ADMISSIBLE SEMI-LINEAR REPRESENTATIONS

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Let k be an algebraically closed field[1] of characteristic zero, F be a universal domain over k, i.e., an algebraically closed extension of k of countable transcendence degree, and $G_{F/k}$ be the field automorphism group of F over k with the base of open subgroups $\{G_{F/k(x)} \mid x \in F\}$.

An F-semi-linear representation of $G_{F/k}$, i.e., an F-vector space V endowed with an additive semi-linear ($g(fv) = gf \cdot gv$ for any $f \in F$, $g \in G$ and $v \in V$) G-action $G \times V \to V$, is called *admissible* if it is smooth (with open stabilizers) and $\dim_{F^U} V^U < \infty$ for any open subgroup $U \subseteq G$.

In this note we study the category $\mathcal{A}$ of admissible F-semi-linear representations of $G_{F/k}$.

Clearly, $\mathcal{A}$ is an additive category and it is shown in [R] that it is a tensor category. Theorem 3.5 asserts that the category $\mathcal{A}$ is abelian.

This is deduced from the acyclicity of the objects of $\mathcal{A}$: $H^1_{\mathrm{smooth}}(G_{F/k}, V) = 0$ for any object V of $\mathcal{A}$, cf. Proposition 3.4, meaning that F is a projective object of $\mathcal{A}$.

Other results of this note are motivated by the following conjecture.

Conjecture. Let $\mathcal{C}$ be the category of smooth F-semi-linear representations of $G_{F/k}$.

(1) The functors $\mathrm{Hom}_{\mathcal{C}}(\bigotimes^q_F \Omega^1_F, -)$ are exact on $\mathcal{A}$ for any $q \geq 0$.
(2) Irreducible objects of $\mathcal{A}$ are direct summands of the tensor algebra $\bigotimes^{\bullet}_F \Omega^1_{F/k}$.

It follows from this conjecture that if V, V' are irreducible objects of $\mathcal{A}$ and $\mathrm{Ext}^j_{\mathcal{A}}(V, V') \neq 0$ then there is a surjection $V' \otimes_F \bigotimes^j_F \Omega^1_{F/k} \to V$ and $\mathrm{Ext}^j_{\mathcal{A}}(\Omega^{j+q}_{F/k}, \Omega^q_{F/k}) = \mathrm{Hom}_k(\mathrm{Sym}^j_k \Omega^1_k, k)$. If $k = \overline{\mathbb{Q}}$ is the field of algebraic numbers then $\mathcal{A}$ is semi-simple.

Let $n \geq 1$ be an integer, $K = k(\mathbb{P})$ be the function field of an n-dimensional projective k-space $\mathbb{P}$ and $G = \mathrm{Aut}(\mathbb{P}/k)$ be its automorphism group. Fix a k-field embedding $K \hookrightarrow F$.

We show, in particular, that if V is an admissible F-semi-linear representation of $G_{F/k}$ with no sub-objects isomorphic to F then any irreducible subquotient of the K-semi-linear representation $V^{G_{F/K}}$ of G is a direct summand of $\bigotimes^{\geq 1}_K \Omega^1_{K/k}$ (this is Theorem 2.5 and Proposition 3.4 below). This is an evidence for the part (2) of the conjecture. Lemma 3.9 gives an evidence for the above mentioned corollary of the conjecture.

0.1. **Notation.** Let k, F and $G_{F/k}$ be as above. We fix a transcendence basis $x_1, x_2, x_3, \dots$ of F over k. For each $n \geq 1$ set $Y_n = \mathbf{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathbb{A}^n_k = \mathbf{Spec} k[x_1, \dots, x_n] \subset \mathbb{P}^n_k = \mathbb{P}(Q_n) = \mathbf{Proj} k[X_0, \dots, X_n]$ with $x_j = X_j/X_0$, $K_n = k(x_1, \dots, x_n)$, $G_n = \mathrm{Aut}(\mathbb{P}^n_k/k)$. Let $\mathrm{Aff}_n = G_n \cap \mathrm{Aut}(\mathbb{A}^n_k/k)$ be the affine subgroup. Let $H_n = \mathrm{Aff}_n \cap \mathrm{Aut}(\mathbb{A}^n_k/\mathbb{A}^{n-1}_k)$ be the subgroup fixing the coordinates $x_1, \dots, x_{n-1}$ on $\mathbb{A}^n_k$. Let $T_n \subset G_n$ be the maximal torus acting freely on Y_n. Let $L_n \subset \mathbb{A}^n_k$ be the line of common zeroes of $x_1, \dots, x_{n-1}$.

For a field extension L/K we denote by $\mathrm{Der}(L/K)$ the Lie K-algebra of derivations of L over K. The subgroup of ℓ-th roots of unity in k is denoted by μ_ℓ.

[1] In fact, we shall only use the existence in k of all ℓ-primary roots of unity for a prime ℓ.

1. Equivariantness of irreducible PGL-sheaves

Let $\mathfrak{S}\mathfrak{L}_n^u$ be the category of finite-dimensional K_n-semi-linear G_n-representations whose restrictions to the maximal torus T_n in G_n are induced by unipotent representations.

In [R], for $n>1$, a fully faithful functor $\mathfrak{S}\mathfrak{L}_n^u \xrightarrow{\mathcal{S}} \{\text{coherent } G_n\text{-sheaves on } \mathbb{P}_k^n\}$ is constructed. (One says that $\mathcal{V}$ is a G_n-sheaf if it is endowed with a collection of isomorphisms $\alpha_g : \mathcal{V} \xrightarrow{\sim} g^*\mathcal{V}$ for each $g \in G_n$ satisfying the chain rule: $\alpha_{hg} = g^*\alpha_h \circ \alpha_g$ for any $g, h \in G_n$.) The composition of $\mathcal{S}$ with the generic fibre functor is the identical full embedding of $\mathfrak{S}\mathfrak{L}_n^u$ to the category of finite-dimensional K_n-semi-linear G_n-representations.

In this section we show that the category $\mathfrak{S}\mathfrak{L}_n^u$ is abelian and its irreducible objects are generic fibres of irreducible coherent G_n-*equivariant* sheaves on $\mathbb{P}_k^n$, i.e., direct summands of $\mathrm{Hom}_{K_n}((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n} \Omega^1_{K_n/k})$ for appropriate M.

Lemma 1.1. *The category $\mathfrak{S}\mathfrak{L}_n^u$ is closed under taking K_n-semi-linear subquotients.*

Proof. Let $V \in \mathfrak{S}\mathfrak{L}_n^u$ and $0 \to V_1 \to V \xrightarrow{\pi} V_2 \to 0$ be a short exact sequence of K_n-semi-linear representations of G_n. Then the sequences $0 \to V_1^{T_n^{\mathrm{tors}}} \to V^{T_n^{\mathrm{tors}}} \to V_2^{T_n^{\mathrm{tors}}}$ and $0 \to V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n \to V = V^{T_n^{\mathrm{tors}}} \otimes_k K_n \xrightarrow{\pi'} V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n$ are also exact. As the surjection π factors through π', we see that $V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n$, and therefore, $V_1 = V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n$, i.e., that restriction of V_1 (resp., of V_2) to T_n are induced by a sub- (resp., quotient) representation of $V^{T_n^{\mathrm{tors}}}$. $\square$

Lemma 1.2. *Let E be the total space of a vector bundle on $\mathbb{P}_k^n$, $\mathrm{Aut}_{\mathrm{lin}}(E)$ be the group of automorphisms of E over k inducing linear transforms between the fibres, and $\tau : G_n \to \mathrm{Aut}_{\mathrm{lin}}(E)$ be an irreducible G_n-structure on E, i.e., a homomorphism splitting the projection $\mathrm{Aut}_{\mathrm{lin}}(E) \to G_n$. Then the Zariski closure $\overline{\tau(G_n)}$ is reductive.*

Proof. Let Aut_τ be the kernel of the projection $\overline{\tau(G_n)} \xrightarrow{\pi} G_n$.

For each point $p \in \mathbb{P}_k^n$ let $\rho_p : R_p := \pi^{-1}(\mathrm{Stab}_p) \to \mathrm{GL}(E_p)$ be the natural representation.

As we suppose that E is an irreducible G_n-bundle, ρ_p is irreducible, since otherwise $B := \overline{\tau(G_n)}B_p \subset E$ is a G_n-subbundle for any proper R_p-invariant k-subspace $B_p \subset E_p$.

In particular, ρ_p is trivial on the unipotent radical of R_p. The unipotent radical of any algebraic group contains the unipotent radical of its arbitrary normal subgroup, so ρ_p is trivial on the unipotent radical of Aut_τ. As the action of Aut_τ on E is faithful, $\bigcap_p \ker \rho_p|_{\mathrm{Aut}_\tau} = \{1\}$, i.e., Aut_τ is reductive. As G_n is also reductive, so is $\overline{\tau(G_n)}$. $\square$

We need the following particular case of Théorème 8.16 of [BT]. For a commutative finite k-algebra A denote by $R_{A/k}$ the Weil functor of restriction of scalars on A-schemes, cf. [DG], I, §1, 6.6.

Theorem 1.3. *Let G be a simply connected absolutely almost simple k-group, and G' be a reductive k-group. Let $\tau : G(k) \to G'(k)$ be a homomorphism with Zariski dense image. Let $G'_1, \dots, G'_m$ be the almost simple normal subgroups of G'.*

Then there exist finite field extensions k_i/k, field embeddings $\varphi_i : k \to k_i$, a special isogeny $\beta : \prod_{i=1}^m R_{k_i/k}{}^{\varphi_i}G \to G'$ (here ${}^{\varphi_i}G := G \times_{k,\varphi_i} k_i$) and a homomorphism $\mu : G(k) \to Z_{G'}(k)$ such that $\beta(R_{k_i/k}{}^{\varphi_i}G) = G'_i$ and $\tau(h) = \mu(h) \cdot \beta(\prod_{i=1}^m \varphi_i^\circ(h))$ for any $h \in G(k)$ (here $\varphi_i^\circ : G(k) \to R_{k_i/k}{}^{\varphi_i}G(k)$ is the canonical homomorphism). $\square$

Corollary 1.4. *Under assumptions of Theorem 1.3, for any torus $T \subset G$ the Zariski closure of $\tau(T(k))$ is a torus in G'.* $\square$

Proposition 1.5. *Let V be an irreducible subquotient of an object of $\mathfrak{SL}^u_n$ for $n \geq 2$. Then V is a direct summand of $\mathrm{Hom}_{K_n}((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n} \Omega^1_{K_n/k})$ for an appropriate M.*

Proof. The functor $\mathcal{S}$, mentioned in the beginning of this §, associates to V a coherent G_n-sheaf $\mathcal{V}$ on $\mathbb{P}^n_k$ with generic fibre V.

Let, as before, T_n be a maximal torus in G_n and $Y_n \subset \mathbb{P}^n_k$ be the n-dimensional T_n-orbit. As $V^{T_n^{\mathrm{tors}}} = \Gamma(Y_n, \mathcal{V})^{T_n^{\mathrm{tors}}}$, cf. [R], is a unipotent representation of T_n, Lemma 1.2 and Corollary 1.4 imply that $\Gamma(Y_n, \mathcal{V})^{T_n^{\mathrm{tors}}}$ is a trivial representation of T_n.

As G_n is generated by a finite number of conjugates of T_n, we see that $\mathcal{V}$ is equivariant, i.e., the action $G_n \times \mathrm{tot}(\mathcal{V}) \to \mathrm{tot}(\mathcal{V})$ is a morphism of k-varieties. The generic fibres of irreducible G_n-equivariant sheaves on $\mathbb{P}^n_k$ are exactly of the desired type. $\square$

Remark. There can exist, a priori, non-equivariant irreducible coherent G_n-sheaves on $\mathbb{P}^n_k$, e.g. the one induced by a non-rational representation of G_n is seemingly of this type.

2. "Positivity"

In this section we show that for any admissible F-semi-linear representation V of $G_{F/k}$ any irreducible subquotient of the K_n-semi-linear representation $V^{G_{F/K_n}}$ of G is a direct summand of $\bigotimes^{\bullet}_{K_n} \Omega^1_{K_n/k}$.

It is shown in [R] that any finite-dimensional K_n-semi-linear G_n-representation extendable to $\mathrm{End}(K_n/k)$, e.g. $V^{G_{F/K_n}}$, is an object of $\mathfrak{SL}^u_n$. By Proposition 1.5, we only need to eliminate the negative twists by $\Omega^n_{K_n/k}$ in irreducible subquotients of $V^{G_{F/K_n}}$.

To do that we show first that the generic fibres of irreducible coherent G_n-equivariant sheaves are determined by their restrictions to a subgroup $\mathrm{Aff}_n = G_n \cap \mathrm{Aut}(\mathbb{A}^n_k/k)$.

Lemma 2.1 ([R], Lemma 2.2)**.** *Let P be a subgroup of $\mathrm{Aut}(K/k)$ such that the H-orbit of any element of $K - k$ spans a k-subspace in K of dimension $> r^2$. Then the map $H^1(P, \mathrm{GL}_r k) \to H^1(P, \mathrm{GL}_r K)$ is injective.* $\square$

Lemma 2.2. *Let Aff_n be the group of affine transformations of an affine space $\mathbb{A}^n_k$ with the function field K_n. Then the natural morphism*

$$(1)\quad \{\text{rational } k\text{-linear } \mathrm{Aff}_n\text{-representations}\} \xrightarrow{\otimes_k K_n} \{K_n\text{-semi-linear } \mathrm{Aff}_n\text{-representations}\},$$

injective by Lemma 2.1, transforms isomorphism classes of irreducible k-representations of Aff_n to isomorphism classes of irreducible K_n-semi-linear representations of Aff_n.

Proof. Let W be an irreducible k-representations of Aff_n, and $U \subset W \otimes_k K_n$ a non-zero K_n-semi-linear subrepresentation of Aff_n. Let $\alpha = \sum_{j=1}^N w_j \alpha_j \in U$ be a non-zero element with minimal possible N. Multiplying α by an element of K_n we may assume that all α_j are polynomials: $\alpha = \sum_I w'_I z^I$. The elements of the unipotent radical of Aff_n, i.e., σ such that $\sigma x - x \in k$ for any linear function x on $\mathbb{A}^n_k$, act trivially on W.

Applying an appropriate composition of difference operators $\sigma - \tau$ for some σ, τ in the unipotent radical of Aff_n to α, we get a non-zero element of W. As any non-zero element of W generates $W \otimes_k K_n$, this means that $U = W \otimes_k K_n$. $\square$

Corollary 2.3. *Let Aff_n, $\mathbb{A}^n_k$ and K_n be as in Lemma 2.2. Then there is a natural injection*

$$(2)\quad \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible } G_n\text{-subrepresentations in} \\ \bigoplus_M \mathrm{Hom}_K((\Omega^n_{K_n/k})^{\otimes M}, \bigotimes^{\bullet}_{K_n} \Omega^1_{K_n/k}) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible rational } k\text{-linear} \\ \mathrm{Aff}_n\text{-representations} \end{array} \right\}$$

such that its composition with the morphism (1) is the restriction map. $\square$

Let W be an $(n+1)$-dimensional k-vector space, $L \subset W$ be a one-dimensional subspace, $H_{\rm lin} = \ker[\mathrm{GL}(W,L) \to \mathrm{GL}(W/L)] \cong k^\times \ltimes \mathrm{Hom}(W/L, L)$.

Lemma 2.4. *For any Young diagram λ with no columns of height $n+1$ one has*

$$(S^\lambda W^\vee \otimes_k (\det W)^{\otimes s})^{H_{\rm lin}} = \begin{cases} S^\lambda(W/L)^\vee & \text{if } s = 0, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Denote by $X = AF(W) \cong \mathrm{GL}(W)/R_u(B)$, the variety of complete affine flags in W. An affine flag is a filtration $W_\bullet = (0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{n+1} = W)$ with $\dim_k W_j = j$ and a collection of $l_j \in W_j/W_{j-1} - \{0\}$.

Let $Y \cong \mathrm{GL}(W)/B$ be the variety of complete linear flags in W. Then the natural projection $X \xrightarrow{\pi} Y$ is a principal $(\mathbb{G}_m)^{n+1}$-bundle, and there is a decomposition $\pi_*\mathcal{O}_X = \bigoplus_\mu \mathcal{M}(\mu)$ into a direct sum of invertible sheaves on Y, where μ runs over the group $\mathbb{Z}^{n+1}$ of characters of $(\mathbb{G}_m)^{n+1}$, so $\mathcal{O}(X) = \bigoplus_\mu \Gamma(Y, \mathcal{M}(\mu))$.

Set $X^\circ = \{(V_\bullet, l_\bullet) \in X \mid V_n \cap L = 0 \Leftrightarrow \text{“}l_{n+1} \in L\text{”}\}$. Then reduction modulo L defines a principal $L^n \times \mathbb{G}_m$-bundle $X^\circ \to AF(W/L)$, $(l_1, \dots, l_{n+1}) \mapsto (l_1, \dots, l_n)$.

Let $X^\circ \xrightarrow{\text{“}l_{n+1}\text{”}} L - \{0\}$ be the natural H- (or $\mathbb{G}_m$-) equivariant map, and $\bar{l}_{n+1}$ be the composition of “l_{n+1}” with an isomorphism $L - \{0\} \cong \mathbb{G}_m$.

Set $SH := H_{\rm lin} \cap \mathrm{SL}(W) = \mathrm{Hom}(W/L, L)$. Then $\mathcal{O}(X)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(X^\circ)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))[\bar{l}_{n+1}, \bar{l}_{n+1}^{-1}]$, so $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$, where $\mu' \in \mathbb{Z}^n$ is the restriction of μ to the first n multiples of $(\mathbb{G}_m)^{n+1}$.

For any μ this is an irreducible representation of $\mathrm{GL}(W/L)$, and thus, $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$ if $\mathcal{O}(X)(\mu) \neq 0$.

As any irreducible representation of $\mathrm{SL}(W)$ coincides with $\mathcal{O}(X)(\mu)$ for some μ, this implies that $(S^\lambda W^\vee)^{SH} = S^\lambda(W/L)^\vee$. $\square$

Theorem 2.5. *For any F-semi-linear $G_{F/k}$-representation $V \in \mathcal{A}$ any irreducible subquotient of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ is a direct summand of $\bigotimes^\bullet_{K_n} \Omega^1_{K_n/k}$.*

Proof. Let $K' = k(W)$ be purely transcendental over K_n of degree 1. By Proposition 1.5 and Corollary 2.3, the restrictions to $\mathrm{Aff}(W)$ (or to its reductive quotient $\mathrm{GL}(W)$) of irreducible subquotients of the K'-semi-linear $\mathrm{PGL}(W \oplus k)$-representation $V^{G_{F/K'}}$ are of type $(S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K'$ for a Young diagram λ with no columns of height $n+1$ and some integer s.

Let $H \subset \mathrm{Aff}(W)$ be the subgroup fixing the functionals $x_1, \dots, x_n$ in $W^\vee$ vanishing on L. Let Aff_u be the unipotent radical of $\mathrm{Aff}(W)$, i.e. the group of translations of W.

Then the restrictions to Aff_n of the irreducible subquotients of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ are contained in $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K')^H$. As $H \cap \mathrm{Aff}_u = \langle 1 \rangle_k \cong k$, we get $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K')^{H \cap \mathrm{Aff}_u} = (S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_n$, so $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K')^H = (S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H \otimes_k K_n$.

By Lemma 2.4, $(S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H$ coincides with $S^\lambda(W/L)^\vee$ if $s = 0$, and vanishes otherwise. This means that any representation of Aff_n obtained this way is a direct summand of the tensor algebra of the representation $(W/L)^\vee = (\Omega^1_{K_n/k})^{\{\text{translations}\}}$ of $\mathrm{GL}_n k$. As any irreducible subquotient U of the K_n-semi-linear G_n-representation $V^{G_{F/K_n}}$ is determined by its restriction $U|_{\mathrm{Aff}_n}$ to Aff_n and $U|_{\mathrm{Aff}_n}$ is a direct summand of $\bigotimes^\bullet_{K_n} \Omega^1_{K_n/k}$, the same holds for U. $\square$

3. Extensions in $\mathfrak{S}\mathfrak{L}_n^u$ and in $\mathcal{A}$

Denote by $\mathrm{Aff}_n^{(\ell)}\mathbb{Q}$ the subgroup of Aff_n consisting of the $\mathbb{Q}(\mu_\ell)$-affine substitutions of $x_1, \dots, x_n$ with Jacobian in μ_ℓ, and by $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ the evident subgroup of index ℓ.

Lemma 3.1. *Let $n \geq 2$ and $\ell \geq 2$. Then restriction of any object $V \in \mathfrak{S}\mathfrak{L}_n^u$ to $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ is induced by a rational representation of its reductive quotient $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ over k. Irreducible rational representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ over k induce irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ over K_n. In particular, any extension in $\mathfrak{S}\mathfrak{L}_n^u$ splits as an extension of K_n-semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$.*

Proof. It is shown in [R] that $H^0(U_0, -)$ is the fibre functor on $\mathfrak{S}\mathfrak{L}_n^u$, where U_0 is the unipotent radical of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$, so $V = V^{U_0} \otimes_k K_n$ and the restrivction of V to $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ is induced by a k-linear representation V^{U_0} of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ for any $V \in \mathfrak{S}\mathfrak{L}_n^u$.

By Proposition 1.5, the irreducible subquotients of V restricted to $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ are induced by rational irreducible representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$, so the irreducible subquotients of V^{U_0} are rational and induce irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ over K_n.

It follows from Theorems 1.3 and 3.8 that the k-linear representation V^{U_0} of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ is semi-simple. $\square$

Remarks. 1. Let $V = \Omega^1_{K_n}/\Lambda \otimes_k K_n$, where $\Lambda \subset \Omega^1_k$ is a proper k-subspace. Let the extension $0 \to V \to U \to K_n \to 0$ be given by the cocycle $(\omega_\sigma = d\log\frac{\sigma\omega}{\omega}) \in Z^1(G_n, V)$, where $\omega = dx_1 \wedge \cdots \wedge dx_n \in \Omega^n_{K_n/k}$. Then the restriction of (ω_σ) to $\mathrm{GL}_n k$ is non-trivial.

2. The convolution with the Euler vector field $\sum_{j=1}^n x_j\partial/\partial x_j$ defines a $\mathrm{GL}_n k$-equivariant morphism $\Omega^1_{K_n/k} \to K_n$ given by $dx_j \mapsto x_j$. It is non-split for $n \geq 3$, since $(\Omega^1_{K_n/k})^{\mathrm{SL}_n\mathbb{Q}} = 0$.

Lemma 3.2. *Let $n, \ell \geq 2$ and s be some integers such that ℓ does not divide $s+1$. Let $V = S^\lambda_{K_n}\Omega^1_{K_n/k} \otimes_{K_n} (\Omega^n_{K_n/k})^{\otimes s}$ for a Young diagram λ with columns of height $< n$.*

Then $(V^{\mathcal{H}_n})^{\mathrm{Aff}_{n-1}^{(\ell)}\mathbb{Q}} = V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$, where $\mathcal{H}_n := G_{K_n/K_{n-1}} \cap \mathrm{Aff}_n^{(\ell)}\mathbb{Q}$.

Proof. Let W be the k-span of $dx_1, \dots, dx_n$ in $\Omega^1_{K_n/k}$. Then $S^\lambda_{K_n}\Omega^1_{K_n/k} = S^\lambda_k W \otimes_k K_n$ and $\Omega^n_{K_n/k} = \det_k W \otimes_k K_n$. Set $S\mathcal{H}_n = \mathcal{H}_n \cap \mathrm{SAff}_n^{(\ell)}\mathbb{Q}$. Then $\mathcal{H}_n \cong \mu_\ell \ltimes S\mathcal{H}_n$, and therefore, $V^{\mathcal{H}_n} = (V^{S\mathcal{H}_n})^{\mu_\ell}$.

One has $V^{S\mathcal{H}_n} = (S^\lambda_k W \otimes_k K_n)^{S\mathcal{H}_n} \otimes_k (\det_k W)^{\otimes s}$. As the unipotent radical of $S\mathcal{H}_n$ acts trivially on $S^\lambda_k W$ and fixes exactly K_{n-1} in K_n, if $n \geq 1$, we get $V^{S\mathcal{H}_n} = (S^\lambda_k W)^{S\mathcal{H}_n} \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1} = S^\lambda_k(W^{S\mathcal{H}_n}) \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1}$. Then $V^{\mathcal{H}_n} = S^\lambda_k(W^{S\mathcal{H}_n}) \otimes_k ((\det_k W)^{\otimes s})^{\mu_\ell} \otimes_k K_{n-1} = \begin{cases} S^\lambda_{K_{n-1}}\Omega^1_{K_{n-1}/k} \otimes_{K_{n-1}} (\Omega^{n-1}_{K_{n-1}/k})^{\otimes s} & \text{if } \ell | s \\ 0 & \text{otherwise.} \end{cases}$

On the other hand, $V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = (S^\lambda_k W \otimes_k (\det_k W)^{\otimes s} \otimes_k K_n)^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$ coincides with $(S^\lambda_k W \otimes_k (\det_k W)^{\otimes s})^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$, since the unipotent radical of $\mathrm{Aff}_n^{(\ell)}\mathbb{Q}$ acts trivially on $S^\lambda_k W \otimes_k (\det_k W)^{\otimes s}$ and fixes exactly k in K_n. Thus, for $n \geq 1$, we get $V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = \begin{cases} k & \text{if } \lambda = 0 \text{ and } \ell | s, \\ 0 & \text{otherwise.} \end{cases}$ This implies that $(V^{\mathcal{H}_n})^{\mathrm{Aff}_{n-1}^{(\ell)}\mathbb{Q}} = \begin{cases} k & \text{if either } \lambda = 0 \text{ and } \ell | s, \text{ or } S^\lambda = \wedge^{n-1} \text{ and } \ell | (s+1), \\ 0 & \text{otherwise} \end{cases}$ for $n \geq 2$. $\square$

Lemma 3.3 ([R], Lemma 7.1). *Let $n > m \ge 0$ be integers and H be a subgroup of $G_{F/k}$ preserving K_n and projecting onto a subgroup of $G_{K_n/k}$ containing the permutation group of the set $\{x_1, \dots, x_n\}$. Then the subgroup in $G_{F/k}$ generated by G_{F/K_m} and H is dense.* $\square$

For any $U \in \mathcal{A}$ and $m \ge 0$ set $U_m = U^{G_{F/K_m}}$.

Proposition 3.4. *If $U \in \mathcal{A}$ and there is a subquotient of $U_n \in \mathfrak{S}\mathfrak{L}_n^u$ isomorphic to K_n then there is an embedding $F \hookrightarrow V$ in $\mathcal{A}$. One has $H^1_{\mathrm{smooth}}(G_{F/k}, V) = 0$ for any $V \in \mathcal{A}$.*

Proof. By Lemma 3.3, $U^{G_{F/k}} = U_{n+1}^{\mathrm{Aff}^{(\ell)}_{n+1}\mathbb{Q}} \cap U_n$ for any even $\ell \ge 2$. By Theorem 2.5, Lemma 3.1 and Lemma 3.2, $(U_{n+1}^{\mathcal{H}_{n+1}})^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}} = U_{n+1}^{\mathrm{Aff}^{(\ell)}_{n+1}\mathbb{Q}}$ for any sufficiently big ℓ, so, as $U_n \subseteq U_{n+1}^{\mathcal{H}_{n+1}}$, one has $U^{G_{F/k}} = U_n^{\mathrm{Aff}^{(\ell)}_n\mathbb{Q}}$, and thus, $U^{G_{F/k}} \neq 0$ if there is a subquotient of $U_n \in \mathfrak{S}\mathfrak{L}_n^u$ isomorphic to K_n.

Clearly, $H^0(G, -) = \mathrm{Hom}_G(\mathbb{Q}, -) = \mathrm{Hom}_{\mathcal{C}}(F, -)$ and $H^1_{\mathrm{smooth}}(G, -) = \mathrm{Ext}^1_{\mathcal{S}m_G}(\mathbb{Q}, -) = \mathrm{Ext}^1_{\mathcal{C}}(F, -)$ on $\mathcal{C}$, so we have to show that any smooth F-semi-linear extension $0 \to V \to U \to F \to 0$ splits. Fix $n > 1$ such that U_n surjects onto K_n. By Theorem 2.5 and Lemma 3.1 the semi-linear representation U_n of $\mathrm{Aff}^{(\ell)}_n\mathbb{Q}$ over K_n splits as $K_n \oplus V_n$, and thus, $U^{G_{F/k}}$ projects onto k. $\square$

Theorem 3.5. *The category $\mathcal{A}$ is abelian. The functor $H^0(G_{F/L}, -)$ is exact on $\mathcal{A}$ for any subfield L in F containing k.*

Proof. Let $V \in \mathcal{A}$ and $V \xrightarrow{\pi} V'$ be a surjection of F-semi-linear representations of $G_{F/k}$. By Proposition 3.4, for any $K \subset F$ of finite type over k and any $v \in (V')^{G_{F/\overline{K}}} - \{0\}$, the extension $0 \to \ker \pi \to \pi^{-1}(F \cdot v) \to F \to 0$ of F-semi-linear representations of $G_{F/\overline{K}}$ splits. This implies that the natural projection $V^{G_{F/K}} \xrightarrow{\pi_K} (V')^{G_{F/K}}$ is surjective, and thus, V' is also an admissible semi-linear representation.

The functor $H^0(G_{F/L}, -)$ on $\mathcal{A}$ is the composition of the forgetful functor $\Phi : \mathcal{A}_k \to \mathcal{C}_L$, the functor $H^0(G_{F/\overline{L}}, -)$ on $\mathcal{C}_L$ and the exact functor $H^0(G_{\overline{L}/L}, -)$ on $\mathcal{S}m_{G_{\overline{L}/L}}$. If L is of finite transcendence degree over k then the forgetful functor Φ factors through $\mathcal{A}_{\overline{L}}$, so the composition $H^0(G_{F/\overline{L}}, -) \circ \Phi$ is exact. If L is of infinite transcendence degree over k then $H^0(G_{F/\overline{L}}, -)$ induces an equivalence of categories $\mathcal{S}m_{G_{F/k}} \xrightarrow{\sim} \mathcal{S}m_{G_{\overline{L}/k}}$, so $H^0(G_{F/\overline{L}}, -)$ is also exact. $\square$

Corollary 3.6. $H^1_{\mathrm{smooth}}(G, \Omega^{\bullet}_{F/k,\mathrm{closed}}) = H^1_{\mathrm{smooth}}(G, \Omega^{\bullet}_{F/k,\mathrm{exact}}) = 0.$

Proof. This follows from Proposition 3.4 and the following pair of short exact sequences $0 \to \Omega^q_{F/k,\mathrm{closed}} \to \Omega^q_{F/k} \to \Omega^{q+1}_{F/k,\mathrm{exact}} \to 0$ and $0 \to \Omega^q_{F/k,\mathrm{exact}} \to \Omega^q_{F/k,\mathrm{closed}} \to H^q_{dR/k}(F) \to 0$. $\square$

3.1. **Extensions in $\mathfrak{S}\mathfrak{L}_n^u$.** Now we need the following particular case of Bott's theorem.

Theorem 3.7 (Bott). *If $\mathcal{V}$ is an irreducible G_n-equivariant coherent sheaf on $\mathbb{P}^n_k$ then there exists at most one $j \ge 0$ such that $H^j(\mathbb{P}^n_k, \mathcal{V}) \neq 0$. If $H^j(\mathbb{P}^n_k, \mathcal{V})^{G_n} \neq 0$ then $\mathcal{V} \cong \Omega^j_{\mathbb{P}^n_k/k}$.* $\square$

As well as the following elaboration of Remark 8.19 of [BT] and [T], §5.1.

Theorem 3.8 ([LR], Theorem 3). *Let G be a simple simply connected Chevalley k-group, and $\mathcal{G}$ be a connected algebraic k-group. Let $\tau : G(k) \to \mathcal{G}(k)$ be a homomorphism with*

Zariski dense image. Assume that the unipotent radical $\mathcal{G}_u$ of $\mathcal{G}$ is commutative and the composition $G(k) \xrightarrow{\tau} \mathcal{G}(k) \to G'(k)$, where $G' = \mathcal{G}/\mathcal{G}_u$, is induced by a rational k-morphism $\lambda : G \to G'$. Let $A = k[\varepsilon_1, \dots, \varepsilon_r]/(\varepsilon_1^2, \dots, \varepsilon_r^2)$ and $\mathcal{H} = R_{A/k}(G \times_k A) \cong G \ltimes \mathfrak{g}^{\oplus r}$, where $\mathfrak{g} = \mathrm{Lie}(G)$ is the adjoint representation of G, and $r = \dim \mathcal{G}_u / \dim G'$.

Then $\mathcal{G}_u$ splits into a direct sum of r copies of the adjoint representation of G';

there exist derivations $\delta_1, \dots, \delta_r$ of k and a k-isogeny $\mu : \mathcal{H} \to \mathcal{G}$ such that $\tau = \mu \circ \eta_\delta$, where $\eta_\delta : G(k) \to \mathcal{H}(k)$ is induced by the ring embedding $id + \sum_{j=1}^r \delta_j \varepsilon_j : k \to A$. □

Lemma 3.9. *Let $n \geq 2$. Suppose that $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, V_\circ) \neq 0$ for some irreducible object $V_\circ$ of $\mathfrak{SL}^u_n$. Then either $V_\circ \cong \Omega^1_{K_n/k}$, or $V_\circ \cong \mathrm{Der}(K_n/k)$. One has $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, \Omega^1_{K_n/k}) = k$ and $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, \mathrm{Der}(K_n/k)) = \mathrm{Der}(k)$.*

Proof. Let $\mathcal{V} = \mathcal{S}(V_\circ)$ be the irreducible coherent G_n-equivariant sheaf on $\mathbb{P}(Q) = \mathbb{P}^n_k$ with the generic fibre $V_\circ$, and $V \in \mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, V_\circ)$.

Suppose that the short exact sequence $0 \to \mathcal{V} \to \mathcal{S}(V) \to \mathcal{O} \to 0$ of coherent sheaves on $\mathbb{P}(Q)$ splits. Let E be the total space of $\mathcal{S}(V) \cong \mathcal{O} \oplus \mathcal{V}$. Then, as $\mathrm{Aut}(\mathcal{S}(V), \mathcal{V}) \cong (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}^n_k, \mathcal{V})$, the G_n-structure on V corresponds to a splitting of the sequence

$$(3) \qquad 1 \longrightarrow (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}(Q), \mathcal{V}) \longrightarrow \mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\mathcal{V})) \longrightarrow G_n \longrightarrow 1.$$

As $H^1(G_n, \mathbb{G}_m \times \mathbb{G}_m) = 1$, Theorem 3.8 (with $G = \mathrm{SL}_{n+1}k$, $G' = G_n$ and $\mathcal{G}_u \subseteq \Gamma(\mathbb{P}^n_k, \mathcal{V})$) implies that a non-standard splitting of (3) can exist only if $\Gamma(\mathbb{P}^n_k, \mathcal{V})$ is isomorphic to the adjoint representation of G_n, i.e., if $\mathcal{V} \cong \mathcal{T}_{\mathbb{P}^n_k/k}$. The identity $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, \mathrm{Der}(K_n/k)) = \mathrm{Der}(k)$ follows also from Theorem 3.8.

If $\mathcal{V} \cong \Omega^1_{\mathbb{P}^n_k/k}$ then the target of the homomorphism $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, \mathcal{V}_\circ) \xrightarrow{\alpha} \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_{\mathbb{P}^n_k}, \mathcal{V}) = k$ induced by the functor $\mathcal{S}$ is generated by the class of the Euler extension $0 \to \Omega^1_{\mathbb{P}(Q)/k} \to Q^\vee \otimes_k \mathcal{O}_{\mathbb{P}(Q)}(-1) \to \mathcal{O}_{\mathbb{P}(Q)} \to 0$. Let E be the total space of the vector bundle with the sheaf of sections $Q^\vee \otimes_k \mathcal{O}(-1)$. Any G_n-structure on the middle term of this extension corresponding to an element of $\mathrm{Ext}^1_{\mathfrak{SL}^u_n}(K_n, \Omega^1_{K_n/k})$ is a splitting of the short exact sequence

$$(4) \qquad 1 \longrightarrow \mathrm{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega^1_{\mathbb{P}(Q)/k}) \longrightarrow \mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\Omega^1_{\mathbb{P}(Q)/k})) \longrightarrow G_n \longrightarrow 1.$$

As any point of $\mathbb{P}(Q)$ determines a hyperplane in $Q^\vee$, the group $\mathrm{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega^1_{\mathbb{P}(Q)})$ coincides with the subgroup of $\mathrm{GL}(Q)$ stabilizing all hyperplanes in $Q^\vee$, i.e., with the centre $\mathbb{G}_m$ of $\mathrm{GL}(Q)$. Then $\mathrm{Aut}_{\mathrm{lin}}(E, \mathrm{tot}(\Omega^1_{\mathbb{P}(Q)/k}))$ is a central $\mathbb{G}_m$-extension of G_n, so the splitting of (4) is unique and corresponds to the usual G_n-equivariant structure. □

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