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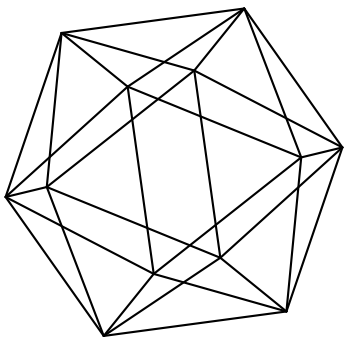

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CHARACTERISTIC-FREE BASES AND BGG RESOLUTIONS OF UNITARY SIMPLE MODULES FOR QUIVER HECKE AND CHEREDNIK ALGEBRAS

C. BOWMAN, E. NORTON, AND J. SIMENTAL

Abstract. We provide a homological construction of unitary simple modules of Cherednik and Hecke algebras of type A via BGG resolutions, solving a conjecture of Berkesch–Griffeth–Sam. We vastly generalize the conjecture and its solution to cyclotomic Cherednik and Hecke algebras over arbitrary ground fields, compute characteristic-free bases for this family of simple modules, and calculate the Betti numbers and Castelnuovo–Mumford regularity of certain symmetric linear subspace arrangements.

Introduction

In [BGG75], Bernstein–Gelfand–Gelfand utilise resolutions of simple modules by Verma modules to prove certain beautiful properties of finite-dimensional Lie algebras. Such resolutions (now known as BGG resolutions) have had spectacular applications in the study of the Laplacian on Euclidean space [Eas05], complex representation theory and homology of Kac–Moody algebras [GL76], statistical mechanics and conformal field theories [GJSV13, MS94, MW03], and they provide graded free resolutions (in the sense of commutative algebra) for determinantal varieties [Las78, EH04]. Remarkably, such resolutions have never been used in the study of symmetric and general linear groups in positive characteristic — or indeed anywhere in modular representation theory!

One of the most important problems in Lie theory is to classify and construct unitary simple representations. For Lie groups, this ongoing project draws on techniques from Dirac cohomology [HPV17], Kazhdan–Lusztig theory [Yee14], and the Langlands Program [Vog00], and has provided profound insights into relativistic quantum mechanics [Wig39]. The Cherednik algebra of a complex reflection group, W, is an important Lie theoretic object which possesses hallmarks from the classical theory: a triangular decomposition and a category $\mathcal{O}$ with a highest weight theory [GGOR03], analogues of translation functors [Los17], induction and restriction functors [BE09a] with associated Harish–Chandra series [LSA18], and Kazhdan–Lusztig theory [RSVV16] (for $W = G(\ell, 1, n)$). Both the unitary representations of real reductive groups [HC75, HC76a, HC76b] and those of Cherednik algebras [Che18] are of huge importance in algebraic harmonic analysis.

For Cherednik algebras of symmetric groups, $H_{1/e}(\mathfrak{S}_n)$, the simple unitary representations $L(\lambda)$ of $H_{1/e}(\mathfrak{S}_n)$ were classified by Etingof, Griffeth and Stoica [ES09] by a combinatorial condition on the partition λ of n labeling the "highest weight" of $L(\lambda)$. In the spirit of classical results in Lie theory, Berkesch, Griffeth, and Sam subsequently conjectured that any unitary simple $L(\lambda)$ admits a BGG resolution [ZGS14, Conjecture 4.5].

The primary purpose of this paper is to prove Berkesch–Griffeth–Sam's conjecture and thus *homologically construct* the unitary simple $H_{1/e}(\mathfrak{S}_n)$-modules:

Theorem A. *Associated to any simple unitary $H_{1/e}(\mathfrak{S}_n)$-module, $L(\lambda)$, we have a complex $\mathbf{C}_\bullet(\lambda) = \bigoplus_{\lambda \trianglerighteq \nu} \Delta(\nu)\langle \ell(\nu) \rangle$ with differential given by an alternating sum over all "one-column homomorphisms". This complex is exact except in degree zero, where $H_0(\mathbf{C}_\bullet(\lambda)) = L(\lambda)$.*

In contrast to classical papers on BGG resolutions and unitary representations, which usually (but not always! [BS11]) employ ideas from algebraic geometry, our methods are completely algebraic and moreover yield several geometric results. Namely, each standard module $\Delta(\nu)$ is a free $\mathbb{C}[x_1, \dots, x_n]$-module, and as a consequence we obtain $\mathfrak{S}_n$-equivariant, graded free resolutions (in the sense of commutative algebra) for the e-equals variety

$$X_{e,1,n} := \mathfrak{S}_n\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 = \cdots = z_e\},$$

and for the following algebraic varieties when $n = ke$:

$$X_{e,k,n} := \mathfrak{S}_n\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{ie+1} = \cdots = z_{(i+1)e} \text{ for } 0 \leqslant i < k\}.$$

We hence provide formulae for the graded Betti numbers and calculate the Castelnuovo–Mumford regularity of these varieties – a notoriously difficult problem in general [DS02, TT15]. Moreover, we also provide formulae for these invariants in the cyclotomic case, where the equalities in the equations defining the above varieties become equalities *up to multiplication by an ℓth root of unity.* Finally, we remark that the Cherednik algebra approach to geometric resolutions was inspired by the Lie theoretic construction of Lascoux's resolutions of determinantal varieties (via parabolic BGG resolutions of unitary modules) [EH04, ZGS14]; it would be interesting to find a purely geometric proof of the resolutions of our varieties by analogy with [Las78].

A key ingredient to our proofs is to work in the 2-categorical setting of *diagrammatic Cherednik algebras* of [Web17]. The diagrammatic calculus is easier for calculation and benefits from a graded structure. The diagrammatic approach allows us to generalize the original conjecture to higher levels and arbitrary ground fields; we prove this more general version. We recast the combinatorial condition in type A for $L(\lambda)$ to be unitary [ES09] as, the partition λ lies in the *fundamental alcove* of the dominant chamber in an affine type A alcove geometry. In our BGG resolution, $\Delta(\nu)$ appears in homological degree d if and only if ν is obtained from λ by reflecting across d walls (increasing the distance from the fundamental alcove by 1 at each step). This alcove model vastly generalizes to the set of all ℓ-partitions whose components *each have at most h columns*, $\mathscr{P}_n^\ell(h)$. For any multipartition lying in the fundamental alcove we then construct a BGG resolution of the corresponding simple $H_c(G(\ell,1,n))$-module. We remark that Griffeth has obtained a combinatorial description of the ℓ-partitions that label unitary irreducible modules for $H_c(G(\ell,1,n))$, [Gri], and it would be interesting to compare this condition to the one arising from the alcove model.

Working with quiver Hecke algebras furthermore allows us to obtain our results over an arbitrary field, $\Bbbk$. The search for an effective description of the dimensions of simple representations of symmetric groups over arbitrary fields is a centre of gravity for much research in modular Lie theory [RW16, LW18a, LW18b]. We show that our resolutions for unitary simples remain stable under reduction modulo p — in other words the beautiful properties of unitary modules extend beyond the confines of characteristic zero (a necessary condition for the definition of unitary modules via bilinear forms to make sense) to "p-unitary simples" for arbitrary fields. Finally, in Theorem 6.4 and Proposition 6.2 we obtain a simple closed form for the Mullineux involution, M, on unitary simples and *explicitly construct this isomorphism* — to our knowledge, this is the first time such an isomorphism has been explicitly constructed (outside of the semisimple case).

This pivots the impact of our result from Cherednik algebras and geometry of subspace arrangements, to modular representations of the symmetric group. As our main result is the first of its kind for symmetric groups we state it now in this simplified form. For the far more general statement concerning all cyclotomic quiver Hecke algebras, see Theorems 4.2 and 4.3.

Theorem B. *Let $\Bbbk$ be a field of characteristic $p>0$. For $D_n^{\Bbbk}(\lambda)$ a "p-unitary simple" we have an associated $\Bbbk\mathfrak{S}_n$-complex $C_\bullet(\lambda)=\bigoplus_{\lambda\trianglerighteq\nu} S_n(\nu)\langle\ell(\nu)\rangle$ with differential given by an alternating sum over all "one-column homomorphisms". This complex is exact except in degree zero, where*

$$H_0(C_\bullet(\lambda)) = D_n^{\Bbbk}(\lambda).$$

Moreover, the simple $\Bbbk\mathfrak{S}_n$-module $D_n^{\Bbbk}(\lambda)$ has basis $\{c_{\mathsf{s}} \mid \mathsf{s}\in \mathrm{Std}_p(\lambda)\}$ where $\mathrm{Std}_p(\lambda)\subseteq\mathrm{Std}(\lambda)$ is the set of p-restricted tableaux. The action on this basis is given in Theorem 4.3. We have that $D_n^{\Bbbk}(\lambda)\otimes\mathrm{sgn}\cong D_n^{\Bbbk}(\lambda_{\mathrm{M}})$ under the map $: c_{\mathsf{s}}\mapsto c_{\mathsf{s}_{\mathrm{M}}}$.

We thus provide the first instances of BGG resolutions anywhere in modular representation theory of finite and algebraic groups; in particular the first homological construction of a family of simple modules for symmetric groups. For Hecke algebras of type B, the simplest examples of our resolutions have appeared in work of mathematical physicists concerning Virasoro and blob algebras [GJSV13, MS94, MW03] and in Brundan–Stroppel's founding work on categorical representation theory [BS10, BS11]. The bases of our "p-unitary simples" first appeared in work of Kleshchev, Ram, and Ruff [Kle96a, Ram03, Ruf06], where they received the names of "completely splittable" and "calibrated" (when $\gcd(p,e)=1$ or $p=e$). We remark that our results for the Hecke algebras depend only on the quantum parameter $e\in\mathbb{N}$ and are entirely independent of the characteristic of the underlying field (for $\ell=1$ we set $e=p$ in Theorems 4.2 and 4.3 to obtain Theorem B for symmetric groups).

The partitions and multipartitions we consider (namely those lying in the fundamental alcove) have no restriction on their e-weight; calculating the composition series of the corresponding Specht modules for symmetric groups and Hecke algebras in positive characteristic is far beyond the current realms of conjecture (which at present have been stretched as far as $w(\lambda) < p^3$ for $h = 3$ by Lusztig–Williamson [LW18a]). Over $\mathbb{C}$ calculating the composition series of these Specht modules is theoretically possible using Kazhdan–Lusztig theory — however it quickly becomes computationally impossible — we provide examples of series of Specht modules (of rank n as $n \to \infty$) for which the length of the composition series tends to infinity. Thus Theorem B provide the *only* contexts in which we can hope to understand unitary simple modules (see Section 4).

1. Diagrammatic algebras

Fix a charge $(e; \kappa_0, \dots, \kappa_{\ell-1}) \in \mathbb{N} \times \mathbb{Z}^\ell$. We now recall the diagrammatic Cherednik and Hecke algebras of [Web17] and results concerning their representation theory from [BC18, BCS17, BS18].

We define a partition, λ, of n to be a finite weakly decreasing sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \dots$, equals n. An ℓ-partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of n is an ℓ-tuple of partitions such that $|\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = n$. We will denote the set of ℓ-partitions of n by $\mathscr{P}^\ell_n$. Given $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^\ell_n$, the Young diagram of λ is the set of nodes,

$$\{(r, c, m) \mid 1 \leqslant c \leqslant \lambda^{(m)}_r\}.$$

We refer to a node (r, c, m) as being in the rth row and cth column of the mth component of λ. Given a box, (r, c, m), we define the content of this box to be $\mathsf{ct}(r, c, m) = \kappa_m + c - r$ and we define its residue to be $\mathrm{res}(r, c, m) \equiv \mathsf{ct}(r, c, m) \pmod e$. We refer to a box of residue $i \in \mathbb{Z}/e\mathbb{Z}$ as an i-box. We refer to a node of residue $i \in I$ as an i-node. Given $\lambda \in \mathscr{P}^\ell_n$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable respectively addable) boxes of the Young diagram of λ so that the resulting diagram is the Young diagram of a ℓ-partition. Given $i \in I$, we let $\mathrm{Rem}_i(\lambda) \subseteq \mathrm{Rem}(\lambda)$ and $\mathrm{Add}_i(\lambda) \subseteq \mathrm{Add}(\lambda)$ denote the subsets of boxes of residue $i \in I$.

Each charge gives rise to a different ordering on $\mathscr{P}^\ell_n$, and hence a different Fock space, a different (diagrammatic) Cherednik algebra, and a different lens through which to study the quiver Hecke algebra [Bow]. These charged ordering on $\mathscr{P}^\ell_n$ are given as follows:

Definition 1.1. *Given $\kappa \in \mathbb{Z}^\ell$ such that $0 \leqslant \kappa_i - \kappa_{i+1} < e$ we write $(r, c, m) \lhd_\kappa (r', c', m')$ if $\mathrm{res}(r, c, m) = \mathrm{res}(r', c', m')$ and either*

(i) $\mathsf{ct}(r, c, m) < \mathsf{ct}(r', c', m')$ or
(ii) $\mathsf{ct}(r, c, m) = \mathsf{ct}(r', c', m')$ and $m > m'$

For $\lambda, \mu \in \mathscr{P}^\ell_n$, we write $\mu \leqslant_\kappa \lambda$ if there is a bijective map $A : [\lambda] \to [\mu]$ such that either $A(r, c, m) \lhd_\kappa (r, c, m)$ or $A(r, c, m) = (r, c, m)$ for all $(r, c, m) \in \lambda$.

Definition 1.2. *Given $\lambda \in \mathscr{P}^\ell_n$, we define a* tableau *of shape λ to be a filling of the boxes of the Russian array of λ with the numbers $\{1, \dots, n\}$. We define a* standard tableau *to be a tableau in which the entries increase along the rows and columns of each component. We let $\mathrm{Std}(\lambda)$ denote the set of all standard tableaux of shape $\lambda \in \mathscr{P}^\ell_n$.*

Definition 1.3. *Given $\lambda \in \mathscr{P}^\ell_n$ we let $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda)$ be the tableau obtained by placing the entry n in the least dominant removable box $(r, c, m) \in \lambda$ and then placing the entry $n-1$ in the least dominant removable box of $\lambda \setminus \{(r, c, m)\}$ and continuing in this fashion.*

Definition 1.4. *For $h \geqslant 0$ we say that $\kappa \in \mathbb{Z}^\ell$ is h-*admissible *if $h < \kappa_i - \kappa_j < e$ for $0 \leqslant i < j < \ell$.*

Remark 1.5. *For $h \geqslant 0$ and $\kappa \in \mathbb{Z}^\ell$ h-admissible the dominance ordering in Definition 1.1 is a coarsening of the usual c-function ordering on the Fock spaces of Foda–Leclerc–Okado–Thibon–Welsh [Web17, Bow]. This is the only ordering for which we have a closed form for a labelling of the simple modules for the quiver Hecke algebra [FLO+99] (in other words, a labelling of the component of the $\widehat{\mathfrak{sl}}_e$ crystal containing the empty ℓ-partition). Notice that this combinatorial ordering and the algebraic structures it underlies are completely different from the classical dominance order and the (graded) cyclotomic q-Schur algebra [DJM98, HM10, SW11]; see [Bow, Web17] for more details.*

Given $h \in \mathbb{N}$, we let $\mathscr{P}_n^\ell(h)$ denote the subset of $\mathscr{P}_n^\ell$ consisting of those ℓ-partitions which have at most h columns in each component, that is

$$\mathscr{P}_n^\ell(h) = \{\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \mid \lambda_1^{(m)} \leqslant h \text{ for } 1 \leqslant m \leqslant \ell\}.$$

Given $\lambda \in \mathscr{P}_n^\ell$, we define its residue sequence, $\mathrm{res}(\lambda)$ to be the sequence obtained by recording the residues of the boxes of λ according to the dominance ordering on boxes.

Definition 1.6. *Let $\epsilon \ll \frac{1}{n\ell}$. To $(r,c,m) \in \lambda$ we associate $\mathbf{I}^\kappa_{(r,c,m)} = \mathsf{ct}(r,c,m) - m/\ell - (r+c)\epsilon$. Given $\lambda \in \mathscr{P}_n^\ell$, we let $\mathbf{I}^\kappa_\lambda$ denote the disjoint union over the $\mathbf{I}^\kappa_{(r,c,m)}$ for $(r,c,m) \in \lambda$. We define a κ-diagram of type $G(\ell,1,n)$ to be a frame $\mathbb{R} \times [0,1]$ with distinguished solid points on the northern and southern boundaries given by $\mathbf{I}^\kappa_\mu$ and $\mathbf{I}^\kappa_\lambda$ for some $\lambda, \mu \in \mathscr{P}_n^\ell$ and a collection of solid strands each of which starts at a northern point and ends at a southern point. Each solid strand carries a residue, $i \in \mathbb{Z}/e\mathbb{Z}$, say (and we refer to this as a solid i-strand). We further require that each solid strand has a mapping diffeomorphically to $[0,1]$ via the projection to the y-axis. Each solid strand is allowed to carry any number of dots. We draw*

- *(i) a "ghost i-strand" 1 unit to the right of each solid i-strand and a a "ghost dot" 1 unit to the right of each solid dot;*
- *(ii) vertical red lines with x-coordinate $\kappa_m - m/\ell \in \mathbb{Q}$ each of which carries the residue κ_m in $\mathbb{Z}/e\mathbb{Z}$ for $1 \leqslant m \leqslant \ell$ which we call a red κ_m-strand.*

Finally, we require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings.

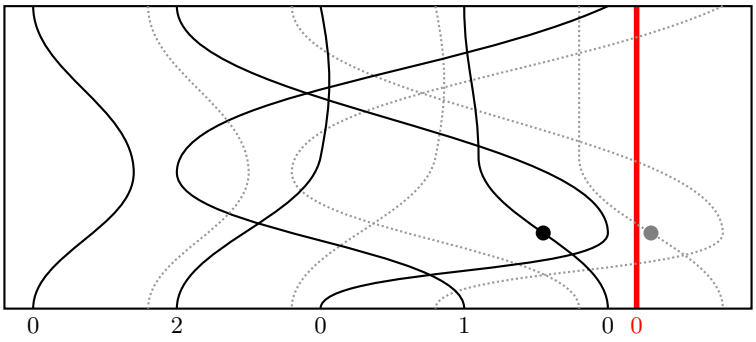
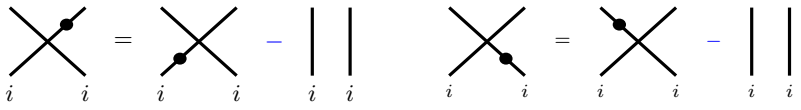


FIGURE 1. A κ-diagram for $\ell = 1$ and $\kappa \in \mathbb{Z}$ with northern and southern loading $\mathbf{I}^\kappa_\omega$ for $\omega = (1^5)$.

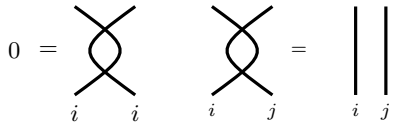
Definition 1.7 (Definition 4.1 [Web17])**.** *We let $\mathbf{A}_n(\kappa)$ denote the $\Bbbk$-algebra spanned by all κ-diagrams modulo the following local relations (here a local relation means one that can be applied on a small region of the diagram). The product $d_1 d_2$ of two diagrams $d_1, d_2 \in A_n(\kappa)$ is given by putting d_1 on top of d_2. This product is defined to be 0 unless the southern border of d_1 matches the northern border of d_2, in which case we obtain a new diagram in the obvious fashion.*

- *(D1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which avoids tangencies, double points and dots on crossings.*
- *(D2) Any solid dot can pass through a crossing of solid or ghost i- and j-strands for $i \neq j$. Namely:*

- *(D3) We can pass a solid dot through a crossing of two like-labelled solid or ghost strands at the expense of an error term:*



(D4) For double-crossings of solid strands with $i \neq j$, we have the following local relation:

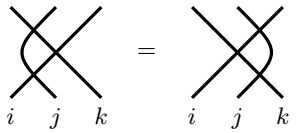


Performing relation (D4) implicitly involves undoing the a corresponding double-crossing of ghost strands at the same time (which we do not picture) and vice versa.

(D5) If $j \neq i-1$, then we can freely pass ghosts through solid strands. We have the following local relations:

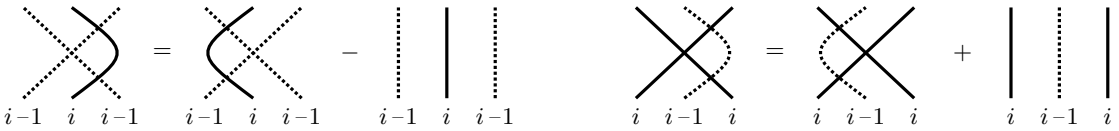
(D6) On the other hand, in the case where $j = i-1$, we have the following local relations:

(D7) Solid strands can move through crossings of solid strands freely. In other words, solid strands satisfy the naive braid relation:



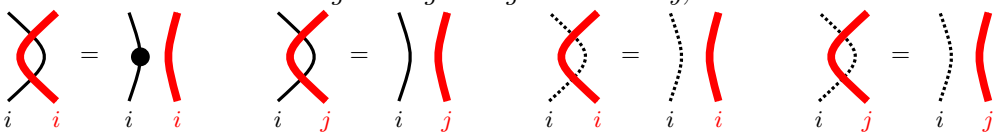
for any $i, j, k \in I$. Performing relation (D7) implicitly involves resolving a braid of three ghost strands at the same time (which we do not picture) and vice versa.

(D8) We can pull a solid i-strand through a ghost-crossing at the expense of an error term.



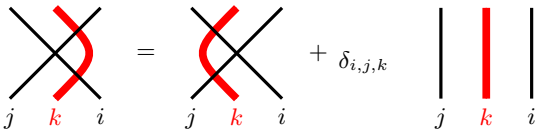
Ghost strands and ghost dots may pass through red strands freely. For $i \neq j$, the solid i-strands may pass through red j-strands freely. If the red and solid strands have the same label, a dot is added to the solid strand when straightening. Diagrammatically, we have that

(D9)



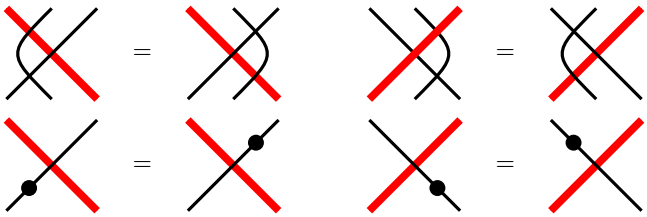
for $i \neq j$ and their mirror images. All solid crossings and dots can pass through red strands, with a correction term.

(D10)



(D11)

(D12)



(D13) Any idempotent in which a solid strand is n units to the rightof the rightmost red-strand is referred to as unsteady and set to be equal to zero.

Given $\lambda \in \mathscr{P}_n^\ell$ and $\underline{i} \in I^n$, we have an associated idempotent $1_\lambda^{\underline{i}}$ given by the diagram with northern and southern points $\mathbf{I}_\lambda^\kappa$, no crossing strands, and northern/southern residue sequence of the diagram given by $\underline{i} \in I^n$. If the residue sequence is equal to that of the partition, $\mathrm{res}(\lambda)$, then we let $1_\lambda := 1_\lambda^{\mathrm{res}(\lambda)}$. We define the diagrammatic Cherednik algebra, $A_n(\kappa)$ to be the algebra

$$A_n(\kappa) := \mathsf{E}^+ \mathbf{A}_n(\kappa) \mathsf{E}^+ \quad \text{where} \quad \mathsf{E}^+ = \textstyle\sum_{\lambda \in \mathscr{P}_n^\ell} 1_\lambda.$$

Definition 1.8. *Let $\lambda, \mu \in \mathscr{P}_n^\ell$. A λ-tableau of weight μ is a bijective map $\mathsf{T} : \lambda \to \mathbf{I}_\mu^\kappa$. We let $\mathcal{T}(\lambda, \mu)$ denote the set of all tableaux of shape λ and weight μ. We say that a tableau T is* semistandard *if it satisfies the following additional properties*

(*i*) $\mathsf{T}(1,1,m) < \kappa_m$,
(*ii*) $\mathsf{T}(r,c,m) < \mathsf{T}(r-1,c,m) - 1$,
(*iii*) $\mathsf{T}(r,c,m) < \mathsf{T}(r,c-1,m) + 1$.

We denote the set of all semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}(\lambda,\mu)$. Given $\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$, we write $\mathrm{Shape}(\mathsf{T}) = \lambda$. We let $\mathrm{SStd}_n^+(\lambda,\mu) \subseteq \mathrm{SStd}_n(\lambda,\mu)$ denote the subset of tableaux which respect residues. In other words, if $\mathsf{T}(r,c,m) = (r',c',m')$ for $(r,c,m) \in \lambda$ and $(r',c',m') \in \mu$, then $\kappa_m + c - r = \kappa_{m'} + c' - r' \pmod e$.

Definition 1.9. *Given S a tableau of shape λ and weight μ, we let C_S denote any diagram tracing out the bijection $\mathsf{S} : [\lambda] \to \mathbf{I}_\mu^\kappa$ using the minimal number of crossings. Given S, T a pair of tableau of shape λ (and possibly distinct weights) we set $C_{\mathsf{ST}} = C_\mathsf{S} C_\mathsf{T}^*$ where C_T^* is the diagram obtained from C_T by flipping it through the horizontal axis.*

Theorem 1.10. *The R-algebra $\mathbf{A}_n(\kappa)$ is a quasi-hereditary graded cellular algebra with basis*

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\mu), \mathsf{T} \in \mathrm{SStd}(\lambda,\nu), \lambda,\mu,\nu \in \mathscr{P}_n^\ell\}$$

and the subalgebra $A_n(\kappa)$ is a quasi-hereditary graded cellular algebra with basis

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}^+(\lambda,\mu), \mathsf{T} \in \mathrm{SStd}^+(\lambda,\nu), \lambda,\mu,\nu \in \mathscr{P}_n^\ell\}.$$

For both algebras, the involution is given by $*$ and the ordering on $\mathscr{P}_n^\ell(h)$ is that of Definition 1.1. We denote the corresponding left cell-modules for $\mathbf{A}_n(\kappa)$ and $A_n(\kappa)$ by $\mathbf{\Delta}(\lambda)$ and $\Delta(\lambda)$ respectively. The modules $\mathbf{\Delta}(\lambda)$ and $\Delta(\lambda)$ have simple heads, denoted by $\mathbf{L}(\lambda)$ and $L(\lambda)$ respectively.

The standard tableaux $\mathrm{Std}(\lambda)$ form the predictable subset of semistandard tableaux of weight $\omega = (1^n)$ as follows. Let $\lambda \in \mathscr{P}_n^\ell$. If $\mathsf{s} \in \mathrm{Std}(\lambda)$ is such that $\mathsf{s}(r_k, c_k, m_k) = k$ for $1 \leqslant k \leqslant n$, then we let $\mathsf{S} \in \mathcal{T}(\lambda,\omega)$ denote the tableau $\mathsf{S} : \lambda \to \omega$ determined by $\mathsf{S}(r_k,c_k,m_k) = \mathbf{I}_{(k,1,\ell)}^\kappa$. We have a bijective map $\varphi : \mathrm{Std}(\lambda) \to \mathcal{T}(\lambda,\omega)$. given by $\varphi(\mathsf{s}) = \mathsf{S}$.

Definition 1.11. *We define the* Schur idempotent, *E_ω, and quiver Hecke algebra, $R_n(\kappa)$, as follows*

$$\mathsf{E}_\omega = \sum_{\underline{i} \in I^n} 1_\omega^{\underline{i}} \quad \text{and} \quad R_n(\kappa) := \mathsf{E}_\omega \mathbf{A}_n(\kappa) \mathsf{E}_\omega.$$

Theorem 1.12 ([Bow])**.** *The algebra $R_n(\kappa)$ admits a graded cellular structure with respect to the poset $(\mathscr{P}_n^\ell, \rhd)$, the basis*

$$\{c_{\mathsf{st}} := C_{\varphi(\mathsf{s}),\varphi(\mathsf{t})} \mid \lambda \in \mathscr{P}_n^\ell, \mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda)\},$$

and the involution $*$. We denote the left cell-module by $S_n(\lambda) = \{c_\mathsf{s} \mid \mathsf{s} \in \mathrm{Std}(\lambda)\}$

When proving results on homomorphisms, the algebra $A_n(\kappa)$ is smaller than $\mathbf{A}_n(\kappa)$ and much easier for computation. We shall then (trivially) inflate these results to $\mathbf{A}_n(\kappa)$ and apply the Schur functor to obtain the corresponding result for $R_n(\kappa)$.

We have an embedding $R_n(\kappa) \hookrightarrow R_{n+1}(\kappa)$ and $R_{n+1}(\kappa)$ is free as a $R_n(\kappa)$-module. We let

$$\mathrm{res}_n^{n+1} : R_{n+1}(\kappa)\text{-mod} \to R_n(\kappa)\text{-mod} \qquad \mathrm{ind}_n^{n+1} : R_n(\kappa)\text{-mod} \to R_{n+1}(\kappa)\text{-mod}$$

denote the obvious restriction and induction functors. We post-compose these functors with the projection onto a block in the standard fashion. This amounts to multiplying by an idempotent

$$\mathsf{E}_r = \sum_{\underline{i} = (i_1,\ldots,i_{n-1},r)} 1_\omega^{\underline{i}}$$

for $r \in I$. We hence decompose these restriction/induction functors into

$$r\text{-res}_n^{n+1} = \mathsf{E}_r \circ \text{res}_n^{n+1} \quad r\text{-ind}_{n-1}^n = \mathsf{E}_r \circ \text{ind}_{n-1}^n$$

Finally we recall a simple case of [Bow, Theorem 6.1]. If λ has precisely one removable r-box, $\Box \in \text{Rem}_r(\lambda)$, then we set $\mathsf{E}_r(\lambda) = \lambda - \Box$ and we have that $r\text{-res}_{n-1}^n(S_n(\lambda)) = S_{n-1}(\lambda - \Box)$.

Remark 1.13. *The algebra $\mathbf{A}_n(\kappa)$ admits a $\mathbb{Z}$-grading* [Web17, Defintion 4.2]. *We do not recall this explicitly here, but in Section 2 we shall encode this grading using the path combinatorics of* [BC18]. *We let t be an indeterminate over $\mathbb{Z}_{\geqslant 0}$. If $M = \oplus_{k\in\mathbb{Z}} M_k$ is a graded $\Bbbk$-module, we write* $\dim_t(M) = \sum_{k\in\mathbb{Z}}(\dim_{\Bbbk}(M_k))t^k$.

Finally, we briefly recall from [BC18, Theorem 4.9] that for the above three algebras, there are graded Morita equivalences relating the subcategories of $\mathbf{A}_n(\kappa)$-mod, $A_n(\kappa)$-mod, and $R_n(\kappa)$-mod whose simple constituents are labelled by $\mathscr{P}_n^\ell(h)$. In particular

$$\text{Hom}_{A_n(\kappa)}(\Delta(\mu), \Delta(\lambda)) \cong \text{Hom}_{\boldsymbol{A}_n(\kappa)}(\boldsymbol{\Delta}(\mu), \boldsymbol{\Delta}(\lambda)) \cong \text{Hom}_{R_n(\kappa)}(S_n(\mu), S_n(\lambda)) \tag{1.1}$$

for $\lambda, \mu \in \mathscr{P}_n^\ell(h)$. This allows us to focus on the algebra $A_n(\kappa)$ where we have the benefit of a highest weight theory which is intimately related to the underlying alcove geometry. Both isomorphisms are simply given by idempotent truncation from $\mathbf{A}_n(\kappa)$.

2. Alcove geometries and bases of diagrammatic algebras

In Subsection 2.1, we recall the alcove geometry controlling the subcategories of representations for quiver Hecke and Cherednik algebras of interest in this paper. In Subsection 2.2 we cast the semistandard tableaux for diagrammatic Cherednik algebras in this geometry; this path-combinatorial framework will be essential for our proofs. In Subsection 2.3 we cast the standard tableaux combinatorics of the quiver Hecke algebra in this geometry — this allows us to define the e-restricted tableaux which we will prove provide bases of simple $R_n(\kappa)$-modules in Section 4.

2.1. **The alcove geometry.** Fix integers $h, \ell \in \mathbb{Z}_{>0}$ and $e \geqslant h\ell$. For each $1 \leqslant i \leqslant h$ and $0 \leqslant m < \ell$ we let ε_{hm+i} denote a formal symbol, and define an ℓh-dimensional real vector space,

$$\mathbb{E}_{h,\ell} = \bigoplus_{\substack{1\leqslant i\leqslant h\\ 0\leqslant m<\ell}} \mathbb{R}\varepsilon_{hm+i}$$

We have an inner product $\langle\ ,\ \rangle$ given by extending linearly the relations

$$\langle \varepsilon_{hm+i}, \varepsilon_{ht+j}\rangle = \delta_{i,j}\delta_{t,m}$$

for all $1 \leqslant i, j \leqslant h$ and $0 \leqslant m, t < \ell$, where $\delta_{i,j}$ is the Kronecker delta. We let Φ and Φ_0 denote the root systems of type $A_{\ell h-1}$ and $A_{h-1} \times A_{h-1} \times \ldots A_{h-1}$ respectively which consist of the roots

$$\{\varepsilon_{hm+i} - \varepsilon_{ht+j} \mid 1 \leqslant i, j \leqslant h \text{ and } 0 \leqslant m, t < \ell \text{ with } (i, m) \neq (j, t)\} \text{ and}$$

$$\{\varepsilon_{hm+i} - \varepsilon_{hm+j} \mid 1 \leqslant i, j \leqslant h \text{ with } i \neq j \text{ and } 0 \leqslant m < \ell\}$$

respectively. We identify $\lambda \in \mathscr{P}_n^\ell(h)$ with a point in $\mathbb{E}_{h\ell}$ via the transpose map

$$(\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \mapsto \sum_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant h}} (\lambda^{(m)})_i^T \varepsilon_{h(m-1)+i}, \tag{2.1}$$

(where the T denotes the transpose partition). Given $r \in \mathbb{Z}$ and $\alpha \in \Phi$ we let $s_{\alpha,re}$ denote the reflection which acts on $\mathbb{E}_{h,\ell}$ by

$$s_{\alpha,re}x = x - (\langle x, \alpha\rangle - re)\alpha$$

and we let W^e and W_0^e denote the groups generated by the reflections

$$\mathcal{S} = \{s_{\alpha,re} \mid \alpha \in \Phi, r \in \mathbb{Z}\} \qquad \mathcal{S}_0 = \{s_{\alpha,0} \mid \alpha \in \Phi_0\}$$

respectively. For $e \in \mathbb{Z}_{>0}$ we assume that $\kappa \in I^\ell$ is h-admissible. We shall consider a shifted action of the Weyl group W^e on $\mathbb{E}_{hl}$ by the element

$$\rho := (\rho_1, \rho_2, \ldots, \rho_\ell) \in I^{h\ell},\ \rho_i := (e - \kappa_i, e - \kappa_i - 1, \ldots, e - \kappa_i - h + 1) \in I^h,$$

that is, given an an element $w \in W^e$, we set $w \cdot_\rho x = w(x+\rho) - \rho$. We let $\mathbb{E}(\alpha, re)$ denote the affine hyperplane consisting of the points

$$\mathbb{E}(\alpha, re) = \{x \in \mathbb{E}_{h,\ell} \mid s_{\alpha,re} \cdot x = x\}.$$

Note that our assumption that $\kappa \in I^\ell$ is h-admissible implies that the origin does not lie on any hyperplane. Given a hyperplane $\mathbb{E}(\alpha, re)$ we remove the hyperplane from $\mathbb{E}_{h,\ell}$ to obtain two distinct subsets $\mathbb{E}^>(\alpha, re)$ and $\mathbb{E}^<(\alpha, re)$ where the origin $\circledcirc \in \mathbb{E}^<(\alpha, re)$. The dominant Weyl chamber, denoted $\mathbb{E}^\circledcirc_{h,\ell}$, is set to be

$$\mathbb{E}^\circledcirc_{h,\ell} = \bigcap_{\alpha \in \Phi_0} \mathbb{E}^<(\alpha, 0).$$

Definition 2.1. *Let $\lambda \in \mathbb{E}_{h,\ell}$. There are only finitely many hyperplanes lying between the point $\lambda \in \mathbb{E}_{h,\ell}$ and the point $\nu \in \mathbb{E}_{h,\ell}$. For a root $\varepsilon_i - \varepsilon_j \in \Phi$, we let $\ell_{\varepsilon_i - \varepsilon_j}(\lambda, \nu)$ denote the total number of these hyperplanes which are perpendicular to $\varepsilon_i - \varepsilon_j \in \Phi$ (including any hyperplanes upon which λ or ν lies). We let $\ell(\lambda, \nu) = \sum_{1 \leqslant i < j \leqslant h\ell} \ell_{\varepsilon_i - \varepsilon_j}(\lambda, \nu)$. We let $\ell(\lambda) := \ell(\lambda, \circledcirc)$ for $\circledcirc$ the origin and refer to $\ell(\lambda)$ simply as the* length *of the point $\lambda \in \mathbb{E}_{h,\ell}$.*

Definition 2.2. *Given $n \in \mathbb{N}$, we define the* fundamental alcove *to be*

$$\mathcal{F}^\ell_n(h) = \{\lambda \in \mathscr{P}^\ell_n \mid \ell(\lambda) = 0\}.$$

The name "fundamental alcove" is clearly inspired by classical Lie theory. We stress that Specht modules from the fundamental alcove can become arbitrarily complicated and that understanding the decomposition numbers $d_{\lambda\mu} = [S_n(\lambda) : D_n(\mu)]$ for $\lambda \in \mathcal{F}^\ell_n(h)$ and $\mu \lhd \lambda$ is, in general, a hopelessly difficult task (see Remark 4.4 and Example 4.5). This might surprise classical Lie theorists, but this is because we are working in the Ringel dual setting, see [BC18] for more details.

Definition 2.3. *Given a map $s : \{1, \dots, n\} \to \{1, \dots, \ell h\}$ we define points $\mathsf{S}(k) \in \mathbb{E}_{h,\ell}$ by $\mathsf{S}(k) = \sum_{1 \leqslant i \leqslant k} \varepsilon_{s(i)}$ for $1 \leqslant k \leqslant n$. We define the associated path of length n in $\mathbb{E}_{h,\ell}$ by*

$$\mathsf{S} = (\mathsf{S}(0), \mathsf{S}(1), \mathsf{S}(2), \dots, \mathsf{S}(n)),$$

where we fix all paths to begin at the origin, so that $\mathsf{S}(0) = \circledcirc \in \mathbb{E}_{h,\ell}$. We let $\mathsf{S}_{\leqslant k}$ denote the subpath of S of length k corresponding to the restriction of the map s to the domain $\{1, \dots, k\} \subseteq \{1, \dots, n\}$. We let Shape$(\mathsf{S})$ *denote the point in $\mathbb{E}_{h,\ell}$ at which S terminates.*

Remark 2.4. *Let S be a path which passes through a hyperplane $\mathbb{E}_{\alpha,re}$ at point $\mathsf{S}(k)$ (note that k is not necessarily unique). Let T be the path obtained from S by applying the reflection $s_{\alpha,re}$ to all the steps in S after the point $\mathsf{S}(k)$. In other words, $\mathsf{T}(i) = \mathsf{S}(i)$ for all $1 \leqslant i \leqslant k$ and $\mathsf{T}(i) = s_{\alpha,re} \cdot \mathsf{S}(i)$ for $k \leqslant i \leqslant n$. We refer to the path T as the reflection of S in $\mathbb{E}_{\alpha,re}$ at point $\mathsf{S}(k)$ and denote this by $s^k_{\alpha,re} \cdot \mathsf{S}$. We write $\mathsf{S} \sim \mathsf{T}$ if the path T can be obtained from S by a series of such reflections.*

Definition 2.5. *Let T denote a fixed path from $\circledcirc$ to $\nu \in \mathscr{P}^\ell_n(h)$. We let* $\mathrm{Path}_n(\lambda, \mathsf{T})$ *denote the set of paths from the origin to λ obtainable by applying repeated reflections to T, in other words*

$$\mathrm{Path}_n(\lambda, \mathsf{T}) = \{\mathsf{S} \mid \mathsf{S}(n) = \lambda, \mathsf{S} \sim \mathsf{T}\}.$$

We let $\mathrm{Path}^+_n(\lambda, \mathsf{T}) \subseteq \mathrm{Path}_n(\lambda, \mathsf{T})$ denote the set of paths which at no point leave the dominant Weyl chamber, in other words

$$\mathrm{Path}^+_n(\lambda, \mathsf{T}) = \{\mathsf{S} \in \mathrm{Path}_n(\lambda, \mathsf{T}) \mid \mathsf{S}(k) \in \mathbb{E}^\circledcirc_{h,\ell} \text{ for all } 1 \leqslant k \leqslant n\}.$$

Definition 2.6. *Given a path $\mathsf{S} = (\mathsf{S}(0), \mathsf{S}(1), \mathsf{S}(2), \dots, \mathsf{S}(n))$, we define $d(\mathsf{S}(k), \mathsf{S}(k-1))$ as follows. For $\alpha \in \Phi$ we set $d_\alpha(\mathsf{S}(k), \mathsf{S}(k-1))$ to be*

- $+1$ *if $\mathsf{S}(k-1) \in \mathbb{E}(\alpha, re)$ and $\mathsf{S}(k) \in \mathbb{E}^<(\alpha, re)$;*
- -1 *if $\mathsf{S}(k-1) \in \mathbb{E}^>(\alpha, re)$ and $\mathsf{S}(k) \in \mathbb{E}(\alpha, re)$;*
- 0 *otherwise.*

We set $\deg(\mathsf{S}(0)) = 0$ and define

$$d(\mathsf{S}(k-1), \mathsf{S}(k)) = \sum_{\alpha \in \Phi} d_\alpha(\mathsf{S}(k-1), \mathsf{S}(k)) \quad \text{and} \quad \deg(\mathsf{S}) = \sum_{1 \leqslant k \leqslant n} d(\mathsf{S}(k), \mathsf{S}(k-1)).$$

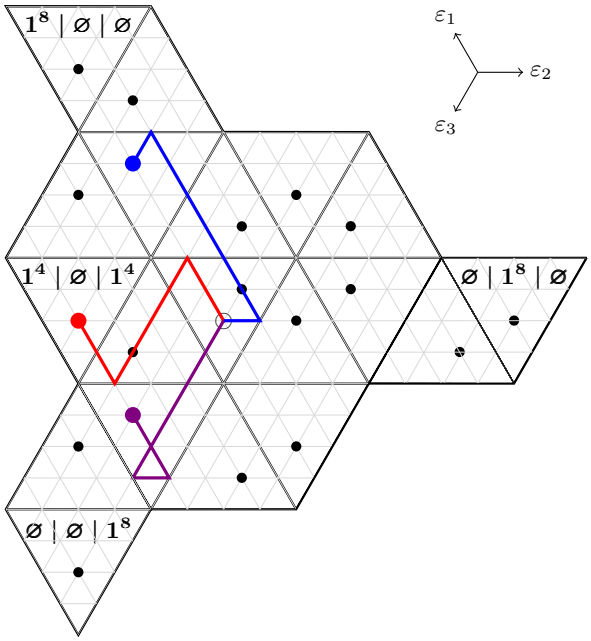


FIGURE 2. The black points label the 3-partitions of a block of $R_8(0,1,2)$ with $e = 4$. The origin is labelled as $\odot$. There are three separate paths drawn on the diagram belonging to $\mathrm{Path}^+((1^4 \mid \varnothing \mid 1^4), \mathsf{T}^{(1^8|\varnothing|\varnothing)})$, $\mathrm{Path}^+((1^6 \mid 1 \mid 1), \mathsf{T}^{(\varnothing|1^8|\varnothing)})$, and $\mathrm{Path}^+((1^2 \mid 1 \mid 1^5), \mathsf{T}^{(\varnothing|\varnothing|1^8)})$. These are coloured red, blue, and violet respectively. We have labelled some of the points in the diagram for reference.

2.2. **Semistandard tableaux as paths.** Let $e > h\ell$. We now provide path-theoretic bases for the diagrammatic Cherednik algebra. Let $\mu \in \mathscr{P}^\ell_n(h)$. We define the component word of μ to be the series of ℓ-partitions

$$\varnothing = \mu^{(0)} \xrightarrow{+X_1} \mu^{(1)} \xrightarrow{+X_2} \mu^{(2)} \xrightarrow{+X_3} \cdots \xrightarrow{+X_{n-1}} \mu^{(n-1)} \xrightarrow{+X_n} \mu^{(n)} = \mu$$

where $X_k = (r_k, c_k, m_k)$ is the least dominant removable node of the partition $\mu^{(k)} \in \mathscr{P}^\ell_k(h)$. Using the component word of μ, we define a distinguished path T^μ from the origin to μ as follows

$$\mathsf{T}^\mu = (+\varepsilon_{X_1}, +\varepsilon_{X_2}, \ldots, +\varepsilon_{X_n}).$$

For $\lambda \in \mathscr{P}^\ell_n(h)$, we let

$$\mathsf{S} = (+\varepsilon_{Y_1}, +\varepsilon_{Y_2}, \ldots, +\varepsilon_{Y_n}) \in \mathrm{Path}(\lambda, \mathsf{T}^\mu).$$

From S, we obtain a tableau $\overline{\mathsf{S}} \in \mathcal{T}(\lambda, \mu)$ by setting $\overline{\mathsf{S}}(X_k) = \mathbf{I}_{Y_k}$. We freely identify paths and tableaux in this manner (and so we drop the overline). Under this identification, we obtain a bijection $\mathrm{SStd}^+(\lambda, \mu) \leftrightarrow \mathrm{Path}^+(\lambda, \mathsf{T}^\mu)$ and hence we can rewrite the basis of Theorem 1.10 in terms of paths (see [BC18, Theorem 5.21]) as follows. For $\lambda \in \mathscr{P}^\ell_n(h)$ we have that

$$\Delta(\lambda) = \{C_\mathsf{S} \mid \mathsf{S} \in \mathrm{Path}^+(\lambda, \mathsf{T}^\mu), \mu \in \mathscr{P}^\ell_n(h)\}. \tag{2.2}$$

Definition 2.7. *Let $\lambda, \mu \in \mathscr{P}^\ell_n(h)$ and suppose that $\lambda \rhd \mu$. Then we let $\mathsf{T}^\mu_\lambda \in \mathrm{Path}(\lambda, \mathsf{T}^\mu)$ denote the unique path satisfying*

$$\deg(\mathsf{T}^\mu_\lambda) = \ell(\mu) - \ell(\lambda).$$

The above definition is well-defined by [BC18, Proposition 7.4] and these paths will be very useful later on. Examples of this path/tableau for three distinct pairs (λ, μ) are given in Figure 2.

Remark 2.8. *If $e = h\ell$, then all the results of this paper go through unchanged modulo minor edits to the proofs. Annoyingly, the definition of the degree and reflections of paths require some tinkering (akin to the case $e = \infty$ case covered in detail in* [BC18, Section 6.4]*). In what follows, we only discuss the case $e > h\ell$ explicitly. For Cherednik algebras of symmetric groups, we provide an explicit and independent proof of our main result in quantum characteristic $e = h$ in Subsection 7.2.*

2.3. **Standard tableaux as paths.** Given $\lambda \in \mathscr{P}_n^\ell(h)$, a tableau $\mathsf{t} \in \mathrm{Std}(\lambda)$ is easily identified with the series of partitions $\mathsf{t}(k)$ for $0 \leqslant k \leqslant n$, which in turn determine a path in $\mathbb{E}_{h,\ell}^+$ via the map in equation (2.1). This provides path-theoretic bases of Specht modules $S_n(\lambda)$ for $\lambda \in \mathscr{P}_n^\ell(h)$. We now restrict our attention to $\lambda \in \mathcal{F}_n^\ell(h) \subseteq \mathscr{P}_n^\ell(h)$ and define the subset of e-restricted standard λ-tableaux which will index the basis of the simple module $D_n(\lambda)$ for $\lambda \in \mathcal{F}_n^\ell(h)$.

Definition 2.9. *Given $\lambda \in \mathcal{F}_n^\ell(h)$, we say that $\mathsf{s} \in \mathrm{Std}(\lambda)$ is e-restricted if $\mathsf{s}(k) \in \mathcal{F}_k^\ell(h)$ for all $1 \leqslant k \leqslant n$. We let $\mathrm{Std}_e(\lambda)$ denote the set of all e-restricted tableaux of shape λ.*

Given $\lambda \in \mathcal{F}_n^\ell(h)$, we say that a node $\square \in \mathrm{Rem}(\lambda)$ is good if $\lambda - \square \in \mathcal{F}_{n-1}^\ell(h)$ (we remark that this is easily seen to coincide with the classical definition of a good node). We let $\mathcal{F}_h(\lambda)$ denote the set of all good removable nodes of λ. The following result is obvious, but will be essential for the proof of our main theorem.

Proposition 2.10. *Given $\lambda \in \mathcal{F}_n^\ell(h)$, we have that $\langle c_\mathsf{s}, c_\mathsf{t} \rangle = \delta_{\mathsf{s},\mathsf{t}}$ for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_e(\lambda)$. Furthermore,*

$$\Bbbk\{c_\mathsf{s} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\} \subseteq D_n(\lambda) \quad \text{and} \quad \mathrm{Std}_e(\lambda) \leftrightarrow \bigsqcup_{\square \in \mathcal{F}_h(\lambda)} \mathrm{Std}_e(\lambda - \square).$$

Proof. For $\lambda \in \mathcal{F}_n^\ell(h)$, we have that $\mathsf{s} \in \mathrm{Std}_e(\lambda)$ if and only if $\mathsf{s}{\downarrow_{n-1}} \in \mathrm{Std}_r(\nu)$ for some $\nu \in \mathcal{F}_{n-1}^\ell(h)$; the bijection follows. To see that $\{c_\mathsf{s} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\} \subseteq D_n^\Bbbk(\lambda)$ and that $\langle c_\mathsf{s}, c_\mathsf{t} \rangle = \delta_{\mathsf{s},\mathsf{t}}$ for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_e(\lambda)$, it is enough to show that

$$1_\omega^{\mathrm{res}(\mathsf{s})} S_n(\nu) \neq 0 \text{ implies } \nu \rhd \lambda \text{ or } \nu = \lambda \text{ and } 1_\omega^{\mathrm{res}(\mathsf{s})} S_n(\lambda) = c_\mathsf{s} \tag{2.3}$$

for $\mathsf{s} \in \mathrm{Std}_e(\lambda)$. To see this, assume that c_s for $\mathsf{s} \in \mathrm{Std}_e(\lambda)$ belongs to some simple composition factor $L(\nu)$ of $S_n(\lambda)$ for $\nu \neq \lambda$; in which case $\lambda \rhd \nu$ and

$$1_\omega^{\mathrm{res}(\mathsf{s})} L(\nu) \subseteq 1_\omega^{\mathrm{res}(\mathsf{s})} S_n(\nu) \neq 0$$

which gives us our required contradiction. Now we turn to the proof of equation (2.3). If $\nu \trianglelefteqslant \lambda$, then $\nu \in \mathscr{P}_n^\ell(h)$. Given $\mathsf{t} \in \mathrm{Std}(\nu)$ with $\mathsf{t}(r_k, c_k, m_k) = k$, we identify t with the path

$$(+\varepsilon_{hm_1+c_1}, +\varepsilon_{hm_2+c_2}, + \ldots, +\varepsilon_{hm_n+c_n}).$$

Given $\mathsf{t} \in \mathrm{Std}(\nu)$, we have that

$$\mathrm{Path}^+(\lambda, \mathsf{t}) = \{\mathsf{u} \in \mathrm{Std}(\lambda) \mid \mathrm{res}(\mathsf{u}) = \mathrm{res}(\mathsf{t})\}.$$

Given any $\mathsf{s} \in \mathrm{Std}_e(\lambda)$, we have that $\mathsf{s}(k) \in \mathcal{F}_k^\ell(h)$ for all $1 \leqslant k \leqslant n$ and hence $\mathsf{s}(k) \notin \mathbb{E}(\alpha, me)$ for any $\alpha \in \Phi$, $m \in \mathbb{Z}$. Hence

$$\mathsf{s} \notin \bigcup_{\substack{\nu \in \mathscr{P}_n^\ell(h) \\ \mathsf{t} \in \mathrm{Std}(\nu), \mathsf{t} \neq \mathsf{s}}} \mathrm{Path}^+(\lambda, \mathsf{t}) \tag{2.4}$$

and the result follows. □

Example 2.11. *Let $h = 1$ and $\ell = 3$ and $\kappa = (0, 1, 2) \in (\mathbb{Z}/4\mathbb{Z})^3$ as in Figure 2. The unique $\lambda \in \mathcal{F}_n^3(h)$ is given by $\lambda = ((1^3), (1^3), (1^2))\}$. The tableau t^λ is the unique element of $\mathrm{Std}_e(\lambda)$ and hence $D_8(\lambda)$ is 1-dimensional.*

3. One column homomorphisms

In Subsection 3.1 we construct the maps which will provide the backbone of our BGG complexes. We then consider how these homomorphisms compose (in terms of "diamonds") and it is these in-depth diagrammatic calculations that provide the technical crux of the paper: In Subsection 3.2 we classify the diamonds in terms of pairs of reflections in the alcove geometry; in Subsection 3.3 we localise to consider the μ weight-spaces of cell-modules $\Delta(\lambda)$ for μ, λ two points in a given diamond; and finally in Subsection 3.4 we use these results to prove that, within a diamond, composition of homomorphisms commutes (up to scalar multiplication by ± 1) or is zero (for degenerate diamonds).

3.1. **One column homomorphisms.** Let $e > h\ell$. Given $1 \leqslant i < j \leqslant h\ell$ and $\alpha, \beta \in \mathscr{P}_n^\ell(h)$, we suppose that $\ell(\alpha) = \ell(\beta) - 1$ and that $\beta \rhd \alpha$. Then there exists a unique hyperplane $\mathbb{E}(\varepsilon_i - \varepsilon_j, \mu_{ij}e)$ for $1 \leqslant i, j \leqslant h\ell$ and $\mu_{ij} \in \mathbb{Z}$ such that $s_{i-j,\mu_{ij}e}(\alpha) = \beta$. By definition, this amounts to removing a series of nodes from the jth column of α and adding them in the ith column of α to obtain $\beta \in \mathscr{P}_n^\ell$ or vice versa. By not assuming that $i < j$, we can use the notation

$$s_{i-j,\mu_{ij}e}(\alpha) = \beta$$

to always mean that β is obtained by removing a series of nodes from the jth column of α and adding them in the ith column of β. There are two distinct cases to consider. The most familiar case (to many Lie theorists) is that in which $\ell_{\varepsilon_i-\varepsilon_j}(\alpha, \beta) = 1$. In other words $\mu_{ij} \in \mathbb{Z}$ is the *unique* value such that

$$\alpha \in \mathbb{E}^{>}(\varepsilon_i - \varepsilon_j, \mu_{ij}e) \qquad \beta \in \mathbb{E}^{<}(\varepsilon_i - \varepsilon_j, \mu_{ij}e).$$

We refer to such pairs (α, β) as maximal pairs. These pairs include those related by a reflection "through a common alcove wall". The other case (which should be familiar to those who study blob and Virasoro algebras) is that in which

$$\ell_{\varepsilon_i-\varepsilon_j}(\alpha, \beta) = 2\ell_{\varepsilon_i-\varepsilon_j}(\alpha) - 1$$

and so $\mathbb{E}(\varepsilon_i - \varepsilon_j, \mu_{ij}e)$ is just one of many hyperplanes lying between α and β; these pairs (α, β) correspond to pairs which are as far away as possible in the alcove geometry. We refer to such pairs (α, β) as minimal pairs (and they only exist for $\ell > 1$). We wish to distinguish between such pairs. Therefore, for a minimal (respectively maximal) pair we set $m_{ij} := \mu_{ij}$ (respectively $M_{ij} := \mu_{ij}$). We have that $m_{ij} \in \{0, 1\}$ for any pair $\alpha, \beta \in \mathscr{P}_n^\ell(h)$.

Example 3.1. *Let $h = 1$, $\ell = 3$ and $\kappa = (0, 1, 2)$ as in Figure 2. The pair $((1^8 \mid \varnothing \mid \varnothing), (1^2 \mid \varnothing \mid 1^6))$ is a minimal pair. There are three hyperplanes parallel to $\mathbb{E}_{\varepsilon_1-\varepsilon_3,0}$ separating these two points.*

Remark 3.2. *Note that, near the origin, it is possible that a reflection is both maximal and minimal. For example, consider the pair $(1^6 \mid \varnothing \mid 1^2)$ and $(1^4 \mid \varnothing \mid 1^4)$ pictured in Figure 2.*

Theorem 3.3. *Let $e > h\ell$ and suppose that $\kappa \in I^\ell$ is h-admissible. Let $\alpha, \beta \in \mathscr{P}_n^\ell(h)$ be such that $\ell(\beta) = \ell(\alpha) - 1$. Let $1 \leqslant i, j \leqslant h\ell$ and $\mu_{ij} \in \mathbb{N}$ be such that $s_{i-j,\mu_{ij}e}(\alpha) = \beta$. We have that*

$$\alpha \setminus \alpha \cap \beta = \{X_1, X_2, \dots, X_k\} \quad \text{and} \quad \beta \setminus \alpha \cap \beta = \{Y_1, Y_2, \dots, Y_k\}$$

with $X_a \rhd X_{a+1}$ (respectively $Y_a \rhd Y_{a+1}$) for $1 \leqslant a < k$ is a sequence of nodes belonging to the jth column of α (respectively ith column of β). There is a unique $\mathsf{T}^\alpha_\beta \in \mathrm{SStd}^+(\beta, \alpha)$, as follows

$$\mathsf{T}^\alpha_\beta(\square) = \begin{cases} \square & \text{if } \square \in \alpha \cap \beta \\ Y_k & \text{if } \square = X_k. \end{cases}$$

We have $\mathrm{Hom}_{A_n(\kappa)}(\Delta(\alpha), \Delta(\beta)) = \Bbbk\{\varphi^\alpha_\beta\}$ where φ^α_β is determined by $\varphi^\alpha_\beta(C_{\mathsf{T}^\alpha}) = C_{\mathsf{T}^\alpha_\beta}$. We define $\mathsf{t}^\alpha_{\alpha\cap\beta} \in \mathrm{Std}(\alpha)$ and $\mathsf{t}^\beta_{\alpha\cap\beta} \in \mathrm{Std}(\beta)$ to be the unique standard tableaux of given shape determined by

$$\mathsf{t}^\alpha_{\alpha\cap\beta}(r, c, m) = \mathsf{t}^{\alpha\cap\beta}(r, c, m) = \mathsf{t}^\beta_{\alpha\cap\beta}(r, c, m)$$

for $(r, c, m) \in \alpha \cap \beta$. We have that $\phi^\alpha_\beta(c_{\mathsf{t}^\alpha_{\alpha\cap\beta}}) = c_{\mathsf{t}^\alpha_{\alpha\cap\beta}}$ determines the corresponding unique homomorphism in $\mathrm{Hom}_{R_n(\kappa)}(S_n(\alpha), S_n(\beta))$. The homomorphisms φ^α_β and ϕ^α_β are both of degree t^1.

Proof. For the statement for $A_n(\kappa)$ see [BC18, Corollary 10.12]. By the definition of $\mathsf{t}^\alpha_{\alpha\cap\beta}$, we have that $\mathrm{Path}(\lambda, \mathsf{t}^\alpha_{\alpha\cap\beta}) = \emptyset$ unless $\lambda \trianglerighteq \alpha$. Therefore $e(\mathsf{t}^\alpha_{\alpha\cap\beta})\Delta(\lambda) = 0$ unless $\lambda \trianglerighteq \alpha$. Therefore $c_{\mathsf{t}^\alpha_{\alpha\cap\beta}} \in L(\alpha)$ and thus it is enough to define a homomorphism, ϕ^α_β say, by where it sends $c_{\mathsf{t}^\alpha_{\alpha\cap\beta}}$. Now, we have that

$$C_{\varphi(\mathsf{t}^\alpha_{\alpha\cap\beta})} C_{\mathsf{T}^\alpha_\beta} = C_{\varphi(\mathsf{t}^\beta_{\alpha\cap\beta})} \in \mathbf{A}_n(\kappa)$$

and so the result follows by applying the Schur idempotent. □

3.2. **Diamonds formed by pairs of one-column morphisms.** We wish to consider all possible ways of composing a pair of such one-column homomorphisms. Let $\alpha, \beta \in \mathscr{P}_n^\ell(h)$ be such that $\beta \rhd \alpha$ and $\ell(\alpha) = \ell(\beta) + 2$. There are six such cases to consider which we now list. The first five cases of homomorphisms should be familiar to all Lie theorists. We first consider the cases in which α, β differ in precisely three columns. In other words, α, β belong to a plane $\mathbb{R}\{\varepsilon_j - \varepsilon_i, \varepsilon_k - \varepsilon_i\}$ for some $1 \leqslant i, j, k \leqslant h\ell$ and (without loss of generality) we can assume that

$$\langle \alpha, \varepsilon_i \rangle > \langle \alpha, \varepsilon_j \rangle > \langle \alpha, \varepsilon_k \rangle. \tag{3.1}$$

(1) We have $\beta := s_{k-i,\mu_{ki}e} s_{j-i,\mu_{ji}e}(\alpha)$ and $\gamma := s_{j-i,\mu_{ji}e}(\alpha)$. There are two subcases
   (a) $\delta := s_{k-j,\mu_{kj}e}(\alpha) \in \mathscr{P}_n^\ell$;
   (b) $\delta := s_{k-j,\mu_{kj}e}(\alpha) \notin \mathscr{P}_n^\ell$;
(2) we have $\beta := s_{k-i,\mu_{ki}e} s_{k-j,\mu_{kj}e}(\alpha)$ and $\gamma := s_{k-j,\mu_{kj}e}(\alpha)$. There are two subcases
   (a) $\delta := s_{j-i,\mu_{ji}e}(\alpha) \in \mathscr{P}_n^\ell$;
   (b) $\delta := s_{j-i,\mu_{ji}e}(\alpha) \notin \mathscr{P}_n^\ell$;
(3) $\delta := s_{j-i,\mu_{ji}e}(\alpha)$ and $\gamma := s_{k-i,\mu_{ki}e}(\alpha)$ $\beta := s_{k-j,\mu_{kj}e}(\delta) = s_{j-i,\mu_{ji}e}(\gamma)$, all belong to $\mathscr{P}_n^\ell$;
(4) $\delta := s_{k-j,\mu_{kj}e}(\alpha)$, and $\gamma := s_{k-i,\mu_{ki}e}(\alpha)$ $\beta := s_{j-i,\mu_{ji}e}(\delta) = s_{k-j,\mu_{kj}e}(\gamma)$, all belong to $\mathscr{P}_n^\ell$.

We now assume that α and β differ in four columns (so that we cannot picture them belonging to a plane). Without loss of generality, we assume that

$$\langle \alpha, \varepsilon_i \rangle > \langle \alpha, \varepsilon_j \rangle \qquad \langle \alpha, \varepsilon_k \rangle > \langle \alpha, \varepsilon_l \rangle.$$

This is the case in which

(5) $\gamma := s_{j-i,\mu_{ji}e}(\lambda)$, $\delta := s_{l-k,\mu_{lk}e}(\alpha)$, and $\beta := s_{l-k,\mu_{lk}e}(\gamma) = s_{j-i,\mu_{ji}e}(\delta)$ all belong to $\mathscr{P}_n^\ell$.

Finally, we have one additional case to consider in which α and β differ *only* in 2 columns. In other words α and β belong to a line $\mathbb{R}\{\varepsilon_i - \varepsilon_j\}$.

(6a) We have $\beta = s_{i-j,(1-m_{ji})e} s_{j-i,m_{ji}e}(\alpha)$ and $\gamma := s_{j-i,M_{ji}e}(\alpha)$ belong to $\mathscr{P}_n^\ell$ and
   (i) $\delta := s_{j-i,m_{ji}e}(\alpha)$ does not belong to $\mathscr{P}_n^\ell$;
   (ii) For $\ell > 1$ we have that $\delta := s_{j-i,m_{ji}e}(\alpha)$ does belong to $\mathscr{P}_n^\ell$;
(6b) For $\ell > 1$ we have that $\beta := s_{i-j,m_{ji}e} s_{j-i,M_{ji}e}(\alpha)$, and $\gamma := s_{j-i,M_{ji}e}(\alpha)$ and $\delta := s_{j-i,m_{ji}e}(\alpha)$ belong to $\mathscr{P}_n^\ell$ and cannot be written in the form specified in case (6a).

In the "does belong to $\mathscr{P}_n^\ell$" cases, we get a diamond in the complex and so we refer to these 4-tuples as diamonds and (α, β) as a diamond pair. In the "does not belong to $\mathscr{P}_n^\ell$" case, we get a single strand in the complex, and so we refer to these 3-tuples as degenerate-diamonds or strands. The first four cases can be pictured by projecting into the plane $\mathbb{R}\{\varepsilon_j - \varepsilon_i, \varepsilon_k - \varepsilon_i\}$ as depicted in Figure 3.

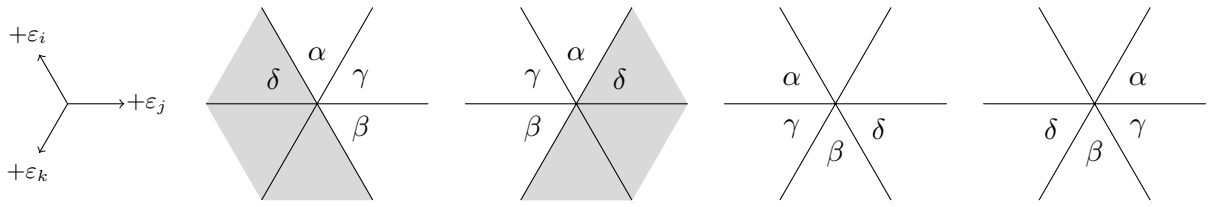


FIGURE 3. The first four cases of diamond pairs (α, β).

In the first two cases, the lightly coloured-in region denotes the "missing" region of (1b) and (2b). For diamonds formed entirely of maximal pairs, the pictures in Figure 3 consists only of six e-alcoves and their walls; thus the hyperplanes pictured are the *only* hyperplanes between α and β. See Figure 2 and equation (3.4) for such an example. For (degenerate) diamonds involving one or two minimal pairs, there can be many other hyperplanes between α and β which are not pictured. See Figure 2 and equation (3.3) for such an example. The fifth case arises from a pair of orthogonal reflections and cannot be pictured in 2-dimensional space, however it is also the easiest case and so we do not lose much by being unable to picture it. The subcases of (6) for which $\ell > 1$ are easily

pictured and should be familiar to those who work with Virasoro and blob algebras. See Figure 2 and equation (3.2) for such an example (many further examples can be found in [BCS17]).

Example 3.4. *Let $h = 1$ and $\ell = 3$ and $\kappa = (0, 1, 2)$ as in Figure 2. The diamond consisting of*

$$\alpha = (1^8 \mid \varnothing \mid \varnothing) \quad \beta = (1^2 \mid \varnothing \mid 1^6) \quad \gamma = (1^6 \mid \varnothing \mid 1^2) \qquad \delta = (1^2 \mid \varnothing \mid 1^6). \tag{3.2}$$

is as in case (6a). The diamond consisting of the 3-partitions,

$$\alpha = (\varnothing \mid 1^8 \mid \varnothing) \quad \beta = (1^6 \mid 1 \mid 1) \quad \gamma = (1^7 \mid \varnothing \mid 1) \qquad \delta = (\varnothing \mid 1^7 \mid 1). \tag{3.3}$$

is as in case (4) and is a mixture of minimal and maximal pairs. Let

$$\alpha = (\varnothing \mid \varnothing \mid 1^8) \quad \beta = (1^2 \mid 1 \mid 1^5) \quad \gamma = (\varnothing \mid 1^3 \mid 1^5) \quad \delta = (1^2 \mid \varnothing \mid 1^6). \tag{3.4}$$

The diamond $(\alpha, \beta, \gamma, \delta)$ is as in case (4) and consists solely of maximal pairs.

Remark 3.5. *We added a clause so that cases (6a) and (6b) are mutually exclusive. Without that clause, these cases would have a non-trivial intersection for points near the origin (see Remark 3.2). We have added this clause as these two subcases are genuinely different, see Proposition 3.8 below.*

Definition 3.6. *Let (α, β) be a non-degenerate diamond pair. We define the (α, β)-*vertex *to be*

$$\xi = (\alpha \cap \beta \cap \gamma \cap \delta) \in \mathscr{P}_n^\ell(h).$$

In case (6), there exists $0 < y < e$ and $x \geqslant 0$ such that $\langle \alpha - \xi, \varepsilon_i \rangle = xe + y$. In cases (1), (2) and (4), we let W_ξ denote the copy of $\mathfrak{S}_3$ generated by the reflections through the hyperplanes $\mathbb{E}(j - i, \mu_{ji}e)$, $\mathbb{E}(k - j, \mu_{kj}e)$, and $\mathbb{E}(k - i, \mu_{ki}e)$. Given $s \in W_\xi$ we let

$$x = \langle \alpha - \xi, \varepsilon_i \rangle = \langle s(\alpha) - \xi, \varepsilon_{s(i)} \rangle \qquad y = \langle \alpha - \xi, \varepsilon_j \rangle = \langle s(\alpha) - \xi, \varepsilon_{s(j)} \rangle.$$

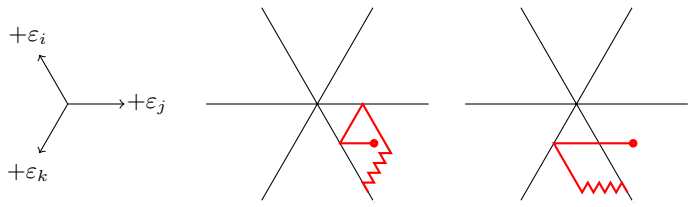
We let $\{X_1^{s(\alpha)}, X_2^{s(\alpha)}, \dots, X_x^{s(\alpha)}\}$ denote the final x nodes of the $s(i)$th column of $s(\alpha)$ and let $\{Y_1^{s(\alpha)}, Y_2^{s(\alpha)}, \dots, Y_y^{s(\alpha)}\}$ denote the final y nodes of the $s(j)$th column of $s(\alpha)$.

Remark 3.7. *In cases (1), (2) and (4) we have that $0 < y < e$ and $\operatorname{res}(X_k) = \operatorname{res}(Y_k)$ for $1 \leqslant k \leqslant \min\{x, y\}$.*

3.3. **Paths in diamonds.** We shall now consider reflections of the corresponding paths in the hyperplanes described in our 6 cases above. We remark that each of these paths passes through each hyperplane *at most once*. Therefore, we simplify our notation of Remark 2.4 by dropping the superscript on the reflection. We now consider the (dominant) paths in $\operatorname{Path}(\beta, \alpha)$. In case (1a) there at two paths

$$\mathsf{S}_\beta^\alpha := s_{j-k,\mu_{kj}e} s_{k-i,\mu_{ki}e}(\mathsf{T}^\alpha) \text{ and } \mathsf{T}_\beta^\alpha := s_{j-i,\mu_{ji}e} s_{k-j,\mu_{kj}e}(\mathsf{T}^\alpha)$$

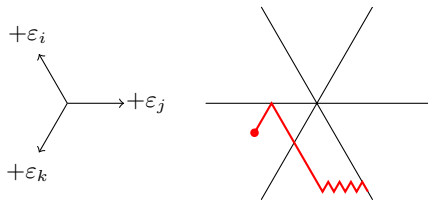
of degrees 0 and 2 respectively, which are both dominant. Generic examples of such paths (drawn from the point at which they meet the hyperplane $\mathbb{E}(k - j, \mu_{kj}e)$ onwards) are pictured below



and are of degree 0 and 2 respectively. In case (2a) there is a unique path

$$\mathsf{T}_\beta^\alpha := s_{k-i,\mu_{ki}e} s_{k-j,\mu_{kj}e}(\mathsf{T}^\alpha)$$

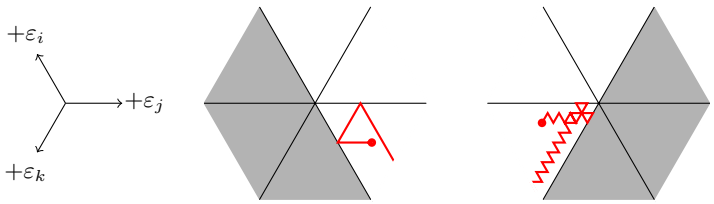
which is of degree 2 and dominant. A generic example of such a path is pictured below.



In each of cases $(1b)$ and $(2b)$ there is a single path

$$s_{j-k,\mu_{jk}e}s_{k-i,\mu_{ki}e}(\mathsf{T}^\alpha) \text{ and } s_{j-i,\mu_{ji}e}s_{k-i,\mu_{ki}e}(\mathsf{T}^\alpha)$$

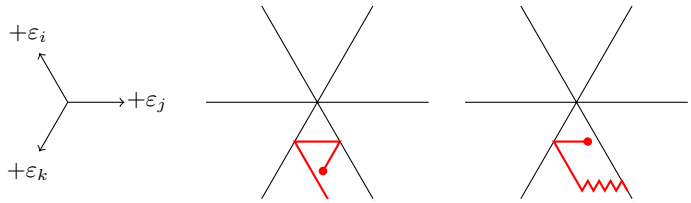
of degree 2, *neither* of which is dominant. These are pictured below



In each of cases (3) and (4) there is a unique path

$$\mathsf{T}^\alpha_\beta := s_{k-j,\mu_{kj}e}s_{j-i,\mu_{ji}e}(\mathsf{T}^\alpha) \qquad \mathsf{T}^\alpha_\beta := s_{j-i,\mu_{ij}e}s_{k-j,\mu_{kj}e}(\mathsf{T}^\alpha)$$

respectively, of degree 2. Generic examples of such paths are pictured below



In case (5), the reflections are orthogonal and there is a unique (dominant) path and if we assume (without loss of generality) that $\langle\alpha,\varepsilon_i\rangle > \langle\alpha,\varepsilon_k\rangle$, then this path is given by

$$\mathsf{T}^\alpha_\beta := s_{l-k,\mu_{lk}e}s_{j-i,\mu_{ji}e}(\mathsf{T}^\alpha)$$

and is of degree 2. In case $(6a)$ we have $(x-1)$ distinct dominant paths of degree 0 given as follows,

$$\mathsf{S}_\chi = \begin{cases} s_{i-j,(M_{ij}-\chi-1)e}s_{i-j,(M_{ij}-\chi)e}(\mathsf{T}^\alpha) & \text{for } m_{ij}=1 \\ s_{i-j,(M_{ij}+\chi+1)e}s_{i-j,(M_{ij}+\chi)e}(\mathsf{T}^\alpha) & \text{for } m_{ij}=0 \end{cases}$$

for $1 \leqslant \chi < x$ (for x as in Definition 3.6); we also have a unique path of degree 2 given by

$$\mathsf{T}^\alpha_\beta = s_{j-i,(1-m_{ji})e}s_{i-j,m_{ij}e}(\mathsf{T}^\alpha) \in \mathrm{Path}(\beta,\mathsf{T}^\alpha)$$

which is dominant if and only if we are in case $(6a)(ii)$. In case $(6b)$ we have a unique (dominant) path

$$\mathsf{T}^\alpha_\beta = s_{i-j,(2m_{ji}-M_{ji})e}s_{j-i,m_{ji}e}(\mathsf{T}^\alpha)$$

of degree 2. Using equation (2.2), we now summarise the above as follows.

Proposition 3.8. *Let $\alpha,\beta \in \mathscr{P}^\ell_n(h)$ be a pair such that $\beta \rhd \alpha$ and $\ell(\alpha) = \ell(\beta)+2$. We have that*

$$\dim_t(1_\alpha\Delta(\beta)) = \begin{cases} 0 & \text{in cases } (1b) \text{ and } (2b) \\ t^2+1 & \text{in case } (1a) \\ t^2 & \text{in cases } (2a),(3),(4),(5) \text{ and } (6b) \\ x-1 & \text{in case } (6a)(i) \\ t^2+x-1 & \text{in case } (6a)(ii) \end{cases}$$

where $x \in \mathbb{Z}_{>0}$ is defined in Definition 3.6 and t is the indeterminate over $\mathbb{Z}_{\geqslant 0}$ from Remark 1.13.

Example 3.9. *In Figure 2, we have that the tableaux*

$$\mathsf{T}^{(1^8|\varnothing|\varnothing)}_{(1^4|\varnothing|1^4)} = s^{(6)}_{1-3,M_{13}} \circ s^{(2)}_{3-1,M_{31}}(\mathsf{T}^{(1^8|\varnothing|\varnothing)}) \quad \mathsf{T}^{(1^8|\varnothing|\varnothing)}_{(1^4|\varnothing|1^4)} = s^{(7)}_{3-1,m_{31}} \circ s^{(1)}_{1-2,M_{12}}(\mathsf{T}^{(1^8|\varnothing|\varnothing)})$$

$$\text{and} \quad \mathsf{T}^{(1^8|\varnothing|\varnothing)}_{(1^4|\varnothing|1^4)} = s^{(6)}_{1-2,m_{12}} \circ s^{(5)}_{3-2,m_{32}}(\mathsf{T}^{(1^8|\varnothing|\varnothing)})$$

are as in cases $(6a)$, (4), and (4) respectively and are all of degree t^2.

3.4. **Compositions of one-column homomorphisms in diamonds.** We now consider the composition of the one-column homomorphisms in terms of the path basis constructed in Proposition 3.8. Let $\mathsf{T} \in \mathcal{T}(\lambda, \mu)$ and $\mathsf{T}(X) = \mathbf{I}^{\kappa}_{Y} \in \mathbb{Z}[\epsilon]$ for $X \in \lambda$, $Y \in \mu$; we abuse notation by writing either $\mathsf{T}(X) = Y$ or $\mathsf{T}(X) = \mathbf{I}^{\kappa}_{Y}$. From Proposition 3.8, we deduce the immediate corollary.

Corollary 3.10. *The composition of two one-column homomorphisms is zero in the degenerate cases, in other words cases (1b), (2b) and (6a)(i).*

Proof. Cases (1b) and (2b) are clear. Case (6a)(i) follows because the composition of two homomorphism of degree t^1 must be a vector of degree t^2 and no such vector exists (by Proposition 3.8). □

Proposition 3.11. *Let $\alpha, \beta \in \mathscr{P}^{\ell}_{n}(h)$ be a pair such that $\beta \rhd \alpha$ and $\ell(\alpha) = \ell(\beta) + 2$. Assume that we are not in one of degenerate cases of Corollary 3.10. We have that*

- $C_{\mathsf{T}^{\alpha}_{\beta}} = C_{\mathsf{T}^{\alpha}_{\delta}} C_{\mathsf{T}^{\delta}_{\beta}}$ *in all remaining cases;*
- $C_{\mathsf{T}^{\alpha}_{\beta}} = C_{\mathsf{T}^{\alpha}_{\gamma}} C_{\mathsf{T}^{\gamma}_{\beta}}$ *in all remaining cases except* (1a), (6), *and* (4).

Proof. For $\gamma \in \mathscr{P}^{\ell}_{n}$ (similarly for $\delta \in \mathscr{P}^{\ell}_{n}$) it is clearly enough to show that

$$\mathsf{T}^{\alpha}_{\gamma}\mathsf{T}^{\gamma}_{\beta} = \mathsf{T}^{\alpha}_{\beta} \in \mathcal{T}(\beta, \alpha) \tag{3.5}$$

on the level of bijective maps : $\beta \to \alpha$, and furthermore that if

$$(r,c,m) \rhd (r',c',m') \text{ and } \mathsf{T}^{\gamma}_{\beta}(r,c,m) \lhd \mathsf{T}^{\gamma}_{\beta}(r',c',m') \text{ implies } \mathsf{T}^{\alpha}_{\gamma}\mathsf{T}^{\gamma}_{\beta}(r,c,m) \lhd \mathsf{T}^{\alpha}_{\gamma}\mathsf{T}^{\gamma}_{\beta}(r',c',m') \tag{3.6}$$

for any two nodes $(r,c,m), (r',c',m') \in \beta \setminus \xi$ of the same or adjacent residue. This is simply by the definition of the bases elements corresponding to these tableaux (and the fact that double-crossings between strands of non-adjacent distinct residues can be removed by relation (D5)). The cases listed in the above proposition are precisely those for which equation (3.5) and (3.6) are both true (in other words, 3.5 and 3.6 both hold in all cases except in cases (1a), (4), and (6) for the product $\mathsf{T}^{\alpha}_{\gamma}\mathsf{T}^{\gamma}_{\beta}$ — which will be discussed separately).

We shall consider case (2), as the other cases are identical. It is clear that $\mathsf{S}(r,c,m) = (r,c,m)$ if $(r,c,m) \in \xi$ for $\mathsf{T}^{\lambda}_{\mu}$ for $\lambda, \mu \in \{\alpha, \beta, \gamma, \delta\}$. Thus it remains to consider the restriction of these bijections to : $\beta \setminus \xi \to \alpha \setminus \xi$ (via both $\gamma \setminus \xi$ and $\delta \setminus \xi$). We have that

$$\mathsf{T}^{\gamma}_{\beta}(X^{\beta}_{p}) = Y^{\gamma}_{p} \quad \mathsf{T}^{\gamma}_{\beta}(X^{\beta}_{y+q}) = X^{\gamma}_{y+q} \quad \mathsf{T}^{\gamma}_{\beta}(Y^{\beta}_{p}) = X^{\gamma}_{p}$$
$$\mathsf{T}^{\alpha}_{\gamma}(Y^{\gamma}_{p}) = Y^{\alpha}_{p} \quad \mathsf{T}^{\alpha}_{\gamma}(X^{\gamma}_{y+q}) = X^{\alpha}_{y+q} \quad \mathsf{T}^{\alpha}_{\gamma}(X^{\gamma}_{p}) = X^{\alpha}_{p}$$

and

$$\mathsf{T}^{\delta}_{\beta}(X^{\beta}_{p}) = X^{\delta}_{p} \quad \mathsf{T}^{\delta}_{\beta}(X^{\beta}_{y+q}) = X^{\delta}_{y+q} \quad \mathsf{T}^{\delta}_{\beta}(Y^{\beta}_{p}) = Y^{\delta}_{p}$$
$$\mathsf{T}^{\alpha}_{\delta}(X^{\delta}_{p}) = Y^{\alpha}_{p} \quad \mathsf{T}^{\alpha}_{\delta}(X^{\delta}_{y+q}) = X^{\alpha}_{y+q} \quad \mathsf{T}^{\alpha}_{\delta}(Y^{\beta}_{p}) = X^{\alpha}_{p}$$

and

$$\mathsf{T}^{\alpha}_{\beta}(X^{\beta}_{p}) = Y^{\alpha}_{p} \quad \mathsf{T}^{\alpha}_{\beta}(X^{\beta}_{y+q}) = X^{\alpha}_{y+q} \quad \mathsf{T}^{\alpha}_{\beta}(Y^{\beta}_{p}) = X^{\alpha}_{p}$$

for $1 \leqslant p \leqslant y$ and $1 \leqslant q \leqslant x - y$. Therefore equation (3.5) holds. To see that equation (3.6) holds, one requires the following observation

$$X^{\beta}_{j} \rhd Y^{\beta}_{j} \qquad Y^{\gamma}_{j} \rhd X^{\gamma}_{j} \qquad X^{\delta}_{j} \rhd Y^{\delta}_{j} \qquad Y^{\alpha}_{j} \rhd X^{\alpha}_{j}$$

for all $1 \leqslant j \leqslant y$; one can apply this observation to each of the above tableaux in turn. Thus equation (3.6) holds, as required. □

It remains to consider the γ subcases of (1), (4), and (6) not considered above. In all these cases, we shall see that equation (3.5) and (3.6) fail. Thus, we must apply some relations in order to rewrite each product-diagram in the required form. We let $A^{\rhd\beta}_{n}(\kappa) = \sum_{\lambda \rhd \beta} A_n(\kappa) 1_\lambda A_n(\kappa)$. Given $(r,c,m) \in \alpha$, we let $\mathsf{y}(r,c,m)1_\alpha$ denote the diagram 1_α with a dot added on the vertical solid strand with x-coordinate given by $\mathbf{I}^{\kappa}_{(r,c,m)}$. Following [BK09], we set $y_k = \mathsf{y}(k,1,\ell)$.

Proposition 3.12. *Let (α, β) be a diamond pair as in case* (1a). *Then*

$$C_{\mathsf{T}^{\alpha}_{\gamma}} C_{\mathsf{T}^{\gamma}_{\beta}} = -\mathsf{y}(X^{\alpha}_{y}) 1_\alpha C_{\mathsf{S}^{\alpha}_{\beta}} = (-1)^{y+1} C_{\mathsf{T}^{\alpha}_{\beta}} + A^{\rhd\beta}_{n}(\kappa). \tag{3.7}$$

Proof. In case $(1a)$, we have $\mathsf{T}^\alpha_\gamma \circ \mathsf{T}^\gamma_\beta = \mathsf{S}^\alpha_\beta \in \mathcal{T}(\beta, \alpha)$ as bijective maps. However, the corresponding product of diagrams has a single double-crossing of non-zero degree; this is between the strand from X^β_{y+1} on the southern edge to X^α_{y+1} on the northern edge and the strand from Y^β_y on the southern edge to X^α_y on the northern edge. In particular, $\mathrm{res}(X^\beta_{y+1}) = \mathrm{res}(Y^\beta_y) - 1$, and

$$X^\beta_{y+1} \lhd Y^\beta_y \qquad \mathsf{T}^\gamma_\beta(X^\beta_{y+1}) = X^\gamma_{y+1} \rhd Y^\gamma_y = \mathsf{T}^\gamma_\beta(Y^\beta_y) \qquad \mathsf{T}^\alpha_\gamma(Y^\gamma_y) = X^\alpha_y \rhd X^\alpha_{y+1} = \mathsf{T}^\alpha_\gamma(X^\gamma_{y+1}).$$

For $1 \leqslant p, p' \leqslant y$ the strand from X^β_{y+p} on the southern edge to X^α_{y+p} on the northern edge double-crosses with the strand from $Y^\beta_{p'}$ on the southern edge to $X^\alpha_{p'}$ on the northern edge; since $y < e$, we can remove all of the double-crossings for $(p, q) \neq (1, 1)$ using relation (D5). We now resolve the final double crossing (for $(p, q) = (1, 1)$) using relation the leftmost equality of (D6) and hence obtain

$$C_{\mathsf{T}^\alpha_\gamma} C_{\mathsf{T}^\gamma_\beta} = \mathsf{y}(X^\alpha_{y+1}) 1_\alpha C_{\mathsf{S}^\alpha_\beta} - \mathsf{y}(X^\alpha_y) 1_\alpha C_{\mathsf{S}^\alpha_\beta}.$$

Concerning the former diagram: we pull the dot down the strand and encounter no like-crossings on the way; hence this term is equal to zero. It remains to prove the second equality in equation (3.7). We let $\mathsf{U}_y \in \mathcal{T}(\alpha, \beta)$ denote the map

$$\mathsf{U}_y(r, c, m) = \begin{cases} X^\alpha_y & \text{for } (r, c, m) = X^\beta_y \\ Y^\alpha_y & \text{for } (r, c, m) = Y^\beta_y \\ \mathsf{S}^\alpha_\beta(r, c, m) & \text{otherwise.} \end{cases}.$$

We claim that

$$\mathsf{y}(X^\alpha_y) 1_\alpha C_{\mathsf{S}^\alpha_\beta} = -C_{\mathsf{U}_y} + A^{\rhd\beta}_n(\kappa).$$

To see this, pull the dot at the top of the diagram $\mathsf{y}(X^\alpha_y) 1_\alpha C_{\mathsf{S}^\alpha_\beta}$ down the strand on which it lies (from X^α_y on northern edge to Y^β_y on the southern edge) towards the bottom of the diagram. By Definition 1.7, we can do this freely until we encounter a like-crossing of the form in relation (D3). Such a crossing involves the aforementioned strand (between points X^α_y and Y^β_y on the northern and southern edges) and some vertical strand of the same residue. Such a vertical strand either (i) corresponds to a step of the form $+\varepsilon_m$ for $m \notin \{i, j, k\}$ or (ii) is the vertical strand from X^β_y on the southern edge to Y^α_y on the northern edge. In the former case, the resulting error term belongs to $A^{\rhd\beta}_n(\kappa)$. In the latter case, we apply relation (D3) to move the dot past the crossing at the expense of acquiring an error term, which is equal to $-C_{\mathsf{U}_y}$. Finally (in the diagram which has a dot) we continue pulling the dot reaches the bottom of the diagram, the resulting diagram again belongs to $A^{\rhd\beta}_n(\kappa)$. Thus the only non-zero term acquired in this process is $-C_{\mathsf{U}_y}$ and the claim holds. If $y = 1$, then $\mathsf{U}_y = \mathsf{T}^\alpha_\beta$ and we are done. Suppose that $y > 1$. Consider

- (i) the solid strand from Y^β_{y-1} on the southern edge to X^α_{y-1} on the northern edge
- (ii) the solid strand from X^β_{y-1} on the southern edge to Y^α_{y-1} on the northern edge
- (iii) the ghost strand from Y^β_y on the southern edge to Y^α_y on the northern edge.

These three strands together form a triple-crossing as on the right-hand side of the rightmost equation in relation (D8). Applying relation (D8), we can undo the crossing (at the expense of multiplication by minus one and an error term with the same number of crossings). Consider the error term: We are free to pull the ghost strand (of the strand connecting Y^β_y and Y^α_y) to the left to obtain a diagram which belongs to $A^{\rhd\beta}_n(\kappa)$. That leaves one remaining non-zero diagram which differs from $-C_{\mathsf{U}_y}$ in that we have undone the aforementioned triple-crossing; to summarise

$$\mathsf{y}(X^\alpha_y) 1_\alpha C_{\mathsf{S}^\alpha_\beta} = C_{\mathsf{U}_{y-1}} + A^{\rhd\beta}_n(\kappa) \quad \text{with} \quad \mathsf{U}_y(r, c, m) = \begin{cases} X^\alpha_{y-1} & \text{for } (r, c, m) = X^\beta_{y-1} \\ Y^\alpha_{y-1} & \text{for } (r, c, m) = Y^\beta_{y-1} \\ \mathsf{U}_y(r, c, m) & \text{otherwise.} \end{cases}$$

Repeat this argument until all y crossings have been resolved, the results follows. □

Proposition 3.13. *Let (α,β) be a diamond pair as in case (4). Then*

$$C_{\mathsf{T}^\alpha_\gamma} C_{\mathsf{T}^\gamma_\beta} = (-1)^x C_{\mathsf{T}^\alpha_\beta} + A_n^{\triangleright\beta}(\kappa).$$

Proof. We have that

$$\mathsf{T}^\alpha_\gamma \circ \mathsf{T}^\gamma_\beta(r,c,m) = \begin{cases} X^\alpha_p = \mathsf{T}^\alpha_\beta(Y^\beta_p) & \text{if } (r,c,m) = X^\beta_p \text{ for } 1 \leqslant p \leqslant x \\ Y^\alpha_p = \mathsf{T}^\alpha_\beta(X^\beta_p) & \text{if } (r,c,m) = Y^\beta_p \text{ for } 1 \leqslant p \leqslant x \\ Y^\alpha_q = \mathsf{T}^\alpha_\beta(Y^\beta_q) & \text{if } (r,c,m) = Y^\beta_q \text{ for } x+1 \leqslant q \leqslant y \\ \mathsf{T}^\alpha_\beta(r,c,m) & \text{for } (r,c,m) \in \xi \end{cases}$$

Consider

(*i*) the solid strand from Y^β_x on the southern edge to X^α_x on the northern edge;
(*ii*) the solid strand from X^β_x on the southern edge to Y^α_x on the northern edge;
(*iii*) the ghost strand of the strand from Y^β_{x+1} on the southern edge to Y^α_{x+1} on the northern edge.

These strands together form a crossing as on the righthand-side of the rightmost equation in relation (D8). Undoing this crossing we obtain an error term (corresponding to the diagram on the lefthand-side of the rightmost equality in relation (D8)) which belongs to $A_n^{\triangleright\beta}(\kappa)$ and another (non-zero) term. One can then repeat the above argument with the latter diagram (except replacing the subscript 'x' with '$x-1$'). Continuing in this fashion, we obtain the required result. □

Proposition 3.14. *Let (α,β) be a diamond pair as in case (6b). Then*

$$C_{\mathsf{T}^\alpha_\gamma} C_{\mathsf{T}^\gamma_\beta} = (-1)^y C_{\mathsf{T}^\alpha_\beta} + A_n^{\triangleright\beta}(\kappa).$$

Proof. We let $\{X^\alpha_1, X^\alpha_2, \dots, X^\alpha_{ex}, Y^\alpha_1, \dots, Y^\alpha_y\}$ denote the final $xe+y$ nodes of the ith column of α. We let $\{X^\beta_1, X^\beta_2, \dots, X^\beta_{ex}\}$ denote the final xe nodes of the jth column of β and $\{Y^\beta_1, \dots, Y^\beta_y\}$ denote the final y nodes of the ith column of β. We have that

$$\mathsf{T}^\alpha_\gamma \circ \mathsf{T}^\gamma_\beta(r,c,m) = \begin{cases} \mathsf{T}^\alpha_\beta(Y^\beta_p) = X^\alpha_p & \text{for } (r,c,m) = X^\beta_p \text{ and } 1 \leqslant p \leqslant y \\ \mathsf{T}^\alpha_\beta(X^\beta_p) = Y^\alpha_p & \text{for } (r,c,m) = Y^\beta_p \text{ and } 1 \leqslant p \leqslant y \\ \mathsf{T}^\alpha_\beta(X^\beta_q) = X^\alpha_q & \text{for } (r,c,m) = X^\beta_q \text{ and } y < q \leqslant ex \\ \mathsf{T}^\alpha_\beta(r,c,m) & \text{for } (r,c,m) \in \xi. \end{cases}$$

Consider

(*i*) the solid strand from Y^β_y on the southern edge to X^α_y on the northern edge;
(*ii*) the solid strand from X^β_y on the southern edge to Y^α_y on the northern edge;
(*iii*) the ghost of the strand from X^β_{y+1} on the southern edge to X^α_{y+1} on the northern edge.

These strands together form a crossing as on the righthand-side of the latter equality in relation (D8). Undoing this crossing we obtain an error term (corresponding to the diagram on the lefthand-side of the latter equality in relation (D8)) which belongs to $A_n^{\triangleright\beta}(\kappa)$ and another (non-zero) term. One can then repeat the above argument with the latter diagram (except replacing the subscript 'y' with '$y-1$'). Continuing in this fashion, we obtain the required result. □

Proposition 3.15. *Let (α,β) be a diamond pair as in case (6a)(ii). Then*

$$C_{\mathsf{T}^\alpha_\gamma} C_{\mathsf{T}^\gamma_\beta} = (-1)^{e(x+1)+y} C_{\mathsf{T}^\alpha_\beta} + A_n^{\triangleright\beta}(\kappa).$$

Proof. We first fix some notation. We denote the final e nodes at the end of the jth column of β by $X^\beta_1, \dots, X^\beta_e$. We denote the final $e(x-1)+y$ nodes at the end of the ith column of β by $X^\beta_{e+1}, \dots, X^\beta_{ex}, Y^\beta_1, \dots, Y^\beta_y$. We let $X^\alpha_1, X^\alpha_2, \dots, X^\alpha_{ex}, Y^\alpha_1, \dots, Y^\alpha_y$ denote the final $ex+y$ nodes at the end of the ith column of α. Given $\sigma \in \mathfrak{S}_x$ we define $\mathsf{U}_\sigma \in \mathcal{T}(\beta,\alpha)$ as follows,

$$\mathsf{U}_\sigma(X^\beta_{ep-q}) = \mathsf{U}_\sigma(X^\beta_{e\sigma(p)-q}) \qquad \mathsf{U}_\sigma(Y^\beta_t) = Y^\alpha_t \tag{3.8}$$

for $1 \leqslant p, \leqslant x$, $0 \leqslant q < e$, $0 \leqslant t < y$ and such that $\mathsf{U}_\sigma(r,c,m) = (r,c,m)$ for $(r,c,m) \in \xi$. We have that $\mathsf{T}^\alpha_\beta = \mathsf{U}_{\mathrm{id}}$ for $\mathrm{id} \in \mathfrak{S}_x$ and $\mathsf{S}_\chi = \mathsf{U}_\sigma$ for $\sigma = s_1 s_2 \dots s_\chi$ for $1 \leqslant \chi < x$ (and so any element of $\mathrm{SStd}(\beta,\alpha)$ can be written in the form of equation (3.8)).

We now state a claim that will provide the crux of the proof. Set $\sigma = s_1 s_2 \dots s_\chi$ for $1 \leqslant \chi \leqslant x$. Given $1 \leqslant r < \chi$, we refer to the strand in C_{U_σ} from X^β_e on the southern edge to $X^\alpha_{\chi e}$ on the northern as the principal strand. Let $C^r_{\mathsf{U}_\sigma}$ denote the diagram obtained from C_{U_σ} by placing a dot on the principal strand at any point in the interval $(\mathbf{I}^\kappa_{X^\alpha_{er}}, \mathbf{I}^\kappa_{X^\alpha_{er+1}}) \times [0,1]$. For $\sigma \neq s_1$, we claim that

$$C^r_{\mathsf{U}_\sigma} = C^{r-1}_{\mathsf{U}_\sigma} + (-1)^{e+1} C^{r-1}_{\mathsf{U}_{\sigma'}} \tag{3.9}$$

$$C^r_{\mathsf{U}_{s_1}} = C^{r-1}_{\mathsf{U}_{s_1}} + (-1)^{e} C_{\mathsf{U}_{s_1}} \tag{3.10}$$

modulo $A^{\rhd\beta}_n(\kappa)$ where $\sigma' = s_1 s_2 \dots s_{r-1} s_{r+1} \dots s_\chi$. Diagrammatically, we can think of our claim as simply a beefed-up version of relation (D3) in which we consider crossings involving collections of strands (each of size $e > 1$). We let $i = \mathrm{res}(X^\beta_e)$.

We now prove the claim. First apply relation (D3) to pull the dot through the crossing i-strands and hence obtain $C^{r-1}_{\mathsf{U}_\sigma}$ plus another term with a minus sign. For this latter diagram, the ghost of the principal i-strand can be pulled to the left through the crossing solid $(i+1)$-strands as in relation (D8). We hence obtain two diagrams: one with the same number of crossings, and one in which the crossing of $(i+1)$-strands has been undone. The former is zero modulo the stated ideal. The latter diagram now has a crossing of two solid $(i+2)$-strands and a ghost $(i+1)$-strand as in relation (D8). Repeating as necessary, this process terminates with a diagram (occurring with coefficient $(-1)^e$) which traces out the bijection of $\mathsf{U}_{\sigma'}$ but with many double-crossings.

- If $\sigma = s_1$, then all of these double-crossings are of degree zero;
- If $\sigma \neq s_1$, then precisely one of these double-crossings has non-zero degree: that between the solid strand from X^β_e on the southern edge to X^α_{re} on the northern edge and the ghost of the strand from X^β_{re-e+1} on the southern edge to $X^\alpha_{\chi e-e+1}$ on the northern edge.

In the latter case, we resolve this double-crossing as in relation (D8) and obtain two diagrams: one is of the required form and the other belongs to the stated ideal. In either case, the claim holds. Having proven our claim, we are now ready to prove the result. We have that

$$\mathsf{T}^\alpha_\gamma \circ \mathsf{T}^\gamma_\beta(r,c,m) = \begin{cases} \mathsf{U}_{s_1 \dots s_{x-1}}(X^\beta_t) = X^\alpha_{xe-e+t} & \text{for } (r,c,m) = Y^\beta_t \text{ and } 1 \leqslant t \leqslant y \\ \mathsf{U}_{s_1 \dots s_{x-1}}(Y^\beta_t) = Y^\alpha_t & \text{for } (r,c,m) = X^\beta_t \text{ and } 1 \leqslant t \leqslant y \\ \mathsf{U}_{s_1 \dots s_{x-1}}(r,c,m) & \text{otherwise.} \end{cases}$$

Therefore, using y applications of (D8) we obtain a diagram which traces out the same bijection as $\mathsf{U}_{s_1 \dots s_{x-1}}$ (modulo error terms). However the resulting diagram contains a single degree 2 double-crossing between the solid strand from X^β_e to X^α_{ex} (on the southern and northern edges, respectively) with the ghost of the strand from Y^β_1 to Y^α_1 (on the southern and northern edges, respectively). Resolving this crossing using relation (D6), we obtain that

$$C_{\mathsf{T}^\alpha_\gamma} C_{\mathsf{T}^\gamma_\beta} = (-1)^{y+1} C_{\mathsf{U}_{s_1 \dots s_{x-1}}} + A^{\rhd\beta}_n(\kappa).$$

We now successively apply equation (3.9) a total of $x-1$ times, followed by a single application of equation (3.10). The error terms all belong to $A^{\rhd\beta}_n(\kappa)$ and the result follows. □

Theorem 3.16. *Let $\alpha, \beta \in \mathscr{P}^\ell_n(h)$ be a non-degenerate diamond pair. We have that*

$$\varphi^\alpha_\delta \circ \varphi^\delta_\beta = \varphi^\alpha_\beta = \varepsilon_{\alpha,\beta,\gamma,\delta} \varphi^\alpha_\gamma \circ \varphi^\gamma_\beta \quad \textit{where} \quad \varepsilon_{\alpha,\beta,\gamma,\delta} = \begin{cases} (-1)^{y+1} & \textit{in cases } (1a) \\ (-1)^{x} & \textit{in cases } (4) \\ (-1)^{y} & \textit{in cases } (6b) \\ (-1)^{e(x+1)+y} & \textit{in case } (6a)(ii) \\ 1 & \textit{otherwise.} \end{cases}$$

Moreover the map φ^α_β is determined by $\varphi^\alpha_\beta(C_{\mathsf{T}^\alpha}) = C_{\mathsf{T}^\alpha_\beta}$ and $\dim_{\Bbbk}(\mathrm{Hom}_{\mathbf{A}_n(\kappa)}(\mathbf{\Delta}(\alpha), \mathbf{\Delta}(\beta))) = t^2$.

Proof. This theorem is mostly a restatement of the earlier results of this section (proved in the $A_n(\kappa)$ setting) using equation (1.1). To verify that the homomorphism space is 1-dimensional, it remains to check that $C_{\mathsf{S}} \in L(\beta)$ for $\mathsf{S} \in \mathrm{SStd}^+(\beta, \alpha)$ for each S such that $\deg(\mathsf{S}) = 0$. We will not need the dimension result in what follows and so we leave this as an exercise for the reader. □

4. A characteristic-free BGG-complex in the quiver Hecke algebra and characteristic-free bases of simple modules

We are now ready to prove the main result from the introduction over $\Bbbk$ an arbitrary field. Given $\alpha \in \mathcal{F}^\ell_n(h)$, we define an associated $R_n(\kappa)$-complex and show that this complex forms a BGG resolution of $D_n(\alpha)$. We simultaneously construct bases and representing matrices for $D_n(\alpha)$ and completely determines its restriction along the tower of cyclotomic quiver Hecke algebras.

Proposition 4.1. *Let $e > h\ell$. For $\lambda \in \mathcal{F}^\ell_n(h)$, we set*

$$\mathbf{C}_\bullet(\lambda) := \bigoplus_{\lambda \trianglerighteqslant \mu} \mathbf{\Delta}_n(\mu)[\ell(\mu)]$$

where for $\ell \geqslant 0$ we define the differential is the homomorphism of graded degree t^1 given as follows

$$\boldsymbol{\delta}_\ell = \sum_{\lambda \trianglerighteqslant \mu \triangleright \nu} \varepsilon(\mu, \nu) \varphi^\mu_\nu$$

for $\varepsilon(\mu, \nu) \in \Bbbk \setminus \{0\}$ assigned arbitrarily, providing that each diamond in the complex satisfies

$$\varepsilon_{\alpha,\beta,\gamma,\delta} = -\varepsilon(\alpha, \gamma)\varepsilon(\alpha, \delta)\varepsilon(\gamma, \beta)\varepsilon(\delta, \beta). \tag{4.1}$$

In which case, we have that $\mathrm{Im}(\boldsymbol{\delta}_{\ell+1}) \subseteq \ker(\boldsymbol{\delta}_\ell)$, in other words $\mathbf{C}_\bullet(\lambda)$ is a complex.

Proof. This is a standard argument using Theorem 3.16 and the fact that if $\ell(\alpha) = \ell(\beta) + 2$, then there exists at most two ℓ-partitions, γ and δ say, such that $\alpha \rhd \gamma, \delta \rhd \beta$. □

We now apply the Schur functor to the above to obtain a complex of modules in the quiver Hecke algebra as follows,

$$C_\bullet(\lambda) := \mathsf{E}_\omega \mathbf{C}_\bullet(\lambda) = \bigoplus_{\lambda \trianglerighteqslant \mu} S_n(\mu)[\ell(\mu)] \qquad \text{with} \qquad \mathsf{E}_\omega \boldsymbol{\delta}_\ell = \delta_\ell.$$

Theorem 4.2. *Let $e > h\ell$, let $\kappa \in I^\ell$ be h-admissible, let $\Bbbk$ be a field, and $\lambda \in \mathcal{F}^\ell_n(h)$. The $R_n(\kappa)$-complex $C_\bullet(\lambda)$ is exact except in degree zero, where*

$$H_0(C_\bullet(\lambda)) = D_n(\lambda).$$

We have $D_n(\lambda) = \Bbbk\{c_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\}$ and $\mathrm{rad}(S_n(\lambda)) = \Bbbk\{c_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}(\lambda) \setminus \mathrm{Std}_e(\lambda)\}$. Furthermore,

$$\mathrm{res}^n_{n-1}(D_n(\lambda)) = \bigoplus_{\square \in \mathcal{F}_h(\lambda)} D_n(\lambda - \square).$$

Proof. We assume, by induction, that if $\lambda \in \mathcal{F}^\ell_{n-1}(h)$, then the complex $C_\bullet(\lambda)$ forms a BGG resolution and that $\{c_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\}$ forms a basis of the simple module $D_n(\lambda)$. We now assume that $\lambda \in \mathcal{F}^\ell_n(h)$ and consider the complex $C_\bullet(\lambda)$. We have that

$$\mathrm{res}^n_{n-1}(C_\bullet(\lambda)) = \bigoplus_{r \in I} \mathsf{E}_r(C_\bullet(\lambda)).$$

We now consider one residue at a time. As λ belongs to an alcove, we have that λ (and any $\mu \lhd \lambda$) has either 0 or 1 removable r-boxes for each $r \in I$. We let $\mathsf{E}_r(\lambda)$ denote the unique ℓ-composition (respectively ℓ-partition) which differs from λ by removing an r-node. For each residue, there are two possible cases.

- We have that $\mathsf{E}_r(\lambda)$ lies on an alcove wall or $\mathsf{E}_r(\lambda) \notin \mathscr{P}^\ell_n$. By restriction, we have that $\mathrm{Im}(\mathsf{E}_r\delta_{\ell+1}) \subseteq \ker(\mathsf{E}_r\delta_\ell)$ and so $\mathsf{E}_r(C_\bullet(\lambda))$ forms a complex. We have that $\mathsf{E}_r(\lambda)$ is fixed by reflection through some hyperplane and the ℓ-compositions of n which dominate $\lambda \in \mathcal{F}^\ell_n(h)$ come in pairs (μ^+, μ^-) with $\mu^- \rhd \mu^+$ and $\ell(\mu^+) = \ell(\mu^-) + 1$ and furthermore such that

$$\mathsf{E}_r(\mu^+) = \mathsf{E}_r(\mu^-) = \mu \in \widetilde{\mathfrak{S}}_{h\ell} \cdot (\mathsf{E}_r(\lambda)).$$

We have that

$$\mathsf{E}_r(S_n(\mu^+)) = \mathsf{E}_r(S_n(\mu^-)) = \begin{cases} 0 & \text{if either } \mu^+ \notin \mathscr{P}_n^\ell \text{ or } \mu^- \notin \mathscr{P}_n^\ell \\ S_{n-1}(\mu) & \text{otherwise} \end{cases}$$

Thus our complex $\mathsf{E}_r(C_\bullet(\lambda))$ decomposes into two chains of identical modules as follows,

$$\mathsf{E}_r(C_\bullet(\lambda)) = \bigoplus_{\mu \lhd \lambda - \square} S_n(\mu)[\ell(\mu) - 1]) \bigoplus_{\mu \lhd \lambda - \square} S_n(\mu)[\ell(\mu)]. \tag{4.2}$$

Given $\mu \lhd \lambda - \square$, the restriction of $\phi^{\mu^+}_{\mu^-} \in \mathrm{Hom}_{R_n}(S_n(\mu^+), S_n(\mu^-))$ is equal to

$$1_\mu \in \mathrm{End}_{R_n(\kappa)}(S_n(\mu)) \tag{4.3}$$

by the construction of $\phi^{\mu^+}_{\mu^-}$ in Theorem 3.3 and [Bow, Theorem 6.1]. By restriction, we have

$$\mathrm{Im}(\mathsf{E}_r(\delta_{\ell+1})) \subseteq \ker(\mathsf{E}_r(\delta_\ell))$$

and by equation (4.3), we have that $\mathsf{E}_r\delta_{\ell+1} = \sum_{\ell(\mu)=\ell+1} 1_\mu + \dots$ and so the complex is exact. We conclude that $H(\mathsf{E}_r(C_\bullet(\lambda)) = 0$.

- We have that $\mathsf{E}_r(\lambda) \in \mathcal{F}^\ell_{n-1}(h)$. Then $\mathsf{E}_r(C_\bullet(\lambda))$ is given by

$$\mathsf{E}_r\left(\bigoplus_{\lambda \trianglerighteq \mu} (S_n(\mu))[\ell(\mu)]\right)$$

with homomorphisms $\mathsf{E}_r\delta_\ell : \mathsf{E}_r C^\ell_n \to \mathsf{E}_r C^{\ell-1}_n$. We have that

$$\mathsf{E}_r S_n(\mu)[\ell(\mu)] = S_{n-1}(\mu - \square)[\ell(\mu - \square)]$$

if $\mathrm{Rem}_i(\mu) \neq \emptyset$ and is zero otherwise. In the non-zero case, this is simply because $\mu - \square$ belongs to the same alcove as μ (and therefore the lengths coincide) for $\mu \trianglelefteq \lambda$. Now, for a pair μ, μ' with $\square \in \mathrm{Rem}_i(\mu)$ and $\square' \in \mathrm{Rem}_i(\mu')$, we have that $\mathsf{E}_r\phi^\mu_{\mu'} = \phi^{\mu-\square}_{\mu'-\square'}$ (again by the construction of $\phi^\mu_{\mu'}$ in Theorem 3.3 and [Bow, Theorem 6.1]). Thus $\mathsf{E}_r(C_\bullet(\lambda)) = C_\bullet(\lambda - \square)$ and the righthand-side is exact except $H_0(C_\bullet(\lambda - \square)) = D_n(\lambda - \square)$ by our inductive assumption. Thus $\mathsf{E}_r(H_0(C_\bullet(\lambda))) = H_0(C_\bullet(\lambda - \square)) = D_n(\lambda - \square)$ and $\mathsf{E}_r(H_j(C_\bullet(\lambda))) = 0$ for all $j > 0$.

Putting all of the above together, we have shown that

$$\mathrm{res}^n_{n-1}(H_j(C_\bullet(\lambda))) = \begin{cases} \bigoplus_{\square \in \mathcal{F}_h(\lambda)} D_n(\lambda - \square) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Now, since $\mathrm{Head}(S_n(\lambda)) = D_n(\lambda) \notin \mathrm{Im}(\delta_1)$, we are able to conclude that

$$\mathrm{res}^n_{n-1}(D_n(\lambda)) \subseteq \bigoplus_{\square \in \mathcal{F}_h(\lambda)} D_n(\lambda - \square). \tag{4.5}$$

Conversely, we have that

$$|\mathrm{Std}_e(\lambda)| = \sum_{\square \in \mathcal{F}_h(\lambda)} |\mathrm{Std}_e(\lambda - \square)| \tag{4.6}$$

by Proposition 2.10. By induction, the righthand-side of equation (4.6) is equal to the dimension of the righthand-side of equation (4.5). The lefthand-side of equation (4.6) is a lower bound for the dimension of the lefthand-side of equation (4.5). Putting these two things together, we deduce that

$$\mathrm{res}^n_{n-1}(D_n(\lambda)) = \bigoplus_{\square \in \mathcal{F}_h(\lambda)} D_n(\lambda - \square) \tag{4.7}$$

and furthermore, the set $\{c_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\}$ does indeed form a basis of $D_n(\lambda)$; to obtain the basis of the radical, recall that $e_{\mathsf{t}} L(\mu) = 0$ for $\lambda \rhd \mu$ and $\mathsf{t} \in \mathrm{Std}_e(\lambda)$. Putting equation (4.4) and equation (4.7) together, we have that

$$\mathrm{res}^n_{n-1}(H_j(C_\bullet(\lambda))) = \begin{cases} \mathrm{res}^n_{n-1} D_n(\lambda) & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \qquad H_j(C_\bullet(\lambda)) = \begin{cases} D_n(\lambda) & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

where the second equality follows because $\mathrm{res}^n_{n-1} D_n(\mu) \neq 0$ for any $\lambda \trianglerighteq \mu$ (even though $\mathsf{E}_r(D_n(\mu)) = 0$ is possible for a given $r \in I$, as seen above). □

Note that the restriction rule was used as the starting point in [Kle96a], where Kleshchev obtains results concerning the dimensions of simple modules. Weirdly, our proof deduces that the homology of the complex is equal to $D^{\Bbbk}_n(\lambda)$, that the basis $D^{\Bbbk}_n(\lambda)$ is of the stated form, and the restriction of the simple module is of the stated form all at once!

Theorem 4.3. *For $\lambda \in \mathcal{F}^\ell_n(h)$ the action of $R_n(\kappa)$ on $D_n(\lambda) = \Bbbk\{c_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}_e(\lambda)\}$ is as follows:*

$$y_k(c_{\mathsf{s}}) = 0 \qquad 1^{\underline{i}}_{\omega}(c_{\mathsf{s}}) = \delta_{\underline{i},\mathrm{res}(\mathsf{s})} \qquad \psi_r(c_{\mathsf{s}}) = \begin{cases} c_{\mathsf{s}_{k\leftrightarrow k+1}} & \text{if } |\mathrm{res}(\mathsf{s}(r)) - \mathrm{res}(\mathsf{s}(r+1))| > 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathsf{s}_{k\leftrightarrow k+1}$ is the tableau obtained from s by swapping the entries k and $k+1$. In particular, the subalgebra $\langle y_k, 1^{\underline{i}}_{\omega} \mid 1 \leqslant k \leqslant r, \underline{i} \in I^n\rangle \leqslant R_n(\kappa)$ acts semisimply on $D_n(\lambda)$. The weight-spaces of $D_n(\lambda)$ are all 1-dimensional and $D_n(\lambda)$ is concentrated in degree zero only. Finally, the cellular bilinear form is given by $\langle c_{\mathsf{s}}, c_{\mathsf{t}}\rangle = \delta_{\mathsf{s},\mathsf{t}}$ for $\mathsf{s},\mathsf{t} \in \mathrm{Std}_e(\lambda)$.

Proof. The statements not relating to the action follow from Proposition 2.10 and equation (2.4) and Theorem 4.2. The action of the idempotents is obvious. The other zero-relations all follow because the product has non-zero degree (and the module $D_n(\lambda)$ is concentrated in degree 0). Finally, assume $|\mathrm{res}(\mathsf{s}(r)) - \mathrm{res}(\mathsf{s}(r+1))| > 1$. The strands terminating at $(r,1,\ell)$ and $(r+1,1,\ell)$ on the northern edge either do or do not cross. In the former case, we can resolve the double crossing in $\psi_r c_{\mathsf{s}}$ without cost by our assumption on the residues and the result follows. The latter case is trivial. Finally, notice that $\mathsf{s}_{k\leftrightarrow k+1} \in \mathrm{Std}_e(\lambda)$ under the assumption that $|\mathrm{res}(\mathsf{s}(r)) - \mathrm{res}(\mathsf{s}(r+1))| > 1$. □

Remark 4.4. *Let $p > 0$. Combinatorially computing the composition series of $S_n(\lambda)$ for $\lambda \in \mathscr{P}^1_n(h)$ for arbitrary primes seems to be an impossible task [Wil17b]. If we assume that $p \gg h$ is suitably large then we can use Kazhdan–Lusztig theory to combinatorially calculate $\dim_{\Bbbk}(D_n(\lambda))$, this requires (as a minimum) that* **all partitions** *$\mu \in \mathscr{P}^1_n(h)$ such that $\mu \lhd \lambda$ belong to the first p^2-alcove [RW16]. This is equivalent to the requirement that the p-weight of λ (defined in Section 5) is less than p. For $h = 3$ this combinatorics has been conjecturally extended (in terms of billiards in an alcove geometry) to the first p^3-alcove [LW18a]. We stress that there is no restriction on the p-weight of $\lambda \in \mathcal{F}^1_n(h)$. Therefore understanding the composition series of unitary Specht modules is well beyond the current state of the art. Thus our two descriptions of the simple modules $D_n(\lambda)$ for $\lambda \in \mathcal{F}^1_n(h)$ provide the only contexts in which these modules can currently be hoped to be understood.*

Example 4.5 ([BC18, Proposition 7.6])**.** *Let $\ell \geqslant 2$, $e = \ell+1$, and $\kappa = (0,1,2,\dots,\ell-1) \in (\mathbb{Z}/e\mathbb{Z})^\ell$, and $\Bbbk$ be arbitrary. We have that $\lambda := ((n),(n),\dots,(n)) \in \mathcal{F}^\ell_{n\ell}(h)$ and that*

$$[S_{n\ell}((n),(n),\dots,(n)) : D_{n\ell}(\nu)] = t^{\ell(\nu)} + \dots \tag{4.8}$$

modulo terms of lower order degree. Therefore every simple module $D(\nu)$ for $\lambda \rhd \nu \in \mathscr{P}^\ell_{n\ell}(1)$ appears with multiplicity at least 1. Therefore as $n \to \infty$, the number of composition factors of $S_{n\ell}((n),(n),\dots,(n))$ tends to infinity and so is impossible to compute. In contrast, the module $D_{n\ell}(\nu)$ is 1-dimensional and easily seen to be spanned by c_{t^λ} for t^λ as in Definition 1.3.

5. Symmetric group combinatorics: e-abaci

We now discuss how the combinatorial description of resolutions simplifies for (diagrammatic) Cherednik algebras of symmetric groups. In this case, we choose to emphasize the abacus presentation of partitions. We first recall this classical combinatorial approach, then flesh out the notion of homological degree introduced in [ZGS14] that is key to [ZGS14, Conjecture 4.5], and finally identify all this as the level 1 case of the alcove geometry already studied in the previous sections.

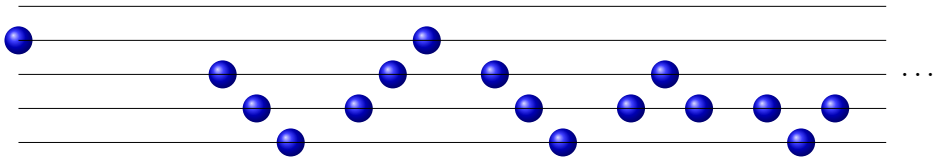
5.1. **The abacus of a partition.** Let $\lambda \in \mathscr{P}^1_n(h)$. Then λ can be encoded by an abacus with at least h beads, where each bead stands for a column of λ. This is simply a sequence of spaces and beads which records the shape of the border of λ, since knowing the border of λ is the same as knowing λ. We form the $\mathbb{Z}$-abacus $\mathcal{A}^h_{\mathbb{Z}}(\lambda)$ with h beads by walking along the border from the top

right corner to the bottom left corner of the Young diagram of λ, writing a space every time we walk down and a bead every time we walk left.

Example 5.1. *The $\mathbb{Z}$-abaci of $(3^4,1),(3^3,2,1^2)\in\mathscr{P}^1_{13}(3)$ with 3 beads are as follows*

Fix $e\geqslant 2$. We obtain an e-abacus $\mathcal{A}_e(\lambda)$ by looping the $\mathbb{Z}$-abacus around e horizontal runners. This can be described as follows: subdivide $\mathcal{A}^h_{\mathbb{Z}}(\lambda)$ into segments of length e starting from the leftmost position, then rotate each segment counterclockwise by ninety degrees so that it is vertical. The partition is now written on e horizontal runners. Thus the runners of our e-abacus resemble a musical staff, and $\mathcal{A}_e(\lambda)$ resembles sheet music. Like a staff, the runners of $\mathcal{A}_e(\lambda)$ are bounded to the left. We let them extend infinitely to the right, because we want to think of being able to move beads in that direction by adding boxes or e-strips at the bottom of the Young diagram of a partition. In French a musical score is called a *partition*, so we may say that our abaci are written in the French style.

Example 5.2. *A 5-abacus.*

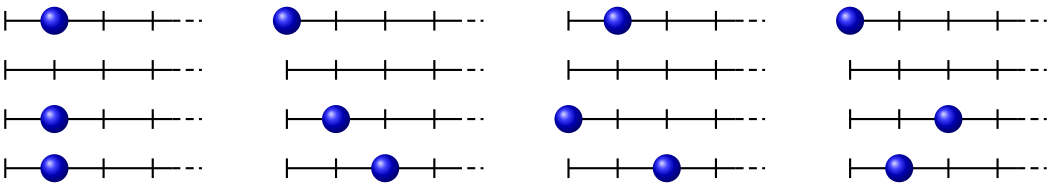


We let $w_e(\lambda)$ denote the total number of vacant spots which have a bead to their right and refer to this as the e-weight. If $w(\rho)=0$ then we say that ρ is an e-core. Given a partition λ, we define the e-core of λ to be the partition obtained by moving all beads on $\mathcal{A}_e(\lambda)$ as far left as possible. We let

$$\Lambda(\rho,w):=\{\mu\vdash|\rho|+we\mid e\text{-core}(\mu)=\rho\}$$

for ρ an e-core.

Example 5.3. *The 4-abaci with 3 beads of $(3^4,1)$, $(3^3,2,1^2)$, $(3,2^5)$ and $(3^3,1^4)\in\mathscr{P}^1_{13}(3)$ are as follows*



We have that $w_4(\lambda)=3$ and 4-core$(\lambda)=(1)$ for each of these examples.

Remark 5.4. *Note that for $\mu\in\mathscr{P}^1_n(h)$, its removable box of highest content has content $h-k$, where k is the position of the first bead in the $\mathbb{Z}$-abacus $\mathcal{A}^h_{\mathbb{Z}}(\mu)$. In particular, this bead sits in runner k mod e in the e-abacus $\mathcal{A}^h_e(\mu)$. Thus, in order to make the labels of the runners of the e-abaci in μ correspond in a nice way to the contents of addable and removable i-boxes of the partitions, one should label the runners from bottom to top by $h-1,h-2,\dots,1,0,e-1,e-2,\dots,h+1,h$. With this convention, removing a box of content i mod e corresponds to moving a bead on runner $i-1$ down to runner i; and adding a box of content i mod e corresponds to moving a bead on runner i up to runner $i-1$.*

5.2. **e-unitary partitions and posets.** We recall the definition of e-unitary partitions from [ZGS14] and show that these are precisely the partitions in $\mathcal{F}^1_n=\cup_{h\geqslant 1}\mathcal{F}^1_n(h)$ studied in this paper.

Definition 5.5. [ES09, ZGS14] *Fix $e\geqslant 2$. Suppose λ has exactly h columns and form $\mathcal{A}^h_{\mathbb{Z}}(\lambda)$ the abacus on h beads. We call λ an e-unitary partition if all the beads on $\mathcal{A}^h_{\mathbb{Z}}(\lambda)$ lie in an interval of width e. In particular, $\mathcal{A}_e(\lambda)$ has at most one bead on each runner. Given an e-unitary partition λ, we let $\mathrm{Po}_e(\lambda)$ denote the set of all the e-abaci obtained from λ by successively moving a bead on some runner one step to the right so long as we also move a bead on a different runner one step to the left.*

Proposition 5.6. *The set $\mathcal{F}_n^1 = \cup_{h\geqslant 1}\mathcal{F}_n^1(h)$ is precisely equal to the set of e-unitary partitions.*

Proof. Suppose $\lambda \in \mathcal{P}_n^1$ has exactly h columns and let γ_h, γ_1 denote the positions of the leftmost and rightmost beads on $\mathcal{A}_{\mathbb{Z}}^h(\lambda)$. Now simply note that $\gamma_1 - \gamma_h \leqslant e-1$ if and only if $\langle \lambda+\rho, \varepsilon_1 - \varepsilon_h\rangle < e-1$ if and only if $\lambda \in \mathcal{F}_n^1(h)$. □

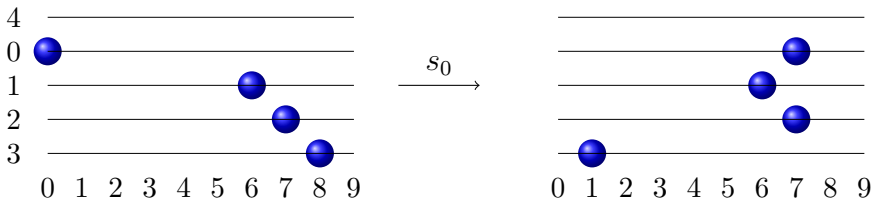
Example 5.7. *When $e = 4$, $(3^4, 1)$ is a 4-unitary partition, and $(3^3, 2, 1^2), (3, 2^5), (3^3, 1^4) \in \mathrm{Po}_4(\lambda)$.*

Remark 5.8. *If $e = h$ then λ is e-unitary if and only if $\lambda = (e^k)$ for some $k \in \mathbb{N}$. If λ is an e-unitary partition, then any $\mu \in \mathrm{Po}_e(\lambda)$ is always e-restricted unless $\lambda = (e^k)$ and $\mu = \lambda$.*

If an e-abacus $\mathcal{A}_e(\mu)$ has at most one bead on each runner, let b_i be the unique bead on the runner labeled i if such a bead exists, and let $\beta_i \in \mathbb{Z}_{\geqslant 0}$ be the horizontal position of b_i. Sometimes by abuse of notation we might just refer to β_i as a bead.

5.3. **The affine and extended affine symmetric group actions.** There is a natural action of the affine symmetric group $\widetilde{\mathfrak{S}}_h$ on $\mathrm{Po}_e(\lambda)$ when we take the presentation of $\widetilde{\mathfrak{S}}_h$ given by generators s_i, $i \in \mathbb{Z}/h\mathbb{Z}$, subject to the relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ if $|i-j| > 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (where all subscripts are taken mod h). $\mathfrak{S}_h = \langle s_1, \ldots, s_{h-1}\rangle$ acts by permutation of the h runners containing beads, while s_0 switches the top and bottom beads in the abacus, then moves the bottom bead one step to the right and the top bead one step to the left. From the description of $\mathrm{Po}_e(\lambda)$ in Definition 5.5, $\widetilde{\mathfrak{S}}_h$ acts transitively on $\mathrm{Po}_e(\lambda)$.

Example 5.9. *Illustration of the action of s_0:*



The extended affine symmetric group $\widehat{\mathfrak{S}}_h$ is the semidirect product $\mathbb{Z}^h \rtimes \mathfrak{S}_h$. There is a natural action of $\widehat{\mathfrak{S}}_h$ on the set of e-abaci with exactly one bead on a fixed subset of h runners, and no beads on the other runners: $\mathbb{Z}^h$ acts as the group of horizontal translations of the beads on their runners, and $\mathfrak{S}_h$ as permutations of the h runners containing the beads. This action is locally nilpotent for the subgroup $\mathbb{Z}_{<0}^h$ consisting of left translations of the beads. In terms of partitions, the meaning is as follows: let ρ be an e-core of some unitary partition; equivalently, $\mathcal{A}_e(\rho)$ has its beads pushed all the way to the left and they are concentrated in the leftmost column of the e-abacus. Let $\mathcal{P}_e(\rho)_h$ be the union of all $\mathrm{Po}_e(\lambda)$, λ an e-unitary partition such that the e-core of λ is ρ and λ has h columns. Let $\epsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^h$ with the 1 in the i'th position. Then ϵ_i acts on $\mu \in \mathcal{P}_e(\rho)_h$ by shifting the bead on the i'th runner containing a bead one unit to the right; on the Young diagram of μ it adds an e-rimhook whose arm-length is at most $h-1$. Observe that $\mathcal{P}_e(\rho)_h$ is generated by $\mathcal{A}_e(\rho)$ under the action of $\widehat{\mathfrak{S}}_h$:

$$\widehat{\mathfrak{S}}_h \cdot \mathcal{A}_e(\rho) = \mathcal{P}_e(\rho)_h$$

$\mathcal{P}_e(\rho)_h$ is naturally identified with the monoid $\mathbb{Z}_{\geqslant 0}^h$ as a left $\widehat{\mathfrak{S}}_h$-module by identifying an abacus $\mathcal{A} \in \mathcal{P}_e(\rho)_h$ with the h-tuple of its beads' positions $(\beta_1, \ldots, \beta_h) \in \mathbb{Z}_{\geqslant 0}^h$.

5.4. **The homological degree statistic.** Let λ be an e-unitary partition. We recall the homological degree statistic on $\mathrm{Po}_e(\lambda)$ introduced by Berkesch-Griffeth-Sam.

Definition 5.10. *Suppose $\mathcal{A}$ is an e-abacus with at most one bead on each runner. A disorder of $\mathcal{A}$ is an unordered pair $\{i, j\}$ such that runners i and j both contain a bead, satisfying $\beta_i > \beta_j$ and b_j is above b_i. In other words, a pair of beads of $\mathcal{A}$ yields a disorder if one bead is above and strictly to the left of the other bead.*

Definition 5.11. [ZGS14, Definition 4.3] *Let $\mu \in \mathrm{Po}_e(\lambda)$. The homological degree of μ, written $\mathrm{hd}(\mu)$, is the sum of the differences of all horizontal positions of beads in $\mathcal{A}_e(\mu)$ minus the number of disorders of $\mathcal{A}_e(\mu)$:*

$$\mathrm{hd}(\mu) = \sum_{\substack{i,j \in \mathbb{Z}/eZ \\ b_i, b_j \neq \emptyset \\ b_j \text{ is below } b_i}} |\beta_i - \beta_j| - \#\{\text{disorders of } \mathcal{A}_e(\mu)\}$$

Example 5.12. *In Example 5.9, let ν denote the partition whose abacus is on the left, and let $\mu = s_0(\nu)$ as in the picture. Then $\mathcal{A}_5(\nu)$ has 6 disorders and $\mathrm{hd}(\nu) = 1+2+8+1+7+6-6 = 19$; $\mathcal{A}_5(\mu)$ has 1 disorder and $\mathrm{hd}(\mu) = 6+5+6+1+1-1 = 18$. Observe that s_0 changed the homological degree by 1.*

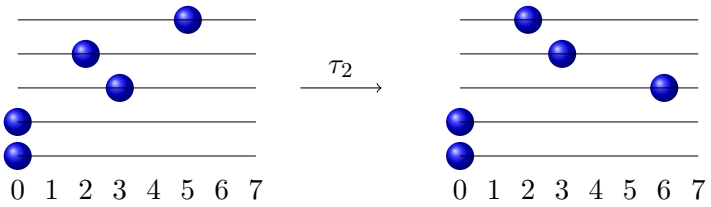
5.5. **Homological degree produced recursively by elements of $\widehat{\mathfrak{S}}_e$.** Notice that empty runners of $\mathcal{A}_e(\mu)$ play no role in $\mathrm{hd}(\mu)$; if the empty runners are removed from $\mathcal{A}_e(\mu)$, the homological degree remains the same. For simplicity of the formulas and exposition, we therefore work in the case that there are no empty runners, that is, $h = e$ columns and $\lambda = (e^k)$ for some $k \in \mathbb{N}$. The empty runners can be put back in at the end.

Our first characterization of the homological degree produces this statistic recursively starting from the empty partition by applying sequences of special elements $\tau_i \in \widehat{\mathfrak{S}}_e$, $i = e-1, e-2, \ldots, 1, 0$, in a non-increasing order with respect to i.

Definition 5.13. *Let $\tau_i \in \widehat{\mathfrak{S}}_e$ be defined as follows: τ_i fixes the bottom i runners; on the top $e-i$ runners, it first cyclically rotates the beads in the topwards direction, then shifts one space to the right the bead on the $(e-i)$'th runner from the top.*

Each τ_i is the "affine generator" of the subgroup $\widehat{\mathfrak{S}}_{e-i}$ of $\widehat{\mathfrak{S}}_e$ which fixes the bottom i runners: τ_i together with $\mathfrak{S}_{e-i}$ generates $\widehat{\mathfrak{S}}_{e-i}$ [LT00, Section 2.1]. We are interested in applying τ_i to abaci whose bottom i runners have their beads pushed all the way to the left.

Example 5.14. *Consider the 5-abacus of $(3^{11}, 2^3, 1^{11})$. Then τ_2 acts as follows:*



Observe that τ_2 increased the homological degree of the abacus by 2.

Suppose τ is a partition all of whose parts are of size at most $e-1$, and which may contain parts of size 0, so $\tau = ((e-1)^{a_{e-1}}, (e-2)^{a_{e-2}}, \ldots, 1^{a_1}, 0^{a_0})$. Thus τ fits inside an $e-1$ by k box, where k is the total number of parts of τ. Now identify τ with the element of $\widehat{\mathfrak{S}}_e$ given by the composition of operators $\tau_0^{a_0}\tau_1^{a_1}\ldots\tau_{e-1}^{a_{e-1}}$. By abuse of notation we will also call this element τ. The proof of the following lemma is straightforward:

Lemma 5.15. *Let $\tau = ((e-1)^{a_{e-1}}, (e-2)^{a_{e-2}}, \ldots, 1^{a_1}, 0^{a_0})$ with $\sum_{i=0}^{e-1} a_i = k \in \mathbb{Z}_{\geqslant 0}$. Then $\tau(\mathcal{A}_e(\emptyset)) = \mathcal{A}_e(\mu)$ with $\mu \in \mathrm{Po}_e(e^k)$. Any $\mu \in \mathrm{Po}_e(e^k)$ is produced in this way from a unique such τ, and we have:*

$$\mathrm{hd}(\mu) = \sum_{i=0}^{e-1} i a_i = |\tau|$$

Let λ be an arbitrary e-unitary partition. By removing the empty runners from the e-abaci in $\mathrm{Po}_e(\lambda)$, there is likewise a natural bijection between the partitions μ in $\mathrm{Po}_e(\lambda)$ and partitions τ which fit inside an $(h-1)$ by k box,

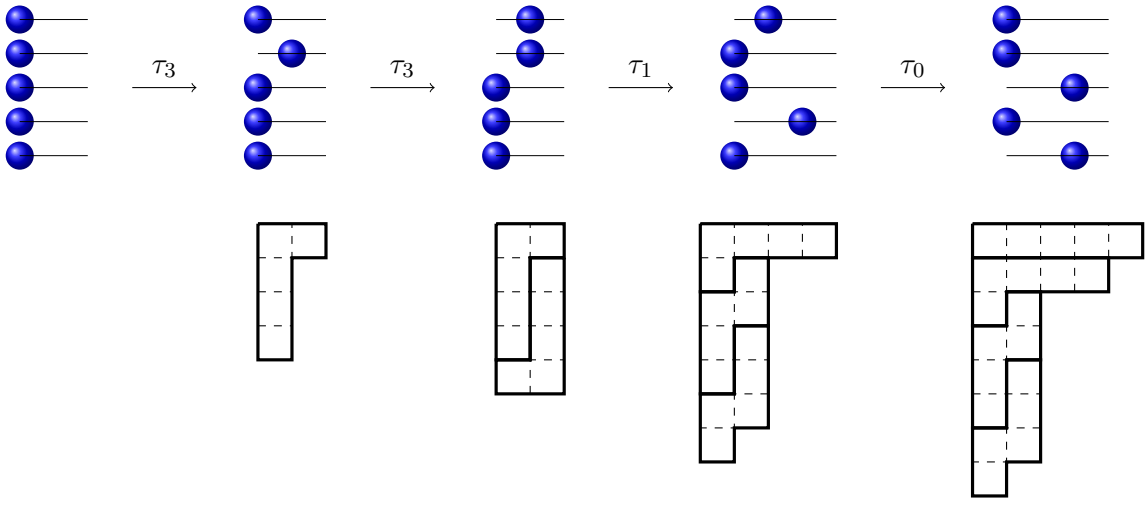
$$\{\mu \in \mathrm{Po}_e(\lambda)\} \overset{\Phi}{\xleftarrow{\simeq}} \{\tau \subset (h-1)^k\},$$

given by $\Phi(\tau) = \tau(\emptyset)$ (where τ on the right-hand-side is the corresponding element of $\widehat{\mathfrak{S}}_h$ as described above). This bijection identifies $\mathrm{hd}(\mu)$ with $|\tau|$.

Remark 5.16. *Such a bijection turns up elsewhere in representation theory: notably, partitions τ which fit inside an $(h-1)$ by k box also parametrize (1) the simple and standard modules of a regular block $\mathcal{B}^{\mathfrak{p}}$ of parabolic category $\mathcal{O}^{\mathfrak{p}}$ for $\mathfrak{gl}(h-1+k)$ with respect to the maximal parabolic $\mathfrak{gl}(h-1)\times\mathfrak{gl}(k)$ [Str09]; (2) the Schubert cells in the Grassmannian $\mathsf{Gr}(k,h-1+k) = \mathsf{Gr}(h-1,h-1+k)$. The category $\mathcal{B}^{\mathfrak{p}}$ is equivalent to perverse sheaves on the Grassmannian [Bra02, Str09], explaining the coincidence of (1) and (2). Let $L_{(h-1)^k}$ denote the simple module in $\mathcal{B}^{\mathfrak{p}}$ labeled by $\tau = (h-1)^k$, the unique maximal element of the poset (the poset structure is given by inclusion of Young diagrams). The bijection following Lemma 5.15 identifies the characters of unitary $L(\lambda) \in [\mathcal{O}_{1/e}(\mathfrak{S}_n)]$ and $L_{(h-1)^k} \in [\mathcal{O}^{\mathfrak{p}}]$. Moreover, $L_{(h-1)^k}$ has a BGG resolution [BH09] and via the bijection Φ we obtain a natural bijection between the Verma modules appearing in degree i of the respective resolutions in $\mathcal{O}_{1/e}(S_n)$ and $\mathcal{O}^{\mathfrak{p}}$. However, the categories $\mathcal{O}_{1/e}(\mathfrak{S}_n)_{\leqslant\lambda}$ and $\mathcal{B}^{\mathfrak{p}}$ are not equivalent if $k>2$, and as a poset $\mathrm{Po}_e(\lambda)$ has "extra edges" coming from the $\widetilde{\mathfrak{S}}_h$-action if $k>2$. The difference between the resolutions of $L(\lambda)$ and $L_{(h-1)^k}$ thus manifests itself in the maps in the complex.*

5.6. **Homological degree via rimhooks of minimal leg-length.** Consider again the case there is exactly one bead on every runner of the abacus. By the definition of τ_i, it follows that the effect of applying $\tau = ((e-1)^{a_{e-1}}, (e-2)^{a_{e-2}}, \dots, 1^{a_1}, 0^{a_0})$ to the empty partition is to build a Young diagram λ by successively dropping e-rimhooks which meet the leftmost column (with leg-lengths $e-1$ (a_{e-1} times), $e-2$ (a_{e-2} times) and so on) Tetris-style on top of the partition constructed so far, then letting the boxes slide down the columns so that the result is a partition. This can change the shape of the previous rimhooks that were added, but not the set of their leg-lengths. Thus we obtain a second combinatorial explanation of the homological degree: *if $e=h$ then $\mathrm{hd}(\lambda)$ is the sum of the leg-lengths of the e-rimhooks of minimal leg-length composing λ. If $e>h$ then $\mathrm{hd}(\lambda)$ is the sum of the leg-lengths of the e-rimhooks of minimal leg-length composing λ minus $(e-h)k$, where $k = e$-weight(λ).* This can be restated in a uniform way by considering the arm-lengths instead of the leg-lengths of the rimhooks: $\mathrm{hd}(\mu)$ is equal to $(h-1)k$ minus the sum of armlengths of the (minimal leg-length) rimhooks removed.

Example 5.17. *Let $e=h=5$ and $\tau = (3,3,1,0)$. Then $\tau(\emptyset) = (5,4,2^5,1) =: \lambda$ and $\mathrm{hd}(\lambda) = 7$. We show the process of applying τ on abaci and partitions and the four 5-rimhooks of minimal leg-lengths $3,3,1,0$ which compose λ:*



5.7. **Homological degree is the length function.** We now give a third combinatorial description of the homological degree by identifying it with the length function on $\widetilde{\mathfrak{S}}_h$. This unifies the combinatorics of abaci with that of alcove geometries and allows us to describe the BGG complex in type A in terms of abaci. Let λ be a unitary partition, and suppose λ has $h<e$ columns.

Lemma 5.18. *The following are equivalent for $\nu,\mu \in \mathrm{Po}_e(\lambda)$, $\mu \rhd \nu$:*

- *$\ell(\nu) = \ell(\mu)+1$ and ν is obtained from μ by moving a column of boxes as in Theorem 3.3;*

- *$t\mu = \nu$ for some transposition $t \in \widetilde{\mathfrak{S}}_h$ acting on abaci as above, subject to the following conditions on the beads β_i of $\mathcal{A}_e(\mu)$:*
  - *(1) if $t \in \mathfrak{S}_h$ and swaps runners i and j, then for each runner k between runners i and j, $\beta_k \notin [\beta_i, \beta_j]$;*
  - *(2) if t is conjugate to s_0 and acts nontrivially on runners i and j, runner i below runner j, then: for each runner k below runner i, $\beta_k \notin [\beta_i, \beta_j + 1]$ and for each runner ℓ above runner j, $\beta_\ell \notin [\beta_{i-1}, \beta_j]$.*

Therefore, the homological degree statistic on $\mathrm{Po}_e(\lambda)$ coincides with the length function on $\mathrm{Po}_e(\lambda)$ coming from the $\widetilde{\mathfrak{S}}_h$ alcove geometry.

Proof. This is a translation of 1-column moves from the language of Young diagrams into the language of abaci. A direct computation using Definition 5.11 shows that the conditions for a transposition t to increase the homological degree by 1 are exactly those given by (1) and (2). □

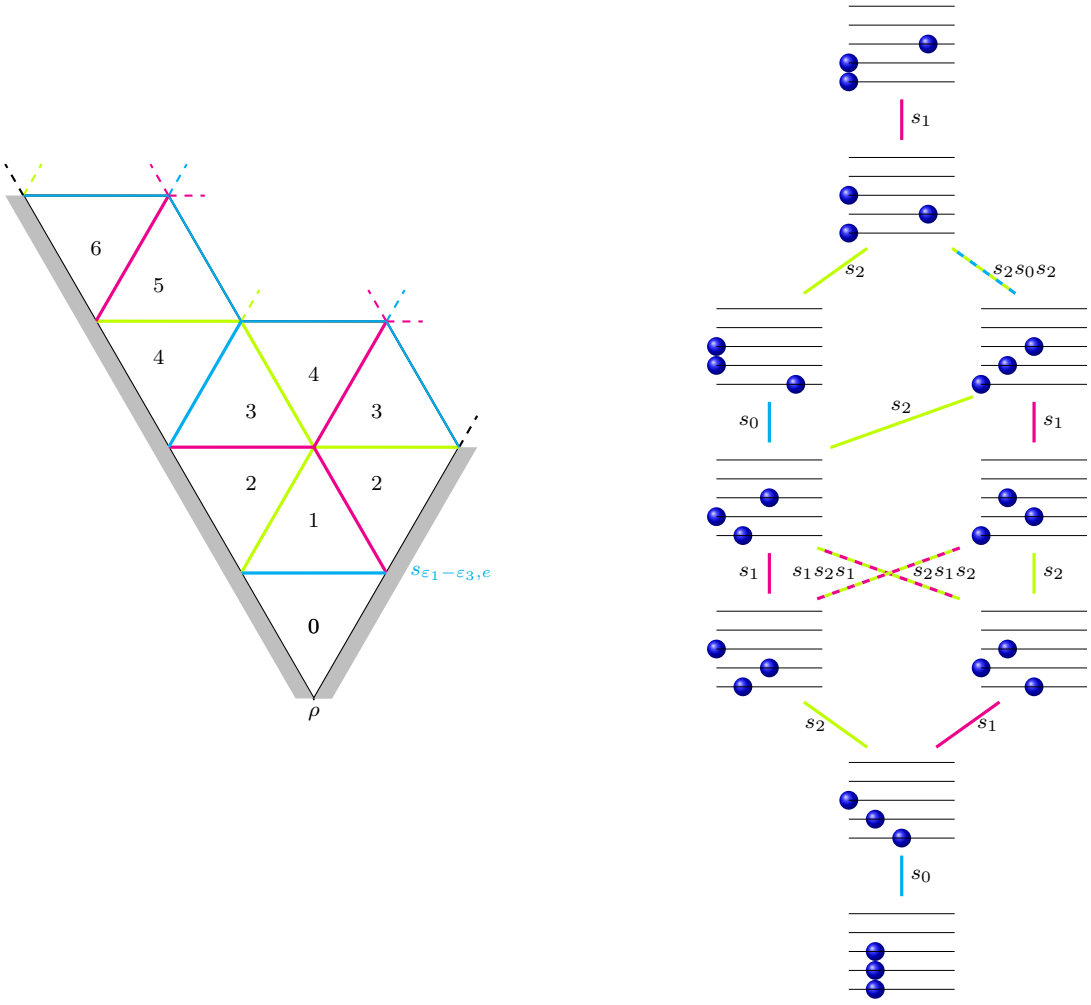
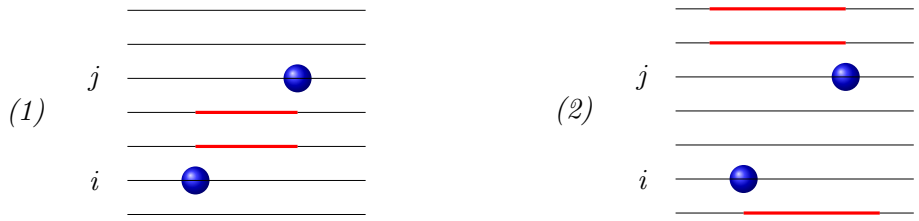


FIGURE 4. On the left we have the alcoves corresponding to partitions in $\mathrm{Po}_e(\lambda)$ when $h = 3$ and $e = 5$. The fundamental alcove is at the bottom and contains $(3^5) \in \mathcal{F}_{15}(3)$. Each alcove contains a number indicateing the length/homological degree for a point in that alcove. The grey region denotes the non-dominant region. The dotted lines indicate that we tile one sixth of $\mathbb{R}^2$ when we let $n \to \infty$. Crossing a wall of color i corresponds to applying s_i to the partition in that alcove, with i: 0, 1, 2. On the right-hand side, we have extracted the poset $\mathrm{Po}_e(\lambda)$. The homological degree increases from the bottom (where it is zero) to the top (where it is 6). The edges of the poset are coloured and decorated so as to facilitate comparison between the two pictures.

The superficial difference between the two $\widetilde{\mathfrak{S}}_h$ actions on $\mathrm{Po}_e(\lambda)$, the one coming from abaci, the other from alcove geometry, is simply the difference between generating sets for $\widetilde{\mathfrak{S}}_h$: the generators

s_i in the former case may be identified with the reflections across the hyperplane walls bordering the fundamental alcove in the latter case. Since all s_i play a symmetrical role, we can cyclically relabel the simple reflections s_i so that s_0 is the reflection across the unique wall that must be crossed to get out of the fundamental alcove while staying in $\mathrm{Po}_e(\lambda)$.

Example 5.19. *The conditions (1) and (2) on abaci in Lemma 5.18 in a picture: applying t will increase the homological degree by 1 if and only if no bead lies in the red regions of the runners.*



6. The Mullineux map on unitary simple modules

We first recall the Mullineux involution on the quiver Hecke algebra of the symmetric group: Let M denote the R_n-automorphism determined by

$$\mathrm{M} : e(i_1, i_2, \dots, i_n) \mapsto e(-i_1, -i_2, \dots, -i_n) \quad \mathrm{M} : \psi_r \mapsto \psi_r \quad \mathrm{M} : y_k \mapsto y_k \tag{6.1}$$

for $0 \leqslant k \leqslant n$ and $0 \leqslant r < n$ and $\underline{i} = (i_1, \dots, i_n) \in I^n$. Given a simple module $D_n^{\Bbbk}(\lambda)$, we let $D_n^{\Bbbk}(\lambda)^{\mathrm{M}}$ denote the module with the same underlying vector space but with the multiplication defined by twisting the action with the involution M. The relationship between these two simples was the subject of a conjecture of Mullineux [Mul79]. The combinatorics of this relationship is fiendishly complicated in general and is only understood on the level of the *labels* of simple modules. The purpose of this section is to examine the effect of the Mullineux map on the simple modules $D_n^{\Bbbk}(\lambda)$ for $\lambda \in \mathcal{F}_n^1$. We show that the set of these simples is preserved under the Mullineux involution. Moreover, we construct an explicit Mullineux isomorphism in terms of the bases and representing matrices of these simples given in Theorems 4.2 and 4.3 — we remark that this is the time the Mullineux isomorphism has been explicitly constructed (outside of the trivial semisimple case). Furthermore we shall see that the Mullineux combinatorics drastically simplifies on unitary e-regular partitions λ and that we can easily compute $\mathrm{M}(\lambda)$ on the e-abacus of λ. We define the unitary branching graph, $\mathcal{Y}$, to have vertices on level k given by

$$\mathcal{Y}_k = \{\lambda \mid \lambda \text{ is } e\text{-restricted and } \lambda \in \mathcal{F}_k^1 \}$$

and edges connecting levels k and $k+1$ given by

$$\mathcal{E}_{k,k+1} = \{\lambda \to \mu \mid \lambda \in \mathcal{Y}_k, \mu \in \mathcal{Y}_{k+1} \text{ and } \lambda = \mu - \square \text{ for } \square \text{ a good node}\}.$$

We first discuss how the abaci of an e-core ρ and its transpose ρ^t are obtained from one another when ρ has at most $e-1$ columns. Recall the basics of abaci from Section 5.1. First, note that if ρ has at most $h < e$ columns then ρ^t has at most $e-h$ columns. Now, let $\mathcal{A}_e^h(\rho)$ denote the e-abacus of ρ written with h beads, and perform the following procedure on it: (1) swap the empty spots and the beads in the first column (so that the resulting abacus has $e-h$ beads), then (2) flip this abacus upside down. The resulting abacus, $\mathcal{A}_e^{e-h}(\rho^t)$, is the e-abacus of ρ^t written with $e-h$ beads.

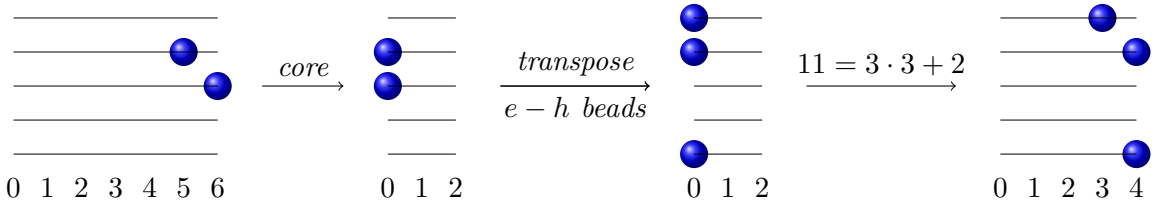
Definition 6.1. *Let $\lambda \in \mathcal{F}_n(h)$ for some $1 \leqslant h < e$ and let ρ be the e-core of λ. Write $w(\lambda) = (e-h)q + r$ for some $q \geqslant 0$, $0 \leqslant r < e-h$. Define λ_{M} to be the partition with abacus obtained from $\mathcal{A}_e^{e-h}(\rho^t)$ by moving the bottom r beads $(q+1)$-units to the right, and the top $e-h-r$ beads q units to the right.*

Proposition 6.2. *If $\lambda \in \mathcal{Y}_n$, then $\lambda_{\mathrm{M}} \in \mathcal{Y}_n$. Specifically: in the case $\lambda = \rho$, we have $\rho_{\mathrm{M}} = \rho^t$. Otherwise, we have $\lambda_{\mathrm{M}} \in \mathcal{F}_n(e-h)$.*

Proof. If $\lambda = \rho$ is an e-core, then $w(\lambda) = 0$ and algorithm just stops after the step where we take the transpose of ρ. The abacus $\mathcal{A}_e^{e-h}(\rho^t)$ clearly satisfies the criterion for unitarity (Definition 5.5) since all of its beads are concentrated in the first column. If $w(\lambda) > 0$, so λ is not an e-core, we

must move the bottom-most bead of $\mathcal{A}_e^{e-h}(\rho^t)$ at least one unit to the right to obtain $\mathcal{A}_e^{e-h}(\lambda_{\mathrm{M}})$. This guarantees that $\mathcal{A}_e^{e-h}(\lambda_{\mathrm{M}})$ does not start with a bead, and since $\mathcal{A}_e^{e-h}(\lambda_{\mathrm{M}})$ has $e-h$ beads, we conclude that λ_{M} has precisely $e-h$ columns. Finally, by construction, λ_{M} satisfies the conditions of Definition 5.5. □

Example 6.3. *Let $e=5$, $h=2$, $\lambda=(2^{28},1^3)$, $w(\lambda)=11$. We obtain $\lambda_{\mathrm{M}}=(3^{19},1^2)$ as follows:*



Theorem 6.4. *The map $\mathrm{M}:\mathcal{Y}_k\to\mathcal{Y}_k$ for $k\geqslant 0$ is a well-defined graph involution. Given*

$$\mathsf{s}=(\lambda^{(0)}\xrightarrow{r_1}\lambda^{(1)}\xrightarrow{r_2}\dots\xrightarrow{r_n}\lambda^{(n)})$$

we let s_{M} denote the path

$$\mathsf{s}_{\mathrm{M}}=(\lambda_{\mathrm{M}}^{(0)}\xrightarrow{-r_1}\lambda_{\mathrm{M}}^{(1)}\xrightarrow{-r_2}\dots\xrightarrow{-r_n}\lambda_{\mathrm{M}}^{(n)}).$$

We have that $D_n^{\Bbbk}(\lambda_{\mathrm{M}})\cong(D_n^{\Bbbk}(\lambda))^{\mathrm{M}}$ and that the isomorphism is determined by $:c_{\mathsf{s}}\mapsto c_{\mathsf{s}_M}$.

Proof. The Mullineux involution M is characterized as the unique involution on e-regular partitions mapping $\emptyset$ to $\emptyset$ and such that $\mathrm{M}(\tilde{f}_i(\lambda))=\tilde{f}_{-i}(\mathrm{M}(\lambda))$ [Kle96b, FK97, BO98]. We want to identify λ_{M} with $\mathrm{M}(\lambda)$ for all vertices λ of $\mathcal{Y}$. By construction we have that λ_{M} is also vertex of $\mathcal{Y}_k$ whenever λ is, and that $(\lambda_{\mathrm{M}})_{\mathrm{M}}=\lambda$. It is clear that $\emptyset_{\mathrm{M}}=\emptyset$. Thus if $i\in\{0,\dots,e-1\}$ is such that $\tilde{f}_i(\lambda)\in\mathcal{F}_n^1$, we need to show that

$$\left(\tilde{f}_i(\lambda)\right)_{\mathrm{M}}=\tilde{f}_{-i}(\lambda_{\mathrm{M}}).$$

We remark that, if $\lambda\in\mathcal{F}_n^1$, then $\tilde{f}_i(\lambda)$ adds the leftmost addable box of content residue i, if any. In order to keep track of the action of $\tilde{f}_i$ on abaci, we follow the conventions of Remark 5.4, so we label the runners of an e-abacus with h beads, at most one bead per runner, from bottom-to-top by $h-1,h-2,\dots,1,0,e-1,\dots,h$. This is done so that the labels of the runners correspond nicely to the contents of addable/removable boxes. Note that the labeling of runners changes in the process of constructing λ_{M}, when $\mathcal{A}_e(\rho)$ with h beads is replaced by $\mathcal{A}_e(\rho^t)$ with $e-h$ beads. The abacus $\mathcal{A}_e(\lambda)$ has a bead (resp. empty space) on runner i if and only if $\mathcal{A}_e(\lambda_{\mathrm{M}})$ has an empty space (resp. bead) on runner $-i-1$. Finally, observe that if the top runner is labeled m in these conventions, that $\tilde{f}_m$ increases the weight w of a partition by at most 1 but all $\tilde{f}_i$, $i\neq m$, do not increase the weight. Now set $\hat{\rho}$ to be the core of $\tilde{f}_i(\lambda)$. We consider two cases.

Case 1. $w(\tilde{f}_i(\lambda))=w(\lambda)$. So either $i\neq h$ or $i=h$ and λ is a core. In the latter case, $\tilde{f}_i(\lambda)$ is also a core, and both $(\tilde{f}_i(\lambda))_{\mathrm{M}}$ and $\tilde{f}_{-i}(\lambda_{\mathrm{M}})$ coincide with the transpose of $\tilde{f}_i(\lambda)$. In the former case, the abaci $\mathcal{A}_e(\rho)$ and $\mathcal{A}_e(\hat{\rho})$ coincide on all runners except those labeled by i and $i-1$. Thus, $\mathcal{A}_e(\rho^t)$ and $\mathcal{A}_e(\hat{\rho}^t)$ only differ on runners $-i-1$ and $-i$: $\mathcal{A}_e(\hat{\rho}^t)$ has a bead on runner $-i-1$ and an empty space on runner $-i$, while the opposite is true for $\mathcal{A}_e(\rho^t)$. Thus, $(\tilde{f}_i(\lambda))_{\mathrm{M}}$ is obtained from λ_{M} by sliding the bead on runner $-i$ up runner $-i-1$. But this is exactly how we obtain $\tilde{f}_{-i}(\lambda_{\mathrm{M}})$ from λ_{M}. We are done in this case.

Case 2. $w(\tilde{f}_i(\lambda))=w(\lambda)+1$. So $i=h$, and the abacus of $\tilde{f}_i(\lambda)$ is obtained from that of λ by moving the bead on the top runner (labeled h) down to the bottom runner (labeled $h-1$) and then one unit right. Just as in the first case, the abaci $\mathcal{A}_e(\rho^t)$ and $\mathcal{A}_e(\hat{\rho}^t)$ only differ on runners $-h$ and $-h-1$. Note that these are the top and bottom runners of the abacus, respectively. Write division with remainder $w=w(\lambda)=(e-h)q+r$, so that λ_{M} is obtained from $\mathcal{A}_e(\rho^t)$ by moving the bottom r beads $q+1$ units to the right, and the remainder $e-h-r$ beads q units to the right. We have a subdivision into two further cases.

Case 2.1. $r<e-h-1$. So $w+1=(e-h)q+(r+1)$ is division with remainder, and $(\tilde{f}_i(\lambda))_{\mathrm{M}}$ is obtained from $\mathcal{A}_e(\hat{\rho}^t)$ by moving the bottom $r+1$ beads $q+1$ units to the right, and the remaining beads q units to the right. Note that the beads $2,\dots,r+1$ of $\mathcal{A}_e(\hat{\rho}^t)$ coincide with the beads

$1, \dots, r$ of $\mathcal{A}_e(\rho^t)$. Thus, $(\tilde{f}_i(\lambda))_{\mathrm{M}}$ is obtained from λ_{M} by taking the bead in the top runner, moving it down to the bottom runner and sliding one unit to the right. This is precisely $\tilde{f}_{-i}(\lambda_{\mathrm{M}})$.

Case 2.2. $r = e - h - 1$. So $w + 1 = (e - h)(q + 1)$. Here, $(\tilde{f}_i(\lambda))_{\mathrm{M}}$ is obtained from $\mathcal{A}_e(\hat{\rho}^t)$ by moving *all* beads $q + 1$ units to the right, while λ_{M} is obtained from $\mathcal{A}_e(\rho^t)$ by moving all beads $q + 1$ units to the right, except the one in the top runner, that we only move q units to the right. So we see that, again, $(f_i(\lambda))_{\mathrm{M}}$ is obtained from λ_{M} by taking the bead in the top runner, moving it down to the bottom runner and sliding one unit to the right. So $(\tilde{f}_i(\lambda))_{\mathrm{M}} = \tilde{f}_{-i}(\lambda_{\mathrm{M}})$.

This proves that the involution $\mathcal{Y} \to \mathcal{Y}$ given by $\lambda \mapsto \lambda_{\mathrm{M}}$ coincides with the Mullineux involution restricted to $\mathcal{Y}$. Now, the bases of $D^{\Bbbk}_n(\lambda)$ for $\lambda \in \mathcal{F}^1_n$ are given by the paths in the unitary branching graph terminating at said vertices. By Theorem 4.3 we can match up these bases through the action of the idempotents under the twisting by the Mullineux map (see equation (6.1)). □

Example 6.5. *Let $e = 7$. We have that $\mathrm{M}(3^{10}, 2^4) = (4^8, 1^3)$. We depict these partitions, and the manner in which they can constructed via adding rim 7-hooks in Figure 5. Furthermore, we provide an example of $\mathtt{t} \in \mathrm{Std}_7(3^{10}, 2^4)$ and $\mathtt{t}_{\mathrm{M}} \in \mathrm{Std}_7(4^8, 1^3)$. Note that the map on the level of tableau preserves the rim hooks drawn in the two diagrams!!*

1	2	3
4	8	9
5	11	15
6	12	18
7	13	19
10	14	20
16	17	21
22	23	24
25	29	30
26	32	
27	33	
28	34	
31	35	

1	4	5	6
2	8	11	12
3	9	15	18
7	10	16	22
13	14	17	23
19	20	21	24
25	26	27	28
29	33	34	35
30			
31			
32			

FIGURE 5. A pair of tableaux $\mathtt{t} \in \mathrm{Std}_7(3^{10}, 2^4)$ and $\mathtt{t}_{\mathrm{M}} \in \mathrm{Std}_7(4^8, 1^3)$ indexing basis elements swapped under the isomorphism $D^{\Bbbk}_{35}(3^{10}, 2^4)^{\mathrm{M}} \cong D^{\Bbbk}_{35}(4^8, 1^3)$.

7. The rational Cherednik algebra of the symmetric group over $\mathbb{C}$

For the remainder of the paper, we restrict our attention to the field $\mathbb{C}$ and rational Cherednik algebras of type $G(1, 1, n)$. Let $\mathfrak{S}_n$ be the symmetric group on n elements. The group $\mathfrak{S}_n$ acts on the algebra of polynomials in $2n$ non-commuting variables $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots y_n\rangle$. Fix a number $c \in \mathbb{C}$. The *rational Cherednik algebra* $H_c(\mathfrak{S}_n)$ is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle \rtimes \mathfrak{S}_n$ by the relations

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0, \qquad [y_i, x_j] = c(ij)\ (i \neq j), \qquad [y_i, x_i] = 1 - c\sum_{j \neq i}(ij)$$

where (ij) denotes the transposition in $\mathfrak{S}_n$ that switches i and j, see [EG02]. H_c has three distinguished subalgebras: $\mathbb{C}[\underline{y}] := \mathbb{C}[y_1, \dots, y_n]$, $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$, and the group algebra $\mathbb{C}\mathfrak{S}_n$. The *PBW theorem* [EG02, Theorem 1.3] asserts that multiplication gives a vector space isomorphism

$$\mathbb{C}[\underline{x}] \otimes \mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[\underline{y}] \xrightarrow{\cong} H_c$$

called the *triangular decomposition* of H_c, by analogy with the triangular decomposition of the universal enveloping algebra of a semisimple Lie algebra.

We define the category $\mathcal{O}_c(\mathfrak{S}_n)$ to be the full subcategory consisting of all finitely generated H_c-modules on which $y_1, \dots, y_n$ act locally nilpotently. Category $\mathcal{O}_c$ is not always very interesting. By [DdJO94], see also [BE09a, Section 3.9], $\mathcal{O}_c$ is semisimple (and equivalent to the category of

representations of $\mathfrak{S}_n$) unless $c = r/e$, with $\gcd(r; e) = 1$ and $1 < e \leqslant n$. Equivalences of categories reduce the study of $\mathcal{O}_{r/e}(\mathfrak{S}_n)$ to $\mathcal{O}_{1/e}(\mathfrak{S}_n)$, for $1 < e \leqslant n$ [Rou08]. For the rest of the paper we work with $\mathcal{O}_{1/e}(\mathfrak{S}_n)$. It will be convenient to set

$$\mathcal{O}_{1/e} := \bigoplus_{n \geqslant 0} \mathcal{O}_{1/e}(\mathfrak{S}_n).$$

The category $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ is Morita equivalent (over $\mathbb{C}$) to $A_n(\kappa)$ for any value of $\kappa \in I^\ell$ [Web17, Theorem A]. Thus, $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ is a highest weight category with respect to the dominance ordering $\rhd$ on $\mathscr{P}^1_n$. The standard modules are constructed as follows. Extend the action of $\mathfrak{S}_n$ on $S_n(\lambda)$ to an action of $\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n$ by letting $y_1, \dots, y_n$ act by 0. The algebra $\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n$ is a subalgebra of $H_{1/e}$ and we define

$$\Delta(\lambda) := \mathrm{Ind}^{H_{1/e}}_{\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n} S_n(\lambda) := H_{1/e} \otimes_{\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n} S_n(\lambda) = \mathbb{C}[\underline{x}] \otimes S_n(\lambda)$$

where the last equality is *only as* $\mathbb{C}[\underline{x}]$*-modules* and follows from the triangular decomposition. We let $L(\lambda)$ denote the unique irreducible quotient of $\Delta(\lambda)$.

Any module $M \in \mathcal{O}_{1/e}(\mathfrak{S}_n)$ is finitely generated over the algebra $\mathbb{C}[\underline{x}]$ and, as such, it has a well-defined support $\mathrm{supp}(M) \subseteq \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[\underline{x}])$. We now explain a way to compute the supports of simple modules in $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ that was obtained in [Wil17a]. To do this, for any $i = 0, \dots, \lfloor n/e \rfloor$, denote by X_i the variety

$$X_i := \mathfrak{S}_n\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 = z_2 = \cdots z_e, z_{e+1} = \cdots = z_{2e}, \cdots, z_{(i-1)e+1} = \cdots = z_{ie}\}$$

By its definition, X_i is a $\mathfrak{S}_n$-stable subvariety of $\mathbb{C}^n$. Note that $X_0 = \mathbb{C}^n$, and these subvarieties form a chain $X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_{\lfloor n/e \rfloor}$. Now recall that a partition λ is said to be *e-restricted* if $\lambda_i - \lambda_{i+1} < e$ for every $i \geqslant 0$, that is, if no two consecutive parts of λ differ by more than $e - 1$ parts. By the division algorithm, for any partition λ there exist unique partitions μ, ν such that $\lambda = e\mu + \nu$ and ν is e-restricted. Then, according to [Wil17a, Theorem 1.6],

$$\mathrm{supp}(L_{1/e}(\lambda)) = X_{|\mu|}$$

So, for example, $L_{1/e}(\lambda)$ has full support if and only if λ is e-restricted. On the other hand, if e divides n, then $L_{1/e}(\lambda)$ has minimal support if and only if $\lambda = e\mu$, where μ is a partition of n/e.

The categories $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ come equipped with induction and restriction functors

$$\mathrm{Res}^n_{n-1} : \mathcal{O}_{1/e}(\mathfrak{S}_n) \rightleftarrows \mathcal{O}_{1/e}(\mathfrak{S}_{n-1}) : \mathrm{Ind}^n_{n-1}$$

that were constructed by Bezrukavnikov and Etingof in [BE09a]. Their definition is quite technical and will not be needed. In fact, Bezrukavnikov and Etingof constructed restriction functors for any parabolic subgroup of $\mathfrak{S}_n$, [BE09a]. It follows from their construction that M has full support if and only if it is not killed by restriction to any parabolic subgroup. We will use this property below without further mention.

7.1. **Unitary modules.** For $\lambda \in \mathscr{P}^1_n$, fix a positive-definite, $\mathfrak{S}_n$-invariant Hermitian form on the irreducible representation $S_n(\lambda)$. A standard argument shows that this form can be extended to a Hermitian form $(\cdot, \cdot)$ on the standard module $\Delta_{1/e}(\lambda)$, which is H_c-invariant in that $(y_i v, v') = (v, x_i v')$ for every $v, v' \in \Delta_{1/e}(\lambda)$ and $i = 1, \dots, n$. The simple module $L_{1/e}(\lambda)$ is the quotient of $\Delta_{1/e}(\lambda)$ by the radical of this form. In particular, $L_{1/e}(\lambda)$ is equipped with a H_c-invariant, non-degenerate Hermitian form. We say that $L_{1/e}(\lambda)$ is *unitary* if this form is positive-definite.

Theorem 7.1. [ES09] *The Hermitian form on $L(\lambda)$ is positive-definite if and only if λ is an e-unitary partition. Thus $L(\lambda)$ is unitary if and only if $\lambda \in \mathcal{F}^1_n$*

Applying the KZ functor to these simples, we obtain the complete set of simple unitary modules for the Hecke algebra. We emphasise that the simples labelled by $\lambda = (e^k)$ for some $k \geqslant 0$ do not survive under the KZ functor and so there are fewer unitary simples for the Hecke algebras.

Theorem 7.2 ([Sto, Corollary 4.5])**.** *The simple R_n-module $D^{\mathbb{C}}_n(\lambda)$ is unitary if and only if λ is e-restricted and $\lambda \in \mathcal{F}^1_n$*

Remark 7.3. *We would like to say some words on higher levels. Associated to the group $G(\ell,1,n)$ there is a rational Cherednik algebra $H_c(G(\ell,1,n))$, where $c=(c_0,c_1,\dots,c_{\ell-1})$ is now a collection of ℓ complex numbers. The definition of a unitary module goes through unchanged. We let $\ell=2$ and taking the charge $c_0=1/e$ and $c_1=0$ so that we are, essentially, working with rational Cherednik algebras associated to the Weyl group of type D. We have checked (using Griffeth's classification of unitary modules [Gri]) that if $\lambda\in\mathcal{F}_n^2$ then $L(\lambda)$ is indeed a unitary module.*

7.2. **Changing quantum characteristics.** Having constructed a BGG resolution for any unitary module with $h<e$ columns, we proceed to relate these complexes to each other for various e, and to construct the complex in the special case $h=e$ for the unitary module $L(e^k)$. As observed in Section 5, the e-abacus of any unitary module which is not of the form $L(e^k)$ will contain empty runners; removing the empty runners produces the h-abacus of a partition of the form (h^k), with $h<e$ and k equal to the weight of the block containing λ. So we may try using the runner removal Morita equivalences of Chuang-Miyachi which upgrade the combinatorial operation "removing runners" to an equivalence of highest weight categories [CM10]. Given an e-core partition ρ and $k\in\mathbb{N}$, let $n:=|\rho|+ek$ and set

$$\Lambda(\rho,k):=\{\lambda\mid\lambda\in\mathscr{P}_n^1, e\text{-core}(\lambda)=\rho, w(\lambda)=k\}\subseteq\mathscr{P}_n^1,$$
$$\Lambda_h^+(\rho,k):=\{\lambda\in\Lambda(\rho,k)\mid\lambda\in\mathscr{P}_n^1(h)\}\subseteq\Lambda(\rho,k)\}$$
$$\Lambda_h^-(\rho,k):=\{\lambda\mid\lambda^T\in\mathscr{P}_n^1(h), e\text{-core}(\lambda)=\rho^T, w(\lambda)=k\}\subseteq\Lambda(\rho^T,k).$$

Notice that the transpose map gives a bijection between the sets $\Lambda_h^-(\rho,k)$ and $\Lambda_h^+(\rho,k)$; under this map the partial ordering on the sets is reversed. Let $\mathcal{O}_{1/e}(\rho,k)$ denote the block of category $\mathcal{O}_{1/e}$ corresponding to $\Lambda(\rho,k)$. Note that the set $\Lambda_h^-(\rho,k)$ is co-saturated in $\Lambda(\rho^T,k)$ so we can consider the quotient category of $\mathcal{O}_{1/e}(\rho^T,k)$ by the Serre subcategory spanned by simples whose label does *not* belong to $\Lambda_h^-(\rho,k)$. We denote this quotient by $\mathcal{O}^-_{1/e,h}(\rho,k)$ This is a highest weight category, with standard objects $\Delta_h^-(\nu):=\pi(\Delta_{1/e}(\nu))$, where $\nu\in\Lambda_h^-(\rho,k)$ and $\pi:\mathcal{O}_{1/e}(\rho^T,k)\to\mathcal{O}^-_{1/e,h}(\rho,k)$ is the quotient functor. We remark that π admits a left adjoint $\pi^!:\mathcal{O}^-_{1/e,h}(\rho,k)\to\mathcal{O}_{1/e}(\rho^T,k)$, and $\pi^!(\Delta^-(\nu))=\Delta(\nu)$ for $\nu\in\Lambda_h^-(\rho,k)$.

Given $\nu\in\Lambda_h^-(\rho,k)$ we set $\mathcal{A}_h^{CM}(\nu):=\mathcal{A}_h(\nu^T)$. Let $\mathsf{r}=(r_0,\dots,r_{h-1},r_h)\in\mathbb{Z}_{\geqslant0}^{h+1}$, and construct a partition ν^+ as follows. In the abacus $\mathcal{A}_h^{CM}(\nu)$, insert r_i empty runners between runners $i-1$ and i (so r_0 and r_h are the number of empty runners inserted at the top and bottom of the abacus, respectively). This creates a new e-abacus, $\mathcal{A}$, with $e:=h+r_0+\cdots+r_h$ runners. We denote by ν^+ the unique partition such that $\mathcal{A}=\mathcal{A}_e^{CM}(\nu^+)$. We let $\rho=\varnothing^+$. We have a bijection

$$\mathsf{R}:\Lambda^-(\varnothing,k)\to\Lambda^-(\rho,k)$$

given by $\mathsf{R}:\nu\mapsto\nu^+$ and we let $\mathsf{R}^{-1}:\nu\mapsto\nu^-$ denote the inverse. We are now able to recall the main result of Chuang–Miyachi.

Theorem 7.4 ([CM10])**.** *The categories $\mathcal{O}^-_{1/h,h}(\varnothing,k)$ and $\mathcal{O}^-_{1/e,h}(\rho,k)$ are equivalent as highest weight categories. Moreover, the equivalence*

$$\underline{\mathsf{R}}:\mathcal{O}^-_{1/h,h}(\varnothing,k)\to\mathcal{O}^-_{1/e,h}(\rho,k)$$

sends the standard module $\Delta_h^-(\nu)$ to the standard module $\Delta_h^-(\nu^{\mathsf{R}})$.

Note, however, that we cannot apply the above theorem directly since we are interested in the subcategories $\mathcal{O}^+_{1/h,h}(\varnothing,k)$ and $\mathcal{O}^+_{1/e,h}(\rho^T,k)$ rather than the quotient categories $\mathcal{O}^-_{1/h,h}(\varnothing,k)$ and $\mathcal{O}^-_{1/e,h}(\rho,k)$, where $\mathcal{O}^+_{1/h,h}(\varnothing,k)$ denotes the Serre subcategory spanned by the simples whose label belongs to $\Lambda^+(\varnothing,k)$, and similarly for $\mathcal{O}^+_{1/e,h}(\rho,k)$. Let us fix this. Following [GGOR03, Section 4], we note that the rational Cherednik algebra $H_{1/e}:=H_{1/e}(\mathfrak{S}_n)$ has finite global dimension and is isomorphic to its opposite algebra; an explicit isomorphism is given by $w\mapsto w^{-1}$, $x\mapsto x$, $y\mapsto -y$. In particular, the functor $\mathrm{RHom}_{H_{1/e}}(\bullet,H_{1/e})$ gives an equivalence $D^b(H_{1/e}\text{-mod})\to D^b(H_{1/e}\text{-mod}^{\mathrm{opp}})$. Let us denote by D the functor $\mathrm{RHom}_{H_{1/e}}(\bullet,H_{1/e})[n]$. The following theorem

summarizes various results of [GGOR03, Section 4.3.2]. We denote by $D^b(\mathcal{O}_{1/e}(\mathfrak{S}_n))$ the subcategory of $D^b(H_{1/e}\text{-mod})$ consisting of complexes with homology in $\mathcal{O}_{1/e}$, and by $\mathcal{O}_{1/e}^{\Delta}$ the category of objects in $\mathcal{O}_{1/e}$ that admit a Δ-filtration.

Theorem 7.5. *The functor* D *induces a derived equivalence* $\mathsf{D} : D^b(\mathcal{O}_{1/e}(\mathfrak{S}_n)) \to D^b(\mathcal{O}_{1/e}(\mathfrak{S}_n)^{\mathrm{opp}})$ *as well as an equivalence of exact categories* $\mathsf{D} : \mathcal{O}_{1/e}(\mathfrak{S}_n)^{\Delta} \to (\mathcal{O}_{1/e}(\mathfrak{S}_n)^{\Delta})^{\mathrm{opp}}$. *For a partition* $\lambda \vdash n$, $\mathsf{D}(\Delta(\lambda)) = \Delta(\lambda^T)$ *(where both sides of the equation are interpreted as complexes concentrated in degree* 0*).*

By abuse of notation, we will write $\mathsf{D} : D^b(\mathcal{O}_{1/e}) \to D^b(\mathcal{O}_{1/e}^{\mathrm{opp}})$ for $\bigoplus_{n\geqslant 0} \mathrm{RHom}_{H_{1/e}(\mathfrak{S}_n)}(\bullet, H_{1/e}(\mathfrak{S}_n))[n]$. Let us mention a property of D that will be important later. The following is an immediate consequence of [Los17, Lemma 2.5] and the definition of a perverse equivalence [Los17, Section 1.4].

Lemma 7.6. *For every* $n \geqslant 0$*, the functor* D *induces a (contravariant!) abelian autoequivalence of the category of* minimally supported *modules in category* $\mathcal{O}_{1/e}(\mathfrak{S}_n)$.

Let $\lambda \in \mathcal{F}_n^1 \subseteq \mathscr{P}_n^1(h)$ be such that $(h^k)^+ = \lambda$, where k is the e-weight of λ and h the number of nonempty runners in $\mathcal{A}_e(\lambda)$. Define the functor $\tilde{\mathsf{R}}^-$ via the following composition

$$\tilde{\mathsf{R}}^- := \mathsf{D}\pi^!\underline{\mathsf{R}}^{-1}\pi\mathsf{D} : D^b(\mathcal{O}_{1/e}(\rho, k)) \to D^b(\mathcal{O}_{1/h}(\varnothing, k))).$$

Each functor in the composition defining $\tilde{\mathsf{R}}^-$ takes Vermas to Vermas, and is either an equivalence of Δ-filtered categories or exact on Δ-filtered categories while being an isomorphism on spaces of homomorphisms between Vermas. It follows that for $\mu \in \mathrm{Po}_e(\lambda)$, $\tilde{\mathsf{R}}^-\Delta(\mu) = \Delta(\mu^-)$, and that $\tilde{\mathsf{R}}^-$ takes a complex to a complex and sends nonzero maps to nonzero maps (however, we cannot conclude from this that $\tilde{\mathsf{R}}^-$ takes a resolution to a resolution). Define $C_\bullet(h^k) = \tilde{\mathsf{R}}^-(C_\bullet(\lambda))$. By construction, this is a complex whose ℓ-th term is given by

$$C_\ell(h^k) = \bigoplus_{\substack{\mu\in \mathrm{Po}_e(\lambda)\\ \mathrm{hd}(\mu)=\ell}} \Delta(\mu^-) = \bigoplus_{\substack{\tau\in \mathrm{Po}_h(h^k)\\ \mathrm{hd}(\tau)=\ell}} \Delta(\tau)$$

and which has a map $\Delta(\tau) \to \Delta(\tau')$ whenever $\mathrm{hd}(\tau) = \ell$, $\mathrm{hd}(\tau') = \ell - 1$, and $\mathcal{A}_h(\tau) = t\mathcal{A}_h(\tau')$ for some transposition $t \in \widetilde{\mathfrak{S}}_h$. $C_\bullet(h^k)$ is a complex that looks identical to $C_\bullet(\lambda)$ but with the partitions μ relabeled by μ^-, and in particular $L(h^k)$ is the head of $C_0(h^k) = \Delta(h^k)$.

The following theorem answers [ZGS14, Conjecture 4.5] in the affirmative.

Theorem 7.7. *If* $L(\lambda) \in \mathcal{O}_{1/e}(\mathfrak{S}_n)$ *is unitary then* $L(\lambda)$ *has a BGG resolution* $C_\bullet(\lambda)$ *whose* ℓ*th term is given by*

$$C_\ell(\lambda) = \bigoplus_{\substack{\mu\in \mathrm{Po}_e(\lambda)\\ \mathrm{hd}(\mu)=\ell}} \Delta(\mu)$$

Proof. We have already shown the conjecture holds if $h < e$. Let $n = ke$ for some $k > 0$ and take (e^k), the unique unitary partition of n with e columns. Choose any $e' > e$ and any unitary partition $\lambda \in \mathcal{P}^u_{e'}$ with e columns and e'-weight k. Let $C_\bullet(\lambda)$ be the BGG resolution of $L(\lambda)$ and apply $\tilde{\mathsf{R}}^-$ to it. By the remarks above, $\tilde{\mathsf{R}}^-(C_\bullet(\lambda)) = C_\bullet(e^k)$ is the desired complex and $L(e^k)$ is the head of $C_0(e^k)$. We need to show that $C_\bullet(e^k)$ is exact except in degree 0, where $H_0(C_\bullet) = L(e^k)$.

As in the proof of the $h < e$ case, if $\lambda \in \mathrm{Po}_e(e^k)\backslash\{(e^k)\}$, then λ is e-restricted. Thus $E_i(L(\lambda)) \neq 0$ for some $i \in \mathbb{Z}/e\mathbb{Z}$, so if $L(\lambda)$ is a composition factor of a homology group $H_j(C_\bullet)$ then $E_i(C_\bullet)$ will fail to be exact. Similarly, it holds (by basic properties of highest weight categories) that $L(e^k)$ occurs exactly once in the composition series of all the C_j, when $j = 0$.

Next, $E_i(L(e^k)) = 0$ since e^k has a single removable box and it is never a good removable box. Thus, it suffices to check that $E_i(C_\bullet)$ is exact for each $i \in \mathbb{Z}/e\mathbb{Z}$. This is identical to the argument used in the $h < e$ cases for those i such that $E_i(L(\lambda)) = 0$. □

We also make the observation that resolutions of unitary modules are, in a manner of speaking, independent of e. Let h be the number of columns of λ and let k be the e-weight of λ.

Corollary 7.8. *Let $\lambda \in \mathcal{F}_n^1$. The shape of the BGG complex $C_\bullet(\lambda)$ depends only on $h, k \in \mathbb{N}$.*

Proof. $\tilde{\mathsf{R}}^-$ identifies $C_\bullet(\lambda)$ with $C_\bullet(h^k)$, thus sends a resolution of $L(\lambda)$ to a resolution of $L(h^k)$. □

7.3. **Ringel duality and more BGG resolutions.** We can also construct some new BGG resolutions as corollaries of Theorem 7.7 via Ringel duality. These resolutions will also be used in the study subspace arrangements in Subsection 8.1.3. The character of $L(ek) = L(\mathrm{triv}) \in \mathcal{O}_{1/e}(\mathfrak{S}_{ek})$ is dual to the character of $L(e^k)$ in the sense that its character is obtained from that of $L(e^k)$ by taking the transpose of each partition labelling a Verma module [EGL15, Remark 5.1]:

$$L(ek) = \sum_{\substack{\mu \in \mathrm{Po}_e(e^k) \\ \mathrm{hd}(\mu) = \ell}} (-1)^\ell \Delta(\mu^T).$$

This is every bit as much an alternating sum character formula as that of $L(e^k)$, so we may naturally ask whether its character formula also comes from a BGG resolution.

Let $C_\bullet$ be the BGG resolution of $L(e^k)$. We apply Ringel duality to construct a complex, $\mathsf{D}(C_\bullet)$, in the principal block $\mathcal{O}(\varnothing, k) \subset \mathcal{O}_{1/e}(\mathfrak{S}_{ek})$. The complex $\mathsf{D}(C_\bullet)$ is obtained from $C_\bullet$ by replacing $\Delta(\mu)$ with $\Delta(\mu^T)$ for all $\mu \in \mathrm{Po}_e(e^k)$ and reversing the direction of all the arrows (since D is a contravariant functor which takes Vermas to Vermas). By [EGL15], the alternating sum of the terms of $\mathsf{D}(C_\bullet)$ in the Grothendieck group $[\mathcal{O}(\varnothing, k)]$ coincides with the character of $L(\mathrm{triv}) = L(ek)$.

Corollary 7.9. *$\mathsf{D}(C_\bullet)$ is a BGG resolution of $L(\mathrm{triv}) = L(ek)$.*

Proof. A resolution is quasi-isomorphic to the module it resolves, so in $D^b(\mathcal{O}_{1/e}(\mathfrak{S}_{ek}))$, $L(e^k)$ is isomorphic to its resolution $C_\bullet$. Since the Ringel duality D is a derived self-equivalence of $D^b(\mathcal{O}_{1/e}(\mathfrak{S}_{ek}))$ [GGOR03], this implies $\mathsf{D}(C_\bullet) \simeq \mathsf{D}(L(e^k))$ in $D^b(\mathcal{O}_{1/e}(\mathfrak{S}_{ek}))$. We know that at the end of the complex we have: $\Delta(ek-1, 1) \to \Delta(ek) \to 0$, and so $L(ek) = \mathrm{Head}(\Delta(ek))$ must occur in the homology of $\mathsf{D}(C_\bullet)$. Therefore $L(ek)$ is a composition factor of $\mathsf{D}(L(e^k))$.

We claim that $\mathsf{D}(L(e^k)) = L(ek)$. This follows from [Los17, Lemma 2.5] which states that D is a perverse equivalence with respect to the filtration by dimensions of support: in particular, D is a self-equivalence of the semi-simple subcategory spanned by the minimal support modules $L(e\sigma)$. Since $\mathsf{D}^2 = Id$, it follows that D must permute the minimal support simple modules $L(e\sigma)$, $\sigma \vdash k$. We have already seen that $\mathsf{D}(L(e^k) = \mathsf{D}(L(e(1^k)))$ contains $L(ek) = L(e(k))$ as a composition factor; it follows that $\mathsf{D}(L(e^k)) = L(ek)$.

To conclude, $\mathsf{D}(C_\bullet)$ is equivalent to $L(ek)$ in $D^b(\mathcal{O}_{1/e}(\mathfrak{S}_{ek}))$, where $L(ek)$ is considered as a complex concentrated in degree 0. Hence $H_i(\mathsf{D}(C_\bullet)) = \delta_{i0} L(ek)$, as required. □

Let π denote the quotient functor which kills the subcategory generated by $\{L(\nu) \mid \nu \text{ has more than } e \text{ rows}\}$.

Corollary 7.10. *$\pi\mathsf{D}(C_\bullet)$ is a BGG resolution of $\underline{L}(\mathrm{triv}) = \underline{L}(ek)$ in the quotient category $\pi(\mathcal{O}_{1/e}(\mathfrak{S}_{ke}))$. By adding an arbitrary configuration of $a \in \mathbb{Z}_{>0}$ empty runners to the abacus, $\mathsf{R}\pi\mathsf{D}(C_\bullet)$ is a BGG resolution of $\mathsf{R}\underline{L}(\mathrm{triv})$ in $\mathsf{R}\pi(\mathcal{O}_{1/e}(\mathfrak{S}_{ke}))$.*

Proof. The quotient functor π is exact, sends $\Delta(\mu)$ to the standard module $\underline{\Delta}(\mu)$, and sends $L(\mu)$ to the simple module $\underline{L}(\mu)$. The first claim then follows from Corollary 7.9, implying the second claim by Theorem 7.4. □

7.4. **Computation of Lie algebra and Dirac cohomology.** BGG resolutions for classical and affine Lie algebras over $\mathbb{C}$ are closely related to the computation of Lie algebra cohomology [Kos61, GL76, BH09]. Recently, a version of Lie algebra cohomology (and homology) for rational Cherednik algebras over $\mathbb{C}$ was constructed in [HW18]; $\mathfrak{h}^* := \bigoplus \mathbb{C}x_i$ plays the role of the nilradical $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, and the complex reflection group W plays the role of the Cartan subalgebra.

Theorem 7.11. *Let $\lambda \in \mathcal{F}_n^1 \subseteq \mathscr{P}_n^1(h)$. We have that*

$$H_i(\mathfrak{h}^*, L(\lambda)) = \bigoplus_{\substack{\mu \in \mathrm{Po}_e(\lambda) \\ \mathrm{hd}(\mu) = i}} S_n(\mu).$$

Proof. This follows immediately from our main theorem and [HW18, Proposition 6.1]. □

Likewise, if $L(\lambda) \in \mathcal{O}_c(G(\ell,1,n))$ where c corresponds to the rank e and charge $\underline{s} = (\kappa_1, \kappa_2, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ for the Fock space, and $\lambda \in \mathcal{F}_n^\ell$, then

$$H_i(\mathfrak{h}^*, L(\lambda)) = \bigoplus_{\substack{\mu \trianglelefteq \lambda \\ \ell(\mu)=i}} S_n(\mu).$$

This also computes the Lie algebra cohomology $H^i(\mathfrak{h}^*, L(\lambda))$. Indeed, by Poincaré duality (cf. [HW18, Proposition 2.7]), we get

$$H^i(\mathfrak{h}^*, L(\lambda)) = H_{n-i}(\mathfrak{h}^*, L(\lambda)) \otimes \wedge^n \mathfrak{h},$$

where $n := \dim \mathfrak{h}$. A consequence of the computation of Lie algebra cohomology for unitary modules admitting a BGG resolution is that this immediately gives the computation of the Dirac cohomology $H_D(L(\lambda))$. This is defined as the usual Dirac cohomology, where the Dirac operator $D \in H_{1/e}(\mathfrak{S}_n) \otimes \mathfrak{c}$ has been constructed in [Ciu16]. Here $\mathfrak{c}$ is the Clifford algebra associated to $\bigoplus \mathbb{C}x_i \oplus \bigoplus \mathbb{C}y_j$ with its natural nondegenerate bilinear form $(x_i, y_j) = \delta_{ij}$. For a module $M \in \mathcal{O}_{1/e}(\mathfrak{S}_n)$, the algebra $H_{1/e}(\mathfrak{S}_n) \otimes \mathfrak{c}$ acts on the space $M \otimes \wedge^\bullet \mathfrak{h}$, and the Dirac cohomology is defined to be, as usual, $\ker(D)/\ker(D) \cap \mathrm{im}(D)$. This is a representation of $\widetilde{W}$, a certain double-cover of the group W. Then, by [HW18, Theorem 5.1], $H_D(L(\lambda)) = \bigoplus_{\mu \leqslant \lambda} S_n(\mu) \otimes \chi$, where χ is a 1-dimensional character of the double cover $\widetilde{W}$. We refer to [HW18] for details.

8. Graded free resolutions of algebraic varieties, Betti numbers, and Castelnuovo–Mumford regularity

We now consider the consequences of our results for computing minimal resolutions of linear subspace arrangements. Easy examples of ideals whose resolutions we compute include the braid arrangements of type A and type D. Such minimal resolutions are difficult to compute geometrically [Las78]. As a consequence, we prove a combinatorial formula for the Betti numbers of the ideal of the m-equals arrangement predicted in [ZGS14]. We also calculate the Castelnuovo–Mumford regularity for the coordinate ring of these arrangements, a notoriously difficult problem in general (see [DS02, TT15]).

It is pointed out in [EH04] that BGG resolutions via parabolic Verma modules for Lie algebras can be used to provide commutative algebra resolutions of determinantal ideals by viewing the coordinate ring as a unitarizable highest weight module. We employ our Cherednik algebra resolutions in an analogous fashion. The first of these commutative algebra resolutions, given in Subsection 8.1.1, was predicted in [ZGS14] and concerns the smallest ideal, $I_{e,1,n}$, of the polynomial representation (this is the vanishing ideal of the subspace arrangement, $X_{e,1,n}$, consisting of e equal coordinates for $e \leqslant n$). We then provide a cyclotomic generalisation of this resolution in Subsection 8.1.2. The third resolution, given in Subsection 8.1.3, concerns the smallest quotient, $\mathbb{C}[X_{e,k,n}]$, of the polynomial representation (this is the coordinate ring of the subspace arrangement, $X_{e,k,n}$, consisting of k clusters of e equal coordinates for $ke = n$); the ideal vanishing on this space was studied in [ZGS14], however since neither this ideal nor its quotient is unitary (in general) the authors did not predict any resolution arising via Cherednik algebras.

8.1. **Commutative algebra.** Let us discuss the consequences that the existence of the BGG resolution has for the study of graded modules over $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[\underline{x}]$. First of all, for every $\mu \vdash n$, the standard module $\Delta_{1/e}(\mu)$ is free as a $\mathbb{C}[\underline{x}]$-module. So the resolution $C_\bullet(\lambda)$ is, in fact, a free resolution of $L_{1/e}(\lambda)$ when we view all involved modules as $\mathbb{C}[\underline{x}]$-modules.

An observation now is that every module in category $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ automatically acquires a grading compatible with the usual grading on $\mathbb{C}[\underline{x}]$, as follows. Consider the deformed Euler element[1] $\mathsf{eu} := \frac{1}{2}\sum_{i=1}^n x_i y_i + y_i x_i \in H_{1/e}$. This is a grading element of $H_{1/e}$ in the sense that $[\mathsf{eu}, x_i] =$

[1]We remark that our Euler element 'eu' differs from the one used in [ZGS14] by the constant $n(e-n+1)/2e$

$x_i, [\mathsf{eu}, y_i] = -y_i$, and $[\mathsf{eu}, w] = 0$ for $w \in \mathfrak{S}_n$. Any module in category $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ is now graded by generalized eigenspaces for eu:

$$M = \bigoplus_{a \in \mathbb{C}} M_a, M_a := \{m \in M : (\mathsf{eu} - a)^k m = 0 \text{ for } k \gg 0\}.$$

Note that, since the grading on M was defined using an element of $H_{1/e}$, every morphism in category $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ has degree 0. In particular, this grading is *different* from the grading of objects in $\mathcal{O}_{1/e}(\mathfrak{S}_n)$ that has been used so far in this paper. The grading by generalized eigenspaces of eu, however, is better-suited for the purposes of commutative algebra.

A priori, $M \in \mathcal{O}_{1/e}(\mathfrak{S}_n)$ is only $\mathbb{C}$-graded, but in our case we can do better. Since $[\mathsf{eu}, w] = 0$ for $w \in \mathfrak{S}_n$, eu may be seen as an endomorphism of the $\mathfrak{S}_n$-module $S_n(\tau) \cong 1 \otimes S_n(\tau) \subseteq \mathbb{C}[\underline{x}] \otimes \tau = \Delta_{1/e}(\tau)$. Thus, eu acts by a scalar c_τ on $S_n(\tau)$, and by the definition of $\Delta_{1/e}(\tau)$ we get that $\Delta_{1/e}(\tau)_a \neq 0$ if and only if $a = c_\tau + k$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Moreover,

$$\Delta_{1/e}(\tau)_{c_\tau + k} = \mathbb{C}[\underline{x}]_k \otimes S_n(\tau)$$

where $\mathbb{C}[\underline{x}]_k$ denotes the subspace of homogeneous polynomials of degree k in the variables $x_1, \dots, x_n$. We will write $\mathbb{C}[\underline{x}] \otimes S_n(\lambda)$ to refer to the $\mathbb{C}[\underline{x}]$-module $\Delta_{1/e}(\lambda)[c_\lambda]$, where the brackets denote the usual grading shift. Thus, $\mathbb{C}[\underline{x}] \otimes S_n(\lambda)$ is $\mathbb{Z}_{\geqslant 0}$-graded, and $(\mathbb{C}[\underline{x}] \otimes S_n(\lambda))_k = \mathbb{C}[\underline{x}]_k \otimes S_n(\lambda)$.

Now consider the resolution of the graded $\mathbb{C}[\underline{x}]$-module $L_{1/e}(\lambda)[c_\lambda]$, where the i-th term of the complex is given by

$$\bigoplus_{\substack{\mu \in \mathrm{Po}_e(\lambda) \\ \mathrm{hd}(\mu) = i}} (\mathbb{C}[\underline{x}] \otimes S_n(\mu))[c_\lambda - c_\mu]$$

We remark that, since λ and μ belong to the same block of category $\mathcal{O}_{1/e}(\mathfrak{S}_n)$, $c_\lambda - c_\mu$ is actually an integer. Of course, this is the same as the BGG resolution $C_\bullet(\lambda)$, but we write it in this way to emphasize that we are only interested in the $\mathbb{C}[\underline{x}]$-module structure. By abuse of notation, we also denote this complex by $C_\bullet(\lambda)$. Note that $(\mathbb{C}[\underline{x}] \otimes S_n(\mu))[c_\lambda - c_\mu] = \Delta_{1/e}(\mu)[c_\lambda]$, from where it follows that all the maps in the complex have degree 0 as maps of graded $\mathbb{C}[\underline{x}]$-modules. In particular, $C_\bullet(\lambda)$ is a graded-free resolution of $L_{1/e}(\lambda)[c_\lambda]$.

The value of c_λ can be expressed in terms of the content of the boxes of λ, namely

$$c_\lambda = \frac{n}{2} - \frac{1}{e} \sum_{\square \in \lambda} \mathrm{column}(\square) - \mathrm{row}(\square)$$

It follows from Section 5.6 or from Lemma 5.18 that if $\mathrm{hd}(\mu) < \mathrm{hd}(\nu)$ then $c_\mu < c_\nu$. In particular, when viewing the differential in the resolution $C_\bullet(\lambda)$ as matrices with coefficients in $\mathbb{C}[\underline{x}]$, no nonzero entry of the differential is a degree 0 element of $\mathbb{C}[\underline{x}]$. It follows immediately that:

Lemma 8.1. *The complex $C_\bullet(\lambda)$ is a minimal graded free resolution of $L_{1/e}(\lambda)[c_\lambda]$.*

Lemma 8.1 implies a combinatorial formula for computing many interesting invariants of the module $L_{1/e}(\lambda)[c_\lambda]$. In the rest of this section, if $L_{1/e}(\lambda)$ is unitary we write:

$$n := |\lambda|, \quad k := e\text{-weight}(\lambda), \quad h := \#\text{columns}(\lambda)$$

Recalling the basics of abaci in Section 5.1 this means that the abacus $\mathcal{A}_e(\lambda)$ has h nonempty runners and there are k vacant spaces in $\mathcal{A}_e(\lambda)$ with some bead to their right.

Proposition 8.2. *Suppose $L_{1/e}(\lambda)$ is unitary. Then,*

(1) $\beta_{i,j} = \displaystyle\sum_{\substack{\mu \in \mathrm{Po}_e(\lambda) \\ c_\lambda - c_\mu = -j \\ \mathrm{hd}(\mu) = i}} \dim(S_n(\mu))$,

(2) $pdim(L_{1/e}(\lambda)) = (h-1)k$,

(3) $depth(L_{1/e}(\lambda)) = n - (h-1)k$

where $\beta_{i,j}$ denotes the (i,j)-graded Betti number of $L_{1/e}(\lambda)[c_\lambda]$, and pdim stands for the projective dimension as a graded $\mathbb{C}[\underline{x}]$-module.

Proof. Statement (1) is clear from the form of the resolution $C_\bullet(\lambda)$. The maximal homological degree of a partition in $\mathrm{Po}_e(\lambda)$ is acquired by sliding all the beads to the left and then sliding the highest bead k spaces to the right. (2) follows from here. Finally, by the Auslander-Buchsbaum formula, (3) is equivalent to (2). □

Another consequence of Lemma 8.1 and the fact that the function c_λ is strictly increasing on homological degrees, is the computation of the Castelnuovo-Mumford regularity of the module $L_{1/e}(\lambda)[c_\lambda]$. Recall that, by definition, the regularity of a module M is

$$\mathrm{reg}(M) := \max\{j : \text{there exists } i \text{ such that } \beta_{i,i+j}(M) \neq 0\}$$

In other words, for a minimal graded-free resolution $C_\bullet$ of M, for each $i = 0, \dots, \mathrm{pdim}(M)$, let n_i be the maximum degree of a generator of C_i, and $m_i := n_i - i$. Then, $\mathrm{reg}(M) = \max_i\{m_i\}$. The Castelnuovo-Mumford regularity is a measure of the computational complexity of the module M and it is, in general, incredibly difficult to compute, cf. [DS02, TT15].

Proposition 8.3. *Suppose $L_{1/e}(\lambda)$ is unitary. Let $\mu_0 \in \mathrm{Po}_e(\lambda)$ be obtained by, first, sliding all beads of $\mathcal{A}_e(\lambda)$ to the left, and then, sliding the upmost beat k spaces to the right. Then,*

$$\mathrm{reg}(L_{1/e}(\lambda)[c_\lambda]) = (c_{\mu_0} - c_\lambda) - (h-1)k$$

Proof. As in the paragraph above the statement of the proposition, let us denote by n_i the maximum degree of a generator of $C(\lambda)_i$, and $m_i := n_i - i$. Note that $n_i := \max\{c_\mu - c_\lambda : \mu \in \mathrm{Po}_e(\lambda), \mathrm{hd}(\mu) = i\}$. Since the c-function is increasing in homological degree, the sequence (n_i) is increasing and therefore the sequence (m_i) is nondecreasing. So the regularity of $L_{1/e}(\lambda)[c_\lambda]$ is $m_{\mathrm{pdim}(L_{1/e}(\lambda))}$. Since $\mathrm{pdim}(L_{1/e}(\lambda)) = (h-1)k$, the result follows. □

Example 8.4. *Consider $e = 5$, $n = 15$ and $\lambda = (3^4, 2, 1)$. Then, pdim$L_{1/e}(\lambda) = 4$, so $L_{1/e}(\lambda)$ is not Cohen-Macaulay and a minimal graded-free resolution of $L_{1/e}(\lambda)[c_\lambda]$ is*

$$0 \to (3,2,1^{10})[-9] \to (3,2^2,1^7)[-5] \quad \to (3,2^6)[-3] \oplus (3^3,1^6)[-3] \to (3^3,2^2,1^2)[-1] \to (3^4,2,1)$$
$$\to L_{1/e}(\lambda)[c_\lambda] \to 0$$

where for brevity, we write $\mu[d]$ in place of $(\mathbb{C}[\underline{x}] \otimes S_n(\mu))[d]$. From the resolution, we see that $\mathrm{reg}(L_{1/e}(\lambda)[c_\lambda]) = 5$.

8.1.1. *The e-equals ideal.* We examine these results in the situation where the modules $L_{1/e}(\lambda)$ have a clear geometric meaning. The representation theoretic import of $X_{e,1,n}$ was first noticed and explained in [ZGS14]. Resolutions of the ideals vanishing on these subspace arrangements are given by BGG resolutions of the corresponding unitary module for $H_{1/e}(\mathfrak{S}_n)$.

Let $n = (e-1)p + q$, with $0 \leqslant q < e-1$. Consider the partition $\lambda = ((e-1)^p, q)$ of n. Note that the e-abacus of λ has exactly one empty runner, and the module $L_{1/e}(\lambda)$ is unitary. In fact, it follows from [Wil17a] that $L_{1/e}(\lambda)$ is isomorphic to the socle of the polynomial representation

$$\Delta_{1/e}(\mathrm{triv}) \cong \mathbb{C}[x_1, x_2, \dots, x_n]$$

which by [ES09, Theorem 5.10] coincides with the e-equals ideal $I_{e,1,n}$ of functions vanishing on the set

$$X_{e,1,n} := \mathfrak{S}_n\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 = \cdots = z_e\}.$$

Note that $X_{e,1,n}$ is an arrangement of $\binom{n}{e}$ linear subspaces of $\mathbb{C}^n$, each of dimension $n - e + 1$. When $e = 2$, $X_{2,1,n}$ is nothing but the braid arrangement in $\mathbb{C}^n$, which consists of the reflection hyperplanes for the action of $\mathfrak{S}_n$ on $\mathbb{C}^n$. Let us give a set of generators for the ideal $I_{e,1,n}$, following [ES09, FS12]. Consider the partition $\lambda^T = ((p+1)^q, p^{e-1-q})$, which has exactly $e-1$ parts. Now consider the polynomial

$$p_{\lambda^T}(x_1, \dots, x_n) = V(x_1, \dots, x_{\lambda_1^T})V(x_{\lambda_1^T+1}, \dots, x_{\lambda_1^T+\lambda_2^T}) \cdots V(x_{\lambda_1^T+\cdots+\lambda_{e-2}^T+1}, \dots, x_n)$$

where $V(x_1, \dots, x_k)$ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$. Then, the ideal $I_{e,1,n}$ is generated by the $\mathfrak{S}_n$ images of the polynomial p_{λ^T}.

Since $L_{1/e}(\lambda)$ and $L_{1/e}(\mathrm{triv})$ lie in the same block of category $\mathcal{O}_{1/e}$, the weight of the partition λ is $k = \lfloor n/e \rfloor$. Thus, as was observed in [ZGS14], the projective dimension of the algebra of functions

$\mathbb{C}[X_{e,1,n}] = \mathbb{C}[x_1,\dots,x_n]/I_{e,1,n}$ is $\mathrm{pdim}(\mathbb{C}[X_{e,1,n}]) = \mathrm{pdim}(L_{1/e}(\lambda)) + 1 = (e-2)\lfloor n/e\rfloor + 1$. Since $\dim(X_{e,1,n}) = n-e+1$, it follows that $\mathbb{C}[X_{e,1,n}]$ is Cohen-Macaulay if and only if $e = 2$ or $\lfloor n/e\rfloor = 1$. This way, we recover part of [EGL15, Proposition 3.11].

Example 8.5. *Consider $e = 4$, $n = 10$. The minimal submodule in $\Delta_{1/e}(\mathrm{triv})$ is $I_{4,1,10}$, and it is isomorphic to $L_{1/e}(3^3,1)$. Note that $c_\lambda = 23/4$. The resolution of $L_{1/4}(3^3,1)[-23/4]$ is given by*

$$0 \to (2,1^8)[-8] \to (2^2,1^6)[-6] \to (2^5)[-3] \oplus (3^2,1^4)[-2] \to (3^2,2,1^2)[-1] \to (3^3,1)$$
$$\to L_{1/4}(3^3,1)[-23/4] \to 0$$

A resolution of the coordinate ring $\mathbb{C}[x_1,\dots,x_{10}]/I_{4,1,10}$ looks similar, but each term is further shifted by -12 (because $c_{\mathrm{triv}} - c_\lambda = -12$), and the end of the sequence is $(3^3,1)[-12] \to (10) \to \mathbb{C}[\underline{x}]/I_{4,1,10} \to 0$. Note that the regularity of $\mathbb{C}[x_1,\dots,x_{10}]/I_{4,1,10}$ is 15.

Let us now compute the regularity of the subspace arrangement $X_{e,1,n}$.

Proposition 8.6. *The regularity of the $\mathbb{C}[\underline{x}]$-module $\mathbb{C}[X_{e,1,n}]$ is given by*

$$\mathrm{reg}(\mathbb{C}[X_{e,1,n}]) = \begin{cases} \lfloor n/e\rfloor(n-e+1)-1, & \text{if } n/e \in \mathbb{Z} \\ \lfloor n/e\rfloor(n-e+2)-1, & \text{else.} \end{cases}$$

Proof. Let us write $n = (e-1)p + q = ep_1 + q_1$, with $0 \leqslant q < e-1$ and $0 \leqslant q_1 < e$. As above, let $\lambda = ((e-1)^p, q)$ be the partition such that $L_{1/e}(\lambda)$ is isomorphic to the socle of $\Delta_{1/e}(\mathrm{triv})$. Note that the e-core of any partition in the block of triv $= (n)$ is (q_1) and the e-weight is p_1. It then follows from the rimhook description of homological degree in Section 5.6 that the partition μ_0 with highest homological degree in $\mathrm{Po}_e(\lambda)$ is given by adding p_1 vertical strips of length e in the leftmost column to the e-core of λ: thus $\mu_0 = (a, 1^{n-a})$ where $a = q_1$ if $q_1 > 0$ and $a = 1$ if $q_1 = 0$, and $\mathrm{hd}(\mu_0) = (e-2)p_1$. Now it follows by a direct computation that

$$\mathrm{reg}(\mathbb{C}[X_{e,1,n}]) = c_{\mu_0} - c_{\mathrm{triv}} - (e-2)p_1 - 1 = \frac{(n-a)n}{e} - (e-2)p_1 - 1$$

which coincides with the formula in the statement of the proposition. □

8.1.2. *More BGG resolutions and a generalisation of the e-equals ideal.* We take ℓth powers and obtain a generalisation of the e-equals ideal. These subspaces arrangements admit commutative algebra resolutions which can be constructed via BGG-resolutions for the Cherednik algebra of $G(\ell,1,n)$ (which we also construct in this section). Consider the ideal $I_{e,1,n}(\ell)$ of polynomials vanishing on the set

$$X_{e,1,n}(\ell) := \mathfrak{S}_n\{(z_1,\dots,z_n) \in \mathbb{C}^n : z_1^\ell = z_2^\ell = \cdots = z_e^\ell\}.$$

Note that $X_{e,1,n}(\ell)$ is an arrangement of $\ell^e\binom{n}{e}$ linear subspaces of $\mathbb{C}^n$, each of dimension $n-e+1$. When $e = \ell = 2$, $X_{2,1,n}(2)$ is the braid arrangement of type D_n, consisting of reflection hyperplanes for the reflection representation of the Weyl group of type D_n on $\mathbb{C}^n$. To give a set of generators for the ideal $I_{e,1,n}(\ell)$, recall from the previous subsection the partition $\lambda = ((e-1)^p, q)$ and the polynomial $p_{\lambda^T} \in \mathbb{C}[x_1,\dots,x_n]$. According to [FS12, Proposition 2.5], a set of generators of the ideal $I_{e,1,n}(\ell)$ is given by the $\mathfrak{S}_n$-images of $p_{\lambda^T}(x_1^\ell,\dots,x_n^\ell)$.

Our next goal is to construct a graded-free resolution of the algebra of functions $\mathbb{C}[x_1,\dots,x_n]/I_{e,1,n}(\ell)$. In order to do this, we will use the following well-known commutative algebra result.

Lemma 8.7. *Let F_1, F_2, F_3 be free $\mathbb{C}[x_1,\dots,x_n]$-modules of finite rank, with bases $\{v_1^1,\dots,v_{i_1}^1\}$, $\{v_1^2,\dots,v_{i_2}^2\}$ and $\{v_1^3,\dots,v_{i_3}^3\}$, respectively. Let $A : F_1 \to F_2$, $B : F_2 \to F_3$ be morphisms defined in the given bases by matrices $(f_{ij}(x_1,\dots,x_n))$, $(g_{jk}(x_1,\dots,x_n))$, respectively, and define new morphisms $\widetilde{A}, \widetilde{B}$ by the matrices $(f_{ij}(x_1^\ell,\dots,x_n^\ell))$, $(g_{jk}(x_1^\ell,\dots,x_n^\ell))$, respectively. If $\mathrm{im}(A) = \ker(B)$, then $\mathrm{im}(\widetilde{A}) = \ker(\widetilde{B})$.*

Note that, for $\mu \in \mathscr{P}_n^1$, the module $\mathbb{C}[x_1,\dots,x_n] \otimes S_n(\mu)$ has a distinguished basis indexed by $\mathrm{Std}(\mu)$. Thus, if λ is a unitary partition, we can apply Lemma 8.7 to the complex $C_\bullet(\lambda)$ (viewed as a complex of free $\mathbb{C}[x_1,\dots,x_n]$-modules) to obtain a complex $\widetilde{C}_\bullet(\lambda)$, which is exact outside of degree 0. By construction, thanks to [FS12, Proposition 2.5], when $\lambda = ((e-1)^p, q)$, the zeroth

homology of $\widetilde{C}_\bullet(\lambda)$ coincides with the ideal $I_{e,1,\lambda}(\ell)$. Moreover, by multiplying the grading shifts of $C_\bullet(\lambda)$ by ℓ, this obtains a minimal graded-free resolution of $I_{e,1,\lambda}$, and extending by $\mathbb{C}[x_1,\dots,x_n]$, of the algebra of functions $\mathbb{C}[X_{e,1,n}(\ell)]$. We then obtain the following result.

Proposition 8.8. *The projective dimension of $\mathbb{C}[X_{e,1,n}(\ell)]$ coincides with that of $\mathbb{C}[X_{e,1,n}]$, which is $(e-2)\lfloor n/e\rfloor+1$ so that, regardless of ℓ, $\mathbb{C}[X_{e,1,n}(\ell)]$ is Cohen-Macaulay if and only if $e=2$ or $\lfloor n/e\rfloor=1$. The regularity of $\mathbb{C}[X_{e,1,n}(\ell)]$ is given by*

$$\operatorname{reg}(\mathbb{C}[X_{e,1,n}(\ell)]) = \begin{cases} \lfloor n/e\rfloor(\ell(n-1)-e+2)-1 & \text{if } n/e\in\mathbb{Z} \\ \lfloor n/e\rfloor(\ell n-e+2)-1 & \text{else} \end{cases}$$

We have obtained the complex $\widetilde{C}_\bullet(\lambda)$ by means of pure commutative algebra. As it turns out, $\widetilde{C}_\bullet(\lambda)$ is a complex of standard modules for the rational Cherednik algebra of the group $G(\ell,1,n):=\mathfrak{S}_n\ltimes(\mathbb{Z}/\ell\mathbb{Z})^n$ under a special class of parameters. The group $G(\ell,1,n)$ is a complex reflection group, acting naturally on $\mathbb{C}^n$, and the rational Cherednik algebra depends on a function $\widetilde{c}:S\to\mathbb{C}$, where $S\subseteq G(\ell,1,n)$ is the set of reflections and $\widetilde{c}(s)=\widetilde{c}(wsw^{-1})$ for every $s\in S$, $w\in G(\ell,1,n)$. Here, for a complex number $c\in\mathbb{C}$, we will take *any* function $\widetilde{c}$ such that $\widetilde{c}(s)=c$, if $s\in G(\ell,1,n)$ is conjugate to a reflection in $\mathfrak{S}_n$. Any other reflection in $G(\ell,1,n)$ is conjugate to a nonzero element of, say, the first copy of $\mathbb{Z}/\ell\mathbb{Z}$, so we have $\ell-1$ more parameters for $H_{\widetilde{c}}(G(\ell,1,n))$, let us call them $c_1,\dots,c_{\ell-1}$.

The rational Cherednik algebra $H_{\widetilde{c}}(G(\ell,1,n))$ admits a presentation very similar to that of the rational Cherednik algebra $H_c(\mathfrak{S}_n)$ of the symmetric group. We will not give this presentation. Instead, we remark that $H_{\widetilde{c}}(G(\ell,1,n))$ is the subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[x_1,\dots,x_n])$ generated by the functions x_i of multiplication by x_i $(i=1,\dots,n)$, the elements of $G(\ell,1,n)$ (naturally viewed as automorphisms of $\mathbb{C}[x_1,\dots,x_n]$) and the *Dunkl–Opdam operators*:

$$\widetilde{D}_i := \partial_i - c\sum_{j\neq i}\sum_{t=0}^{\ell-1}\frac{1}{x_i-\xi^t x_j}(1-(ij)^t) - \sum_{k=1}^{\ell-1}\frac{2c_k}{(1-\xi^{-k})x_i}(1-\xi_i^k)$$

where $\xi:=\exp(2\pi\sqrt{-1}/\ell)$, $\xi_i\in G(\ell,1,n)$ is the element that acts by multiplication by ξ on the i-th coordinate in $\mathbb{C}^n$, and $(ij)^t\in G(\ell,1,n)$ is $(ij)^t=\xi_i^t\xi_j^{-t}(ij)$. Let us remark that a similar presentation exists for the algebra $H_c(\mathfrak{S}_n)$, the Dunkl operators are now given by

$$D_i = \partial_i - c\sum_{j\neq i}\frac{1}{x_i-x_j}(1-(ij)).$$

We will need the following result, that relates the operators D_i and $\widetilde{D}_i$.

Lemma 8.9. *For $g\in\mathbb{C}[x_1,\dots,x_n]$, denote by $\widetilde{g}:=g(x_1^\ell,\dots,x_n^\ell)$. Then, for any $i=1,\dots,n$:*

$$\widetilde{D}_i(\widetilde{g}) = \ell x_i^{\ell-1}\widetilde{D_i(g)}$$

Proof. First of all, note that $\widetilde{g}$ is invariant under the action of $(\mathbb{Z}/\ell\mathbb{Z})^n$ on $\mathbb{C}[x_1,\dots,x_n]$, and so it follows that

$$\widetilde{D_i(\widetilde{g})} = \partial_i(\widetilde{g}) - c\sum_{j\neq i}\sum_{t=0}^{\ell-1}\frac{\widetilde{g}-(ij)\widetilde{g}}{x_i-\xi^t x_j}.$$

Now let $h(x_1,\dots,x_n)\in\mathbb{C}[x_1,\dots,x_n]$ be such that $g-(ij)g=(x_i-x_j)h$. Note that it follows that $\widetilde{g}-(ij)\widetilde{g}=(x_i^\ell-x_j^\ell)\widetilde{h}$, so

$$\sum_{t=0}^{\ell-1}\frac{\widetilde{g}-(ij)\widetilde{g}}{x_i-\xi^t x_j} = \sum_{t=0}^{\ell}\prod_{\substack{k=0\\k\neq t}}^{\ell-1}(x_i-\xi^k x_j)\widetilde{h} = \ell x_i^{\ell-1}\widetilde{h}$$

and the result follows. □

The algebra $H_{\widetilde{c}}(G(\ell,1,n))$ still admits a triangular decomposition $H_{\widetilde{c}}(G(\ell,1,n))=\mathbb{C}[x_1,\dots,x_n]\otimes\mathbb{C}G(\ell,1,n)\otimes\mathbb{C}[y_1,\dots,y_n]$, where y_i is the Dunkl–Opdam operator $\widetilde{D}_i$. In particular, one can still

define standard modules. For an irreducible representation E of $G(\ell,1,n)$, we have the standard module $\Delta_{\widetilde{c}}(E)$. As a $\mathbb{C}[x_1,\dots,x_n]$-module, $\Delta_{\widetilde{c}}(E)=\mathbb{C}[x_1,\dots,x_n]\otimes E$.

The irreducible representations, $S_n(\lambda)$, of $G(\ell,1,n)$ are indexed by the set $\mathscr{P}_n^\ell$, and each $S_n(\lambda)$ has a natural basis indexed by the set $\mathrm{Std}(\lambda)$. In particular, if $\lambda\in\mathscr{P}_n^1$, we can consider the ℓ-partition $\widetilde{\lambda}\in\mathscr{P}_n^\ell$ given by $\widetilde{\lambda}=(\lambda,\emptyset,\dots,\emptyset)$. The sets $\mathrm{Std}(\widetilde{\lambda})$ and $\mathrm{Std}(\lambda)$ are obviously identified. Moreover, $G(\ell,1,n)$ admits a natural surjection to $\mathfrak{S}_n$, and the irreducible representation $S_n(\widetilde{\lambda})$ of $G(\ell,1,n)$ is simply given by the $\mathfrak{S}_n$-irreducible $S_n(\lambda)$ under this surjection.

Proposition 8.10. *Let $c\in\mathbb{C}$. Then, for any $\lambda,\mu\in\mathscr{P}_n^1$ and any parameter $\widetilde{c}$ as above, there is a natural identification*

$$\sim:\mathrm{Hom}_{H_c(\mathfrak{S}_n)}(\Delta_c(\lambda),\Delta_c(\mu))\ \xrightarrow{\cong}\ \mathrm{Hom}_{H_{\widetilde{c}}(G(\ell,1,n))}(\Delta_{\widetilde{c}}(\widetilde{\lambda}),\Delta_{\widetilde{c}}(\widetilde{\mu}))$$

given as follows. For a standard Young tableau $\mathsf{t}\in\mathrm{Std}(\lambda)$, if $f\in\mathrm{Hom}_{H_c(\mathfrak{S}_n)}(\Delta_c(\lambda),\Delta_c(\mu))$ is given by $f(1\otimes\mathsf{t})=\sum_{\mathsf{s}\in\mathrm{Std}(\mu)}f_{\mathsf{ts}}(x_1,\dots,x_n)\otimes\mathsf{s}$, then $\widetilde{f}(1\otimes\mathsf{t})=\sum_{\mathsf{s}\in\mathrm{Std}(\mu)}f_{\mathsf{ts}}(x_1^\ell,\dots,x_n^\ell)\otimes\mathsf{s}$.

Proof. We need to show, first, that $\widetilde{f}|_{1\otimes S_n(\widetilde{\lambda})}$ is a map of $G(\ell,1,n)$-representations. This follows from the fact that, for any polynomial $g\in\mathbb{C}[x_1,\dots,x_n]$, $g(x_1^\ell,\dots,x_n^\ell)$ is invariant under the action of $(\mathbb{Z}/\ell\mathbb{Z})^n$. Now we need to show that, for any standard Young tableau $\mathsf{t}\in\mathrm{Std}(\lambda)$, $\widetilde{f}(1\otimes\mathsf{t})$ is annihilated by all Dunkl operators $\widetilde{D}_i$. This is a direct consequence of Lemma 8.9.

This shows that $f\mapsto\widetilde{f}$ does define a morphism, which is clearly injective. To show that it is bijective, let $h:\Delta_{\widetilde{c}}(\widetilde{\lambda})\to\Delta_{\widetilde{c}}(\widetilde{\mu})$ be a morphism. In particular, $h|_{1\otimes S_n(\widetilde{\lambda})}$ is a map of $G(\ell,1,n)$-modules. This implies that, if $h(1\otimes\mathsf{t})=\sum_{\mathsf{s}\in\mathrm{Std}(\mu)}h_{\mathsf{ts}}(x_1,\dots,x_n)\otimes\mathsf{s}$, then $h_{\mathsf{ts}}(x_1,\dots,x_n)\in\mathbb{C}[x_1^\ell,\dots,x_n^\ell]$ for every $\mathsf{s}\in\mathrm{Std}(\mu)$. Thanks to Lemma 8.9, this implies that $h=\widetilde{f}$ for some $f:\Delta_c(\lambda)\to\Delta_c(\mu)$. □

Remark 8.11. *If $c\notin 1/2+\mathbb{Z}$, then the existence of an isomorphism between $\mathrm{Hom}_{H_c(\mathfrak{S}_n)}(\Delta_c(\lambda),\Delta_c(\mu))$ and $\mathrm{Hom}_{H_{\widetilde{c}}(G(\ell,1,n))}(\Delta_{\widetilde{c}}(\widetilde{\lambda}),\Delta_{\widetilde{c}}(\widetilde{\mu}))$ follows from* [GGOR03, Proposition 5.9].

By Proposition 8.10 and Lemma 8.7, we have that if λ is a unitary partition of n, then the complex $\widetilde{C}_\bullet(\lambda)$ is actually a complex of standard modules for $H_{\widetilde{c}}(G(\ell,1,n))$, which is exact outside of degree zero, and thus it is a BGG resolution of its zeroth homology.

Remark 8.12. *The zeroth homology of $\widetilde{C}_\bullet(\lambda)$ is* not *necessarily an irreducible $H_{\widetilde{c}}(G(\ell,1,n))$-module. For example, if $\lambda=((e-1)^p,q)$, we have seen that $H_0(\widetilde{C}_\bullet(\lambda))$ is the ideal $I_{e,1,n}(\ell)$. When $\ell=2$, $e<n$ is even and the parameter $\widetilde{c}$ is such that $\widetilde{c}(s)=0$ if s is not conjugate to a reflection in $\mathfrak{S}_n$, then this is an indecomposable, but not irreducible, $H_{\widetilde{c}}(G(\ell,1,n))$-module.*

Remark 8.13. *Even if the zeroth homology of $\widetilde{C}_\bullet(\lambda)$ is irreducible (and thus it necessarily coincides with $L_{\widetilde{c}}(\widetilde{\lambda})$) the natural Hermitian form on $L_{\widetilde{c}}(\widetilde{\lambda})$ does not need to be positive-definite, even if that for $L_c(\lambda)$ is. An example of this is given by taking $\ell=2$, odd n, $e=n$, $\lambda=(e-1,1)$ and the parameter $\widetilde{c}$ as in Remark 8.12 . In this case, $L_{\widetilde{c}}(\widetilde{\lambda})=I_{e,1,n}(2)$, which does not admit an invariant positive-definite Hermitian form, cf.* [FS12, Proposition 7.1]

8.1.3. *The (k,e)-equals ideal.* We now consider the subspace arrangements of k distinct clusters of e equal parameters for $n=ke$. We show that the BGG resolution of $L(\mathrm{triv})$ is a minimal resolution of the coordinate ring of this subspace arrangement and generalise this to type $G(\ell,1,n)$ as before.

Let $n=ke$, as we have seen, in this case we can give a BGG resolution of $L_{1/e}(\mathrm{triv})$. It follows from [ES09, Theorem 5.10] that $\mathrm{rad}(\Delta_{1/e}(\mathrm{triv}))$ is the ideal $I_{e,k,n}$ of functions vanishing on

$$X_{e,k,n}:=\mathfrak{S}_n\{(z_1,\dots,z_n)\in\mathbb{C}^n:z_1=\dots=z_e,z_{e+1}=\dots=z_{2e},\dots,z_{(k-1)e+1}=\dots=z_{ke}\}.$$

Recall that the resolution of $L_{1/e}(\mathrm{triv})$ is obtained as the Ringel dual of the resolution of $L_{1/e}(e^k)$. Thus, the projective dimension of the algebra of functions $\mathbb{C}[X_{e,k,n}]=\mathbb{C}[\underline{x}]/I_{e,k,n}\cong L_{1/e}(\mathrm{triv})[c_{\mathrm{triv}}]$ is $(e-1)k$. By the Auslander–Buchsbaum formula, the depth of $\mathbb{C}[X_{e,k,n}]$ is $n-(e-1)k=k$. So $\mathbb{C}[X_{e,k,n}]$ is always Cohen-Macaulay, and we recover a special case of [EGL15, Proposition 3.11].

Let us now analyze the regularity of $L_{1/e}(\mathrm{triv})[c_{\mathrm{triv}}]$. By an argument similar to the proof of Proposition 8.6, this is given by $c_{(k^e)} - c_{\mathrm{triv}} - (e-1)k$. By a direct computation, this is

$$\mathrm{reg}(L_{1/e}(\mathrm{triv})[c_{\mathrm{triv}}]) = \frac{k(n-e-k+1)}{2}$$

Example 8.14. *Assume $e = 3$, $n = 6$. Then we have that a resolution of $L(\mathrm{triv})[c_{\mathrm{triv}}] = \mathbb{C}[x_1, \cdots, x_6]/I_{3,2,6}$ is given by*

$$0 \to (2^3)[-6] \to (3,2,1)[-5] \to (3^2)[-4] \oplus (4,1^2)[-4] \to (5,1)[-2] \to (6) \to \mathbb{C}[\underline{x}]/I_{3,2,6} \to 0$$

and $\mathrm{reg}(\mathbb{C}[\underline{x}]/I_{3,2,6}) = 2$.

Of course, for $\ell \geqslant 1$ we also have the subspace arrangement

$$X_{e,k,n}(\ell) := \mathfrak{S}_n\{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^\ell = \cdots = z_e^\ell, z_{e+1}^\ell = \cdots = z_{2e}^\ell, \ldots, z_{(k-1)e+1}^\ell = \cdots = z_{ke}^\ell\}$$

And its defining ideal $I_{e,k,n}(\ell)$. Since $I_{e,k,n}$ is the unique maximal submodule in $\Delta_{1/e}(\mathrm{triv})$ and the submodules of this standard module are linearly ordered, the ideal $I_{e,k,n}$ is generated in a single degree. Thus, the exact same argument as that in the proof of [FS12, Proposition 2.5], if $q_1(x_1, \ldots, x_n), \ldots, q_t(x_1, \ldots, x_n)$ are generators of $I_{e,k,n}$ of minimal degree, then $q_1(x_1^\ell, \ldots, x_n^\ell), \ldots, q_t(x_1^\ell, \ldots, x_n^\ell)$ are generators of $I_{e,k,n}(\ell)$. It follows that the complex $\widetilde{C}_\bullet(\mathrm{triv})$ is a minimal graded-free resolution of the algebra of functions $\mathbb{C}[X_{e,k,n}(\ell)]$, and the variety $X_{e,k,n}(\ell)$ is always Cohen-Macaulay. Moreover, the regularity of $\mathbb{C}[X_{e,k,n}(\ell)]$ is given by $\ell(c_{(k^e)} - c_{\mathrm{triv}}) - (e-1)k$, or more explicitly,

$$\mathrm{reg}(\mathbb{C}[X_{e,k,n}(\ell)]) = \frac{k[\ell(n+e-k-1) - 2(e-1)]}{2}.$$

We remark that in general as $H_{\widetilde{c}}(G(\ell,1,n))$-modules, $\mathbb{C}[X_{e,k,n}(n)]$ does *not* coincide with $L_{\widetilde{c}}(\mathrm{triv})$. For example, if $\ell = 2$, $e = n$ is even and $\widetilde{c}(s) = 0$ for a reflection s not conjugate to an element of $\mathfrak{S}_n$, then $L_{\widetilde{1/e}}(\mathrm{triv})$ is finite-dimensional, while $\mathbb{C}[X_{e,1,n}(2)]$ is not.

Remark 8.15. *Changing the parameter of the rational Cherednik algebra to $c = a/e > 0$ with $\gcd(a;e) = 1$ does not change the shape of the resolution $C_\bullet(\lambda)$, so the projective dimension and depth of $L_{a/e}(\lambda)$ are independent of $a \in \mathbb{Z}_{>0}$ when λ is e-unitary. However, the value of c_λ is not independent of $a \in \mathbb{Z}_{>0}$, and we get*

$$\beta_{i,j}(L_{a/e}(\lambda)) = \beta_{i,j/a}(L_{1/e}(\lambda))$$

where we implicitly agree that $\beta_{i,j/a} = 0$ if $j/a \notin \mathbb{Z}$. For any such $a \in \mathbb{Z}_{>0}$ the module $L_{a/e}((m-1)^p, q)$ can be identified with an ideal of $\mathbb{C}[\underline{x}]$ whose vanishing set coincides with $X_{e,1,n}$. This ideal is radical if and only if $a = 1$, cf. [ES09, Theorem 5.10]. *Similar considerations apply to $L_{a/e}(\mathrm{triv})$.*

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