r+1)^2) \sim \frac{(2r+1)^2}{\log((2r+1)^2)}.$$

The quantity of visible numbers is equal to

$$\psi((2r+1)^2) = 4T(r) \sim \frac{24r^2}{\pi^2},$$

so the ratio of prime numbers to visible numbers is equivalent to

$$\frac{\pi^2(2r+1)^2}{24r^2\log((2r+1)^2)} \sim \frac{\pi^2}{6\log(2r+1)^2}.$$

For the hexagonal helix, the number of all primes, contained in the solid s-th hexagon, is equalalent to the number

$$p(3s(s+1)) \sim \frac{(3s(s+1)}{\log(3s(s+1))}.$$

The quantity of visible numbers is equal to

$$\phi(3s(s+1)) = 3T(s) \sim \frac{18s^2}{\pi^2},$$

so the ratio of prime numbers to visible numbers is equivalent to

$$\frac{3\pi^2 s(s+1)}{18s^2\log(3s(s+1))} \sim \frac{\pi^2}{6\log 3s(s+1)}.$$

Since a quantity of nodes at s-th layer is infinetisimal on a quantity of all nodes in solid s-th square (hexagon) when $s \to +\infty$, it follows easily from the facts, mentioned above, that the ratios of prime numbers to visible numbers for square and hexagonal helices are the same asymptotically and

$$\frac{p(n)}{\psi(n)} \sim \frac{p(n)}{\phi(n)} \sim \frac{\pi^2}{6\log n},$$

when $n \to \infty$.

The considerations for the snakelike disposition are the same as for the square helical disposition. The set of visible numbers for the variant of snakelike dispositions coincides with the set of visible numbers for snakelike disposition itself. ■

Let $\Theta(n)$ be a quantity of all natural numbers, which are visible numbers for any of three considered disposition simultaneously, and are not greater than a given number n. Because of considerations above, it would be interesting to find an exact answer to the following question.

Question 1. *What is an asymptotic proportion $\Theta(n)/\theta(n) \sim \Theta(n)/\psi(n) \sim \Theta(n)/\phi(n)$ up to equivalence, when $n \to \infty$?*

Remark 1. *Because of the statements above, this question is equivalent to the following one: what is an asymptotic proportion $\Theta(n)/\phi(n)$, when $n \to \infty$, when $n \to \infty$?*

We get immediately from Theorem 5 or directly from Proposition 8 the following corollary.

Corollary 5. *For any number $c < 1$ and sufficiently large n, one has the following inequality*

$$\min(\Theta(n)/\theta(n), \Theta(n)/\psi(n), \Theta(n)/\phi(n)) > c\frac{36}{\pi^4}.$$

It implies in turn that for any number $C > 1$ and sufficiently large n,

$$p(n)/\Theta(n) < C\frac{\pi^6}{216\log n}.$$

References

[1] Klein F. *Elementarmathematik vom hoheren standpunkte aus erster band.Arithmetik. Algebra. Analysis.* Springer-Verlag, Berlin, 1924.
[2] Berestovskii V.N., and Bumina E.A. *Integer lattices and prime numbers.*(Russian), Communications of Omsk University, 1998, No. 3, pp. 13-15.
[3] Vinogradov I.M. *Foundations of the theory of numbers,* Pergamon, New York, 1955.

Berestovskii Valerii Nikolaevich
Omsk Branch of Sobolev Institute of Mathematics
SO RAN
Pevtsova 13, Omsk, 644099, Russia
E-mail address: berestov@iitam.omsk.net.ru