A New Realization of Matrix Bialgebra Associated to A Hecke Operator

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0. Introduction

Let V be a vector space of finite dimension over a field of characteristic zero. There are two ways of interpreting the symmetric tensor algebra over V. The first one is to consider it as a factor algebra of the tensor algebra $\mathcal{T}(V)$ over V. Thus we have a *relation* between two tensors, like $a \otimes b = b \otimes a$. The second way is to consider it as a *subalgebra* of $\mathcal{T}(V)$ consisting of symmetrized tensors, e.g., for $a, b \in V$, the element $a \otimes b + b \otimes a$ belongs to the symmetric tensor algebra on V. We are thus speaking of two realizations of the symmetric tensor algebras over V. The analogous procedure applies also for the exterior (anti-symmetric) tensor algebra.

For a quantum space associated to a Hecke operator, the two realizations for its (quantum) symmetric and anti-symmetric tensor algebras was first obtained by Gurevich [1]. While the first realization was taken as the definition, the second realizations follows from the general theory of Hecke algebras. The aim of this paper is to give the two realization for the (anti-) symmetric tensor algebras on the quantum semi-group of endormorphism (associated to a Hecke operator).

The main idea in giving the second realization is to construct a projector on the tensor products of V. For the classical case, it is the (anti-) symmetrizer operator, e.g., $a \otimes b \longrightarrow (a \otimes b + b \otimes a)/2$. For the quantum space, it is the quantum (anti-) symmetrizer, constructed in terms of the trivial and signature representations of the Hecke algebras. The direct generalization of the latter operators to the case of quantum semi-groups fails, these operators are no more projectors. Moreover the eigenvalues of these operators a known only for small ranks.

By a simple observation (Equation (1)), we are able to find the eigenvalues of these operators and then we can modify them to get projectors.

The paper is organised as follows. In Section 1 we recall the definitions and the two realizations of the quantum (anti-) symmetric algebras associated to a Hecke operator. In Section 2 we introduce the two operators Φ and Ψ and find out they eigevalues. Then we define the projector $\bar{\Phi}$ and $\bar{\Psi}$ and give the second realization for the quantum (anti-) symmetric algebras on the quantum semi-group of endomorphisms. In the last section we define the (anti-) symmetric algebras on the quantum space of homomorphism of two quantum spaces. The second realization of these algebras is also given.

0.1. PRELIMINARIES

Partitions A partitions λ of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r, ...)$ of non-decreasing non-negative integers, whose sum is n, we write $|\lambda| = n$. The maximal number r, for which $\lambda_r \neq 0$, is called the length of λ. The diagram $[\lambda]$ associated to λ is a matrix, whose first row contains λ_1 elements, second row contains λ_2 elements, and so on. The conjugated to λ, denoted by λ', is the one, whose diagram $[\lambda']$ is obtained from $[\lambda]$ by rotating it 180° along its diagonal.

A standard λ-tableau is the diagram $[\lambda]$, filled by numbers $1, 2, ..., |\lambda|$, in such a way that they increase along rows and columns. The number of standard tableaux is

$$d_\lambda = \prod_{x \in [\lambda]} \frac{n!}{h_\lambda(x)}$$

where the product is taken over all nodes x of the diagram λ, $h_\lambda(x)$ is the hook-length of the node x: if x is in the i-th row and j-th column of $[\lambda]$, then $h_\lambda(x) := \lambda_i + \lambda'_j - i - i + 1$.

Symmetric Groups The symmetric group $\mathfrak{S}_n$ consists of permutations of the set $1, 2, \ldots, n$. It can be regarded as group generated by transposition $v_i = (i, i+1)$, $1 \leq i \leq n-1$, subject to the relations: $v_i^2 = 1, v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}$. The length of $l(w)$ of an element w is the minimal length of the words in v_i expressing w. It equal the number of pairs $1 \leq i < j \leq n$ for which $wi > wj$.

Hecke Algebras The Hecke algebra $\mathcal{H}_n = \mathcal{H}_{n,q}$ is the q-analogue of the group algebra $\mathbb{K}[\mathfrak{S}_n]$. It is generated over $\mathbb{K}$ by 1 and the elements $T_i, 1 \leq i \leq n-1$, subject to the relations $T_i^2 = (q-1)T_i + q$ and $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. When $q = 1$, $\mathcal{H}_{n,1}$ reduces to $\mathbb{K}[\mathfrak{S}_n]$, $T_i \longrightarrow v_i$. $\mathcal{H}_n$ has a basis consisting of $T_w, w \in \mathfrak{S}_n$, $T_1 := 1$, $T_{v_i} := T_i$, $T_w T_u = T_{wu}$ if $l(w) + l(u) = l(wu)$.

We shall always assume that $q^n \neq 1, \forall n > 1$. In this case $\mathcal{H}_{n,q}$ is semisimple.

Representations of the Hecke Algebras Representations of $\mathcal{H}_{n,q}$, for q not being root of unity, can be parameterized by partitions of n. Let S_λ be the simple representation of $\mathcal{H}_n$, corresponding to λ, then the dimension of S_λ is d_λ.

We shall also consider the algebra $\mathcal{H}_n^{\mathrm{op}}$. Its simple comodules are canonically identified with S_λ^*: if ϕ_λ is the representation of $\mathcal{H}_n$ on S_λ, then the representation of $\mathcal{H}_n^{\mathrm{op}}$ on S_λ^* is given by $\bar{\phi}_\lambda(w) := \phi_\lambda(w)^*$.

Let $\mathcal{A}_\lambda$ be the block in $\mathcal{H}_n$, that corresponds to λ. Let $E_\lambda^{ij}, 1 \leq i, j \leq d_\lambda$ be a basis of $\mathcal{A}_\lambda$ such that $E_\lambda^{ij} E_\lambda^{kl} = \delta_k^j E_\lambda^{il}$. Thus, E_λ^{ii} are mutually orthogonal primitive idempontents of $\mathcal{H}_n$. Notice that for $\lambda \neq \mu$, $E_\lambda^{ij} E_\mu^{kl} = 0$.

A bilinear form on $\mathcal{H}_n$ There exists a non-degenerate, symmetric, associative bilinear form on $\mathcal{H}_n$, defined as follows: $(T_u, T_w) := q^{l(u)} \delta_w^u$. Thus, we see that $\{q^{-l(w)} T_{w^{-1}}, w \in \mathfrak{S}_n\}$ is the dual basis to $\{T_w, w \in \mathfrak{S}_n\}$ with respect to this bilinear form. Therefore, the Casimir element $\sum_{w \in \mathfrak{S}_n} q^{-l(w)} T_w \otimes T_{w^{-1}}$ is central:

$$\sum_{w \in \mathfrak{S}_n} q^{-l(w)} T_i T_w \otimes T_{w^{-1}} = \sum_{w \in \mathfrak{S}_n} q^{-l(w)} T_w \otimes T_{w^{-1}} T_i.$$

Since $\mathcal{A}_\lambda$ is simple, the bilinear form restricted on $\mathcal{A}_\lambda$ should satisfy

$$(E_\lambda^{ij}, E_\lambda^{kl}) = \delta_k^j \delta_l^i k_\lambda$$

for certain coefficient k_λ. k_λ is given explicitly by $k_\lambda = \prod_{x\in[\lambda]}[h_\lambda(x)]_q^{-1}$, where $[n]_q := (q^n-1)/(q-1)$. Thus $\{E_\lambda^{ij}, \lambda \vdash n, 1 \le i,j \le d_\lambda\}$ and $\{k_\lambda^{-1}E_\lambda^{ji}, \lambda \vdash n, 1 \le i,j \le d_\lambda\}$ are dual basis. Therefore

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} T_w \otimes T_{w^{-1}} = \sum_{\substack{\lambda\vdash n\\ 1\le i,j\le d_\lambda}} k_\lambda^{-1} E_\lambda^{ij} \otimes E_\lambda^{ji}. \tag{1}$$

Hecke Operators Let V be a finite dimensional vector space over a field $\mathbb{K}$ and R be a invertible operator on $V\otimes V$. R is called Hecke operator if the following conditions are satisfied:

$$R_1R_2R_1 = R_2R_1R_2 \quad \text{where } R_1 := R\otimes \mathrm{id}_V, R_2 := \mathrm{id}_V \otimes R,$$
$$(R+1)(R-q) = 0.$$

It is easy to see that a Hecke operator induces a represetation ρ_n of $\mathcal{H}_{n,q}$ oo $V^{\otimes n}$, for $n>1$: $\rho_n(T)i) = R_i := \mathrm{id}_V^{\otimes i-1} \otimes R \otimes \mathrm{id}_V^{\otimes n-i-1}$. For any $w \in \mathfrak{S}_n$, we denote $R_w := \rho_n(T_w)$.

1. MATRIX QUANTUM SEMIGROUPS OF TYPE A

1.1. QUADRATIC ALGEBRAS Let V be a vector space over $\mathbb{K}$ of finite dimension. Let R be a subspace of $V\otimes V$. Let $\mathcal{A} = \mathcal{A}(V,R)$ be the factor algebra of the tensor algebra on V by the two-sided ideal generated by elements of R:

$$\mathcal{A}(V,R) := \mathcal{T}(V)/(R).$$

Such an algebra is called quadratic algebra [4]. R is called the space of relations of $\mathcal{A}$. The two-sided ideal, generated by R, is usually denoted by $R(\mathcal{A})$. Since the relation on $\mathcal{A}$ are homogeneous, $\mathcal{A}$ inherits the grading from $\mathcal{T}(V)$

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n, \quad \mathcal{A}_n = V^{\otimes n}/R^n(\mathcal{A})$$
$$R^n(\mathcal{A}) = \sum_{i=1}^{n-1} R(\mathcal{A})_i^n, \quad R(\mathcal{A})_i^n := V^{i-1}\otimes R \otimes V^{n-i-1},\ 1\le i\le n-1.$$

The Poincaré (or Hilbert) series of $\mathcal{A}$ is by definition the formal power series $P_{\mathcal{A}}(t) := \sum_{k=0}^{\infty} \dim_{\mathbb{K}} \mathcal{A}_k$. The dual quadratic algebra to $\mathcal{A}$, $\mathcal{A}^!$ is defined to be $\mathcal{A}^! := \mathcal{T}(V^*)/(R(\mathcal{A})^\perp)$. It is easy to see that $(\mathcal{A}_n^!)^* = \bigcap_{i=1}^{n-1} R(\mathcal{A})_i^n$.

1.2. QUANTUM SPACES AND QUANTUM ENDOMORPHISM RINGS Let R be a Hecke operator on V. We define the following quadratic algebras

$$\mathcal{S} = \mathcal{S}_R := \mathcal{T}(V)/(\mathrm{Im}\,(R-q)), \tag{2}$$
$$\Lambda = \Lambda_R := \mathcal{T}(V)/(\mathrm{Im}\,(R+1)). \tag{3}$$

$\mathcal{S}_R$ and Λ_R are considered as the function algebra and the exterior algebra on a quantum space. The function algebra on the quantum semi-group of endomorphisms of this quantum space is defined to be a quadratic algebra on $V^*\otimes V$:

$$\mathcal{E} = \mathcal{E}_R := \mathcal{T}(V^*\otimes V)/(\mathrm{Im}\,(\bar{R}-1)). \tag{4}$$

where $\bar{R} := S_{(23)}(R^{*-1} \otimes R)S_{(23)}$, acting on $(V^* \otimes V)^{\otimes 2}$. One can check that

$$R(\mathcal{E}) = S_{(23)}\left(R(\mathcal{S})^{\perp} \otimes R(\mathcal{S}) \oplus R(\Lambda)^{\perp} \otimes R(\Lambda)\right).$$

Using $S_{(23)}$ we shall identify $\bar{R}$ with $R^{*-1} \otimes R$ acting on $V^{*\otimes 2} \otimes V^{\otimes 2}$.

$\mathcal{E}$ is a bialgebra with the coproduct $\Delta(\mathcal{E}_n) \subset \mathcal{E}_n \otimes \mathcal{E}_n$, i.e., $\mathcal{E}_n$ is a coalgebra. Λ_n, $\mathcal{S}_n$ are $\mathcal{E}_n$-comodules, they are factor comodules of $V^{\otimes n}$. Thus E_n^* is an algebra and acts on these spaces. On the other hand, the operator R induces an action ρ_n on $V^{\otimes n}$. We have a quantum analogue of Schur's double centralizers theorem [2]:

$$\begin{aligned} \mathcal{E}_n^* &\cong \operatorname{End}_{\mathcal{H}_n}(V^{\otimes n}) && (5)\\ \rho_n(\mathcal{H}_n) &\cong \operatorname{End}_{\mathcal{E}_n^*}(V^{\otimes n}) = \operatorname{End}^{\mathcal{E}_n}(V^{\otimes n}). && (6) \end{aligned}$$

As a consequence, simple $\mathcal{E}_n$ comodules are parameterized by primitive idempontents of $\mathcal{H}_n$, which, in their order, are parameterized by partitions of n. For each primitive idempotents E_λ of $\mathcal{H}_n$, $\rho_n(E_\lambda)$ is either zero of a simple E_n-comodule, conjugated idempontents define isomorphism comodules. In particular, as E_n-comodules

$$\Lambda_n \cong \operatorname{Im}\rho_n(Y_n), \quad S_n \cong \operatorname{Im}\rho_n(X_n),$$

where $X_n = ([n]_q!)^{-1}\sum_{w\in\mathfrak{S}_n} T_w$, $Y_n = ([n]_{q^{-1}}!)^{-1}\sum_{w\in\mathfrak{S}_n} q^{-l(w)}T_w$, the symmetrizer and anti-symmetrizer operators. Moreover we have an isomorphism of algebras

$$\mathcal{S} \cong \bigoplus_{n=0}^{\infty} \rho_n(X_n) \quad \Lambda \cong \bigoplus_{n=0}^{\infty} \rho_n(X_n) \tag{7}$$

where the product on the spaces on the right-hand sides of these isomorphisms are given by

$$\begin{aligned} x \in \rho(X_n), y \in \rho(X_m) &\longrightarrow x \cdot y := \rho_{m+n}(X_{m+n})(x \otimes y) && (8)\\ x \in \rho(Y_n), y \in \rho(Y_m) &\longrightarrow x \cdot y := \rho_{m+n}(Y_{m+n})(x \otimes y). && (9) \end{aligned}$$

This is the second realization of $\mathcal{S}$ and Λ, first considered by Gurevich [1]. Further, we have

$$\operatorname{Im}\rho_n(X_n) = \bigcap_{i=1}^{n-1} R(\Lambda)_i^n, \quad \operatorname{Im}\rho_n(Y_n) = \bigcap_{i=1}^{n-1} R(S)_i^n. \tag{10}$$

This means $\mathcal{S}$ is isomorphic to $\Lambda^{!*}$, Λ is isomorphic to $\mathcal{S}^{!*}$ as graded space, here $*$ means graded dual.

A problem arises naturally, to get an analogous realizations for the algebra $\mathcal{E}$.

In studying the Poincaré series of $\mathcal{E}$, Sudbery [5] introduced an operator Φ, acting on $\mathcal{E}_1^{\otimes 3}$. It is an analogue of the symmetrizer X_n, defined for $-\bar{R}$. The study was further developed in [2], where a complete description of the Poincaré series of $\mathcal{E}$ was given, in terms of the Poincaré series of Λ. The following operator plays a role:

$$\Phi^n := \sum_{w\in\mathfrak{S}_n} (-1)^{l(w)}\bar{R}_w. \tag{11}$$

As it has been mentioned, we identity $\bar{R}_w$ with $R_w^{*-1} \otimes R_w$, acting on $V^{*\otimes n} \otimes V^{\otimes n}$. For $n = 3$, Φ^3 is the one introduced by Sudbery. It was shown that

$$\operatorname{Im} \Phi^n = \bigcap_{i=1}^{n-1} R(\mathcal{E})_i^n, \tag{12}$$

for transcendent q, from which follows

$$P_{\mathcal{E}}(t) = P_{\mathcal{S}} \circ P_{\Lambda}, \tag{13}$$

"$\circ$" is the product in the λ-ring $\mathbb{K}_0[[t]]$ of power series with constant coefficient equal to 1. Although, later, Equation (13) was proved for all q not root of unity [3], (12) wasn't proved for non-transcendent q. One of the aims of the next section is to prove (12).

2. ANOTHER REALIZATION FOR THE ALGEBRA $\mathcal{E}$

2.1. THE OPERATOR Φ^n All needed information on the operator Φ^n can be obtained form the following observation. Let us denote $S := qR^{-1}$. Then S is a Hecke operator, too. We have $\bar{R} = -q^{-1} S^* \otimes R$. Let σ_n be the representation of $\mathcal{H}_n^{\mathrm{op}}$ on $V^{*\otimes n}$, $\sigma(T_i) = S_i^*$, thus, $\sigma_n(T_w) = (S_w)^* = {S^*}_{w^{-1}}$. Applying $\sigma_n \otimes \rho_n$ on the identity (1), we get

$$\sum_{s\in\mathfrak{S}_n} q^{-l(w)} {S^*}_{w^{-1}} \otimes R_w = \sum_{s\in\mathfrak{S}_n} \sigma_n(E_\lambda^{ij}) \otimes \rho_n(E_\lambda^{ji}). \tag{14}$$

Therefore

$$\begin{aligned} \sum_{s\in\mathfrak{S}_n} (-1)^{l(w)} \bar{R}_w &= \sum_{s\in\mathfrak{S}_n} q^{l(w)} {S^*}_{w^{-1}} \otimes R_w \\ &= \sum_{s\in\mathfrak{S}_n} k_\lambda^{-1} \sigma_n(E_\lambda^{ij})^* \otimes \rho_n(E_\lambda^{ji}). \end{aligned} \tag{15}$$

Denote $\Phi_\lambda := \sum_{1\le i,j\le d_\lambda} k_\lambda^{-1} \sigma_n(E_\lambda^{ij}) \otimes \rho_n(E_\lambda^{ji})$. Then we have

$$\Phi_\lambda^2 = d_\lambda k_\lambda^{-1} \Phi_\lambda. \tag{16}$$

Thus, the spectrum of Φ^n is contained in $\{d_\lambda k_\lambda^{-1} | \lambda \vdash n\} \cup 0$. Note that if $q = 1$, $d_\lambda k_\lambda^{-1} = n!$.

$V^{\otimes n}$ decomposes into $\mathcal{H}_n$-modules as follows:

$$V^{\otimes n} \cong \bigoplus_{\lambda\vdash n} V_\lambda, \quad V_\lambda \cong \bigoplus_{\substack{\lambda\vdash n \\ 1\le i\le l_\lambda}} S_{i,\lambda}. \tag{17}$$

The module V_λ is determined uniquely, while $S_{i,\lambda} \cong S_\lambda$, $\forall i$, $l_\lambda = \dim \operatorname{Im} \rho_n(E_\lambda)$. Therefore, as $\mathcal{H}_n^{\mathrm{op}} \otimes \mathcal{H}_n$-module, $V^{*\otimes n} \otimes V^{\otimes n}$ decomposes as

$$V^{*\otimes n} \otimes V^{\otimes n} \cong \bigoplus_{\lambda,\mu\vdash n} V_\lambda^* \otimes V_\mu \cong \bigoplus_{\substack{\lambda,\mu\vdash n \\ 1\le i\le l_\lambda \\ 1\le j\le l_\mu}} S_{j,\mu}^* \otimes S_{i,\lambda}. \tag{18}$$

Notice, that as an $\mathcal{H}_n$-module, S_λ is isomorphic to the subspace of $\mathcal{H}_n$ spanned by $\{E_\lambda^{mj} | 1 \le m \le d_\lambda\}$, for each fixed j. Analogously, $\{E_\lambda^{jm} | 1 \le m \le d_\lambda\}$ is isomorphic to S_λ^* as $\mathcal{H}_n^{\mathrm{op}}$-module, for each fixed j. Therefore, we have

Lemma 2.1 *If $l_\lambda := \dim \operatorname{Im} \rho_n(E_\lambda) \neq 0$, i.e., the $\mathcal{E}_n$-comodule corresponding to λ is not zero, then the operator Φ_λ has rank 1 on $S_\lambda^* \otimes S_\lambda$ and vanishes on $S_\mu^* \otimes S_\nu$, for $\mu \neq \lambda$ or $\nu \neq \lambda$.*

Proof. Fix i, j and let $\{E_\lambda^{im} \otimes E_\lambda^{ni} | 1 \leq m, n \leq l_\lambda\}$ be a basis of $S_\lambda^* \otimes S_\lambda$. Then

$$\Phi_\lambda(E_\lambda^{im} \otimes E_\lambda^{ni}) = \delta_n^m k_\lambda^{-1} \sum_{1 \leq k \leq l_\lambda} E_\lambda^{ik} \otimes E_\lambda^{ki}$$

∎

Consequently, as $\mathcal{E}_n^{\mathrm{op}} \otimes \mathcal{E}_n$-comodules

$$\begin{aligned} \operatorname{Im}(\Phi_\lambda) &\cong \operatorname{Im} \sigma_n(E_\lambda) \otimes \operatorname{Im} \rho_n(E_\lambda), \\ \operatorname{Im}(\Phi^n) &\cong \bigoplus_{\lambda \vdash n} \operatorname{Im} \sigma_n(E_\lambda) \otimes \operatorname{Im} \rho_n(E_\lambda). \end{aligned} \tag{19}$$

2.2. THE OPERATOR Ψ^n In (4), we see that $\mathcal{E}_2$ is isomorphic to $\operatorname{Im}(\tilde{R} + q)$, where $\tilde{R} := S_{(23)}\,(R^* \otimes R)\,S_{(23)}$. Let us define

$$\Psi^n := \sum_{s \in \mathfrak{S}_n} q^{-l(w)} \tilde{R}_w.$$

We shall also identify $\tilde{R}$ with $R^* \otimes R$. Let τ be the representation of $\mathcal{H}_n^{\mathrm{op}}$ on $V^{*\otimes n}$, induced by R^*. In analogy to (15), we have

$$\sum_{s \in \mathfrak{S}_n} q^{l(w)} \tilde{R}_w = \sum_{s \in \mathfrak{S}_n} k_\lambda^{-1} \tau_n(E_\lambda^{ij}) \otimes \rho_n(E_\lambda^{ji}).$$

Like for Φ_λ, we have:

Lemma 2.2 *If $l_\lambda \neq 0$, then the operator Ψ^n has rank 1 on $S_\lambda^* \otimes S_\lambda$ and vanishes on $S_\nu^* \otimes S_\nu$, for $\mu \neq \lambda$ or $\nu \neq \lambda$. Consequently, we have and isomorphism of $\mathcal{E}_n^{\mathrm{op}} \otimes \mathcal{E}_n$ comodules*

$$\operatorname{Im} \Psi^n = \bigoplus_{\lambda \vdash n} \operatorname{Im} \tau_n(E_\lambda) \otimes \operatorname{Im} \rho_n(E_\lambda). \tag{20}$$

On the other hand, according to (6),

$$\mathcal{E}_n^* \cong \bigoplus_{\lambda \vdash n} \operatorname{End}_{\mathcal{A}_\lambda}(V^{\otimes n}) \cong \bigoplus_{\lambda \vdash n} \operatorname{Im} \rho_n(E_\lambda) \otimes \operatorname{Im} \rho_n(E_\lambda)^*.$$

Consequently, we have

$$\mathcal{E}_n \cong \bigoplus_{\lambda \vdash n} \operatorname{Im} \rho_n(E_\lambda)^* \otimes \operatorname{Im} \rho_n(E_\lambda). \tag{21}$$

Therefore

$$\mathcal{E}_n \cong \operatorname{Im} \Psi^n, \tag{22}$$

as $\mathcal{E}_n^{\rm cop}\otimes\mathcal{E}_n$. The operator Ψ^n is not a projector, so that, in order to get a second realization for $\mathcal{E}_n$, we need the following operator.

$$\bar\Psi^n := \sum_\lambda k_\lambda d_\lambda^{-1}\tau_n(E_\lambda^{ij})^*\otimes\rho_n(E_\lambda^{ji}).$$

From (16) we see that $\bar\Psi^n$ is a projector, having the same image as Ψ^n. We define a multiplication on $\bigoplus_{n=0}^\infty \operatorname{Im}\Psi^n$, setting, for $a\in V^{*\otimes n}\otimes V^{\otimes n}$ and $b\in V^{*\otimes m}\otimes V^{\otimes m}$, $a\cdot b := \bar\Psi^{n+m}(a\otimes b)$.

Theorem 2.3 *The algebra $\mathcal{E}$ is isomorphic as a graded algebra to $\bigoplus_{n=0}^\infty \operatorname{Im}\Psi^n$.*

Proof. We have $(\tilde R+q)(\bar R-1)=(\bar R-1)(\tilde R+q)=0$. Therefore, if $x\in\operatorname{Im}(\bar R_i-1)$ then $(\tilde R+q)x=0$, consequently, Ψ^n vanishes on $\sum_{i=1}^{n-1}R(E)_i^n$. Then vanishes $\bar\Psi^n$ on $\sum_{i=1}^{n-1}R(E)_i^n$, too. Taking (22) into account, we conclude that $\sum_{i=1}^{n-1}R(E)_i^n$ is precisely the Kernel of $\bar\Psi^n$.

Thus, the linear map $\psi:\mathcal{T}(V^*\otimes V)\longrightarrow\bigoplus_{n=0}^\infty\operatorname{Im}\bar\Psi^n$, $a\longmapsto\bar\Psi^n(a)$ has the following properties

$$\begin{aligned} a\in R(\mathcal{E}) &\Longrightarrow \psi(a)=0\\ a &= \bar\psi(a) \bmod R(\mathcal{E}). \end{aligned} \tag{23}$$

Therefore, ψ factorizes through $R(\mathcal{E})$ to a map $\psi:\mathcal{E}\longrightarrow\bigoplus_{n=0}^\infty\operatorname{Im}\bar\Psi^n$. Let $a\in(V^*\otimes V)^{\otimes m}$, $b\in(V^*\otimes V)^{\otimes n}$. According to (23), we have

$$\bar\Psi^m(a)=a \bmod R(E)^m \quad \bar\Psi^n(b)=a \bmod R(E)^n,$$

hence

$$\bar\Psi^m(a)\otimes\bar\Psi^n(b)=a\otimes b \bmod R(E)^{m+n}.$$

Consequently,

$$\bar\Psi^{m+n}(\bar\Psi^m(a)\otimes\bar\Psi^n(b))=\bar\Psi^{m+n}(a\otimes b), \tag{24}$$

meaning that ψ is an algebra homomorphism and therefore isomorphism $\mathcal{E}\longrightarrow\bigoplus_{n=0}^\infty\operatorname{Im}\bar\Psi^n$. ∎

We now proceed to show that

$$\operatorname{Im}\Phi^n=\bigcap_{i=1}^{n=1}R(\mathcal{E})_1^n. \tag{25}$$

The inclusion "$\subset$" is obviously, for $\Phi^n=(\bar R_i-1)\Phi_i^n$ for certain operator Φ_i^n.

Let us denote $\mathcal{B}_n:=\bigcap_{i=1}^{n=1}R(\mathcal{E})_1^n$. To show the equation, we compare the dimension of $\mathcal{B}_n$ and $\operatorname{Im}\Phi^n$. Accoriding to (19), the dimension of $\operatorname{Im}\Phi^n$ is equal to $\sum_{\lambda\vdash n}\dim\operatorname{Im}\sigma_n(E_\lambda)\dim\operatorname{Im}\rho_n(E_\lambda)$. From the defition of the operator S, we see that $\sigma_n(E_\lambda)\cong\rho_n(E_{\lambda'})$ as vector spaces. Therefore $\dim\operatorname{Im}\Phi^n=\sum_{\lambda\vdash n}l_{\lambda'}l_\lambda$.

On the other hand, since $\mathcal{E}$ is a Koszul algebra [2]

$$P_{\mathcal{E}}(t)P_{\mathcal{E}^!}(-t)=1. \tag{26}$$

Since $\dim(\mathcal{E}_n) = \sum_{\lambda\vdash n} l_\lambda^2$, we also have

$$\dim \bigcap_{i=1}^{n-1} R(\mathcal{E})_i^n = \dim R(\mathcal{E}_n^!)^* = \dim R(\mathcal{E}_n^!) = \sum_{\lambda\vdash n} l_{\lambda'} l_\lambda. \tag{27}$$

Thus, $\dim \operatorname{Im}\Phi^n = \dim \mathcal{B}_n$. Whence (25) follows.

Remark 2.4 The proof of (25) above is not self-contained, it uses the fact that $\mathcal{E}$ is a Koszul algebra. The other proof might be as follows. One shows directly the exactness of the complex

$$0 \to \operatorname{Im}\Phi^n \to \operatorname{Im}\Phi^{n-1} \otimes \mathcal{E}_1 \to \operatorname{Im}\Phi^{n-2} \otimes \mathcal{E}_2 \to \cdots \to \mathcal{E}_n \to 0 \tag{28}$$

for $n > 1$, from which both (25) and the Koszul property of E would follow. The exactness of this complex seems to have close connection with the Littlewood-Richardson coefficients.

Finally, let us denote by $\mathcal{F}$ the quadratic algebra on $V^* \otimes V$ with relation $R(\mathcal{F}) := \operatorname{Im}(\tilde{R} + q)$. Thus, $\mathcal{F}$ can be considered as the exterior algebra over the matrix quantum semi-group. We have

$$R(\mathcal{F}) = \mathcal{S}_{(23)} \left(R(\mathcal{S})^\perp \otimes R(\Lambda) \bigoplus R(\Lambda)^\perp \otimes R(\mathcal{S}) \right). \tag{29}$$

The vector space $V^* \otimes V$ is self dual with respect to the pairing $(\phi \otimes x, \psi \otimes y) := (\phi, y)(\psi, x)$. With respect to this pairing, $\mathcal{F}$ is canonically isomorphic to $E^!$. Therefore $\mathcal{F}$ is Koszul algebra and

$$P_{\mathcal{F}}(t) = P_{\mathcal{E}}(-t)^{-1} = P_\Lambda(t) \circ P_{\mathcal{S}}(t), \tag{30}$$

or equivalently

$$\dim \mathcal{F}_n = \sum_{\lambda\vdash n} l_\lambda l_{\lambda'}. \tag{31}$$

Decompose $(V^* \otimes V)^{\otimes n}$ (which is identified with $V^{*\otimes n} \otimes V^{\otimes n}$) into simple $\mathcal{H}_n^{\mathrm{op}} \otimes \mathcal{H}_n$-modules (with the action given by $\tau_n \otimes \rho_n$). Notice that, as in the case of $\mathcal{E}$, $\mathcal{F}_n$ and $R(\mathcal{F})^n$ are $\mathcal{H}_n^{\mathrm{op}} \otimes \mathcal{H}_n$-modules. Recall that the operator Φ^n also has rank $\sum_{\lambda\vdash n} l_\lambda l_{\lambda'}$ on $V^{*\otimes n} \otimes V^{\otimes n}$ and vanishes on $R(\mathcal{F})^n$. Comparing the dimension we see that $R(\mathcal{F})^n$ is precisely the kernel of Φ^n. Analogously, the image of Ψ^n is $\bigcap_{i=1}^{n-1} R(\mathcal{F})_i^n$. Thus we proved an analogous of Theorem 2.3:

Theorem 2.5 *The algebra $\mathcal{F}$ is a Koszul algebra and can be realized as $\bigoplus_{n=0}^\infty \operatorname{Im}\bar{\Phi}^n$.*

3. The Quantum Spaces of Homomorphisms

Let R and S be Hecke operators on V and W, respectively. We define the quadratic algebra $\mathcal{M} := \mathcal{M}_{RS}$ on $W^* \otimes V$ with

$$R(\mathcal{M}) := \mathcal{S}_{(23)} \left((R_S(\mathcal{S}) \perp \otimes R_R(\mathcal{S}) \oplus R_S(\Lambda)^\perp \otimes R_R(\Lambda) \right), \tag{32}$$

where, as usual, $R_S(\mathcal{S}) := \mathrm{Im}\,(S-q)$, $R_R(\mathcal{S}) := \mathrm{Im}\,(R-q)$, and so on. We have

$$R(M) = \mathrm{Im}\,(\mathcal{S}_{(23)}(S^{*-1}\otimes R)S_{(23)}). \tag{33}$$

As in the previous section, we shall identify the two vector space $(W^*\otimes V)^{\otimes n}$ and $W^{*\otimes n}\otimes V^{\otimes n}$.

Notice that $W^*\otimes V \cong \mathrm{Hom}\,(V,W)^*$. The algebra $\mathcal{M}$ can be interpreted as the function algebra on the quantum space of homomorphisms from the quantum space associated to R to the one associated to S. Let $x_1,\cdots,x_m$ be a basis of V and $\xi^1,\cdots,\xi^n$ be a basis of W^*. Then $\{z^i_j := \xi^i\otimes x_j\}$ form a basis of $W^*\otimes V$. The algebra $\mathcal{M}=\mathcal{M}_{RS}$ is then isomorphic to

$$\mathbb{K}\langle z^i_j | 1\le i\le m, 1\le j\le n\rangle/(SZ_1Z_2 - Z_1Z_2R) \tag{34}$$

where $Z_1 := Z\otimes \mathrm{id}(n)$, $Z_2 := \mathrm{id}(m)\otimes Z$. $\mathcal{M}$ has the following properties: let $A := (a_1,a_2,\ldots,a_m)$ be a point of $\mathcal{S}_S$, i.e., $(R-q)(A\otimes A)=0$, and $T=(t^i_j)$ be a point of $\mathcal{M}_{RS}$, such that a_i and t^k_l commute. Then $A\cdot T$ is a point of $\mathcal{S}_R$. Analogously, if B is a point of Λ_S then $B\cdot T$ is a point of Λ_R.

There is also an interpretation of $\mathcal{M}_{RS}$ from the categorical view-point. Remember that the algebra E can be constructed as the Coend of the functor $\mathfrak{F}$ from the braided monoidal category $\mathcal{V}$, generated by one object v and one morphism $\tau : v^{\otimes 2}\longrightarrow v^{\otimes 2}$, into the category of vector space, such that $\mathfrak{F}(v)=V$ and $\mathfrak{F}(\tau)=R$. We then have, for any vector space X,

$$\mathrm{Nat}\,(\mathfrak{F},\mathfrak{F}\otimes X)\cong \mathrm{Hom}\,(\mathcal{E},X). \tag{35}$$

Let us now consider another function $\mathfrak{G}$, with $\mathfrak{G}(v)=W$ and $\mathfrak{G}(\tau)=S$. Then we have

$$\mathrm{Nat}\,(\mathfrak{F},\mathfrak{G}\otimes X)\cong \mathrm{Hom}\,(\mathcal{M}_{RS},X). \tag{36}$$

The exterior algebra on the quantum space of homomorphism is defined to be

$$\mathcal{N}=\mathcal{N}_{RS} := \mathcal{T}(W^*\otimes V)/\mathrm{Im}\,(S_{(23)}(S^*\otimes R)S_{(23)}+q). \tag{37}$$

There exist coactions of $E^{\mathrm{cop,op}}\otimes E$ on $\mathcal{M}$ and $\mathcal{N}$.

Here, we are interested in the Poincaré series, the Koszul property and the second realization of $\mathcal{M}$ and $\mathcal{N}$. First notice that $\mathcal{N}_{RS}$ is canonically isomorphic to $\mathcal{M}^!_{SR}$. Therefore it is sufficient to study $\mathcal{M}_{RS}$.

Since, on each simple module $S^*_\lambda\otimes S_\mu$ the action of Φ^n_{RS}, if it not trivial, does not depend on the operators R and S. Hence we have

$$\begin{aligned} &R(\mathcal{M})^n = \bigoplus R(\mathcal{M})^n\cap(S^*_\lambda\otimes S_\mu), \quad \text{and}\\ &\mathcal{M}_n\cap(S^*_\lambda\otimes S_\mu) = (S^*_\lambda\otimes S_\mu)\,/(S^*_\lambda\otimes S_\mu)\cap R(\mathcal{M})^n \end{aligned} \tag{38}$$

The operator Φ^n_{RS} is defined as Φ^n with $R^{*-1}\otimes R$ replaced by $S^{*-1}\otimes R$. Ψ^n_{RS} is defined analogously. Lemmas 2.1 and 2.2 hold for Φ^n_{RS} and Ψ^n_{RS}, respectively. We shall denote by $\bar\Phi^n_{RS}$ and $\bar\Psi^n_{RS}$ the corresponding projectors. According the Theorem 2.3, we have, for λ,μ, such that $S^*_\lambda\otimes S_\mu$ contributes in $W^{*\otimes n}\otimes V^{\otimes n}$,

$$(S^*_\lambda\otimes S_\mu)\cap R(M)^n_i = \mathrm{Ker}\,\Phi^n_{RS}\cap(S^*_\lambda\otimes S_\mu), \tag{39}$$

$$\dim\,(S^*_\lambda\otimes S_\lambda)/((S^*_\lambda\otimes S_\lambda)\cap R(M)) = 1. \tag{40}$$

These equations implies that $\dim \mathcal{M}_n = \sum_{\lambda \vdash n} l_\lambda k_{\lambda'}$, where $k_\lambda = \dim \operatorname{Im} \rho_S(E_\lambda)$ and $l_\lambda = \dim \operatorname{Im} \rho_R(E_\lambda)$. Since $\bar{\Psi}^n_{RS}$ is a projector, the map $\bar{\phi}_{RS} : \mathcal{T}(W^* \otimes V) \longrightarrow \bigoplus_{n=0}^\infty \operatorname{Im} \bar{\Phi}^n_{RS}$ induces an isomorphism of algebra $\mathcal{M} \longrightarrow \bigoplus_{n=0}^\infty \operatorname{Im} \bar{\Phi}^n_{RS}$.

On the other hand,the fact, that the action of $S_i^{*-1} \otimes R_i$ on $(S^*_\lambda \otimes S_\mu)$ does not depend on R and S, implies that the lattice induced by $R(\mathcal{M})^n_i \cap (S^*_\lambda \otimes S_\mu), 1 \leq i \leq n-1$ is distributive in $(S^*_\lambda \otimes S_\mu)$ (cf. [2]). Consequently, the lattice generated by $R(\mathcal{M})^n_i, 1 \leq i \leq n-1$ is distributive in $W^{*\otimes n} \otimes V^{\otimes n}$, that is $\mathcal{M}_{RS}$ is a Koszul algebra.

We collect the results obtained in a theorem.

Theorem 3.1 *The algebra $\mathcal{M}_{RS}$ and $\mathcal{N}_{RS}$ are Koszul algebras, their Poincaré series are given by*

$$P_{\mathcal{M}}(t) = P_{\mathcal{S}_R} \circ P_{\mathcal{S}_S}(t), \quad P_{\mathcal{N}}(t) = P_{\Lambda_S} \circ P_{\Lambda_R}(t). \tag{41}$$

There are realizations of $\mathcal{M}$ and $\mathcal{N}$ as subspaces of $\mathcal{T}(W^* \otimes V)$:

$$\mathcal{M} \cong \bigoplus_{n=0}^\infty \operatorname{Im} \bar{\Phi}^n, \quad \mathcal{N} \cong \bigoplus_{n=0}^\infty \operatorname{Im} \bar{\Psi}^n \tag{42}$$

further

$$\operatorname{Im}(\bar{\Phi}^n_{RS}) = \bigcap_{i=1}^{n-1} R(\mathcal{N})^n_i, \quad \operatorname{Im}(\bar{\Psi}^n_{RS}) = \bigcap_{i=1}^{n-1} R(\mathcal{M})^n_i. \tag{43}$$

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