A Vanishing Theorem for Intersection Homology with Twisted Coefficients of Toric Varieties

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Abstract

We prove that $IH_*^{\bar{p}}(\mathbf{X};\mathcal{L}) = \{0\}$ for any toric variety $\mathbf{X}$, any perversity $\bar{p}$ and any local system $\mathcal{L}$ which is strongly non-trivial with respect to the cone complex Σ which defines $\mathbf{X}$. The class of strongly non-trivial local systems includes all non-trivial 1-dimensional local systems. We use a topological definition of $\mathbf{X}$ due to R. MacPherson ([Mac2]) and prove the theorem by reducing to an analogous local statement on each of the elements of a certain (closed) covering $\{\mathbf{X}_\sigma\}_{\sigma\in\Sigma}$ of $\mathbf{X}$.

1 Introduction

The study of the intersection homology groups $IH_*^{\bar{p}}(\mathbf{X};\mathcal{L})$ where $\mathbf{X}$ is a toric variety and $\mathcal{L}$ is a 1-dimensional local system, is of interest beyond the realm of toric varieties. For the trivial local system $\mathcal{L} = \mathbf{Q}_\mathbf{X}$, the ranks of the middle perversity groups $IH_i^{\bar{m}}(\mathbf{X};\mathbf{Q}_\mathbf{X})$ have been computed and were found to be combinatorial invariants of the underlying polytope. This led to the definition of the generalized h-vectors for general convex polytopes ([St]). On the other hand, in their study of generalized hypergeometric functions, Gelfand, Kapranov and Zelevinsky have shown that the irreducible $\mathcal{D}$-modules on a toric variety $\mathbf{X}$ are in $1-1$ correspondence with the middle perversity intersection homology sheaves $IC^{\bar{m}}_{\cdot}(\bar{\mathrm{S}};\mathcal{L})$, where $\bar{\mathrm{S}}$ is the closure of a stratum of $\mathbf{X}$ (and hence also a toric variety) and $\mathcal{L}$ is an irreducible local system on S ([GKZ],[Mac1]). Over $\mathbf{C}$, all irreducible local system are 1-dimensional.

In this paper we prove that under certain conditions on a local system for intersection homology on a toric variety $\mathbf{X}$, the groups $IH_i^{\bar{p}}(\mathbf{X};\mathcal{L})$ all vanish. In prticular, these conditions are satisfied by any non-trivial 1-dimensional local system.

We use a topological definition of $\mathbf{X}$ which, in ignoring the additional algebraic structure, uses only elementary topological constructions and properties of the underlying cone complex. We define a filtration of $\mathbf{X}$ and use spectral sequence arguments to reduce the general theorem to an analogous local theorem on certain subspaces $\mathbf{X}_\sigma \subset \mathbf{X}$ which are in $1-1$ correspondence with the cones σ of the underlying cone complex.

2 Definitions

A *cone complex* (a.k.a. *complete fan*) Σ in $\mathbf{R}^n$ is a decomposition of $\mathbf{R}^n$ into a finite number of closed, convex, rational polyhedral cones, each with apex 0, such that any face of a cone $\sigma \in \Sigma$ is itself in Σ and such that the intersection of any two cones is a common face of both.

Let Σ be a cone complex in $\mathbf{R}^n$. Let $\mathbf{D}$ denote the unit ball in $\mathbf{R}^n$ and let $\mathbf{S}$ be its boundary. $\mathbf{S}$ is decomposed into polyhedral cells by intersecting with the positive dimensional cones of Σ. Define the *dual cell complex* $\mathcal{P} = \mathcal{P}_\Sigma$ to be the n-dimensional polyhedral cell complex whose $(n-1)$-skeleton is the dual cell-block decomposition of $\mathbf{S}$ and whose unique (open) n-cell is the interior of $\mathbf{D}$. The cells of $\mathcal{P}$ are dually paired with the cones of Σ. We denote by $\hat{\sigma} \in \mathcal{P}$ the cell dual to a cone $\sigma \in \Sigma$.

Note: This explicit construction of $\mathcal{P}$ is given only for convenience in the proof of the main theorem. In general, any realization of the abstract cell-complex whose face lattice is dual to that of Σ would yield an isomorphic toric variety.

For every cone $\sigma \in \Sigma$ set $\mathcal{P}_\sigma = \mathcal{P} \cap \sigma$ and $\mathcal{P}_{\partial\sigma} = \cup_{\tau \subset \sigma} \mathcal{P}_\tau$. For each $0 \le k \le n$ set $\mathcal{P}^k = \cup_{\dim\sigma \le k} \mathcal{P}_\sigma$.

Definition 2.1 Let $\mathcal{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ denote the n-torus. Define the *toric variety* $\mathbf{X} = \mathbf{X}_\Sigma$ to be $\mathcal{P} \times \mathcal{T}^n/\sim$, where $(x,\tau) \sim (x',\tau')$ if and only if $x' = x \in \mathrm{relint}\hat{\sigma}$ for some $\sigma \in \Sigma$ and there exist liftings $t, t' \in \mathbf{R}^n$ of τ and τ' such that $t - t' \in \mathrm{span}\sigma$. Thus if $x \in \mathrm{relint}\hat{\sigma}$, then since $\mathrm{span}\sigma$ is a rational subspace of $\mathbf{R}^n$, the torus $\{x\} \times \mathcal{T}^n$ is collapsed in $\mathbf{X}$ to a torus of dimension $n - \dim\sigma = \dim\hat{\sigma}$. There is an obvious projection $\pi : \mathbf{X} \to \mathcal{P}$.

$\mathbf{X}$ has a natural stratification

$$\mathbf{X}_0 \subset \mathbf{X}_2 \subset \ldots \subset \mathbf{X}_{2(n-1)} \subset \mathbf{X}_{2n} = \mathbf{X}$$

where for each k, $\mathbf{X}_{2k}$ is the inverse image under π of the k-skeleton of $\mathcal{P}$. The "non-singular" (open) stratum $\mathbf{X} \setminus \mathbf{X}_{2n-2}$ is equal to $int(\mathbf{D}) \times T^n$ and hence is homeomorphic to $(\mathbf{C}^*)^n$. Fix once and for all a base point $c \in \mathbf{X} \setminus \mathbf{X}_{2n-2}$.

Definition 2.2 A *k-dimensional local system for intersection homology* on a stratified pseudomanifold consists of a local system on the non-singular stratum. In particular, a local system for intersection homology on $\mathbf{X}$ consists of a bundle over $(\mathbf{C}^*)^n$ whose fibre $\mathbf{V}$ is a k-dimensional vector space over a field F (usually C or $\mathbf{Q}$). Equivalently, it consists of a vector space $\mathbf{V}$ and a representation of $\mathbf{Z}^n = \pi_1(\mathbf{X} \setminus \mathbf{X}_{2n-2}, c)$ into $Aut(\mathbf{V})$. Let $\mathbf{V}_x$ denote the fiber of $\mathcal{L}$ over a point $x \in \mathbf{X}$. Any $z \in \mathbf{Z}^n$ determines a monodromy $T_z \in Aut(\mathbf{V}_c)$. We will use the same notation for a local system and for its restriction to a subspace whenever the context is clear.

Definition 2.3 Let Σ be a cone complex in $\mathbf{R}^n$ and let $\sigma \in \Sigma$ be a k-dimensional cone. Let $\mathcal{L}$ be a local system for intersection homology on the associated toric variety $\mathbf{X} = \mathbf{X}_\Sigma$. An ordered basis $B = \{z_1, \ldots, z_n\}$ of $\mathbf{Z}^n$ is a *σ-basis* if $z_i \in \mathrm{span}\sigma$ for $1 \leq i \leq k$ (thus $\{z_1, \ldots, z_k\}$ form a basis of $\mathbf{Z}^n \cap \mathrm{span}\sigma$). We say that $\mathcal{L}$ is *strongly non-trivial with respect to σ* if every $\tau \subseteq \sigma$ admits a τ-basis $\{z_1, \ldots, z_n\}$ of $\mathbf{Z}^n$ such that $T_{z_i} - I : \mathbf{V}_c \to \mathbf{V}_c$ is invertible for some $1 \leq i \leq n$. $\mathcal{L}$ is *strongly non-trivial* (with respect to Σ) if every $\tau \in \Sigma$ admits such a basis.

Remark 2.4 Any non-trivial 1-dimensional local system for intersection homology on $\mathbf{X}$ is strongly non-trivial.
Proof: For any $\sigma \in \Sigma$ choose a basis of $\mathbf{Z}^n \cap \mathrm{span}\sigma$ and complete it to a basis of $\mathbf{Z}^n$. Since $\mathcal{L}$ is non-trivial and 1-dimensional, one of the monodromies corresponding to this basis is equal to multiplication by some constant $d \neq 1$, whence multiplication by $d - 1$ is invertible. □

The main result of this paper is

Theorem 2.5 Let Σ be a cone complex in $\mathbf{R}^n$ and let $\mathbf{X}$ be the associated toric variety. Let $\mathcal{L}$ be a local system for intersection homology on $\mathbf{X}$ which is strongly non-trivial with respect to Σ. Then for any perversity $\bar{p}$ and for all $0 \leq i \leq 2n$, $IH_i^{\bar{p}}(\mathbf{X}; \mathcal{L}) = 0$.

To prove theorem 2.5 we define a filtration of $\mathbf{X}$ and show that in the corresponding spectral sequence the E^1 term vanishes. First, we need the following lemma.

Lemma 2.6 Let $(\mathbf{Y}, y)$ be a stratified pseudomanifold where y is some fixed point in the non-singular stratum and let $(\mathbf{S}^1, s)$ be the circle with base point s. Let $\mathcal{L}$ be a local system for intersection homology on $\mathbf{S}^1 \times \mathbf{Y}$ and let $\mathcal{L}'$ be its restriction to $\{s\} \times \mathbf{Y}$.
Let $\mathbf{V}_{(s,y)}$ be the fiber of $\mathcal{L}$ over the point (s, y) and let $\Phi_y : \mathbf{V}_{(s,y)} \to \mathbf{V}_{(s,y)}$ be the monodromy associated to one of the generators of $\pi_1(S^1 \times \{y\}, (s, y))$.
Then the groups $IH_i^{\bar{p}}(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L})$ all vanish if either
(i) $IH_i^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}') = 0 \; \forall i$, or
(ii) $\Phi_y - I : \mathbf{V}_{(s,y)} \to \mathbf{V}_{(s,y)}$ is invertible.

The proof of lemma 2.6 is deferred to section 4 where we first present the formal definitions and some essential lemmas concerning local systems for intersection homology in the category of piecewise linear geometric chains.

Corollary 2.7 If in addition $\mathbf{Y}'$ is a PL-subspace of $\mathbf{Y}$ and either $IH_i^{\bar{p}}(\{s\} \times \mathbf{Y}, \{s\} \times \mathbf{Y}'; \mathcal{L}') = 0 \; \forall i$ or $\Phi_y - I$ is invertible, then $\forall i$, $IH_i^{\bar{p}}(\mathbf{S}^1 \times \mathbf{Y}, \mathbf{S}^1 \times \mathbf{Y}'; \mathcal{L}) = 0$.
Proof: This follows from lemma 2.6 and the long exact sequence of the pair $(\mathbf{S}^1 \times \mathbf{Y}, \mathbf{S}^1 \times \mathbf{Y}')$. □

Definition 2.8 The filtration of X . For each $\sigma \in \Sigma$ set $\mathbf{X}_\sigma = \pi^{-1}(\mathcal{P}_\sigma)$ and $\mathbf{X}_{\partial\sigma} = \pi^{-1}(\mathcal{P}_{\partial\sigma})$. For each $0 \leq k \leq n$ set $\mathbf{X}^k = \pi^{-1}(\mathcal{P}^k) = \cup_{\dim \sigma = k} \mathbf{X}_\sigma$. This defines a filtration of $\mathbf{X}$ by closed subspaces

$$\mathcal{T}^n = \mathbf{X}^0 \subset \mathbf{X}^1 \subset \ldots \subset \mathbf{X}^n = \mathbf{X}.$$

The essence of the proof of theorem 2.5 stems from the following

Lemma 2.9 Properties of the filtration:
For any k-dimensional cones $\sigma, \tau \in \Sigma$,
(i) $\mathbf{X}_\sigma \cap \mathbf{X}_\tau \subseteq \mathbf{X}^{k-1}$.
(ii) $\mathbf{X}_\sigma \cap \mathbf{X}^{k-1} = \mathbf{X}_{\partial\sigma}$.
(iii) If $k = n$ then $\mathbf{X}_\sigma = c(\mathbf{X}_{\partial\sigma})$ (here $c(Y)$ denotes the topological cone on $\mathbf{Y}$, stratified by the cones on the strata of $\mathbf{Y}$ along with the apex of the cone as the unique 0-stratum).

(iv) If $k < n$ then there is an $(n-1)$-dimensional cone complex Σ' with $\sigma \in \Sigma'$ such that for the associated toric variety $\mathbf{X}'$, $\mathbf{X}_\sigma = \mathbf{S}^1 \times \mathbf{X}'_\sigma$ and $\mathbf{X}_{\partial\sigma} = \mathbf{S}^1 \times \mathbf{X}'_{\partial\sigma}$.

Proof: (i) and (ii) are immediate while (iii) is easy.
Proof of (iv) : Let $\{z_1, \ldots, z_n\}$ be a σ-basis for $\mathbf{Z}^n$. We make the identifications :

$$\mathbf{Z}^{n-1} = \bigoplus_{i=1}^{n-1} \mathbf{Z} z_i, \text{ and} \tag{1}$$

$$\mathbf{R}^{n-1} = \bigoplus_{i=1}^{n-1} \mathbf{R} z_i. \tag{2}$$

Also for each $1 \le i \le n$ let τ_i be the image of $\mathbf{R} z_i$ in $\mathcal{T}^n$, and identify

$$\mathcal{T}^{n-1} = \tau_1 \times \ldots \times \tau_{n-1} \text{ and } \mathcal{T}^n = \mathcal{T}^{n-1} \times \tau_n. \tag{3}$$

Complete σ arbitrarily to a cone complex Σ' in $\mathbf{R}^{n-1}$ and denote by $\mathcal{P}'$ the dual cell complex and by $\mathbf{X}'$ the associated toric variety. Then $\mathcal{P}'_\sigma = \mathcal{P}_\sigma$, and $\mathbf{X}'_\sigma$ is obtained from $\mathcal{P}'_\sigma \times \mathcal{T}^{n-1}$ by collapsing certain subtori of $\mathcal{T}^{n-1}$ over the faces of $\mathcal{P}'$ which $\mathcal{P}'_\sigma$ meets. On the other hand, $\mathbf{X}_\sigma$ is obtained from $\mathcal{P}_\sigma \times \mathcal{T}^n = \mathcal{P}'_\sigma \times \mathcal{T}^{n-1} \times \tau_n$ by collapsing the same subtori of $\mathcal{T}^{n-1}$ over the same points of $\mathcal{P}$. It follows that $\mathbf{X}_\sigma = \mathbf{X}'_\sigma \times \tau_n$, and hence also that $\mathbf{X}_{\partial\sigma} = \mathbf{X}'_{\partial\sigma} \times \tau_n$. □
Note that the sets $\mathbf{X}_\sigma$ are, topologically, strong deformation retracts of the affine toric varieties $\mathbf{X}_{\check\sigma}$ defined in [Da].

The following corollary follows from (i) and (ii) above :

Corollary 2.10 For each $0 \le k \le n$ and for any perversity $\bar{p}$,

$$IH_*^{\bar{p}}(\mathrm{X}^k, \mathrm{X}^{k-1}; \mathcal{L}) = \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma = k}} IH_*^{\bar{p}}(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}; \mathcal{L}) \tag{4}$$

□

Therfore, to prove theorem 2.5 it suffices to show that each of the terms on the right hand side of (4) is trivial. This follows from

Lemma 2.11 Let Σ be a cone complex in $\mathbf{R}^n$, $\mathbf{X}$ the associated toric variety, $\bar{p}$ a perversity and $\mathcal{L}$ a local system for intersection homology on $\mathbf{X}$ which is strongly non-trivial with respect to a cone $\sigma \in \Sigma$. Then $IH_i^{\bar{p}}(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}; \mathcal{L}) = 0 \;\forall i$.

Proof: The proof is by induction on n.
There is only one cone complex Σ in $\mathbf{R}^1$. It has a unique 0-cone σ_0 and two 1-cones σ_1 and σ_2. Let $T : \mathbf{V} \to \mathbf{V}$ be the monodromy corresponding to one of the generators of $\mathbf{Z}$. The strong non-triviality of $\mathcal{L}$ with respect to any one of the cones of Σ is equivalent to the invertability of $T - I$. Now $\mathbf{X}_{\sigma_0} = \mathbf{S}^1$ whence $IH_*(\mathbf{X}_{\sigma_0}; \mathcal{L}) = H_*(\mathbf{S}^1; \mathcal{L})$, and it can easily be seen that $H_0(\mathbf{S}^1; \mathcal{L}) = \operatorname{coker}(T - I)$ and $H_1(\mathbf{S}^1; \mathcal{L}) = \ker(T - I)$, both of which are trivial. Now, since $\mathbf{X}_{\sigma_1} = c(\mathbf{X}_{\sigma_0})$ (by (*iii*) above), it follows from [Bo, p. 29, Prop. 3.1] that $IH_*^{\bar{p}}(\mathbf{X}_{\sigma_1}, \mathbf{X}_{\partial\sigma_1}; \mathcal{L}) = IH_*^{\bar{p}}(\mathbf{X}_{\sigma_1}, \mathbf{X}_{\sigma_0}; \mathcal{L}) = \{0\}$, and similarly for σ_2.

Now let Σ be a cone complex in $\mathbf{R}^n$ with $\mathbf{X}$ the associated toric variety and let $\sigma \in \Sigma$ with $\dim \sigma = k < n$.
Case 1 : there exists a σ-basis $\{z_1, \ldots, z_n\}$ of $\mathbf{Z}^n$ with $T_{z_j} - I$ invertible for some $j > k$. Then we may assume that $j = n$. Use this basis to obtain the decompositions (1), (2) and (3) and write $(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}) = \mathbf{S}^1 \times (\mathbf{X}'_\sigma, \mathbf{X}'_{\partial\sigma})$ accordingly. Then condition (*ii*) of lemma 2.6 is satisfied and the assertion follows from corollary 2.7.
Case 2: for any σ-basis $\{z_1, \ldots, z_n\}$ of $\mathbf{Z}^n$, if $T_{z_i} - I$ invertible for some i then $i \leq k$ (and consequently $z_i \in \operatorname{span}\sigma$). Let $\{z_1, \ldots, z_n\}$ be any σ-basis of $\mathbf{Z}^n$ and write $(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}) = \mathbf{S}^1 \times (\mathbf{X}'_\sigma, \mathbf{X}'_{\partial\sigma})$ accordingly as in part (*iv*) of lemma 2.9. Then the restricted local system $\mathcal{L}' = \mathcal{L}|_{\mathbf{X}'}$ is strongly non-trivial with respect to σ (considered as a cone in Σ'). Thus by the inductive hypothesis, $IH_*^{\bar{p}}(\mathbf{X}'_\sigma, \mathbf{X}'_{\partial\sigma}; \mathcal{L}') = \{0\}$ so that part (*i*) of lemma 2.6 is satisfied, and once again our assertion follows from corollary 2.7.

Finally, suppose $\dim \sigma = n$. Then by (*iii*) above, $\mathbf{X}_\sigma = c(\mathbf{X}_{\partial\sigma})$. By its construction, the filtration of $\mathbf{X}$ restricts to $\mathbf{X}_{\partial\sigma}$ so that corollary 2.10 holds for the restricted spectral sequence namely for each $1 \leq i \leq n - 1$,

$$IH_*^{\bar{p}}(\mathbf{X}^i \cap \mathbf{X}_{\partial\sigma}, \mathbf{X}^{i-1} \cap \mathbf{X}_{\partial\sigma}; \mathcal{L}) = \bigoplus_{\substack{\tau \subset \partial\sigma \\ \dim \tau = i}} IH_*^{\bar{p}}(\mathbf{X}_\tau, \mathbf{X}_{\partial\tau}; \mathcal{L}). \qquad (5)$$

However since $\mathcal{L}$ is strongly non-trivial with respect to σ, it is in particular strongly non-trivial with respect to any cone $\tau \subset \partial\sigma$ and hence it follows from the previous step that all of the terms on the right hand side of (5) vanish whence $IH_*^{\bar{p}}(\mathbf{X}_{\partial\sigma}; \mathcal{L}) = \{0\}$. Thus once again it follows by [Bo, p. 29, Prop. 3.1] that $IH_i^{\bar{p}}(\mathbf{X}_\sigma, \mathbf{X}_{\partial\sigma}; \mathcal{L}) = 0$, $\forall i$. $\square$

3 A counter example for general local systems

In general, the intersection homology with twisted coefficients of a toric variety $\mathbf{X}$ is not trivial. Consider the toric variety $\mathbf{X} \cong S^2$ associated to the unique cone complex Σ in $\mathbf{R}^1$. Let $c \in \mathbf{X} \setminus \mathbf{X}_0 \cong \mathbf{C}^*$ be a base point. Let $\mathcal{L}$ be a k-dimensional local system for intersection homology on $\mathbf{X}$ with fiber $\mathbf{V}$ such that the monodromy $T \in Aut(\mathbf{V}_c)$ associated to one of the generators of $\pi_1(\mathbf{X} \setminus \mathbf{X}_0, c)$ satisfies $0 < \operatorname{rank}(T - I) < k$. One easily computes : $IH_2^{(0)}(\mathbf{X}; \mathcal{L}) = \ker(T - I)$ and $IH_0^{(0)}(\mathbf{X}; \mathcal{L}) = \operatorname{coker}(T - I)$, neither of which is $\{0\}$.

One can construct examples in any (even) dimension, for example using the Künneth theorem ([CGL]) and the previous example and noting that any product of 2-spheres $S^2 \times \cdots \times S^2$ is a toric variety.

4 Local Systems and Intersection Homology

A thorough treatment of local systems for intersection homology in the various categories in which intersection homology is defined can be found in [Macl]. We repeat here the essential definitions in the category of piecewise linear geometric chains.

4.1 Geometric Chains

Definition 4.1 Geometric Prechains.
Let M be a manifold and $\mathcal{L}$ a local system on M, *i.e.* a vector bundle over M whose fiber $\mathbf{V}_x$ over any point $x \in$ M is isomorphic to some (fixed) finite dimensional vector space $\mathbf{V}$ over a field F (usually $\mathbf{Q}$ or $\mathbf{C}$). A *degree-k piecewise linear geometric prechain C in* M *with coefficients in* $\mathcal{L}$ consists of the following data:
1. A piecewise linear subspace $S \subseteq$ M called the *presupport of C*.
2. A Whitney stratification $S = \bigcup S_\alpha$ of S such that each stratum is a piecewise linear subspace of M and such that S is the closure of the union of the k-dimensional strata.
3. For each k-dimensional stratum S_α, a *multiplicity map* $c_\alpha : \tilde{S}_\alpha \to \mathcal{L}|_{S_\alpha}$, where $\tilde{S}_\alpha$ denotes the orientation double cover of S_α. This map is required to satisfy the following property : If $\mathcal{O}_m$ and $\mathcal{O}'_m$ are the two orientations of S_α over some point m, then $c_\alpha(\mathcal{O}_m) = -c_\alpha(\mathcal{O}'_m)$ in $\mathbf{V}_m$.

There is an equivalence relation on prechains under which two prechains are identified if their respective images are equal under some common refinement of stratifications (see [Macl]). We thus define :

Definition 4.2 Geometric Chains. A *piecewise linear geometric k-chain* is an equivalence class of prechains. The complex of piecewise linear geometric chains is denoted $C_*(\mathrm{M};\mathcal{L})$.

4.2 The product with $\mathbf{S}^1$

Let $\mathbf{Y}$ be a stratified pseudomanifold and let y_0 be a point in the non-singular stratum. Let S^1 be the 1-sphere which we regard as the quotient of the interval $[0,1]$ modulo the relation $0 \sim 1$, oriented accordingly. Then $\mathrm{S}^1 \times \mathbf{Y}$ is a stratified pseudomanifold whose i-dimensional strata are the products of S^1 with the $(i-1)$-strata of $\mathbf{Y}$. We represent points in $\mathrm{S}^1 \times \mathbf{Y}$ by pairs (s,y) with $0 \le s < 1$ and $y \in \mathbf{Y}$.

Let $\mathcal{L}$ be a local system for intersection homology on $\mathrm{S}^1 \times \mathbf{Y}$ and let $\mathcal{L}'$ and $\mathcal{L}''$ denote its respective restrictions to $\{0\} \times \mathbf{Y}$ and $\mathrm{S}^1 \times \{y_0\}$.

Definition 4.3 Define a map $\times\mathrm{S}^1 : C_k(\{s\} \times \mathbf{Y};\mathcal{L}') \to C_{k+1}(\mathrm{S}^1 \times \mathbf{Y};\mathcal{L})$ as follows. Let $\xi \in C_k(\{s\} \times \mathbf{Y};\mathcal{L}')$ be represented by a prechain C. Denote by $\mathrm{S}^1 \times C$ the prechain in $\mathrm{S}^1 \times \mathbf{Y}$ whose support is $\mathrm{S}^1 \times supp(C)$ and whose strata are the respective products of $\{0\}$ and of $\mathrm{S}^1 \setminus \{0\}$ with the strata of C. The multiplicities are defined as follows :
Let (s,y) be a point in the $(k+1)$-stratum $(\mathrm{S}^1 \setminus \{0\}) \times S_\alpha$, and let $c_\alpha : \tilde{S}_\alpha \to \mathcal{L}'|_{S_\alpha}$ be the multiplicity map on S_α. The path in $\mathrm{S}^1 \times \mathbf{Y}$ defined by $t \mapsto (st, y)$ $(0 \le t \le 1)$ defines an isomorphism $\Phi_{(s,y)} : \mathbf{V}_{(0,y)} \to \mathbf{V}_{(s,y)}$. Define the multiplicity map $\bar{c}_\alpha$ on $(\mathrm{S}^1 \setminus \{0\}) \times S_\alpha$ by

$$\bar{c}_\alpha(s,y) = \Phi_{(s,y)} \circ c_\alpha(0,y).$$

The multiplicity maps c_α on the strata S_α are unchanged. The geometric chain $\times\mathrm{S}^1(\xi)$ which we will henceforth denote by $\mathrm{S}^1 \times \xi$, is defined to be the equivalence class of prechains represented by $\mathrm{S}^1 \times C$.

The boundary of $\mathrm{S}^1 \times \xi$

For any $y \in \mathbf{Y}$ let $\Phi_y : \mathbf{V}_y \xrightarrow{\cong} \mathbf{V}_y$ be the monodromy corresponding to the path which goes around $\mathrm{S}^1 \times \{y\}$ once in the positive direction. Let ξ be a

k-chain in $\{0\} \times \mathbf{Y}$ and let C be a representative prechain with multiplicity map c_α on each k-stratum S_α. Denote by $(\Phi - I)\xi$ the chain in $\{0\} \times \mathbf{Y}$ represented by the prechain C' whose support and stratification are the same as those of C, and whose multiplicity maps c'_α are given by

$$c'_\alpha(0, y) = (\Phi_y - I) \circ c_\alpha(0, y) = \Phi_y \circ c_\alpha(0, y) - c_\alpha(0, y).$$

The boundary of $\mathbf{S}^1 \times \xi$ is given by:

$$\partial(\mathbf{S}^1 \times \xi) = (\Phi - I)\xi - \mathbf{S}^1 \times \partial\xi.$$

In the proof of lemma 2.6 we will also make use of the chain $(\Phi - I)^{-1}\xi$ which is defined whenever $(\Phi_y - I)$ is invertible for some (equivalently every) $y \in \mathbf{Y}$. It is represented by the prechain C'' whose support and stratification are the same as those of C, and whose multiplicity maps are

$$c''(0, y) = (\Phi_y - I)^{-1} \circ c_\alpha(0, y).$$

Remark 4.4 If $\mathcal{L}$ is 1-dimensional then there is some constant $d \in \mathsf{F}$ such that the monodromies Φ_y are all equal to multiplication by d and hence

$$\partial(\mathbf{S}^1 \times \xi) = (d - 1)\xi - \mathbf{S}^1 \times \partial\xi.$$

Note that if $\mathcal{L}''$ is non-trivial then $d \neq 1$.

Lemma 4.5 The map $\times \mathbf{S}^1$ induces an isomorphism on intersection homology

$$IH^{\bar{p}}_*(\{s\} \times \mathbf{Y}; \mathcal{L}') \to IH^{\bar{p}}_{*+1}(\mathbf{S}^1 \times \mathbf{Y}, \{s\} \times \mathbf{Y}; \mathcal{L})$$

for any perversity $\bar{p}$.

Proof: The proof is virtually identical to [Bo, p. 25, Prop. 2.1], using the canonical isomorphism $IH^{\bar{p}}_*(\mathbf{S}^1 \times \mathbf{Y}, \{s\} \times \mathbf{Y}) \cong IH^{\bar{p}}_*(\mathbf{R} \times \mathbf{Y})$. The introduction of a local system poses no additional problem. □

Corollary 4.6 Let $ID^{\bar{p}}_*(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L}) \subset IC^{\bar{p}}_*(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L})$ be the subcomplex whose $k^{\underline{th}}$ chain group is generated by the chains $\xi_k \in IC^{\bar{p}}_k(\{s\} \times \mathbf{Y}; \mathcal{L}')$ and $\{\mathbf{S}^1 \times \xi_{k-1} \mid \xi_{k-1} \in IC^{\bar{p}}_{k-1}(\{s\} \times \mathbf{Y}; \mathcal{L}')\}$. Then the inclusion $i : ID_*(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L}) \hookrightarrow IC_*(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L})$ induces an isomorphism on homology.

Proof: Consider the two-step filtration $\{s\} \times \mathbf{Y} \subset \mathbf{S}^1 \times \mathbf{Y}$ and the corresponding filtrations of the chain complexes :

$$IC_*^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}') \subset IC_*^{\bar{p}}(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L}) \tag{6}$$

and

$$IC_*^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}') \subset ID_*^{\bar{p}}(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L}). \tag{7}$$

It follows from lemma 4.5 that the inclusion i induces an isomorphism on the E^1 terms of the associated spectral sequences. □

4.3 The proof of lemma 2.6

(i) If $IH_*^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}') = \{0\}$ then by lemma 4.5, the E^1 term vanishes in the spectral sequence associated to the filtered complex (6).
(ii) Now suppose that $\Phi_y - I$ is invertible. By corollary 4.6, any class in $IH_k^{\bar{p}}(\mathbf{S}^1 \times \mathbf{Y}; \mathcal{L})$ is represented by a cycle of the form

$$\psi = \xi_k + \mathbf{S}^1 \times \xi_{k-1}$$

with $\xi_k \in IC_k^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}')$ and $\xi_{k-1} \in IC_{k-1}^{\bar{p}}(\{s\} \times \mathbf{Y}; \mathcal{L}')$. We show that this cycle is a boundary. Since

$$\partial\psi = \partial\xi_k + (\Phi - I)\xi_{k-1} - \mathbf{S}^1 \times \partial\xi_{k-1} = 0,$$

we have in particular that $\partial\xi_k = -(\Phi - I)\xi_{k-1}$ and $\partial\xi_{k-1} = 0$.
Now consider the $(k+1)$-chain

$$\tilde{\psi} = \mathbf{S}^1 \times (\Phi - I)^{-1}\xi_k.$$

Its boundary is given by

$$\partial\tilde{\psi} = (\Phi - I)(\Phi - I)^{-1}\xi_k - \mathbf{S}^1 \times (\Phi - I)^{-1}\partial\xi_k = \xi_k + \mathbf{S}^1 \times \xi_{k-1}.$$

□

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