THIRD MAC LANE COHOMOLOGY

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Abstract. MacLane cohomology is an algebraic version of the topological Hochschild cohomology. Based on the computation of the third author (see Appendix below) we obtain an interpretation of the third Mac Lane cohomology of rings using certain kind of crossed extensions of rings in the quadratic world. Actually we obtain two such interpretations corresponding to the two monoidal structures on the category of square groups.

1. Introduction

Let R be a ring and M a bimodule over R. Then there are three essential cohomology theories associated to the pair (R, M) due to Hochschild, Shukla, and Mac Lane, see [16, 35, 24]. These theories are connected by natural maps ([12])

$$\mathsf{H}^n(R;M) \to \mathsf{SH}^n(R;M) \xrightarrow{\tau^n} \mathsf{HML}^n(R;M).$$

It is known that Mac Lane cohomology coincides with topological Hochschild cohomology ([32]) and coincides also with Baues-Wirsching cohomology of the category $\mathbf{mod}\text{-}R$ of finitely generated free R-modules ([19]). We study the cohomologies in dimension $n = 3$. In this case the elements in $\mathsf{H}^3(R;M)$ are represented by split crossed extensions and elements in $\mathsf{SH}^3(R;M)$ are represented by all crossed extensions of R by M in the monoidal category $(\mathsf{Ab}, \otimes)$ of abelian groups. See [23], [9] and [12]. Here a crossed extension of R by M is an exact sequence in Ab,

$$0 \to M \xrightarrow{\iota} C_1 \xrightarrow{\partial} C_0 \xrightarrow{q} R \to 0,$$

where C_0 is a ring, C_1 is a bimodule over it, q is a ring homomorphism and ι and ∂ are C_0-biequivariant maps satisfying $b\partial(c) = \partial(b)c$ for $b, c \in C_1$.

A similar result for group cohomology in dimension 3 is due to Mac Lane and J. H. C. Whitehead [26].

The main goal of this paper is the construction of appropriate crossed extensions of R by M which represent classes in the third Mac Lane cohomology $\mathsf{HML}^3(R;M)$.

To this end we recall that for any small category $\mathscr{C}$ with a natural system D on it, the Baues-Wirsching cohomology group $H^3(\mathscr{C};D)$ can be represented by linear track extensions of $\mathscr{C}$ by D, see [28, 29, 3]. Hence by the isomorphism

$$\mathsf{HML}^3(R;M) \cong H^3(\mathbf{mod}\text{-}R; \mathsf{Hom}_R(-, - \otimes_R M))$$

elements of $\mathsf{HML}^3(R;M)$ are represented by linear track extensions of $\mathbf{mod}\text{-}R$. Such a description, however, is available for any category $\mathscr{C}$ and does not restrict to the specific nature of Mac Lane cohomology of a ring.

In order to find specific crossed extensions for $\mathsf{HML}^3(R;M)$ we have to proceed from linear algebra to quadratic algebra. Here "linear algebra" is the algebra of rings and modules. A ring is a monoid in the monoidal category $(\mathsf{Ab}, \otimes)$ and a module is an object in Ab together with an action of such a monoid.

In "quadratic algebra" abelian groups are replaced by square groups. In fact, if one considers endofunctors of the category of groups which preserve filtered colimits and reflexive coequalizers, then abelian groups can be identified with linear endofunctors and square groups can be identified with quadratic endofunctors ([10]). The category SG of square groups contains the category Ab as a full subcategory since a linear endofunctor is also quadratic. Composition of functors leads to monoidal structures $\otimes$ and $\square$ in such a way that $(\mathsf{Ab},\otimes)$ is a monoidal subcategory of $(\mathsf{SG},\square)$. There is also another monoidal structure $\underline{\circledcirc}$ on SG such that the identity of SG is a lax monoidal functor $(\mathsf{SG},\underline{\circledcirc})\to(\mathsf{SG},\square)$. Here $\underline{\circledcirc}$ is symmetric while $\square$ is highly nonsymmetric. Compare [8].

Crossed extensions in the monoidal categories $(\mathsf{SG},\square)$ or $(\mathsf{SG},\underline{\circledcirc})$ are defined similarly to the case $(\mathsf{Ab},\otimes)$ above, see section 2. As a main result we prove in this paper the following theorem, compare the more detailed version 2.2.1 below.

1.1. Main theorem.

Theorem 1.1.1. *Elements in the third Mac Lane cohomology group $\mathsf{HML}^3(R;M)$ are in 1-1 correspondence with equivalence classes of linearly generated crossed extensions of R by M in the monoidal category $(\mathsf{SG},\underline{\circledcirc})$, or in the monoidal category $(\mathsf{SG},\square)$.*

Such an interpretation of the group HML^3 was missing for many years; in terms of obstruction theory the problem first arose in the classical paper of Mac Lane [25]. The theorem is based on the quadratic theory developed in [10, 8] and emphasizes importance of the quadratic algebra of square groups. A crucial step in the proof of the theorem relies on the vanishing result achieved by the third named author in the Appendix.

In dimension three, the map τ^3 fits in the exact sequence (see [18], [12])

$$0\to\mathsf{SH}^3(R;M)\xrightarrow{\tau^3}\mathsf{HML}^3(R;M)\xrightarrow{\nu}\mathsf{H}^0(R;{}_2M)\to\mathsf{SH}^4(R;M)\xrightarrow{\tau^4}\mathsf{HML}^4(R;M)$$

where ${}_2M=\{m\in M\mid 2m=0\}$.

As an application of the theorem we describe the connecting homomorphism ν in terms of crossed extensions in SG, see Section 2.5.

It follows from the relationship between $\mathsf{SH}^3(R,M)$ and crossed extensions of rings that $\mathsf{SH}^3(R,M)$ describes homotopy types of those chain algebras C_* with $H_0(C_*)=R$, $H_1(C_*)=M$, and $H_i(C_*)=0$ for $i\neq 0,1$. On the other hand, it follows from the relationship between Mac Lane and topological Hochschild cohomology that $\mathsf{HML}^3(R,M)$ describes homotopy types of ring spectra Λ with $\pi_i(\Lambda)=0$, $i\neq 0,1$, $\pi_0(\Lambda)=R$ and $\pi_1(\Lambda)=M$ [21]. Thus our result shows that crossed extensions of R by M in SG are algebraic models of such ring spectra. It follows that the homomorphism ν is an obstruction for such a ring spectrum to be representable by a chain algebra.

In Section 5 we give an application of our results to the theory of 2-categories.

2. Crossed extensions

We shall apply the following general notion of crossed extension to the monoidal categories $(\mathsf{Ab},\otimes)$, $(\mathsf{SG},\square)$ and $(\mathsf{SG},\underline{\circledcirc})$ where the category SG of square groups is defined in 2.2.

2.1. **Crossed extensions.** Let $(\mathbf{V}, \boxtimes)$ be a monoidal category and let L be a monoid in $\mathbf{V}$. Recall that a *L-biobject* is a tuple (A, l, r), where A is an object in $\mathbf{V}$ and $l : L \boxtimes A \to A$ and $r : A \boxtimes L \to A$ are respectively left and right actions of L on A which are compatible in a natural way. We let ${}_L\mathbf{V}_L$ be the category of L-biobjects in $\mathbf{V}$. In particular the monoid structure on L defines also a structure of a L-biobject on L. In what follows we always consider L as a biobject with this particular structure.

A *crossed L-biobject* is a diagram $C = (\partial : B \to L)$ in the category ${}_L\mathbf{V}_L$ such that the following diagram commutes:

$$\begin{tikzcd}
B \boxtimes B \arrow[r, "\mathsf{Id}\boxtimes\partial"] \arrow[d, "\partial\boxtimes\mathsf{Id}"'] & B \boxtimes L \arrow[d, "r"] \\
L \boxtimes B \arrow[r, "l"'] & L
\end{tikzcd}$$

Let R be a monoid in $(\mathbf{V}, \boxtimes)$, let M be an R-biobject and assume that exact sequences are defined in $\mathbf{V}$. Then a *crossed extension* of R by M in $(\mathbf{V}, \boxtimes)$ is an exact sequence

$$0 \to M \xrightarrow{\iota} C_1 \xrightarrow{\partial} C_0 \xrightarrow{q} R \to 0 \tag{2.1.1}$$

where ∂ is a crossed C_0-biobject as above, q is a morphism of monoids and ι is a morphism in ${}_{C_0}\mathbf{V}_{C_0}$. A morphism between crossed extensions of R by M is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"'] & C_1 \arrow[r, "\partial"] \arrow[d, "f_1"] & C_0 \arrow[r] \arrow[d, "f_0"] & R \arrow[r] \arrow[d, "\mathsf{Id}"] & 0 \\
0 \arrow[r] & M \arrow[r] & C_1' \arrow[r, "\partial'"] & C_0 \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

where f_0 is a morphism of monoids and f_1 is f_0-biequivariant. Let

$$\mathscr{X\!ext}(R; M)^{\mathsf{V},\boxtimes} \tag{2.1.2}$$

be the category of such crossed extensions and morphisms and let

$$\mathrm{Xext}(R; M)^{\mathsf{V},\boxtimes} \tag{2.1.3}$$

be the set of connected components of this category.

One readily checks that crossed extensions in $(\mathsf{Ab}, \otimes)$ are the extensions defined in Section 1. Hence if R is a ring and M is an R-bimodule then one has canonical bijections (see [12], [23, page 42], [9])

$$\mathrm{Xext}(R; M)^{\mathsf{Ab},\otimes} \approx \mathsf{SH}^3(R; M) \tag{2.1.4}$$

and

$$\mathrm{Xext}_{\mathbb{Z}}(R; M)^{\mathsf{Ab},\otimes} \approx \mathsf{H}^3(R; M). \tag{2.1.5}$$

Here $\mathrm{Xext}_{\mathbb{Z}}(R; M)^{\mathsf{Ab},\otimes}$ is the set of connected components of the following subcategory $\mathscr{X\!ext}_{\mathbb{Z}}(R; M)^{\mathsf{Ab},\otimes}$ of $\mathscr{X\!ext}(R; M)^{\mathsf{Ab},\otimes}$: its objects are $\mathbb{Z}$-split crossed extensions (ι, ∂, q) in $(\mathsf{Ab}, \otimes)$, that is, with arrows ι, ∂ and q admitting a $\mathbb{Z}$-splitting; morphisms in $\mathscr{X\!ext}_{\mathbb{Z}}(R; M)^{\mathsf{Ab},\otimes}$ are morphisms (f_0, f_1) in $\mathscr{X\!ext}(R; M)^{\mathsf{Ab},\otimes}$ such that both f_0 and f_1 are $\mathbb{Z}$-split.

2.2. **Square groups.** A *square group* is a diagram

$$A = (\ A_e \xrightarrow{H} A_{ee} \xrightarrow{P} A_e\ )$$

where A_{ee} is an abelian group and A_e is a group. Both groups are written additively. Moreover P is a homomorphism and H is a quadratic map, meaning that the *cross effect*

$$(x \mid y)_H = H(x+y) - H(y) - H(x)$$

is linear in $x, y \in A_e$. In addition the following identities are satisfied

$$(Pa \mid y)_H = 0,$$
$$P(x \mid y)_H = [x, y],$$
$$PHP(a) = P(a) + P(a).$$

Here $[x, y] = -y - x + y + x$, $a, b \in A_{ee}$ and $x, y \in A_e$. It follows from the first two identities that P maps to the center of A_e. The second identity shows also that

$$A^{\mathrm{ad}} := \mathsf{Coker}(P)$$

is abelian. Hence A_e is a group of nilpotence class 2. It follows from the axioms that the function $T = HP - \mathsf{Id}_{A_{ee}}$ is an automorphism of A_{ee} and $T^2 = \mathsf{Id}_{A_{ee}}$. Moreover, the function $\Delta : A_e \to A_{ee}$ is linear, where

$$\Delta(x) = HPH(x) + H(x+x) - 4H(x)$$

and furthermore one has the induced homomorphisms

$$(-,-)_H : A^{\mathrm{ad}} \otimes A^{\mathrm{ad}} \to A_{ee}$$

and

$$\Delta : A^{\mathrm{ad}} \to A_{ee}.$$

We refer to [10] and [8] for more information on square groups. We denote by SG the category of square groups. In what follows we identify abelian groups and square groups with $A_{ee} = 0$. In this way we obtain a full embedding of categories

$$\mathsf{Ab} \subset \mathsf{SG}$$

This inclusion corresponds to the fact that any linear functor is quadratic. The inclusion $\mathsf{Ab} \subset \mathsf{SG}$ has a left adjoint given by $A \mapsto A^{\mathrm{ad}}$.

The category SG has two monoidal structures $\square : \mathsf{SG} \times \mathsf{SG} \to \mathsf{SG}$ [10] and $\underline{\circledcirc} : \mathsf{SG} \times \mathsf{SG} \to \mathsf{SG}$ [8], which are related via a binatural transformation

$$\sigma_{X,Y} : X \square Y \to X \underline{\circledcirc} Y$$

such that the identity functor together with σ defines a lax monoidal functor $\mathsf{Id} : (\mathsf{SG}, \underline{\circledcirc}) \to (\mathsf{SG}, \square)$ [8]. The monoidal category structure $\square$ is highly nonsymmetric, while the monoidal category structure $\underline{\circledcirc}$ is symmetric. For the definitions of the products $\square$ and $\underline{\circledcirc}$ on SG we refer the reader to [10] and [8] respectively. Below we shall, however, describe explicitly the notion of crossed extension in $(\mathsf{SG}, \square)$ and in $(\mathsf{SG}, \underline{\circledcirc})$. Since $(\mathsf{Ab}, \otimes)$ is a monoidal subcategory both in $(\mathsf{SG}, \square)$ and in $(\mathsf{SG}, \underline{\circledcirc})$, we see that a ring R, i. e. a monoid in $(\mathsf{Ab}, \otimes)$, is also a monoid in $(\mathsf{SG}, \square)$ and in $(\mathsf{SG}, \underline{\circledcirc})$. Let M be an R-bimodule. Then crossed extensions

$$0 \to M \xrightarrow{\iota} C_{(1)} \xrightarrow{\partial} C_{(0)} \xrightarrow{q} R \to 0$$

are defined in $(\mathsf{SG},\square)$ and in $(\mathsf{SG},\underline{\circledcirc})$ by (2.1.1). Such an extension is *linearly generated* if R as an additive group is generated by the image of the linear elements of ${C_{(0)}}_e$ in R. Here an element $x \in {C_{(0)}}_e$ is *linear* provided $H(x) = 0$.

As a main result we prove the quadratic analogue of (2.1.4).

Theorem 2.2.1. *Let R be a ring and let M be an R-bimodule. Then there are natural bijections*

$$\mathrm{Xext}_L(R;M)^{\mathsf{SG},\underline{\circledcirc}} \approx \mathrm{Xext}_L(R;M)^{\mathsf{SG},\square} \approx \mathsf{HML}^3(R;M)$$

where the index L indicates the full subcategories of linearly generated crossed extensions. The first bijection is induced by the lax monoidal functor $(\mathsf{SG},\square) \to (\mathsf{SG},\underline{\circledcirc})$.

The proof of this result is given in Section 4.3.

Remark. For the second bijection in the theorem we use the isomorphism

$$\mathsf{HML}^3(R;M) \cong H^3(\mathbf{mod}\text{-}R;D_M)$$

where $\mathbf{mod}$-R is the category of finitely generated free right R-modules and $D_M = \mathsf{Hom}_R(-,-\otimes_R M)$. Here we use the following interpretation of crossed biobjects from 2.1.

Let $(\mathbf{V},\boxtimes)$ be a monoidal category and assume that finite colimits exist in $\mathbf{V}$. Let pair$(\mathbf{V})$ be the category of pairs in $\mathbf{V}$, objects are morphisms $V = \left(V_1 \xrightarrow{\partial} V_0\right)$ in $\mathbf{V}$ and morphisms $V \to W$ are pairs $\alpha = (\alpha_1 : V_1 \to W_1, \alpha_0 : V_0 \to W_0)$ in $\mathbf{V}$ with $\partial\alpha_1 = \alpha_0\partial$. Then $(\mathrm{pair}(\mathbf{V}),\underline{\boxtimes})$ is a monoidal category with $\underline{\boxtimes}$ defined by the diagram with the inner square pushout

$$\begin{tikzcd}
V_1 \boxtimes W_1 \arrow[r, "1\boxtimes\partial"] \arrow[d, "\partial\boxtimes 1"'] \arrow[dr, phantom, "\text{push}"] & V_1 \boxtimes W_0 \arrow[d] \arrow[ddr, bend left, "\partial\boxtimes 1"] & \\
V_0 \boxtimes W_1 \arrow[r] \arrow[drr, bend right, "1\boxtimes\partial"'] & (V\underline{\boxtimes}W)_1 \arrow[dr, "\partial"] & \\
 & & V_0 \boxtimes W_0 := (V\underline{\boxtimes}W)_0.
\end{tikzcd}$$

One readily checks that a crossed L-biobject $C = (\partial : B \to L)$ in 2.1 is the same as a monoid in $(\mathrm{pair}(\mathbf{V}),\underline{\boxtimes})$. Hence the action of the monoid C on an object X in pair$(\mathbf{V})$ is defined, compare section 5.1 in [1]. In this case we call X a *C-module.*

Addendum. For a crossed extension C of R by M in either $(\mathsf{SG},\square)$ or $(\mathsf{SG},\underline{\circledcirc})$ let $\mathbf{mod}$-C be the category of finitely generated free left C-modules. Then $\mathbf{mod}$-C is a linear track extension which represents an element

$$\langle \mathbf{mod}\text{-}C \rangle \in H^3(\mathbf{mod}\text{-}R;D_M)$$

and the bijections from 2.2.1 carry the component of the crossed extension C in $(\mathsf{SG},\underline{\circledcirc})$, resp. in $(\mathsf{SG},\square)$, to the class $\langle\mathbf{mod}\text{-}C\rangle$.

Since a crossed extension C is also a monoid in the category of pairs, we will also call C a "pair algebra". The addendum makes use of modules over such pair algebras.

2.3. **Square rings and quadratic rings.** A monoid in the monoidal category $(\mathsf{SG}, \square)$ is termed a *square ring*, while a monoid in the monoidal category $(\mathsf{SG}, \underline{\circledcirc})$ is termed a *quadratic ring*.

More explicitly (see [5], [10], [6]), to provide a square group Q with a *square ring structure* is the same as to give additionally a multiplicative monoid structure on Q_e. The multiplicative unit of Q_e is denoted by 1. One requires that this monoid structure induces a ring structure on the abelian group Q^{ad} through the canonical projection

$$Q_e \to Q^{\mathrm{ad}}, \quad a \mapsto \bar{a}.$$

Moreover the abelian group Q_{ee} must be a $Q^{\mathrm{ad}} \otimes Q^{\mathrm{ad}} \otimes (Q^{\mathrm{ad}})^{\mathrm{op}}$-module with action denoted by $(\bar{x} \otimes \bar{y}) \cdot a \cdot \bar{z} \in Q_{ee}$ for $\bar{x}, \bar{y}, \bar{z} \in Q^{\mathrm{ad}}$, $a \in Q_{ee}$. In addition the following conditions must be satisfied where $H(2) = H(1+1)$

(i) $x(y+z) = xy + xz$
(ii) $(x+y)z = xz + yz + P((\bar{x} \otimes \bar{y}) \cdot H(z))$
(iii) $(x \mid y)_H = (\bar{y} \otimes \bar{x}) \cdot H(2)$
(iv) $T((\bar{x} \otimes \bar{y}) \cdot a \cdot \bar{z}) = (\bar{y} \otimes \bar{x}) \cdot T(a) \cdot \bar{z}$
(v) $P(a \cdot x) = P(a)x$
(vi) $P((\bar{x} \otimes \bar{x}) \cdot a) = xP(a)$
(vii) $H(xy) = (\bar{x} \otimes \bar{x}) \cdot H(\bar{y}) + H(x) \cdot \bar{y}$

Under the equivalence $\mathsf{Quad}(\mathsf{Gr}) \cong \mathsf{SG}$ square rings correspond to monads on the category of groups, whose underlying functors lie in $\mathsf{Quad}(\mathsf{Gr})$. A *quadratic ring structure* on a square group C is given by a multiplicative monoid structure on C_e and a ring structure on C_{ee}. The multiplicative unit of C_e is denoted by 1. One requires that these structures satisfy the following additional conditions.

(i) $x(y+z) = xy + xz$,
(ii) $(x+y)z = xz + yz + P((y \mid x)_H\, H(z))$.

Thus C^{ad} is a ring. Moreover the maps

$$-T : C_{ee} \to C_{ee},$$
$$(- \mid -)_H : C^{\mathrm{ad}} \otimes C^{\mathrm{ad}} \to C_{ee}$$

are ring homomorphisms, in other words one has

(iii) $(x \mid y)_H (u \mid v)_H = (xu \mid yv)_H$,
(iv) $T(ab) + T(a)T(b) = 0$.

Let us observe that $T(abc) = T(a)T(b)T(c)$. Furthermore the following equations hold

(v) $P(a\Delta(x)) = P(a)x$,
(vi) $P((x \mid x)_H a) = xP(a)$,
(vii) $H(xy) = (x \mid x)_H H(y) + H(x)\Delta(y)$.

It follows from the axioms that $\Delta : C^{\mathrm{ad}} \to C_{ee}$ is a ring homomorphism [8].

Let QR (resp. SR) denote the category of quadratic (resp. square) rings. One has the full embedding of categories $\mathsf{Rings} \subset \mathsf{QR}$ (resp. $\mathsf{Rings} \subset \mathsf{SR}$) which identifies rings with quadratic (resp. square) rings C satisfying $C_{ee} = 0$. This inclusion has a left adjoint given by $R \mapsto R^{\mathrm{ad}}$.

There is also a functor

$$U : \mathsf{QR} \to \mathsf{SR}$$

which assigns to a quadratic ring C a square ring, whose underline square group is the same, while the $C^{\mathrm{ad}} \otimes C^{\mathrm{ad}} \otimes (C^{\mathrm{ad}})^{\mathrm{op}}$-module structure on C_{ee} is given by

$$(\bar{x} \otimes \bar{y})a\bar{z} = (y \mid x)_H a\Delta(z).$$

The initial object in the category of quadratic rings (resp. square rings) is $\mathbb{Z}_{\mathrm{nil}}$, which is given by

$$(\mathbb{Z}_{\mathrm{nil}})_e = \mathbb{Z} = (\mathbb{Z}_{\mathrm{nil}})_{ee}, \quad P = 0, \quad H(x) = \frac{x(x-1)}{2}.$$

We now extend the monoid ring construction to quadratic rings and square rings. For a monoid S one puts

$$\mathbb{Z}_{\mathrm{nil}}[S]_{ee} = \mathbb{Z}[S] \otimes \mathbb{Z}[S],$$

where $\mathbb{Z}[S]$ is the free abelian group generated by S. We take $\mathbb{Z}_{\mathrm{nil}}[S]_e$ to be the free nil_2-group generated by S. The homomorphism P is given by $P(s \otimes t) = [t, s]$, $s, t \in S$, while the quadratic map H is uniquely defined by

$$H(s) = 0, \quad (s \mid t)_H = t \otimes s \quad s, t \in S.$$

One has

$$\mathbb{Z}_{\mathrm{nil}}[S]^{\mathrm{ad}} = \mathbb{Z}[S].$$

There is a unique quadratic (resp. square) ring structure on $\mathbb{Z}_{\mathrm{nil}}[S]$ for which the multiplication on $\mathbb{Z}_{\mathrm{nil}}[S]_e$ extends the multiplication on the monoid S and such that the ring structure (resp. $\mathbb{Z}[S] \otimes \mathbb{Z}[S] \otimes \mathbb{Z}[S]^{\mathrm{op}}$-module structure) on $\mathbb{Z}_{\mathrm{nil}}[S]_{ee} = \mathbb{Z}[S] \otimes \mathbb{Z}[S]$ is the obvious one (resp. is given by $(x \otimes y)(s \otimes t)z = xsz \otimes ytz$). In this case $Q^{\mathrm{ad}} = \mathbb{Z}[S]$ is the usual monoid ring of S. The functor $\mathbb{Z}_{\mathrm{nil}}[-] : \mathsf{Monoids} \to \mathsf{QR}$ (resp. $\mathbb{Z}_{\mathrm{nil}}[-] : \mathsf{Monoids} \to \mathsf{SR}$) is left adjoint to the functor

$$\mathsf{L} : \mathsf{QR} \to \mathsf{Monoids} \quad (\text{resp. } \mathsf{L} : \mathsf{SR} \to \mathsf{Monoids}),$$

where $\mathsf{L}(Q)$ consists of *linear elements* of Q, that is

$$\mathsf{L}(Q) = \{x \in Q_e \mid H(x) = 0\}.$$

The equality $H(xy) = (\bar{x} \otimes \bar{x})H(y) + H(x)\bar{y}$ shows that linear elements indeed form a multiplicative submonoid of Q_e.

2.4. **Quadratic pair algebras and crossed square rings.** Now we consider crossed biobjects in the monoidal categories $(\mathsf{SG}, \square)$ and $(\mathsf{SG}, \underline{\circledcirc})$. Actually we restrict ourselves to considering only those crossed biobjects $\partial : C_1 \to C_0$ which induce isomorphism on ee-level. This is the condition which implies that $\mathsf{Coker}(\partial)$ is a usual ring and $\mathsf{Ker}(\partial)$ is a usual bimodule. This forces us to introduce the following definition.

A *quadratic pair module* (qpm for short) is a morphism of square groups $\partial : C_1 \to C_0$ such that the homomorphism $\partial_{ee} : C_{1ee} \to C_{0ee}$ is an identity map. Thus explicitly a quadratic pair module C is given by a diagram

$$\begin{tikzcd} & C_{ee} \arrow[dl, "P"'] \\ C_1 \arrow[r, "\partial"'] & C_0 \arrow[u, "H"'] \end{tikzcd}$$

where C_1 and C_0 are groups, C_{ee} is an abelian group, P and ∂ are group homomorphisms and H is a quadratic map, and moreover the following identities are satisfied for any $a \in C_{ee}$, $r, s \in C_1$ and $x, y \in C_0$:

$$
\begin{aligned}
PH\partial P(a) &= 2P(a);\\
H(x + \partial P(a)) &= H(x) + H\partial P(a);\\
PH(\partial(r) + \partial(s)) &= PH\partial(r) + PH\partial(s) + [r, s];\\
\partial PH(x + y) &= \partial PH(x) + \partial PH(y) + [x, y].
\end{aligned}
$$

The category of qpm's is denoted by qpm. If C is a qpm, then $\mathsf{Im}(\partial)$ is a normal subgroup of C_0 containing the commutator subgroup of C_0. Thus

$$h_0(C) := \mathsf{Coker}(\partial)$$

is an abelian group. Moreover

$$h_1(C) := \mathsf{Ker}(\partial)$$

is a central subgroup of C_1. We have an exact sequence of square groups

$$0 \to h_1(C) \to C_{(1)} \xrightarrow{(\partial, \mathsf{Id})} C_{(0)} \to h_0(C) \to 0.$$

Here $(C_{(1)})_{ee} = (C_{(0)})_{ee} = C_{ee}$, $(C_{(1)})_e = C_1$ and $(C_{(0)})_e = C_0$. The structural maps are given by

$$
\begin{aligned}
P^{C_{(0)}} &= \partial P, \quad P^{C_{(1)}} = P,\\
H^{C_{(0)}} &= H, \quad H^{C_{(1)}} = H\partial.
\end{aligned}
$$

A qpm together with crossed biobject structure in the monoidal category $(\mathsf{SG}, \underline{\circledcirc})$ is called a *quadratic pair algebra* (shortly qpa). Thus a qpa is a qpm C together with a ring structure on C_{ee} and a quadratic ring structure on $C_{(0)}$. Additionally a two-sided action of C_0 on C_1 is given, which is associative and unital and the following identities are satisfied for all $x, y \in C_0$, $r, s \in C_1$, $a, b \in C_{ee}$:

(i) $P((x|x)_H a\Delta(y)) = xP(a)y$
(ii) $\partial(xry) = x\partial(r)y$
(iii) $\partial(r)s = r\partial(s)$
(iv) $x(r + s) = xr + xs$
(v) $r(x + y) = rx + ry$
(vi) $(x + y)r = xr + yr + P((y|x)_H H\partial(r))$
(vii) $(r + s)x = rx + sx + P((s|r)_{H\partial} H(x))$.

If C is a qpa, then $C_{(0)}$ is a quadratic ring and the multiplication on C_0 yields the multiplication on $h_0(C)$ which equips $h_0(C)$ with a structure of a ring. Moreover $h_1(C)$ is a bimodule over $h_0(C)$.

A qpm together with a crossed biobject structure in the monoidal category $(\mathsf{SG}, \square)$ is called a *crossed square ring* (shortly csr). Thus a csr is a qpm C together with a square ring structure on $C_{(0)}$ and a two-sided action of C_0 on C_1, which is associative and unital and such that the following identities hold for all $x, y, z, t \in C_0$, $r, s \in C_1$, $a, b \in C_{ee}$:

(i) $P((\bar{x} \otimes \bar{x}) \cdot a \cdot y) = x \cdot P(a) \cdot y$
(ii) $\partial(x \cdot r \cdot y) = x \cdot \partial(r) \cdot y$
(iii) $\partial(r)s = r\partial(s)$
(iv) $x(r + s) = xr + xs$
(v) $r(x + y) = rx + ry$

(vi) $(x+y)r = xr + yr + P((\bar{x}\otimes\bar{y})\cdot H\partial r)$
(vii) $(r+s)x = rx + sx + P((\bar{\partial} r\otimes\bar{\partial} s)\cdot Hx)$.

In a crossed square ring the quotient $R = \mathsf{Coker}(\partial)$ has a ring structure and $\mathsf{Ker}(\partial)$ is a bimodule over R. We denote by Csr the category of crossed square rings.

Let R be a ring and M be a bimodule over R. A *quadratic ring extension* (resp. *crossed square ring extension*) of R by M is an exact sequence

$$0 \to M \xrightarrow{i} C_{(1)} \xrightarrow{(\partial,\mathsf{Id})} C_{(0)} \xrightarrow{p} R \to 0$$

where $(\partial,\mathsf{Id}) : C_{(1)} \to C_{(0)}$ is a qpa (resp. csr), the induced homomorphisms $p : \mathsf{Coker}(\partial) \to R$ is an isomorphism of rings and the induced homomorphism $i : M \to \mathsf{Ker}(\partial)$ is an isomorphism of bimodules over $\mathsf{Coker}(p)$. Here M is considered as a bimodule over $\mathsf{Coker}(\partial)$ via the isomorphism $p : \mathsf{Coker}(\partial) \to R$.

Using the definition of the products $\square$ and $\underline{\circledcirc}$ from [10], resp. [8], one readily checks:

Lemma 2.4.1. *A crossed extension of R by M in $(\mathsf{SG},\square)$ is isomorphic to a crossed square ring extension. A crossed extension of R by M in $(\mathsf{SG},\underline{\circledcirc})$ is isomorphic to a quadratic ring extension.*

Hence we have explicitly described the objects in the category of the main theorem 2.2.1.

Example 2.4.2. Let Q be a square ring. One can consider the quotient $Q_{ee}/(\mathsf{Id}-T)$, where as usual $T = HP - \mathsf{Id}$. Let $\tilde{P} : Q_{ee} \to Q_{ee}/(\mathsf{Id}-T)$ be the canonical projection. It is clear that the homomorphism $P : Q_{ee} \to Q_e$ factors through $Q_{ee}/(\mathsf{Id}-T)$. We denote by $\partial : Q_{ee}/(\mathsf{Id}-T) \to Q_e$ the quotient map. Then

$$\begin{tikzcd}
& Q_{ee} \arrow[dl, "\tilde{P}"'] \\
Q_{ee}/(\mathsf{Id}-T) \arrow[r, "\partial"'] & Q_e \arrow[u, "H"']
\end{tikzcd}$$

is a crossed square ring. Thus Theorem 2.2.1 assigns to any square ring Q an element in $\mathsf{HML}^3(Q^{\mathrm{ad}}, Q^{re})$, where $Q^{re} = \mathsf{Ker}(\partial : Q/(\mathsf{Id}-T) \to Q_e)$. In particular, for the square ring $\mathbb{Z}_{\mathrm{nil}}$ one obtains the following crossed square ring

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dl, "1"'] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[u, "H"']
\end{tikzcd}$$

where $H(x) = \frac{x^2-x}{2}$ which defines an element of $\mathsf{HML}^3(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, which is actually the generator.

2.5. **The homomorphism $\nu : \mathsf{HML}^3(R,M) \to \mathsf{H}^0(R, {}_2M)$.** Let R be a ring and let M be a bimodule over R. Take a crossed square ring extension (∂) of R by M

$$0 \longrightarrow M \xrightarrow{\ i\ } C_{(1)} \xrightarrow{(\partial,\mathsf{Id})} C_{(0)} \xrightarrow{\ p\ } R \longrightarrow 0$$

and set

$$\upsilon(w) := PH(2)$$

Since $H(1) = 0$ it follows that $\partial PH(2) = \partial P(1 \mid 1)_H = 0$. On the other hand $2PH(2) = PH\partial PH(2) = 0$. Thus $\upsilon(w) \in {}_2M$. Actually

$$\upsilon(w) \in \mathsf{H}^0(R, {}_2M)$$

and ν yields a well-defined map

$$\nu : \mathrm{Xext}_L(R; M)^{\mathsf{SG},\square} \to \mathsf{H}^0(R; {}_2M).$$

Lemma 2.5.1. *Kernel of ν coincides with the image of*

$$\mathrm{Xext}(R; M)^{\mathsf{Ab},\otimes} \to \mathrm{Xext}_L(R; M)^{\mathsf{SG},\square}.$$

In fact we obtain the lemma directly by the exact sequence in section 1 and the bijections (2.1.4) and 2.2.1. We snow show the lemma more directly in terms of crossed extensions.

Proof. If ∂ is a crossed ring extension then $C_{ee} = 0$ and a fortiori $H = 0$, thus $\nu(\partial) = 0$. Conversely, assume (∂) is a crossed square ring extension with $\nu(\partial) = 0$. Without loss of generality one can assume that $C_{(0)}$ is a monoid square ring $C_{(0)} = \mathbb{Z}_{\mathrm{nil}}[L]$ (see Section 4.3 below). In this case $C_{ee} = \mathbb{Z}[L] \otimes \mathbb{Z}[L]$ and $H(2) = (1 \mid 1)_H = 1 \otimes 1$. The equality $P((\bar{x} \otimes \bar{x} \cdot m \cdot y) = x \cdot P(m) \cdot y$ shows that P factors through $\Lambda^2(\mathbb{Z}[L])$. Thus one gets the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & \Lambda^2(\mathbb{Z}[L]) \arrow[r, "\mathrm{Id}"] \arrow[d, "\tilde{P}"] & \Lambda^2(\mathbb{Z}[L]) \arrow[d, "{[-,-]}"] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"'] & C_1 \arrow[r, "\partial"] \arrow[d] & C_0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\mathsf{Id}"'] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathsf{Coker}\,\tilde{P} \arrow[r] \arrow[d] & \mathbb{Z}[L] \arrow[r] \arrow[d] & R \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Since C_0 is a free nil_2-group on L the commutator map $[-,-]$ is a monomorphism. It follows that $\tilde{P}$ is also a monomorphism, the bottom row is exact and (∂) is equivalent to

$$0 \to M \to \mathsf{Coker}\,\tilde{P} \to \mathbb{Z}[L] \to R \to 0$$

which is a crossed ring extension in $(\mathsf{Ab}, \otimes)$. $\square$

2.6. **Application to ring spectra.** Since Mac Lane cohomology and topological Hochschild cohomology are isomorphic for discrete rings it follows that for any ring spectrum Λ with $\pi_i(\Lambda) = 0$ for $i \neq 0, 1$ there is a well-defined element $k(\Lambda) \in \mathsf{HML}^3(\pi_0(\Lambda), \pi_1(\Lambda))$ known as the first Postnikov invariant (see [21]) and any element in this group comes in this way. Thus linearly generated crossed square rings and quadratic pair algebras can be used to model such ring spectra. The explicit functor from the category of crossed square rings to the category of ring spectra can be constructed as follows. By Corollary 3.7.2 below one can associate to any crossed square ring an internal groupoid in the category of square rings and hence an internal groupoid in the category of algebraic theories (see 3.6 below). Now using the nerve construction one obtains a simplicial object in the category of algebraic theories. Then one can use the well-known construction of Schwede [36] to obtain a ring spectrum in a functorial way.

3. Recollections

3.1. **Preliminaries on double categories and internal categories.** Let $\mathbb{A}$ be a category with finite limits. An *internal category* C in $\mathbb{A}$ consists of the following data: objects C_0 (*object of objects*), C_1 (*object of morphisms*) and morphisms $s, t : C_1 \to C_0$ (*source* and *target*), $i : \mathbb{C}_0 \to \mathbb{C}_1$ (*identity*), $m : C_2 \to C_1$ (*composition*) satisfying associativity and unitality conditions. Here C_2 is defined by the pullback diagram

$$\begin{tikzcd} G_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] & G_1 \arrow[d, "s"] \\ G_1 \arrow[r, "t"'] & G_0 \end{tikzcd}$$

We denote by $\mathsf{Cat}(\mathbb{A})$ the category of internal categories in $\mathbb{A}$. Let us also recall that an internal category C is called an *internal groupoid* provided the diagram

$$\begin{tikzcd} G_2 \arrow[r, "m"] \arrow[d, "p_2"'] & G_1 \arrow[d, "s"] \\ G_1 \arrow[r, "s"'] & G_0 \end{tikzcd}$$

is a pullback diagram. We denote by $\mathsf{Gpd}(\mathbb{A})$ the category of internal groupoids in $\mathbb{A}$.

An internal category in the category of sets Sets is nothing but a small category, while a groupoid object in the category of sets Sets is a groupoid. We write Cat and Gpd instead of $\mathsf{Cat}(\mathsf{Sets})$ and $\mathsf{Gpd}(\mathsf{Sets})$.

Let A be an object of $\mathbb{A}$, then we can consider the internal groupoid A^{dis} with $(A^{dis})_0 = A = (A^{dis})_1$ and $s = t = \mathsf{Id}_A$. An internal category is called *discrete* if it is isomorphic to A^{dis} for some A. We will need also an internal groupoid A^{adis} with $(A^{adis})_0 = A$, $(A^{adis})_1 = A \times A$, where s and t are the projections. An internal category is called *antidiscrete* if it is isomorphic to A^{adis} for some A.

Let $\mathbb{B}$ be a category with finite limits and let $F : \mathbb{A} \to \mathbb{B}$ be a functor which preserves finite limits. Then obviously F yields functors $\mathsf{Cat}(\mathbb{A}) \to \mathsf{Cat}(\mathbb{B})$ and $\mathsf{Gpd}(\mathbb{A}) \to \mathsf{Gpd}(\mathbb{B})$ which will be also denoted by F.

Let us recall that a *double category* is an internal category in the category Cat of small categories. Let $\mathbb{D}$ be a double category with the object category $\mathbb{D}_0$ and morphism category $\mathbb{D}_1$.

We have a functor $\mathsf{Ob} : \mathsf{Cat} \to \mathsf{Sets}$, which assigns to a category $\mathbb{C}$ the set of objects of $\mathbb{C}$. Since Ob preserves inverse limits, for any double category $\mathbb{D}$ we obtain a category $O(\mathbb{D})$, whose morphisms are objects of $\mathbb{D}_1$ and objects are objects of $\mathbb{D}_0$. A double category $\mathbb{D}$ is a *2-category* if $O(\mathbb{D})$ is a discrete category. Equivalently a 2-category is a category enriched in the category Cat. Let us recall how one gets such an enrichment.

Let $\mathbb{D}$ is a 2-category. Then objects of the category $\mathbb{D}_0$ are called simply objects of $\mathbb{D}$, while morphisms of the category $\mathbb{D}_0$ are called simply morphisms of $\mathbb{D}$. Let $f, g : A \to B$ be morphisms of $\mathbb{D}$. Then A and B are also objects in $\mathbb{D}_1$ and we can consider the set of all morphisms $\alpha : A \to B$ in $\mathbb{D}_1$ such that $s(\alpha) = f$ and $t(\alpha) = g$. Such an α is called a 2-morphism from f to g. Thus for objects A and B we have a category $\mathbb{D}(A, B)$ with objects morphisms from A to B in the category $\mathbb{D}_0$ and morphisms from $f : A \to B$ to $g : A \to B$ being all 2-morphisms from f to g.

Conversely, if $\mathbb{B}$ is a a category enriched in the category Cat, then one can consider the following categories $\mathbb{B}_0$ and $\mathbb{B}_1$. The category $\mathbb{B}_0$ has the same objects as $\mathbb{B}$, while morphisms in $\mathbb{B}_0$ are 1-arrows of $\mathbb{B}$. The category $\mathbb{B}_1$ has the same objects as $\mathbb{B}_0$. The morphisms $A \to B$ in $\mathbb{B}_1$ are 2-arrows $\alpha : f \Rightarrow f_1$ where $f, f_1 : A \to B$ are 1-arrows in $\mathbb{B}$. Composition in $\mathbb{B}_1$ is given by $(\beta : x \Rightarrow x_1)(\alpha : f \Rightarrow f_1) := (\beta\alpha : xf \Rightarrow x_1 f_1)$, where

$$\beta\alpha = \beta f_1 + x\alpha = x_1\alpha + \beta f.$$

One furthermore has the source and target functors

$$\begin{tikzcd} \mathbb{B}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbb{B}_0 \end{tikzcd} ,$$

with $s(\alpha : f \Rightarrow f_1) = f$, $t(\alpha : f \Rightarrow f_1) = f_1$, and the "identity" functor $i : \mathscr{T}_0 \to \mathscr{T}_1$ assigning to an 1-arrow f the identity 2-arrow $0_f : f \Rightarrow f$. One easily sees that in this way we obtain a double category such that after applying the functor $\mathsf{Ob} : \mathsf{Cat} \to \mathsf{Sets}$ one gets a discrete category.

3.2. **Preliminaries on Baues-Wirsching cohomology of small categories.** For a small category $\mathscr{C}$ we denote by $\mathscr{F}\mathscr{C}$ the *category of factorizations* of $\mathscr{C}$ [14]. Objects of $\mathscr{F}\mathscr{C}$ are morphisms of $\mathscr{C}$, and a morphism from $\alpha : x \to y$ to $\beta : u \to v$ is a pair $(\nu : u \to x, \psi : y \to v)$ of morphisms in $\mathscr{C}$ such that $\beta = \psi\alpha\nu$, that is, one has a commutative diagram

$$\begin{tikzcd} x \arrow[r, "\alpha"] & y \arrow[d, "\psi"] \\ u \arrow[u, "\nu"] \arrow[r, "\beta"] & v \end{tikzcd}$$

Composition in $\mathscr{F}\mathscr{C}$ is defined by $(\nu, \psi)(\nu', \psi') = (\nu'\nu, \psi\psi')$. A *natural system* on $\mathscr{C}$ is a covariant functor $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$. Now, following [14], one defines the cohomology $\mathsf{H}^*(\mathscr{C}, D)$ as the cohomology of the cochain complex $\mathsf{F}^*(\mathscr{C}, D)$ given by

$$\mathsf{F}^n(\mathscr{C}, D) = \prod_{c_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} c_n} D_{\alpha_1 \cdots \alpha_n}$$

with the coboundary map

$$d : \mathsf{F}^n(\mathscr{C}, D) \to \mathsf{F}^{n+1}(\mathscr{C}, \mathbb{D})$$

given by

$$\begin{aligned} (df)(\alpha_1, \cdots, \alpha_{n+1}) \quad &= (\alpha_1)_* f(\alpha_2, \cdots, \alpha_{n+1}) \\ &+ \textstyle\sum_{i=1}^n (-1)^i f(\alpha_1, \cdots, \alpha_i\alpha_{i+1}, \cdots, \alpha_{n+1}) \\ &+ (-1)^{n+1} (\alpha_{n+1})^* f(\alpha_1, \cdots, \alpha_n). \end{aligned}$$

Here, and in the rest of the paper, we use the following convention. For a diagram $u \xrightarrow{\beta} x \xrightarrow{\alpha} y \xrightarrow{\gamma} v$ and elements $a \in D_\beta$, $b \in D_\gamma$, we write $\alpha_* a$ and $\alpha^* b$ for the image of the elements a and b under the homomorphisms $D(id_u, \alpha) : D_\beta \to D_{\alpha\beta}$ and $D(\alpha, id_v) : D_\gamma \to D_{\gamma\alpha}$ respectively.

We also need the relative cohomologies of small categories. Let $p : \mathscr{K} \to \mathscr{C}$ be a functor which is identity on objects and surjective on morphisms. Let $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ be a natural system on $\mathscr{C}$. We have an induced natural system $p^* D$ on $\mathscr{K}$ given by $g \mapsto D_{pg}$, which we will, abusing notation, still denote by D. Then p yields a monomorphism of cochain complexes $\mathsf{F}^*(\mathscr{C}, D) \to \mathsf{F}^*(\mathscr{K}, D)$. We let $\mathsf{F}^*(\mathscr{C}, \mathscr{K}; D)$ be the cokernel of this homomorphism. The *n-th dimensional relative cohomology* $\mathsf{H}^n(\mathscr{C}, \mathscr{K}; D)$ is defined as the $(n-1)$-th homology of the cochain complex $\mathsf{F}^*(\mathscr{C}, \mathscr{K}; D)$. Then one has an exact sequence

$$0 \to \mathsf{H}^0(\mathscr{C}, D) \to \mathsf{H}^0(\mathscr{K}, D) \to \mathsf{H}^1(\mathscr{C}, \mathscr{K}; D) \to \cdots \to$$
$$\to \mathsf{H}^n(\mathscr{C}, D) \to \mathsf{H}^n(\mathscr{K}, D) \to \mathsf{H}^{n+1}(\mathscr{C}, \mathscr{K}; D) \to \cdots .$$

We have a functor $\mathscr{F}\mathscr{C} \to \mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ which sends an arrow $\alpha : c \to d$ to the pair (c, d). This functor allows us to conclude that any bifunctor gives rise to a natural system. Thus for any bifunctor $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ we have well-defined cohomology groups $H^*(\mathscr{C}, D)$. Among many equivalent definitions of the Mac Lane cohomology [19] for our purposes the most convenient is via the Baues-Wirsching cohomology of small categories [14]. Let R be a ring. Let $\mathbf{Mod}\text{-}R$ be the category of right R-modules and let $\mathbf{mod}\text{-}R$ be the full subcategory of finitely generated free right R-modules. To avoid set-theoretic complications we will assume that objects of $\mathbf{mod}\text{-}R$ are natural numbers and morphisms from y to x, $x, y \in \mathbb{N}$ are $(x \times y)$-matrices with entries in R. We write $f = (f_i^k)$ for a morphism $y \to x$, where $f_i^k \in R, 1 \leqslant i \leqslant x, 1 \leqslant k \leqslant y$.

For an R-R-bimodule M, we denote by $D_M : (\mathbf{Mod}\text{-}R)^{\mathrm{op}} \times \mathbf{Mod}\text{-}R \to \mathsf{Ab}$ the bifunctor given by

$$D_M(X, Y) := \mathsf{Hom}_R(X, Y \otimes_R M), \quad X, Y \in \mathbf{Mod}\text{-}R.$$

Now one defines the *Mac Lane cohomology of R with coefficients in M* by

$$\mathsf{HML}^*(R, M) := \mathsf{H}^*(\mathbf{mod}\text{-}R, D_M).$$

We refer to [19] and Chapter 13 of [23] for relationship between different definitions of Mac Lane cohomology. We use this definition of HML^* also if R is a square ring or a quadratic ring.

3.3. **Third Baues-Wirsching cohomology and linear track extensions.** We recall the relationship between third Baues-Wirsching cohomology and linear track extensions. We start with recalling the definition of track categories.

A *track category* is a groupoid enriched category, i. e. a 2-category such that all of its 2-morphisms are invertible. Equivalently a track category $\mathscr{T}$ is an internal groupoid in the category Cat such that $O(\mathscr{T})$ is a discrete category. We will use the following notation for track categories. Composition of morphisms will be denoted by juxtaposition; for 2-arrows we will use additive notation, so composition is $+$ and identity 2-arrows are denoted by 0. The hom-category for objects A, B of a track category will be denoted by $[\![A, B]\!]$. If there is a 2-arrow $\alpha : f \Rightarrow g$ between maps $f, g \in \mathrm{Ob}([\![A, B]\!])$, we will say that f and g are homotopic and write $f \simeq g$. We have the *homotopy category* $\mathscr{T}_\simeq = \mathscr{T}_0 / \simeq$. Objects of $\mathscr{T}_\simeq$ are objects in $\mathrm{Ob}(\mathscr{T})$, while morphisms of $\mathscr{T}_{\simeq}$ are homotopy classes of morphisms in $\mathscr{T}_0$. A map f in $\mathscr{T}$ is called a *homotopy equivalence* if the class of f in $\mathscr{T}_{\simeq}$ is an isomorphism.

Two track categories $\mathscr{T}$, $\mathscr{T}'$ are called *weakly equivalent* if there is an enriched functor $F : \mathscr{T} \to \mathscr{T}'$ which induces equivalences of hom-groupoids $[\![X,Y]\!]_{\mathscr{T}} \to [\![FX,FY]\!]_{\mathscr{T}'}$ and is *essentially surjective*, i. e. any object of $\mathscr{T}'$ is homotopy equivalent to one of the form FX.

Let $\mathscr{C}$ be a small category and let D be a natural system on $\mathscr{C}$. A *linear track extension of $\mathscr{C}$ by D* denoted by

$$0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \to \mathscr{C} \to 0$$

is a pair $(\mathscr{T},\tau)$. Here $\mathscr{T}$ is a track category equipped with a functor $q : \mathscr{T}_0 \to \mathscr{C}$ which is full and identity on objects. In addition for maps f,g in $\mathscr{T}_0$ we have $q(f) = q(g)$ iff $f \simeq g$. In other words the functor q identifies $\mathscr{C}$ with $\mathscr{T}_{\simeq}$. Furthermore, for each map $f : A \to B$ in $\mathscr{T}_0$ there is given an isomorphism of groups $\tau_f : D_{qf} \to \mathscr{T}(f,f)$, such that for any $\xi : f \Rightarrow g$ and $a \in D_{qf} = D_{qg}$ one has

$$\xi + \tau_f(a) = \tau_g(a) + \xi.$$

Furthermore for any diagram $\xrightarrow{e}\xrightarrow{f}\xrightarrow{h}$ additionally one has

$$h_*\tau_f(a) = \tau_{hf}(h_*a),$$

$$e^*\tau_f(a) = \tau_{fe}(e^*a).$$

For a category $\mathscr{C}$ and a natural system $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ we denote by $\mathsf{Tracks}(\mathscr{C},D)$ the category of all linear track extensions of $\mathscr{C}$ by D, where the morphisms are the obvious ones.

Linear track extensions of categories were first described in the preprint of [3] and the following theorem in a slightly different terminology first was proved in [28] (see also [29]) and was proved by different methods in [4].

Theorem 3.3.1. [28] *For a small category $\mathscr{C}$ and a natural system $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ there exists a natural bijection between the set of connected components of the category $\mathsf{Tracks}(\mathscr{C},D)$ and third cohomology:*

$$\pi_0(\mathsf{Tracks}(\mathscr{C},D)) \cong \mathsf{H}^3(\mathscr{C},D).$$

The proof of Theorem 3.3.1 given in [28] and [29] is based on the following Theorem 3.3.2, which is going to be crucial in this paper as well.

Let $p : \mathscr{K} \to \mathscr{C}$ be a functor which is identity on objects and surjective on morphism. Let $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ be a natural system on $\mathscr{C}$. We denote by $\mathsf{Tracks}(\mathscr{C},\mathscr{K};D)$ the subcategory of $\mathsf{Tracks}(\mathscr{C},D)$ whose objects are track categories $\mathscr{T}$ satisfying $\mathscr{T}_0 = \mathscr{K}$,

$$0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{K} \xrightarrow{q} \mathscr{C} \to 0,$$

whereas morphisms are those morphisms in $\mathsf{Tracks}(\mathscr{C},D)$ which are identity on $\mathscr{K}$.

Theorem 3.3.2. [28],[29] *For a small category $\mathscr{C}$, a bifunctor $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ and a functor $p : \mathscr{K} \to \mathscr{C}$ which is identity on objects and surjective on morphisms, the category $\mathsf{Tracks}(\mathscr{C},\mathscr{K};D)$ is a groupoid and there exists a natural bijection*

$$\pi_0(\mathsf{Tracks}(\mathscr{C},\mathscr{K};D)) \cong \mathsf{H}^3(\mathscr{C},\mathscr{K};D).$$

3.4. **Relative track extensions of algebraic theories.** An *algebraic theory* is a category with finite coproducts. A *morphism of algebraic theories* is a functor preserving finite coproducts. We denote the coproduct by $\vee$. Let $\mathscr{C}$ be an algebraic theory. A natural system $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ is called *cartesian* if for any arrow $f : c = c_1 \vee \cdots \vee c_n \to d$ the natural map

$$D_f \to D_{f_1} \times \cdots \times D_{f_n},$$

given by $x \mapsto ((i_1)^* x, \cdots, (i_n)^* x)$, is an isomorphism. Here $i_k : c_k \to c$ is the standard inclusion and $f_k = i_k \circ f : c_k \to d$. For example if $D : \mathscr{C} \times \mathscr{C}^{\mathrm{op}} \to \mathsf{Ab}$ is a bifunctor such that for all c, d and x from $\mathscr{C}$ one has an isomorphism

$$D(c \vee d, x) \cong D(c, x) \times D(d, x)$$

natural in c, d and x, then the natural system corresponding to D is cartesian.

Let $\mathscr{C}$ be an algebraic theory and let $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ be a cartesian natural system. A track extension

$$0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \xrightarrow{p} \mathscr{C} \to 0$$

is called a *track extension of algebraic theories* if $\mathscr{T}_0$ is an algebraic theory and the functor p is a morphism of algebraic theories.

Lemma 3.4.1. *Let $\mathscr{C}$ be an algebraic theory and let $D : \mathscr{F}\mathscr{C} \to \mathsf{Ab}$ be a cartesian natural system. Let $0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \xrightarrow{p} \mathscr{C} \to 0$ be a track extension of algebraic theories. Then $\mathscr{T}_1$ is also an algebraic theory and $s, t : \mathscr{T}_1 \to \mathscr{T}_0$ are morphisms of algebraic theories.*

Proof. Let $\alpha : f \Rightarrow g$ and $\alpha' : f' \Rightarrow g'$ be tracks, where $f, g : A \to B$ and $f', g' : A' \to B$ are 1-morphisms. We have to show that there is a unique track $(\alpha, \alpha') : (f, g) \Rightarrow (f', g')$ such that $i_A^*(\alpha, \alpha') = \alpha$ and $i_{A'}^*(\alpha, \alpha') = \alpha'$, where $(f, g) : A \vee A' \to B$ is the unique 1-morphism with $i_A(f, g) = f$ and $i_{A'}(f, g) = g$. Here $i_A : A \to A \vee B$ is the canonical inclusion and similarly for $i_{A'}$. First we show the existence of such a track. By assumption $f \simeq g$ and $f' \simeq g'$. Since p preserves finite coproducts, it follows that $(f, f') \simeq (g, g')$. Hence there exists a track $\eta : (f, f') \Rightarrow (g, g')$. Since $i_A^*(\eta) : f \Rightarrow g$, there exists a unique element $x \in D_{pf}$ such that $i_A^*(\eta) = \alpha + \sigma_f(x)$. Similarly there exists a unique element $x' \in D_{pf'}$ such that $i_{A'}^*(\eta) = \alpha' + \sigma_{f'}(x')$. By our assumptions there is a unique element $y \in D_{(pf,pf'}$ such that $i_A(y) = x$ and $i_{A'}(y) = x'$. Then the track $\xi = \eta - \sigma_{(f,f')}(y)$ satisfies the condition required. To prove uniqueness one observes that if ξ and η both satisfy the condition, then they will differ by an element $z \in D_{(pf,pf')}$, whose restrictions to D_{pf} and $D_{pf'}$ are zero, hence it is itself zero and the lemma follows. □

Lemma 3.4.2. *Let $0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \xrightarrow{p} \mathscr{C} \to 0$ be a track extension of algebraic theories and let $\nu : X \to X \vee X$ be an internal cogroup in $\mathscr{T}_0$. Then X is also a cogroup in $\mathscr{T}_1$, where the cogroup structure is given by the morphism $0 : \nu \Rightarrow \nu$.*

Proof. By Lemma 3.4.1 the "identity functor" $\mathscr{T}_0 \to \mathscr{T}_1$ respects finite coproducts and therefore carries a cogroups to cogroups. □

3.5. **Quadratic functors, quadratic categories and square objects.** We now recall the relationship between square groups and quadratic functors. We consider endofunctors $F : \mathsf{Gr} \to \mathsf{Gr}$ of the category of groups with $F(0) = 0$. Additionally we assume that F preserves filtered colimits and reflexive coequalizers. The last condition means that for any simplicial group G_* the canonical homomorphism

$\pi_0(F(G_*)) \to F(\pi_0(G_*))$ is an isomorphism. Such a functor F is completely determined by the restriction of F to the subcategory of finitely generated free groups.

The second cross-effect $F(X|Y)$ of F is a bifunctor defined via the short exact sequence

$$0 \to F(X|Y) \to F(X \vee Y) \to F(X) \times F(Y) \to 0.$$

Here $\vee$ denotes the coproduct in the category of groups and the last map is induced by the canonical projections: $r_1 = (\mathsf{Id}_X, 0) : X \vee Y \to X$ and $r_2 = (0, \mathsf{Id}_Y) : X \vee Y \to Y$. A functor F is called *linear* if the second cross effect vanishes. Moreover, F is called *quadratic* if $F(X|Y)$ is linear in X and Y. Let $\mathsf{lin}(\mathsf{Gr})$ (resp. $\mathsf{Quad}(\mathsf{Gr})$) be the category of such linear (resp. quadratic) endofunctors. Any endofunctor in $\mathsf{lin}(\mathsf{Gr})$ is isomorphic to a functor T of the form $T(X) = A \otimes X_{\mathrm{ab}}$ where A is an abelian group. Therefore there is an equivalence of categories

$$\mathsf{lin}(\mathsf{Gr}) \simeq \mathsf{Ab}.$$

Let $F : \mathsf{Gr} \to \mathsf{Gr}$ be a quadratic functor. We associate with F a square group $\mathsf{cro}(F)$ as follows. We put

$$\mathsf{cro}(F)_e = F(\mathbb{Z}), \quad \mathsf{cro}(F)_{ee} = F(\mathbb{Z} \mid \mathbb{Z}).$$

The homomorphism P of the square group $\mathsf{cro}(F)$ is the restriction of the homomorphism $(\mathsf{Id}, \mathsf{Id})_* : F(\mathbb{Z} \vee \mathbb{Z}) \to F(\mathbb{Z})$. We denote by e_1 and e_2 the canonical free generators of $\mathbb{Z} \vee \mathbb{Z}$. The map H is given by

$$H(x) = \mu_*(x) - p_2(\mu_* x) - p_1(\mu_* x)$$

Here $\mu : \mathbb{Z} \to \mathbb{Z} \vee \mathbb{Z}$ is the unique homomorphism which sends 1 to $e_1 + e_2$, while p_1 and p_2 are endomorphisms of $\mathbb{Z} \vee \mathbb{Z} \to \mathbb{Z} \vee \mathbb{Z}$ such that $p_i(e_i) = e_i$, $i = 1, 2$ and $p_i(e_j) = 0$, if $i \neq j$.

The main result of [10] claims that the functor

$$\mathsf{cro} : \mathsf{Quad}(\mathsf{Gr}) \to \mathsf{SG}$$

is an equivalence of categories. Under this equivalence square rings corresponds to monads on the category of groups, whose underlying functors lie in $\mathsf{Quad}(\mathsf{Gr})$.

Let $\mathbb{C}$ be an algebraic theory with zero object 0. We will say that $\mathbb{C}$ is equipped with a structure of *quadratic theory* if each object C in $\mathbb{C}$ is equipped with a cogroup structure $\nu_C : C \to C \vee C$ and the functor $\mathbb{C}(C, -) : \mathbb{C} \to \mathsf{Gr}$ is quadratic. Thus for all X and Y in $\mathbb{C}$ one has the following short exact sequence of groups

$$0 \to \mathbb{C}(C; X \mid Y) \to \mathbb{C}(C, X \vee Y) \to \mathbb{C}(C, X) \times \mathbb{C}(C, Y) \to 0$$

and $\mathbb{C}(C; X \mid Y)$ is linear in X and Y. This definition is equivalent but not identical to the one given in [5].

Lemma 3.5.1. *Let $\mathscr{C}$ be an additive category, $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ be a biadditive bifunctor and*

$$0 \to D \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \xrightarrow{p} \mathscr{C} \to 0$$

be a linear track extension. If $\mathscr{T}_0$ is a quadratic theory and p preserves finite coproducts, then $\mathscr{T}_1$ is also a quadratic theory.

Proof. By Lemma 3.4.1 the category $\mathscr{T}_1$ is an algebraic theory. It is quite easy to show that the zero object in $\mathscr{T}_0$ remains also a zero object in $\mathscr{T}_1$. By Lemma 3.4.2 any object in $\mathscr{T}_1$ has a canonical cogroup structure. We claim that

$$\mathscr{T}_1(X, Y \mid Z) \cong \mathscr{T}_0(X, Y \mid Z) \times \mathscr{T}_0(X, Y \mid Z),$$

which implies that $\mathscr{T}_1(X,-)$ is a quadratic functor and hence Lemma. To prove the claim we put

$$r_1 = (\mathsf{Id}, 0) : Y \vee Z \to Y \quad \text{and} \quad r_2 = (0, \mathrm{Id}) : Y \vee Z \to Z.$$

By definition of the cross-effect $\mathscr{T}_0(X, Y \mid Z)$ consists of 1-morphisms $f : C \to Y \vee Z$ such that $r_1 f = 0 = r_2 f$. On the other hand $\mathscr{T}_1(X, Y \mid Z)$ consists of tracks $\alpha : f \Rightarrow g$ such that $r_1 f = 0 = r_2 f$, $r_1 g = 0 = r_2 g$ and $r_{1*}\alpha = 0 = r_{2*}\alpha$. Thus our claim is equivalent to the following: suppose $f, g : X \to Y \vee Z$ are 1-morphisms such that $r_1 f = 0 = r_2 f$, $r_1 g = 0 = r_2 g$. Then there exists a unique track $\alpha : f \Rightarrow g$ such that $r_{1*}\alpha = 0 = r_{2*}\alpha$. If α and β both satisfy the assertion, then $\beta = \alpha = \sigma_f(x)$ with $x \in D(X, Y \vee Z) = D(X,Y) \oplus D(X,Z)$. The conditions $r_{1*}\alpha = 0 = r_{2*}\alpha = r_{1*}\beta = r_{2*}\beta$ show that $x = 0$. Hence we proved uniqueness. Now we prove the existence. Since $\mathscr{C}$ is an additive category, p respects coproducts and $r_1 f = 0 = r_2 f$ it follows that $p(f) = 0$ in $\mathscr{C}$. Similarly $p(g) = 0$. In particular $f \simeq g$. Thus there exist a track $\xi : f \Rightarrow g$. Then $r_{1*}(\xi) : 0 \Rightarrow 0$. Hence there exists a unique $y \in D(X,Y)$ such that $r_{1*}(\xi) = \sigma_0(y)$. Similarly $r_{2*}(\xi) = \sigma_0(z)$ for uniquely defined $z \in D(X,Z)$. Since D is biadditive, we have $(y,z) \in D(X, Y \vee Z)$. It is clear that $\alpha = \xi - \sigma_f(y,z)$ satisfies the conditions of the claim. □

3.6. **Square rings and single sorted quadratic theories.** We recall the relationship between square rings and quadratic categories [5]. A quadratic theory is a *single sorted quadratic theory* if the objects of $\mathbb{C}$ are natural numbers and the coproduct on objects corresponds to the addition of natural numbers. Thus each object $\mathbf{n}$ in $\mathbb{C}$ is an n-fold coproduct of $\mathbf{1}$. We additionally require that the cogroup structure on $\mathbf{n}$ is the n-fold coproduct of the cogroup structure on $\mathbf{1}$.

Assume $\mathbb{C}$ is a single sorted quadratic theory. Then one has the square ring $\mathsf{cro}(\mathbb{C})$ with

$$\mathsf{cro}(\mathbb{C})_e = \mathsf{cro}(\mathbb{C}(\mathbf{1}, -))_e = \mathbb{C}(\mathbf{1}, \mathbf{1})$$

and

$$\mathsf{cro}(\mathbb{C})_{ee} = \mathsf{cro}(\mathbb{C}(\mathbf{1}, -))_{ee} = \mathbb{C}(\mathbf{1}; \mathbf{1} \mid \mathbf{1}).$$

The main result of [5] shows that the functor cro from the category of single sorted quadratic theories to the category of square rings is an equivalence of categories. The inverse functor is given by $Q \mapsto \mathbf{mod}\text{-}Q$. Here the objects of the category $\mathbf{mod}\text{-}Q$ are natural numbers, while morphisms from y to x, $x, y \in \mathbb{N}$ are defined by product sets

$$\mathsf{Mor}(y, x) := \Big(\prod_{k=1}^{y} \prod_{i=1}^{x} Q_e\Big) \times \Big(\prod_{k=1}^{y} \prod_{1 \leqslant i < j \leqslant x} Q_{ee}\Big).$$

For a morphism $f : y \to x$ we write $f = (f_i^k, f_{ij}^k)$. If $g = (g_k^s, g_{kl}^s)$ is a morphism $z \to y$, then the composite $fg = ((fg)_i^s, (fg)_{ij}^s)$ is given by

$$(fg)_i^s = f_i^1 \circ g_1^s + \cdots + f_i^y \circ g_y^s + \sum_{k<l} P((\bar f_i^k \otimes \bar f_i^l) \cdot g_{kl}^s)$$

$$(fg)_{ij}^s = \sum_k (f_{ij}^k \cdot \bar g_k^s + \sum_{i<l} ((\bar f_i^k \otimes \bar f_i^l) \cdot g_{kl}^s + (\bar f_i^l \otimes \bar f_j^k) \cdot T g_{kl}^s + \overline{(f_i^l \cdot g_l^s)} \otimes \overline{(f_j^k \cdot g_k^s)} \cdot H(2))$$

Actually $\mathbf{mod}\text{-}Q$ is a single sorted quadratic theory, the group structure on hom's is defined by the formula:

$$(f_i^k, f_{ij}^k) + ({f'}_i^k, {f'}_{ij}^k) = (f_i^k + {f'}_i^k, f_{ij}^k + {f'}_{ij}^k + e_{ij}^k)$$

where
$$e^k_{ij} = (\bar{f}^k_i \otimes \bar{f}'^k_j) \cdot H(2).$$
To get more hints on the category **mod**-Q, we recall that a *right Q-module* [5] is nothing but a right Q-object in the monoidal category $(\mathsf{SG}, \square)$. More explicitly, a right Q-module is a group M together with maps $M \times Q_e \to M$, $(m,x) \to mx$ and $M \times M \times Q_{ee} \to M$, $(m,n,a) \mapsto [m,n]_a$ satisfying the following identities.
$$\begin{aligned}
m1 &= m,\\
(mx)y &= m(xy),\\
m(x+y) &= mx + my,\\
(m+n)x &= mx + nx + [m,n]_{H(a)},\\
mP(a) &= [m,m]_a,\\
[m,n]_{Ta} &= [n,m]_a,\\
[mx,ny]_a &= [m,n]_{(x\otimes y)a},\\
[[m,n]_a, z]_b &= 0.
\end{aligned}$$
Moreover $[m,n]_a$ is linear in m,n and a and lies in the center of M. We denote by **Mod**-Q the category of all right Q-modules. It is a standard fact of universal algebra that the forgetful functor **Mod**-$Q \to \mathsf{Sets}$ has the left adjoint, whose values on a set X is called the free right Q-module generated by the set X. Now one checks directly [5] that the category **mod**-Q is equivalent to the category of finitely generated free right Q-modules.

Let us observe that for $Q = \mathbb{Z}_{\mathrm{nil}}$, the category of right $\mathbb{Z}_{\mathrm{nil}}$-modules is nothing but the category Nil of groups of nilpotence class two. More generally, if S is a monoid then the category of right modules over the square ring $Q = \mathbb{Z}_{\mathrm{nil}}[S]$ is isomorphic to the category of pairs (G, α), where G is a group of nilpotence class two and $\alpha : S \to \mathsf{Hom}(G,G)$ is an action of S on G via group homomorphisms.

3.7. **Internal groupoids and crossed objects.** We describe now internal groupoids in the category of square groups. Actually results obtained in this section are very particular case of much more general results of Gogi Janelidze [17].

Let A and G be square groups. An *action of G on A* is a homomorphism of abelian groups $\xi : A^{\mathrm{ad}} \otimes G^{\mathrm{ad}} \to A_{ee}$.

In particular we have the action of A on A given by $(-,-)_H$, which is called the *adjoint action of A on itself.*

Let ξ be an action of G on A. The *semi-direct* product of G and A denoted $G \rtimes A$ is a square group defined as follows. As a set $(G \rtimes A)_e$ is the cartesian product $G_e \times A_e$ while the group structure is given by
$$(g,x) + (h,y) = (g+h, x+y+P\xi(g,h)).$$
Moreover one puts
$$\begin{aligned}
(G \rtimes A)_{ee} &= G_{ee} \oplus A_e,\\
P(u,a) &= (Pu, Pa),\\
H(g,x) &= (H(g), H(x) - \xi(x,g)).
\end{aligned}$$
One easily sees that
$$[(g,x),(h,y)] = ([g,h], [x,y] + P\xi(x,h) - P\xi(y,g))$$

and

$$((g,x) \mid (h,y))_H = ((g \mid)_H, (x \mid y)_H + HP\xi(x,h) - P\xi(x,h) - P\xi(y,g)).$$

Based on these identities one easily checks that $(G \rtimes A)_e$ is really a square group and one has the following split short exact sequence of square groups

$$0 \to A \to A \rtimes G \to G \to 0$$

with obvious maps. Conversely, let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} G \to 0$$

be a short exact sequence. Assume $j : G \to B$ is a morphism of square groups with $pj = \mathsf{Id}_G$. Then

$$\xi(x,g) := (i_e(x) \mid j_e(g))_H$$

defines an action of G on A and the maps $f_e(g,x) = j_e(g) + i_e(x)$ and $f_{ee}(g,x) = j_{ee}(g) + i_{ee}(x)$ define an isomorphism $f = (f_e, f_{ee}) : G \rtimes A \to B$ of square groups.

A *crossed square group* is a morphism of square groups $\partial : A \to G$ together with an action of G on A such that ∂ is compatible with the action of G, where G acts on itself via the adjoint action and the action of A on A given via ∂ coincides with the adjoint action of A on itself. In other words a homomorphism $\xi : A^{\mathrm{ad}} \otimes G^{\mathrm{ad}} \to A_{ee}$ of abelian groups is given and the following identity holds

$$\partial_{ee}\xi(x,g) = (\partial_e(x), g)_H,$$

$$\xi(x, \partial_e(y)) = (x,y)_H.$$

We denote by $\mathsf{cross(SG)}$ the category of crossed square groups. The following is a specialization of the main result of [17].

Lemma 3.7.1. *Any internal category in the category of square groups is an internal groupoid. Thus* $\mathsf{Cat(SG)} = \mathsf{Gpd(SG)}$ *and there is an equivalence of categories*

$$\mathsf{Gpd(SG)} \cong \mathsf{cross(SG)}.$$

Proof. The first fact is a general property of so called Maltsev categories [17]. The second part can be proved by modifying the argument of Loday in [22] based on our description of split short exact sequences. Alternatively one can check directly that the above definition is a specialization of the general notion of Janelidze and use the main result of [17] on relationship between internal groupoids and crossed objects in so called semi-abelian categories. The checking is an easy exercise because of the explicit description of the coproduct in the category SG of square groups given in [10]. □

Since the functors $(-)_e : \mathsf{SG} \to \mathsf{Gr}$ and $(-)_{ee} : \mathsf{SG} \to \mathsf{Ab}$ preserve limits any internal groupoid X in SG gives rise to two internal groupoids X_e and X_{ee} in the category of groups and abelian groups respectively.

An internal groupoid $X \in \mathsf{Gpd(SG)}$ is called ee-antidiscrete provided X_{ee} is antidiscrete. We denote by $\mathsf{Gpd}_{adee}\mathsf{(SG)}$ the category of ee-antidiscrete internal groupoids in the category of square groups.

Lemma 3.7.2. *The equivalence* $\mathsf{Gpd(SG)} \cong \mathsf{cross(SG)}$ *restricts to an equivalence of categories*

$$\mathsf{Gpd}_{adee}\mathsf{(SG)} \cong \mathsf{qpm}.$$

Proof. We have to show that qpms are exactly crossed square groups $\partial : A \to G$ for which $\partial_{ee} : A_{ee} \to G_{ee}$ is the identity map. But this is clear, because after identification of A_{ee} and G_{ee} via ∂_e, the action ξ of G on A becomes redundant, $\xi(x, g) = (\partial_e(x) \mid x)_H$. $\square$

Let $\mathsf{Gpd}_{adee}(\mathsf{SR})$ denote the category of ee-antidiscrete groupoid objects in the category of square rings. Lemma 3.7.2 implies the following result.

Lemma 3.7.3. *There is an equivalence of categories*

$$\mathsf{Gpd}_{adee}(\mathsf{SR}) \cong \mathsf{Csr}.$$

4. Proof of the main result

4.1. **Relative Mac Lane cohomology.** Let R be a ring and let M be a bimodule over R. Assume also that a surjective morphism $p : Q \to R$ is given from a square ring Q to R. We denote by $\mathscr{H}\!ext(R, Q; M)^{\mathsf{SG},\square}$ the subcategory of the category $\mathscr{H}\!ext(R, M)^{\mathsf{SG},\square}$ whose objects are crossed square ring extensions of the form

$$0 \longrightarrow M \longrightarrow C_{(1)} \xrightarrow{(\partial,\mathsf{Id})} C_{(0)} \xrightarrow{p} R \longrightarrow 0$$

with $C_{(0)} = Q$. Morphism are such morphisms of crossed square ring extensions which are identity on Q

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"'] & C_{(1)} \arrow[r, "(\partial,\mathsf{Id})"] \arrow[d, "f"] & Q \arrow[r, "p"] \arrow[d, "\mathsf{Id}"] & R \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & M \arrow[r] & C'_{(1)} \arrow[r, "(\partial',\mathsf{Id})"] & Q \arrow[r, "p"] & R \arrow[r] & 0.
\end{tikzcd}$$

Then the category $\mathscr{H}\!ext(R, Q; M)^{\mathsf{SG},\square}$ is a groupoid.

Quite similarly, for a given surjective morphism $p : Q \to R$ from a quadratic ring Q to R, we denote by $\mathscr{H}\!ext(R, Q; M)^{\mathsf{SG},\underline{\circledcirc}}$ the subcategory of the category $\mathscr{H}\!ext(R, M)^{\mathsf{SG},\underline{\circledcirc}}$ whose objects are crossed square ring extensions of the form

$$0 \longrightarrow M \longrightarrow C_{(1)} \xrightarrow{(\partial,\mathsf{Id})} C_{(0)} \xrightarrow{p} R \longrightarrow 0$$

with $C_{(0)} = Q$. Then the category $\mathscr{H}\!ext(R, Q; M)^{\mathsf{SG},\underline{\circledcirc}}$ is a groupoid.

Lemma 4.1.1. *Let R be a ring and let L be a monoid and let $p : \mathbb{Z}_{\mathrm{nil}}[L] \to R$ be a surjective morphism of quadratic rings (and hence also a surjective morphism of square rings). Then for any R-bimodule M the functor $\mathscr{H}\!ext(R, M)^{\mathsf{SG},\underline{\circledcirc}} \to \mathscr{H}\!ext(R, M)^{\mathsf{SG},\square}$ yields an equivalence of categories*

$$\mathscr{H}\!ext(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)^{\mathsf{SG},\underline{\circledcirc}} \xrightarrow{\simeq} \mathscr{H}\!ext(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)^{\mathsf{SG},\square}.$$

Proof. It is straightforward to check that the conditions posed on C_1 and ∂ in the definition of quadratic pair algebra and in the definition of crossed square ring are the same provided $C_{(0)} = \mathbb{Z}_{\mathrm{nil}}[L]$. $\square$

Let us turn back to an epimorphism $Q \to R$ for a square ring R. The set of connected components of $\mathscr{H}\!ext(R, Q; M)^{\mathsf{SG},\square}$ has the following cohomological description. In order to give the precise statement we first extend the definition of the Mac Lane cohomology to square rings and then we introduce the relative cohomology groups.

Let Q be a square ring, then Q^{ad} is a ring, which we denote by R. There is an obvious functor

$$q : \mathbf{mod}\text{-}Q \to \mathbf{mod}\text{-}R$$

which is identity on objects and on morphisms it is given by

$$q((f_i^k, f_{ij}^k)) := (\bar{f}_i^k)$$

For any bimodule M over the ring R we let D_M be the bifunctor on $\mathbf{mod}\text{-}R$ given by

$$(X, Y) \mapsto \mathsf{Hom}_R(X, Y \otimes_R M)$$

By abuse of notation we will denote by D_M also the induced bifunctor on $\mathbf{mod}\text{-}Q$. Then we put

$$\mathsf{HML}^*(Q, M) := H^*(\mathbf{mod}\text{-}Q, D_M)$$

Thanks to Section 3.2 we recover for usual rings the classical Mac Lane cohomology. Using the relative cohomology of small categories defined in Section 3.2 one can also define the relative Mac Lane cohomology groups $\mathsf{HML}^*(R, Q; M)$ to be $H^*(\mathbf{mod}\text{-}R, \mathbf{mod}\text{-}Q; D_M)$. Thus one has the following long exact sequence

$$0 \to \mathsf{HML}^0(R; M) \to \mathsf{HML}^0(Q; M) \to \mathsf{HML}^1(R, Q; M) \to \cdots$$
$$\to \mathsf{HML}^n(R; M) \to \mathsf{HML}^n(Q; M) \to \mathsf{HML}^{n+1}(R, Q; M) \to \cdots .$$

The proof of the isomorphisms in Theorem 2.2.1 is based on a computation given in Appendix and on the following result.

Theorem 4.1.2. *Let $p : Q \to R$ be a surjective morphism from a square ring Q to a ring R. Then*

$$\pi_0(\mathscr{Hext}(R, Q; M)^{\mathsf{SG},\square}) \approx \mathsf{HML}^3(R, Q; M).$$

Proof. Let $\mathsf{Tracks}(\mathbf{mod}\text{-}R, \mathbf{mod}\text{-}Q; D_M)$ denote the category of such abelian track categories $\mathscr{T}$ that the corresponding homotopy category $\mathscr{T}_{\simeq}$ is $\mathbf{mod}\text{-}R$, underlying category $\mathscr{T}_0$ is $\mathbf{mod}\text{-}Q$ and the corresponding natural system is given by the bifunctor D_M. We now construct the functor

$$\chi : \mathscr{Hext}(R, Q; M)^{\mathsf{SG},\square} \to \mathsf{Tracks}(\mathbf{mod}\text{-}R, \mathbf{mod}\text{-}Q; D_M)$$

as follows. Let

$$0 \longrightarrow M \longrightarrow \tilde{Q} \xrightarrow{(w,\mathsf{Id})} Q \xrightarrow{p} R \longrightarrow 0,$$

be a crossed square ring extension. The underlying category of the track category $\chi(\omega)$ is $\mathbf{mod}\text{-}Q$. If $\mathbf{f} = (f_i^k, f_{ij}^k)$ and $\mathbf{g} = (g_i^k, g_{ij}^k)$ are morphisms $y \to x$, $x, y \in \mathbb{N}$ in $\mathbf{mod}\text{-}Q$, then a track $\mathbf{f} \Rightarrow \mathbf{g}$ is a collection (h_i^k) of elements in $\tilde{Q}_e$ such that $\partial(h_i^k) = f_i^k - g_i^k$ for all $1 \leqslant i \leqslant x$ and $1 \leqslant k \leqslant y$. Now the result follows from the fact that χ is an isomorphism of categories. The inverse of χ is given as follows. Let $\mathscr{T}$ be a track category such that $\mathscr{T}_{\simeq} = \mathbf{mod}\text{-}R$ and $\mathscr{T}_0 = \mathbf{mod}\text{-}Q$. By Lemma 3.5.1 $\mathscr{T}$ is an internal groupoid in the category of quadratic theories. By applying the functor cro one obtains an internal groupoid in the category of square rings. Moreover, the proof of Lemma 3.5.1 shows that this groupoid is ee-antidiscrete, therefore by Lemma 3.7.3 it defines an object in $\mathscr{Hext}(R, Q; M)^{\mathsf{SG},\square}$, which is the value of the inverse of χ. $\square$

4.2. **A pullback construction.** We now give a construction in the category of crossed square ring extensions which is needed in the proof of Theorem 2.2.1. Let

$$0 \longrightarrow M \longrightarrow C_{(1)} \xrightarrow{(\partial,\mathsf{Id})} C_{(0)} \xrightarrow{p} R \longrightarrow 0,$$

be a crossed square ring extension and let $f : Q_{(0)} \to C_{(0)}$ be a morphism of square rings, such that $p \circ f_e : Q_0 \to R$ is surjective. Based on this data we construct the following crossed square ring

$$\begin{tikzcd} & Q_{ee} \arrow[dl, "P^Q"'] \\ Q_1 \arrow[r, "\partial^Q"'] & Q_0 \arrow[u, "H^{Q_0}"'] \end{tikzcd}$$

where the group Q_1 is defined by the pullback diagram

$$\begin{tikzcd} Q_1 \arrow[r, "\partial^Q"] \arrow[d, "g_e"'] & Q_0 \arrow[d, "f_e"] \\ C_1 \arrow[r, "\partial"'] & C_0 \end{tikzcd}$$

and $P^Q = (P^C \circ f_{ee}, P^{Q_0}) : Q_{ee} \to Q_1$. Then one has the following crossed square ring extension

$$0 \longrightarrow M \longrightarrow Q_{(1)} \xrightarrow{\partial^Q} Q_{(0)} \longrightarrow R \longrightarrow 0.$$

One easily sees that

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"'] & Q_{(1)} \arrow[r, "\partial^Q"] \arrow[d, "g"] & Q_{(0)} \arrow[r, "pf"] \arrow[d, "f"] & R \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & C_{(1)} \arrow[r, "\partial"] & C_{(0)} \arrow[r, "p"] & R \arrow[r] & 0 \end{tikzcd}$$

is a morphism of crossed square ring extensions.

We call this construction the pullback construction and write $f^*\partial$ instead of (∂^Q). Assume now that (∂) is linearly generated and the composite $\mathsf{L}(Q_{(0)}) \to \mathsf{L}(C_{(0)}) \to R$ is surjective, then one easily sees that $(f^*\partial)$ is also linearly generated.

Of course a similar constructions works for quadratic pair algebras.

4.3. **Proof of Theorem 2.2.1.** Let

$$0 \longrightarrow M \longrightarrow \tilde{Q} \xrightarrow{(w,\mathsf{Id})} Q \xrightarrow{q} R \longrightarrow 0,$$

be an object of $\mathscr{H}\!\mathit{ext}_L(R, M)^{\mathsf{SG},\square}$. For simplicity we denote this object by (w). Then it can be also considered as an object of $\mathscr{H}\!\mathit{ext}(R, Q; M)^{\mathsf{SG},\square}$ and therefore (w) defines an element in $\mathsf{HML}^3(R, Q; M)$ thanks to Theorem 4.1.2. Then the boundary homomorphism gives an element in $\mathsf{HML}^3(R, M)$. In this way we get a map

$$\zeta : \pi_0(\mathscr{H}\!\mathit{ext}_L(R, M)^{\mathsf{SG},\square}) \to \mathsf{HML}^3(R; M).$$

Composing it with $\pi_0(\mathscr{H}\!\mathit{ext}_L(R,M)^{\mathsf{SG},\circledcirc}) \to \pi_0(\mathscr{H}\!\mathit{ext}_L(R,M)^{\mathsf{SG},\square})$ we obtain the map

$$\zeta' : \pi_0(\mathscr{H}\!\mathit{ext}_L(R,M)^{\mathsf{SG},\circledcirc}) \to \mathsf{HML}^3(R;M).$$

We have to show that these maps are bijections. Take an $a \in \mathsf{HML}^3(R;M)$. Take any surjective homomorphism $L \to R$ from a free monoid L to the multiplicative monoid of the ring R. It yields a surjective morphism $r : \mathbb{Z}_{\mathrm{nil}}[L] \to R$. Here $\mathbb{Z}_{\mathrm{nil}}[L]$ can be considered as a square ring as well as a quadratic ring. Since $\mathsf{HML}^i(\mathbb{Z}_{\mathrm{nil}}[L]; D_M) = 0$ for $i = 2,3$ (see Theorem A.0.2 in Appendix), we have an isomorphism

$$\partial : \mathsf{HML}^3(R, \mathbb{Z}_{\mathrm{nil}}[L]; M) \cong \mathsf{HML}^3(R;M).$$

Let $b = \partial^{-1}(a) \in \mathsf{HML}^3(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)$ be the element corresponding to a. Thanks to Theorem 4.1.2 the element b defines a crossed square ring extension of R by M

$$0 \longrightarrow M \longrightarrow \tilde{Q} \xrightarrow{(v,\mathsf{Id})} \mathbb{Z}_{\mathrm{nil}}[L] \longrightarrow R \longrightarrow 0$$

which is also linearly generated by construction and therefore is an object of $\mathscr{H}\!\mathit{ext}_L(R,M)^{\mathsf{SG},\square}$. By Lemma 4.1.1 it can be considered also as a quadratic pair algebra extension. Hence ζ and ζ' are surjections. It remains to show that ζ and ζ' are injections as well. Suppose $\zeta(w) = \zeta(w')$ (resp. $\zeta'(w) = \zeta'(w')$). We have to show that (w) and (w') are in the same connected component. Let $\mathsf{L}(Q)$ be the monoid of linear elements in Q. Via q it maps to the multiplicative submonoid $q(\mathsf{L}(Q))$ of R. Take any surjective homomorphism of monoids $F \to q(\mathsf{L}(Q))$ with F a free monoid. It has a lifting to a monoid homomorphism $F \to \mathsf{L}(Q)$, which yields a square (resp. quadratic) ring homomorphism $t : \mathbb{Z}_{\mathrm{nil}}[F] \to Q$. The homomorphism t satisfies all conditions on f in Section 4.2 and hence yields a morphism of crossed square ring extensions (resp. quadratic pair algebra extensions) $t^*(w) \to w$. Thus without loss of generality we can assume that (w) and (w') are chosen in such a way that $Q = \mathbb{Z}_{\mathrm{nil}}[F]$ and $Q' = \mathbb{Z}_{\mathrm{nil}}[F']$. Let L and r be the same as above (see the proof of surjectivity of ζ). Since $L \to R$ is surjective, $q(F) \subset R$ and F is free, there exists a morphism of monoids $F \to L$ such that for the induced morphism $k : Q = \mathbb{Z}_{\mathrm{nil}}[F] \to \mathbb{Z}_{\mathrm{nil}}[L]$ one has $q = r \circ k$. Thus one has the following commutative diagram

$$\begin{tikzcd}
\mathsf{HML}^3(R, \mathbb{Z}_{\mathrm{nil}}[L]; M) \arrow[r] \arrow[d, "k^*"] & \mathsf{HML}^3(R,M) \\
\mathsf{HML}^3(R,Q;M) \arrow[ur] &
\end{tikzcd}$$

Since both morphisms in the diagram with target $HML^3(R,M)$ are isomorphisms, it follows that $k^* : \mathsf{HML}^3(R, \mathbb{Z}_{\mathrm{nil}}[L]; M) \to \mathsf{HML}^3(R,Q;M)$ is also an isomorphism. Considering an extension corresponding to ${k^*}^{-1}(w)$ one sees that there exists a morphism of square ring extensions (resp. quadratic pair algebra extensions)

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"] & \tilde{Q} \arrow[r, "(w,\mathsf{Id})"] \arrow[d] & Q \arrow[r, "q"] \arrow[d, "k"] & R \arrow[r] \arrow[d, "\mathsf{Id}"] & 0 \\
0 \arrow[r] & M \arrow[r] & \bar{Q} \arrow[r, "(\bar{w},\mathsf{Id})"] & \mathbb{Z}_{\mathrm{nil}}[L] \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

In a similar manner we find a morphism of square ring extensions (resp. quadratic pair algebra extensions)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mathsf{Id}"] & \tilde{Q}' \arrow[r, "(w',\mathsf{Id})"] \arrow[d] & Q \arrow[r, "q'"] \arrow[d, "k"] & R \arrow[r] \arrow[d, "\mathsf{Id}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & \bar{Q}' \arrow[r, "(\bar{w'},\mathsf{Id})"] & \mathbb{Z}_{\mathrm{nil}}[L] \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

Since the square ring extensions (resp. quadratic pair algebra extensions) $(\bar{w})$ and $(\bar{w'})$ lie in the same groupoid $\mathscr{H}\!\mathit{ext}(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)^{\mathsf{SG},\square}$ and their classes in $\mathsf{HML}^3(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)$ are the same, it follows that they are isomorphic in the groupoid $\mathscr{H}\!\mathit{ext}(R, \mathbb{Z}_{\mathrm{nil}}[L]; M)^{\mathsf{SG},\square}$. Therefore we have the following diagram in $\mathscr{H}\!\mathit{ext}(R, M)^{\mathsf{SG},\square}$ (resp. $\mathscr{H}\!\mathit{ext}(R, M)^{\mathsf{SG},\underline{\circledcirc}}$):

$$(w') \leftarrow (\bar{w}') \ \cong (\bar{w}) \rightarrow (w),$$

hence the result.

5. Application to theory of 2-categories

In this section we introduce the notion of 2-additive track category, which is the 2-categorical analogue of additive category and we prove a strengthening theorem for such 2-additive track categories.

5.1. **Abelian track categories.** A track category is *abelian* if for any map $f : X \to Y$, the group $\mathrm{Aut}(f)$ of tracks from f to itself is abelian. Any track category which fits in a linear track extension is abelian. Converse is also true: any abelian track category defines a natural system $D = D_{\mathscr{T}}$ on $\mathscr{T}_{\simeq}$ and a linear track extension

$$0 \to D_{\mathscr{T}} \to \mathscr{T}_1 \rightrightarrows \mathscr{T}_0 \to \mathscr{T}_{\simeq} \to 0.$$

The natural system $D_{\mathscr{T}}$ and the linear track extension are unique up to isomorphism (see [7]).

5.2. **Track theories.** A *coproduct* $A \vee B$ in a track category $\mathscr{T}$ is an object $A \vee B$ equipped with 1-morphisms $i_1 : A \to A \vee B$, $i_2 : B \to A \vee B$ such that the induced functor

$$(i_1^*, i_2^*) : [\![A \vee B, X]\!] \to [\![A, X]\!] \times [\![B, X]\!]$$

is an equivalence of groupoids for all objects $X \in \mathscr{T}$. The coproduct is *strong* if the functor (i_1^*, i_2^*) is an isomorphism of groupoids. By duality we have also notion of *product* and *strong product*. A *zero object* in a track category $\mathscr{T}$ is an object 0 such that the categories $[\![0, X]\!]$ and $[\![X, 0]\!]$ are equivalent to the trivial groupoid for all $X \in \mathscr{T}$. A *strong zero object* in a track category $\mathscr{T}$ is an object 0 such that all categories $[\![0, X]\!]$ and $[\![X, 0]\!]$ are trivial groupoids.

A *track theory* (resp. *strong track theory*) is a small track category $\mathscr{T}$ possessing finite coproducts (resp. strong coproducts). Morphisms of track theories are enriched functors which are compatible with coproducts. An equivalence of track theories is a track theory morphism which is a weak equivalence and two track theories are called *equivalent* if they are made so by the smallest equivalence relation generated by these. The following is a particular case of a general result of Power [33]. For a cohomological proof we refer to [13].

Theorem 5.2.1. *Any abelian track theory is equivalent to a strong one.*

If $\mathscr{T}$ is an abelian track theory, then the corresponding category $\mathscr{T}_\simeq$ is an algebraic theory and the natural system $D_\mathscr{T}$ is cartesian. Conversely, if $\mathscr{T}$ is an abelian track category such that $\mathscr{T}_\simeq$ is an algebraic theory and $D_\mathscr{T}$ is cartesian, then $\mathscr{T}$ is an abelian track theory. Moreover an abelian track theory is strong if and only if $\mathscr{T}_0$ is an algebraic theory and the canonical functor $\mathscr{T}_0 \to \mathscr{T}_\simeq$ is a morphism of algebraic theories.

5.3. **2-Additive track categories.** Now we introduce an analogue of additive categories in the 2-world. Let $\mathscr{T}$ be a track theory with zero object. Then for any objects A and B of $\mathscr{T}$, there is a map $p_1 : A \vee B \to A$ and tracks $p_1 i_1 \Rightarrow \mathrm{Id}_A$, $p_1 i_2 \Rightarrow 0$. Similarly for $p_2 : A \vee B \to B$. A *2-additive track category* is an abelian track theory with strong zero object, such that the following conditions hold

i) for any two objects A and B the coproduct $A \vee B$ is also a product via $p_1 : A \vee B \to A$ and $p_2 : A \vee B \to B$

ii) for any morphism $f : A \to B$ there exists a morphism $g : A \to B$ and a track $hd \Rightarrow 0$, where $d : A \to A \vee A$ and $h : A \vee A \to B$ are morphisms with tracks $hi_1 \Rightarrow f$, $hi_2 \Rightarrow g$, $p_1 d \Rightarrow \mathrm{Id}_A$, $p_2 d \Rightarrow \mathrm{Id}_A$.

It is clear that the homotopy category $\mathscr{T}_\simeq$ of a 2-additive track theory is an additive category. The following is a direct consequence of [7] and [20].

Lemma 5.3.1. *Let $\mathscr{T}$ be an abelian track category. Then $\mathscr{T}$ is a 2-additive track category iff $\mathscr{T}_\simeq$ is an additive category and the corresponding natural system $D_\mathscr{T}$ is a biadditive bifunctor.*

It follows that a 2-additive track category determines a triple $(\mathscr{T}_\simeq, D_\mathscr{T}, \mathrm{Ch}(\mathscr{T}) \in H^3(\mathscr{T}_\simeq; D_\mathscr{T}))$. Conversely for an additive category $\mathbb{C}$, a biadditive bifunctor $D : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Ab}$ and an element $a \in H^3(\mathbb{C}; D)$ there exists a 2-additive track category unique up to equivalence such that $\mathscr{T}_\simeq = \mathbb{C}$, $D_\mathscr{T} = D$ and $\mathrm{Ch}(\mathscr{T}) = a$.

5.4. **Strongly and very strongly 2-additive track theories.** As we said Theorem 5.2.1 asserts that any track theory is equivalent to one with strong coproducts. In particular, any 2-additive track category is equivalent to one which possesses strong products. Since the dual of an additive track category is still a track theory, we see that it is also equivalent to one which possesses strong coproducts. Can we always get strong products and coproducts simultaneously? In other words, is every 2-additive track category $\mathscr{T}$ equivalent to a very strongly 2-additive track theory? Here a 2-additive track category is called *very strongly 2-additive* if it admits a strong zero object 0, strong finite coproducts and for any two object A and B the strong coproduct $A \vee B$ is also a strong coproduct via $p_1 : A \vee B \to A$ and $p_2 : A \vee B \to B$. The answer is given by the following result, which also shows that the number 2 plays an important rôle in the theory of 2-categories.

Theorem 5.4.1. *Let $\mathscr{T}$ be a small 2-additive track category with homotopy category $\mathbb{C} = \mathscr{T}_\simeq$ and canonical bifunctor $D = D_\mathscr{T}$. Let ${}_2D$ be the two-torsion part of D. Then there is a well-defined element $\nu(\mathscr{T}) \in H^0(\mathbb{C}; {}_2D)$, which is nontrivial in general and such that $\nu(\mathscr{T}) = 0$ iff $\mathscr{T}$ is equivalent to a very strongly 2-additive track theory. The class $\nu(\mathscr{T})$ for example is zero provided homs of the additive category $\mathbb{C}$ are modules either over $\mathbb{Z}[\frac{1}{2}]$ or over F_2 (the field with two elements).*

Proof. First one observes that a 2-additive track category $\mathscr{T}$ is very strongly 2-additive iff the category $\mathscr{T}_0$ is additive. For simplicity we restrict ourself to the case of single sorted theries. Then $\mathscr{T}_\simeq = \mathbf{mod}\text{-}R$ for a ring R. In this case one has an isomorphism $D(X,Y) \cong \mathsf{Hom}(X, M \otimes_R Y)$ natural in X and Y, where $M = D(R,R)$ is a bimodule over R. We claim that up to equivalence single sorted very strongly 2-additive track categories $\mathscr{T}$ with fixed $\mathscr{T}_\simeq$ and $D_\mathscr{T}$ are in bijection with $\mathsf{SH}^3(R;M)$. Indeed, if

$$0 \to M \to C_1 \xrightarrow{\partial} C_0 \to R \to 0$$

is an object of $\mathsf{Cross}(R,M)$, then we have the following very strongly 2-additive track category $\mathscr{T}$. Objects of $\mathscr{T}$ are the same as the objects of **mod**-R, i. e. natural numbers. For any natural numbers n and m the maps from n to m (which is the same as objects of the groupoid $\mathscr{T}(n,m)$) are $m \times n$-matrices with coefficients in C_0. For $f, g \in Mat_{m\times n}(S)$ the set of tracks $f \to g$ is given by

$$\mathsf{Hom}_{\mathscr{T}(n,m)}(f,g) = \{h \in \mathrm{Mat}_{m\times n}(C_1) \ \mid \ \partial(h) = f - g\}\,.$$

Composition of 1-arrows is given by the usual multiplication of matrices, while composition of tracks is given by the addition of matrices. One easily checks that in this way one really obtains a very strongly 2-additive track theory $\mathscr{T}(\partial)$. It is clear that $\mathscr{T}_\simeq = \mathbf{mod}\text{-}R$, where $R = \mathsf{Coker}(\partial)$ and the bifunctor corresponding to $\mathscr{T}$ is $D = \mathsf{Hom}(-, M \otimes_R -)$. Conversely, assume $\mathscr{T}$ is a single sorted very strongly 2-additive track category with $\mathscr{T}_\simeq = \mathbf{mod}\text{-}R$ and $D_\mathscr{T} = \mathsf{Hom}(-, M \otimes_R -)$. Then $\mathscr{T}_0$ is a single sorted additive category and therefore it is equivalent to **mod**-S, where $S = \mathsf{End}_{\mathscr{T}_0}(1)$. Restriction of the quotient functor $\mathscr{T} \to \mathscr{T}_0$ yields a homomorphism of rings $S \to R$. One defines X to be the set of pairs (h,x), where $x \in \mathsf{Hom}_{\mathscr{T}_0}(1.1)$ and $h : x \Rightarrow 0$ is a track in the groupoid $\mathscr{T}(1,1)$. Moreover we put $\partial = \partial_\mathscr{T}(h,x) = x$. Then X carries a structure of a bimodule over S, and

$$0 \to M \to X \xrightarrow{\partial} S \to R \to 0$$

is a crossed extension and the claim follows from isomorphism (2.1.4). As we said, up to equivalence single sorted 2-additive track categories $\mathscr{T}$ with fixed $\mathscr{T}_\simeq$ and $D_\mathscr{T}$ are in bijection with $\mathsf{HML}^3(R,M)$. Therefore the exact sequence $0 \to \mathsf{SH}^3(R,M) \to \mathsf{HML}^3(R,M) \xrightarrow{\nu} H^0(R, {}_2M)$ together with Proposition 9.1.1 of [12] implies the result. □

Remark. One can describe the function ν in Theorem 5.4.1 as follows. Let $\mathscr{T}$ be a 2-additive track theory. Let $\vee$ denote the weak coproduct in $\mathscr{T}$ and let 0 be the weak zero object. For objects X, Y one has therefore “inclusions” $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$. Since $X \vee Y$ is also a weak product of X and Y in $\mathscr{T}$ it follows that one has also projection maps $p_1 : X \vee Y \to X$ and $p_2 : X \vee Y \to Y$. For each X we choose maps $i_X : X \to X \vee X$ and $t_X : X \vee Y \to Y \vee X$ in such a way that classes of i_X and t_X in $\mathscr{T}_\simeq$ are the codiagonal and twisting maps in the additive category $\mathscr{T}_\simeq$. It follows that there is a unique track

$$\alpha_X : i_X \Rightarrow t \circ i_X$$

such that $p_{i*}(\alpha_X) = 0$ for $i = 1, 2$. Now, let $(1,1) : X \vee X \to X$ be a map which lifts the codiagonal map in $\mathscr{T}_\simeq$. Then $(1,1)_*\alpha_X$ is a track $\mathrm{Id}_X \to \mathrm{Id}_X$ and therefore it differs from the trivial track by an element $\nu(X) \in D(X,X)$. One can prove that the assignment $X \mapsto \nu(X)$ is the expected one.

5.5. **Strongly additive track categories.** A 2-additive track category $\mathscr{T}$ is called *strongly 2-additive* if $\mathscr{T}_0$ is quadratic.

Theorem 5.5.1. *Any 2-additive track theory is equivalent to a strongly 2-additive one.*

Proof. We continue to restrict ourselves to the single sorted case. In this case $\mathscr{T}_{\simeq}$ is the category **mod**-R for a ring R and $D = \mathsf{Hom}_R(-, (-)\otimes_R M)$ for an R-bimodule M. Thus $\mathsf{Ch}((\mathscr{T})) \in \mathsf{HML}^3(R; M)$ and therefore it belongs to $\mathsf{HML}^3(R, Q; M)$ for a square ring Q thanks to the proof of Theorem 2.2.1 given in Section 4.3. Thus the element $\mathsf{Ch}((\mathscr{T}))$ has a realization via track category $\mathscr{T}'$ such that $(\mathscr{T}')_0 = \mathbf{mod}\text{-}Q$ and we are done. □

Appendix A. Cohomology of free monoid square rings

T. Pirashvili

Here we prove the following result.

Theorem A.0.2. *Let L be a free monoid and let $Q = \mathbb{Z}_{\mathrm{nil}}[L]$ be the corresponding monoid square ring. Then for any R-R-bimodule B one has*

$$\mathsf{HML}^2(Q, B) = 0 = \mathsf{HML}^3(Q, B).$$

Proof of Theorem A.0.2 is given in Section A.4. The argument is a modification of the one given in [31].

A.1. **Auxiliary results.** For a ring R we denote by $\mathbf{F}(R)$ or simply by $\mathbf{F}$ the category of all covariant functors from the category **mod**-R of finitely generated free right R-modules to the category **Mod**-R of all right R-modules. It is well known [19] that

$$\mathsf{HML}^*(R, B) \cong \mathsf{Ext}^*_{\mathbf{F}}(\mathsf{Id}, (-)\otimes_R B)$$

We need the following result, which is an easy consequence of Theorem 9.2.1 [12] and the fact that $\mathsf{SH}^i(R, -) = 0$ for all $i \geqslant 2$, provided R is a free ring.

Lemma A.1.1. *Let R be a free ring and let B be an R-R-bimodule. Then one has $\mathsf{HML}^2(R, B) = 0$ and $\mathsf{HML}^3(R, B) \cong \mathsf{H}^0(R, {}_2B)$, where $\mathsf{H}^*(R, -)$ denotes the Hochschild cohomology of R.*

We also need the following vanishing result.

Lemma A.1.2. [27] *Let R be a ring and let*

$$T : (\mathbf{mod}\text{-}R) \times (\mathbf{mod}\text{-}R) \to \mathbf{Mod}\text{-}R$$

be a bifunctor, which is covariant in both variables and $T(0, X) = 0 = T(X, 0)$ for all $X \in \mathbf{mod}\text{-}R$. Then for any additive functor $F : \mathbf{mod}\text{-}R \to \mathbf{Mod}\text{-}R$ one has

$$\mathsf{Ext}^*_{\mathbf{F}}(F, T^d) = 0 = \mathsf{Ext}^*_{\mathbf{F}}(T^d, F),$$

where $T^d(X) = T(X, X)$.

In the following we need the simplicial derived functors of the functor $(-)^{\mathrm{ad}} : Q$-$Mod \to R$-Mod, which are denoted by

$$\mathsf{Tor}^Q_*(-, R) : \mathbf{Mod}\text{-}Q \to \mathbf{Mod}\text{-}R.$$

We recall the definition of these functors. According to [34] the category of simplicial objects in the category of right Q-modules has a closed model category structure where a morphism $f : X_* \to Y_*$ of simplicial objects is a weak equivalence (resp. fibration) when it is so in the category of simplicial sets. Let M be a right Q-module and let X_* be a cofibrant replacement of M. By [34] one can assume that each X_n, $n \geqslant 0$ is a free right Q-module. We also have $\pi_i X_* = 0$ for $i > 0$ and $\pi_0 X_* = M$. Now one puts

$$\mathsf{Tor}^Q_*(M, R) := \pi_*(X^{\mathrm{ad}}_*).$$

It is well known that these are well-defined functors. Since $\mathbf{Mod}\text{-}R \subset \mathbf{Mod}\text{-}Q$, one can consider also the restriction of $\mathsf{Tor}^Q_*(-, R)$ to $\mathbf{Mod}\text{-}(R)$ (see Proposition A.1.3 below).

Proposition A.1.3. *For any square ring Q and for any R-R-bimodule B, one has the following spectral sequence*

$$E^2_{pq} = \mathsf{Ext}^p_{\mathbf{F}}(\mathsf{Tor}^Q_q(-, R), F) \Longrightarrow \mathsf{HML}^{p+q}(Q, B),$$

where $R = Q^{\mathrm{ad}}$, $F(-) = (-) \otimes_R B$.

Proof. Proposition follows immediately from the spectral sequence (8.2.2) and Lemma 8.3.1 of [31].

A.2. **Computation of Tor^Q.** In this section we give a computation of Tor-groups involved in Proposition A.1.3. It is based on Lemma A.2.1 below, which is the specialization of the exact sequence (4.1) of [11]. Let us recall that Eilenberg and Mac Lane [15] defined the quadratic functor

$$\Omega : \mathsf{Ab} \to \mathsf{Ab}$$

such that it commutes with filtered colimits,

$$\Omega(A \oplus B) = \Omega(A) \oplus \Omega(B) \oplus \mathsf{Tor}(A, B)$$

and moreover

$$\Omega(\mathbb{Z}) = 0, \quad \Omega(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

Lemma A.2.1. [11] *Let X_* be a simplicial abelian group, which is degreewise free and has homotopy groups $\pi_i = \pi_i(X_*)$. Then one has*

$$\pi_0(\Lambda^2 X_*) = \Lambda^2(\pi_0)$$

$$0 \to \pi_1 \otimes \pi_0 \to \pi_1(\Lambda^2(X_*)) \to \Omega(\Lambda^2 \pi_0) \to 0$$

$$0 \to \pi_2 \otimes \pi_0 \oplus \Gamma\pi_1 \to \pi_2(\Lambda^2(X_*)) \to \mathsf{Tor}(\pi_1, \pi_0) \to 0$$

Let us recall that if G is a free class two nilpotent group, then one has the following short exact sequence

$$0 \to \Lambda^2(G_{\mathrm{ab}}) \to G \to G_{\mathrm{ab}} \to 0,$$

where the first nontrivial map is induced by $(x, y) \mapsto -x - y + x + y$. Assume now that L is a monoid and $Q = \mathbb{Z}_{\mathrm{nil}}[L]$ is the corresponding monoid square ring. As we already mentioned a right Q-module is the same as a nilpotent group of class two together with an action of L via group homomorphisms. It follows that if X is a free right Q-module, then X is also free as a nilpotent group of class two. Furthermore, X^{ad} in this case is simply X_{ab}, thus we have the following Lemma.

Lemma A.2.2. *Let L be a monoid and let $Q = \mathbb{Z}_{\mathrm{nil}}[L]$ be the monoid square ring. Then, for any free right Q-module X, one has the following short exact sequence*

$$0 \to \Lambda^2(X^{\mathrm{ad}}) \to X \to X^{\mathrm{ad}} \to 0,$$

in the category of modules over the ring $R = Q^{\mathrm{ad}} = \mathbb{Z}[L]$, where the first nontrivial map is induced by $(x, y) \mapsto -x - y + x + y$, and $\Lambda^2(X^{\mathrm{ad}})$ is an R-module via the diagonal action of L.

We would like to use these results in the following situation.

Proposition A.2.3. *Let L be a monoid and let $Q = \mathbb{Z}_{\mathrm{nil}}[L]$ be the monoid square ring. Then, for any free right module M over the ring $R = Q^{\mathrm{ad}}$, one has the following natural isomorphisms*

$$\mathsf{Tor}_0^Q(M, R) \cong M$$
$$\mathsf{Tor}_1^Q(M, R) \cong \Lambda^2(M)$$
$$\mathsf{Tor}_2^Q(M, R) \cong M \otimes \Lambda^2(M)$$
$$\mathsf{Tor}_3^Q(M, R) \cong (M \otimes M \otimes \Lambda^2(M)) \oplus (\Gamma(\Lambda^2(M)))$$

Proof. Let M be a free R-module. Let us take a free simplicial resolution Y_* of M in the category of Q-modules. Thanks to Lemma A.2.2 one has an exact sequence

$$0 \to \Lambda^2 X_* \to Y_* \to X_* \to 0,$$

where $X_* = Y_*^{\mathrm{ad}}$. Since $\pi_i Y_* = 0$ for $i > 0$ and $\pi_0 Y_* = M$ we have $\pi_0 X_* = M$ and $\pi_{i+1} X_* = \pi_i \Lambda^2(X_*)$. Since M is a free abelian group, one can use Lemma A.2.1 to get

$$\pi_1(X_*) \cong \Lambda^2(M), \quad \pi_2(X_*) = M \otimes \Lambda^2(M)$$
$$\pi_3(X_*) \cong \Lambda^2(M) \otimes M^{\otimes 2} \oplus \Gamma(\Lambda^2 M)$$

Comparing with definition of simplicial derived functors we obtain the expected result.

A.3. **Universal quadratic functors.** Let A be an abelian group. We set

$$P(A) = I(A)/I^3(A),$$

where $I(A)$ is the augmentation ideal of the group algebra of A. Let $p : A \to P(A)$ be the map given by $p(a) = (a-1)(\mathsf{mod}\, I^3(A))$. Then p is a *quadratic map*, meaning that the *cross-effect*

$$(a \mid b)_p := p(a + b) - p(a) - p(b)$$

is linear in a and b. Actually p is a universal quadratic function $p : A \to P(A)$ (see [30]). A quadratic map $f : A \to B$ of abelian groups is called *homogeneous* if $f(-a) = f(a)$. It is well known that for any abelian group A there exists a universal homogeneous quadratic function $\gamma : A \to \Gamma(A)$. If A is a module over a monoid ring $R = \mathbb{Z}[L]$, then $P(A)$, $\Gamma(A)$, $A \otimes A$ are also R-modules, where the action of $x \in L$ is given by

$$p(a)x = p(ax), \quad (\gamma(a))x = \gamma(ax), \quad (a \otimes b)x = ax \otimes bx.$$

Lemma A.3.1. *If $F \in \mathbf{F}$ is an additive functor, then*

$$\mathsf{Hom}_{\mathbf{F}}(\Gamma \circ \Lambda^2, F) = 0 = \mathsf{Hom}_{\mathbf{F}}(\Lambda^2, F)$$

Proof. Let us recall that if $T \in \mathbf{F}$ is a functor with $T(0) = 0$, then the second cross-effect of T fits in the decomposition

$$T(A \oplus B) \cong T(A) \oplus T(B) \oplus T(A \mid B).$$

Putting $B = A$ and using the codiagonal morphism $(\mathsf{Id}, \mathsf{Id}) : A \oplus A \to A$ one obtains a natural transformation $\eta_A : T(A \mid A) \to T(A)$. It is clear that any natural transformation from T to an additive functor factors through $\mathsf{Coker}(\eta)$. We first take $T = \Gamma(\Lambda^2)$. Since the second cross-effect of $\Gamma \circ \Lambda^2$ contains as a direct summand the term $\Gamma(A \otimes B)$ and for $A = B$ it maps via η surjectively to $\Gamma(\Lambda^2 A)$, we conclude that there is no nontrivial map from $\Gamma \circ \Lambda^2$ to any additive functor. Similarly for $\mathsf{Hom}_{\mathbf{F}}(\Lambda^2, F)$.

Lemma A.3.2. *Let L be a free monoid and let $R = \mathbb{Z}[L]$ be the corresponding monoid ring. Then*

$$\mathsf{Ext}^p_{\mathbf{F}}(P, F) = 0$$

provided F is additive and $2 \leqslant p \leqslant 4$.

Proof. Since $R = \mathbb{Z}[L]$ is torsion free as an abelian group and F is an additive functor the main result of [30] shows that one has an isomorphism

$$\mathsf{Ext}^p_{\mathbf{F}}(P, F) \cong \mathsf{Ext}^p_{\mathbf{Q}}(P, F),$$

provided $p \leqslant 4$. Here $\mathbf{Q}$ is the abelian category of quadratic functors from **mod**-R to **Mod**-R. For the functor $P \otimes R$, which is given by $X \mapsto P(X) \otimes R$, one has an isomorphism (see [30])

$$\mathsf{Hom}_{\mathbf{Q}}(P \otimes R, T) \cong T(R), \quad T \in \mathbf{Q}$$

It follows that $P \otimes R$ is a projective object in $\mathbf{Q}$. Thus one can use the bar-resolution

$$0 \leftarrow P \leftarrow P \otimes R \leftarrow P \otimes R \otimes R \leftarrow \cdots$$

to get a projective resolution of P in the category $\mathbf{Q}$. In particular one has an isomorphism

$$\mathsf{Ext}^*_{\mathbf{Q}}(P, F) \cong H^*(R, F(R))$$

and the result follows from the fact that the Hochschild cohomology vanishes for free rings in dimensions $\geqslant 2$.

A.4. **Proof of Theorem A.0.2.** We put $F = (-) \otimes_R B \in \mathbf{F}$. Thanks to Proposition A.1.3 one has the following spectral sequence

$$E^2_{pq} = \mathsf{Ext}^p_{\mathbf{F}}(\mathsf{Tor}^Q_q(-, R), F) \Longrightarrow \mathsf{HML}^{p+q}(Q, B).$$

By Proposition A.2.3 restriction of the functor $\mathsf{Tor}^Q_*(-, R)$ to the category **mod**-R is given by

$$\mathsf{Tor}^Q_0(-, R) = \mathsf{Id},$$
$$\mathsf{Tor}^Q_1(-, R) = \Lambda^2,$$
$$\mathsf{Tor}^Q_2(-, R) = \mathsf{Id} \otimes \Lambda^2,$$
$$\mathsf{Tor}^Q_3(-, R) = (\Lambda^2 \otimes \mathsf{Id}^{\otimes 2}) \oplus (\Gamma \circ \Lambda^2),$$

Since F is additive, Lemma A.1.2 shows that

$$E_2^{p0} = \mathsf{Ext}^p_{\mathbf{F}}(\mathsf{Id}, F),$$
$$E_2^{p1} = \mathsf{Ext}^p_{\mathbf{F}}(\Lambda^2, F),$$
$$E_2^{p2} = 0$$

$$E_2^{p3} = \mathsf{Ext}^p_{\mathbf{F}}(\Gamma \circ \Lambda^2, F).$$

We also have

$$E_2^{03} = 0 = E_2^{01}$$

thanks to Lemma A.3.1. Moreover $E_2^{20} = 0$ by Lemma A.1.1. Thus it suffices to show that the following differentials of the spectral sequence

$$d_2 : E_2^{11} = \mathsf{Ext}^1_{\mathbf{F}}(\Lambda^2, F) \to E_2^{30} = \mathsf{Ext}^3_{\mathbf{F}}(\mathsf{Id}, F)$$

and

$$d_2 : E_2^{12} = \mathsf{Ext}^2_{\mathbf{F}}(\Lambda^2, F) \to E_2^{40} = \mathsf{Ext}^4_{\mathbf{F}}(\mathsf{Id}, F)$$

are isomorphisms. Let us observe that in general the differential

$$d_2 : \mathsf{Ext}^p_{\mathbf{F}}(\Lambda^2, F) \to \mathsf{Ext}^{p+2}_{\mathbf{F}}(\mathsf{Id}, F)$$

is given by the cup product with $e \in \mathsf{Ext}^2_{\mathbf{F}}(\mathsf{Id}, \Lambda^2)$ corresponding to the extension

$$0 \leftarrow \mathsf{Id} \leftarrow P^2 \leftarrow \mathsf{Id}^{\otimes 2} \leftarrow \Lambda^2 \leftarrow 0$$

We have $e = e_1 \cup e_2$, where e_1 corresponds to the extension

$$0 \to \Lambda^2 \to \mathsf{Id}^2 \to \mathsf{Sym}^2 \to 0$$

while e_2 corresponds to the extension

$$0 \to \mathsf{Sym}^2 \to P \to \mathsf{Id} \to 0,$$

where Sym^2 is the second symmetric power and the first nontrivial map is induced by the assignment $a \otimes b \mapsto (a \mid b)_p$, while the second map is given by $p(a) \mapsto a$.

It follows from Lemma A.1.2 that the cup product with e_1 yields an isomorphism

$$\mathsf{Ext}^p_{\mathbf{F}}(\Lambda^2, F) \to \mathsf{Ext}^{p+1}_{\mathbf{F}}(\mathsf{Sym}^2, F), \ p \geqslant -1.$$

Similarly Lemma A.3.2 shows that the map

$$\mathsf{Ext}^p_{\mathbf{F}}(\mathsf{Sym}^2, F) \to \mathsf{Ext}^{p+1}_{\mathbf{F}}(\mathsf{Id}, F)$$

induced by the cup product with e_2 is an isomorphism if $2 \leqslant p \leqslant 3$ and we are done.

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