C*–MODULES AND RINGS OF DIFFERENTIAL OPERATORS

MAURICE J. DUPRÉ, JAMES F. GLAZEBROOK, AND EMMA PREVIATO

Abstract. This paper extends Sato's Grassmannian to allow for non-commutative coefficients. An algebraic definition of the Grassmannian of a Banachable algebra permits the identification of a Burchnall-Chaundy and Parshin rings of differential operators.

1. Introduction

Infinite-dimensional Grassmanians have figured prominently in the theoretical advances that brought together integrable equations and moduli spaces. While this is not surprising, in view of the role that finite-dimensional Grassmannians played already in the nineteenth and twentieth centuries as classifying spaces for vector bundles (cf. [25]), the finer analytical and topological aspects that occur in infinite dimension have at times yielded solution of less tractable differential equations (cf. [1]) as well as truly new geometric constructions (cf. [7], [8], [9], [10], [13]) of fundamental interest in manifold theory whether it be in the topological, differential or analytic category.

This work proposes the development of a more technical calibration of this geometric correspondence, with three applications in mind: non commutative coefficients are introduced to create a bridge toward the still-developing field of non-commutative algebraic geometry; the treatment of vector Grassmanians/Baker functions in suitable function spaces is proposed, so that the general geometric strategy can be used to describe evolutions in the moduli space of vector bundles; (3) constructions of C* algebra might be adapted to cover the yet-unclassified algebras of partial differential operators that in principle are attached to higher-dimensional spectral varieties, but cannot be represented as faithfully as was achieved by the group of hidden symmetries of the KP equation (cf. [17], [18], [19], [23], [24]).

More precisely, in this paper we lay out the setting of a Grassmannian which, although being of Sato type, has two new technical features: it allows for non-commutative coefficients, and is restricted by analytic properties which will make it amenable to results of function-theoretic, as opposed to formal, nature. After introductory sections on algebraic Grassmannians, fiber bundles and Banach algebras, we recall the constructions of Sato's formal theory, including Baker functions and KP flows. We recast these in our language, and finally extend them. We prove the necessary propositions along the way, while the theorem that perhaps best encapsules the point of the dictionary is a conjugation theorem for any commutative operator ring into one with constant coefficients (Proposition 8.3.1).

2. Algebraic preliminaries

2.1. **The space of idempotents $P(A)$ and the Grassmannian** $\mathrm{Gr}(A)$**.** Our references for this section are mainly [8] [9] (and references therein).

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Let A be a ring, not necessarily commutative, with identity 1 and whose group of units is denoted by $G(A)$. Let $P(A)$ denote the set of idempotents in A . For suitable A, in fact for the purposes of this paper, we can regard elements of $P(A)$ as projections. There is a natural partial order on $P(A)$ where we say that $p \prec q$ if $qp = p$, and thus "$\prec$ & $\succ$" define an equivalence relation '$\sim$' on $P(A)$. Let $\mathrm{Gr}(A) = P(A)/\sim$ be the set of equivalence classes in $P(A)$ under '$\sim$' . As the set of such equivalence classes, $\mathrm{Gr}(A)$ will be called *the Grassmannian of* A. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi\ : P(A) \longrightarrow \mathrm{Gr}(A)\ . \tag{2.1.1}$$

Let $h : A \longrightarrow B$ be a ring homomorphism. It is straightforward to see that the diagram below is commutative :

$$\begin{array}{ccc} P(A) & \xrightarrow{P(h)} & P(B) \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ \mathrm{Gr}(A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(B)\ . \end{array} \tag{2.1.2}$$

2.2. **Partial isomorphisms.**

Definition 2.2.1. We say that $u \in A$ is *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$ and $vuv = v$, in which case we call v a *relative inverse* (or *pseudoinverse*) for u . In general, such a relative inverse is not unique. Let $W(A)$ denote the set of all partial isomorphisms of the ring A .

If $u \in W(A)$ has a relative inverse v, then clearly $v \in W(A)$ with relative inverse u, and it is easy to see that both vu and uv belong to $P(A)$. Even though v is not uniquely determined by u alone, it is uniquely determined once u, vu and uv are all specified [8]. For $p \in P(A)$, let $W(p, A) \subset W(A)$ denote the subset of all partial isomorphims u in A with a relative inverse v satisfying $vu = p$. Likewise, let $W(A, q)$ denote the subset of all partial isomorphims u in A with a relative inverse v satisfying $uv = q$.

Now for $p, q \in P(A)$, we set

$$\begin{aligned} W(p, A, q) &= W(p, A) \cap W(A, q) \\ &= \{\ u \in qAp : \exists\ v \in pAq\ ,\ vu = p \text{ and } uv = q\ \}\ . \end{aligned} \tag{2.2.1}$$

2.3. **The spaces** $V(p, A)$ **and** $\mathrm{Gr}(p, A)$**.** Recall that elements $x, y \in A$ are said to be *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A . For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$, whereby it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.3.1. A given $u \in W(A)$ is said to be a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$ where p and q are similar. Let $V(A)$ denote the set of all proper partial isomorphims of A .

Whereas the structure of $V(A)$ is interesting in its own right, we shall be mainly concerned here with certain subspaces. Specifically, for a given $p \in P(A)$, let $V(p, A)$ denote the set of all proper partial isomorphisms of A having a relative inverse $v \in W(q, A, p)$ with $q \in \mathrm{Sim}(p, A)$. With reference to (2.2.1) this condition may be expressed as

$$V(p, A) := \bigcup_{q \in \mathrm{Sim}(p, A)} W(p, A, q)\ , \tag{2.3.1}$$

where $V(p,A) = G(A)\cdot p$.

The image of $\mathrm{Sim}(p,A)$ under the map Π defines the space $\mathrm{Gr}(p,A)$ regarded as the Grassmannian naturally associated to $V(p,A)$. We have $\mathrm{Gr}(p,A) = G(A)/G(\Pi(p))$ where $G(\Pi(p))$ denotes the isotropy subgroup of $\Pi(p)$. Further, it can be shown that $\mathrm{Gr}(A)$ is a discrete union of the $\mathrm{Gr}(p,A)$.

For a given ring homomorphism $h : A \longrightarrow B$, there is a restriction of (2.1.2) to a commutative diagram :

(2.3.2)
$$\begin{array}{ccc} V(p,A) & \xrightarrow{P(h)} & V(q,B) \\ \Pi\big\downarrow & & \big\downarrow\Pi \\ \mathrm{Gr}(p,A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(q,B) \end{array}$$

where for $p \in P(A)$, we have set $q = h(p) \in P(B)$.

3. Homogeneous spaces of G(A)

3.1. Subgroups of $G(A)$.

Definition 3.1.1. We say that A is a *Banachable algebra* if A is a topological algebra whose underlying vector space is a Banach space.

In the following, all Banach spaces/algebras are defined over the ground field $\mathbb{C}$. Requiring such a property of A will provide the spaces in question with analytic-manifold structures. To proceed, we record several facts established in [7] [8] [9] concerning the Banach Lie group (of units) $G(A)$.

Given $p \in P(A)$, let $A[p]$ denote the centralizer of p in A . On setting $\hat{p} = 1 - p$ (and noting $\hat{p} \in P(A)$), we have

(3.1.1)
$$A[p] = pAp + \hat{p}A\hat{p} \ ,$$

and it is straightforward to see that $A[p] + pA\hat{p} = pAp + A\hat{p}$, and $A([p])$ is analytically complemented in A by $\hat{p}Ap + pA\hat{p}$. From these relations, we deduce the following subgroups of $G(A)$:

(i) For each $p \in P(A)$, $G(\Pi(p)) := G(A[p] + pA\hat{p}) = G(pAp + A\hat{p})$ is the isotropy subgroup of $\Pi(p)$ in $\mathrm{Gr}(p,A)$.

(ii) For each $p \in P(A)$, $G(A[p]) = A[p] \cap G(A)$ is the isotropy subgroup for *the inner automorphic action* of $G(A)$ on $P(A)$.

(iii) $H(p) = (p + A\hat{p}) \cap G(A)$ is the isotropy subgroup for *the analytic (left) multiplication* of $G(A)$ on $V(p,A)$ and $H(p)$ as an open subset of $p + A\hat{p}$, is an analytic submanifold of $G(A)$.

Observe then the inclusion of subgroups $H(p) \subset G(\Pi(p)) \subset G(A)$, with $H(p)$ open in A .

3.2. Homogeneous fibrations.

Here we record some constructions and properties. It is clear that the constructions are purely algebraic and follow from the development of [8] §3 – §7 and [10] §1 – §2. Their respective proofs can be seen to be independent both of the speciality of the algebra A and the prevailing categorical analyticity :

(1) $P(A)$ is a discrete union of the homogeneous spaces $G(A)/G(A[p])$.

(2) The analytic bundle $\Pi : P(A) \to \mathrm{Gr}(A)$ and its restriction :
$$G(\Pi(p))/G(A[p]) \hookrightarrow G(A)/G(A[p]) \longrightarrow G(A)/G(\Pi(p)) \ .$$

(3) As a homogeneous space, the Stiefel manifold $V(p,A) = G(A)/H(p)$.

(4) As a homogeneous space, the Grassmannian $\mathrm{Gr}(p,A) = G(A)/G(\Pi(p))$.

(5) $\mathrm{Gr}(A)$ is a discrete union of the $\mathrm{Gr}(p, A) = G(A)/G(\Pi(p))$.

(6) The fibration

$$G(\Pi(p))/H(p) \hookrightarrow G(A)/H(p) \longrightarrow G(A)/G(\Pi(p)) \ , \tag{3.2.1}$$

generalizing the well–known *Stiefel bundle* construction in finite dimensions.

3.3. **Specialization to a Banachable algebra.** As shown in [8], the Banachable assumption on A is necessary to endow the space in question with a Banach analytic manifold structure. For A a Banach algebra, the map $\Pi : P(A) \longrightarrow \mathrm{Gr}(A)$ is open. Under the Banachable assumption, $V(p, A)$ may be viewed as a *space of framings* for elements of $\mathrm{Gr}(p, A)$. In particular, we recall from [8] §7 that on setting $G(p) = G(pAp)$, we have $\mathrm{Gr}(p, A) = V(p, A)/G(p)$, and there is a (locally trivial) principal fibration

$$G(p) \hookrightarrow V(p, A) \longrightarrow \mathrm{Gr}(p, A) \ . \tag{3.3.1}$$

Thus the homogeneous fibration (3.2.1) then becomes an analytic locally trivial fibration for the relevant category of differentiability. Also, on setting $\hat{p} = 1 - p$, there is the following expression for the (pointwise) decomposition of tangent spaces :

$$T\mathrm{Gr}(p, A) = pA\hat{p} + \hat{p}Ap \ . \tag{3.3.2}$$

The properties highlighted have special relevance to the application implicit in this paper, namely the description of integrable hierarchies (such as KdV, KP); see Example 3.4.1 below.

3.4. **The spatial correspondence.** Returning to the general case, if $\boldsymbol{\mathcal{A}}$ is a given topological algebra and E is some $\boldsymbol{\mathcal{A}}$–module, then $A = \mathcal{L}_{\boldsymbol{\mathcal{A}}}(E)$ could be taken to be the ring of $\boldsymbol{\mathcal{A}}$–linear transformations of E . An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E . In order to understand the interface between spaces such as $\mathrm{Gr}(p, A)$ and the usual Grassmannians of subspaces (of a vector space E), we will describe a 'spatial correspondence'.

Given a topological algebra $\boldsymbol{\mathcal{A}}$, suppose E is an $\boldsymbol{\mathcal{A}}$–module admitting a decomposition

$$E = F \oplus F^c \ , \qquad F \cap F^c = \{0\} \ , \tag{3.4.1}$$

where F, F^c are closed subspaces of E . We have already noted $A = \mathcal{L}(E)$ as the ring of linear transformations of E . Here $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{array}{ccc} V(p, \mathcal{L}(E)) & \xrightarrow{\varphi} & V(p, E) \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathrm{Gr}(p, \mathcal{L}(E)) & \xrightarrow{=} & \mathrm{Gr}(F, E) \end{array} \tag{3.4.2}$$

where $V(p, E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F . In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a $1:1$ correspondence with those of $\mathrm{Gr}(F, E)$.

For later purposes, if V is an arbitrary vector space admitting e.g. decompositions of the type (3.4.1), then we shall write $\mathrm{Gr}(V)$ for a Grassmannian when the latter is specified. In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a $1:1$ correspondence with those of $\mathrm{Gr}(F, E)$.

Our description so far reveals a framework for considering just about all species of Grassmannians employed in geometry and analysis (including, of course, the usual finite–dimensional ones, but also, notably, the formal models used in [26]). We will proceed to make the necessary specializations as the situation arises.

Example 3.4.1. The following relates to the Grassmann model considered in [26]. Let $\mathbb{C}^\infty = \mathbb{C}^{(\mathbb{N})}$ denote the vector space of all finite sequences $\{z_k\}, z_k \in \mathbb{C}$, and $G_k(\mathbb{C}^\infty)$ the Grassmannian of all k–dimensional linear subspaces of $\mathbb{C}^\infty$. The Stiefel manifold $V_k(\mathbb{C}^\infty)$ is the manifold of all k–frames, that is, the set of all injective linear maps $\mathbb{C}^k \longrightarrow \mathbb{C}^\infty$. Specifically

$$V_k(\mathbb{C}^\infty) = \{\Lambda \in \mathcal{L}(\mathbb{C}^k, \mathbb{C}^\infty) \ : \ \Lambda^{\mathrm{t}} \circ \Lambda \in \mathrm{GL}(k, \mathbb{C})\} \ . \tag{3.4.3}$$

Following e.g.[14](Theorem 47.5), we observe that an embedding $\mathbb{C}^k \longrightarrow \mathbb{C}^\infty$, induces in the inductive limit, the embeddings

$$\begin{aligned} V_k(\mathbb{C}^\infty) &= \varinjlim_N V_k(\mathbb{C}^N) \ , \\ G_k(\mathbb{C}^\infty) &= \varinjlim_N G_k(\mathbb{C}^N) \ . \end{aligned} \tag{3.4.4}$$

The situation considered in [26] is essentially that of $N = 2n$ and $k = n$ (for some n). We will recall several features of this work in the following section(s).

4. The restricted Grassmannians

In this section we exemplify the preceding results by analyzing, in some cases of interest in our study, the fibration $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ when A is a certain Banach algebra. To proceed, we denote by $\mathcal{L}(E, F)$, the Banach space of bounded linear operators between complex Banach spaces E and F, and when $E = F$, $\mathcal{L}(E)$ denotes the resulting complex Banach algebra. The space of Fredholm operators when definable on a space Z, is denoted by $\mathrm{Fred}(Z)$.

4.1. **The Schatten classes.** We refer to [21],[25] for the following facts which are relevant to our situation. Given a pair of Hilbert spaces (H_1, H_2) and an integer α, $1 \leq \alpha < \infty$, we denote by $\mathcal{L}_\alpha(H_1, H_2)$ the α–*Schatten ideal* in $\mathcal{L}(H_1, H_2)$ of adjoinable linear operators $T : H_1 \to H_2$, satisfying

$$\|T\|_\alpha^\alpha = \mathrm{Tr}(T^*T)^{\frac{\alpha}{2}} < \infty \ . \tag{4.1.1}$$

The $\mathcal{L}_\alpha(H_1, H_2)$ are Banach spaces and form an increasing chain of ideals within the compact operators denoted by $\mathcal{L}_\infty(H_1, H_2)$. For instance, the case $\alpha = 1$ corresponds to trace class operators when $H_1 = H_2$, and $\alpha = 2$ corresponds to the Hilbert–Schmidt operators (whereby $\mathcal{L}_2(H_1)$ is a Hilbert space).

Suppose H is a complex separable Hilbert space admitting a decomposition $H = H_1 \oplus H_2$ of the type (3.4.1), where H_1 and H_2 are infinite dimensional closed subspaces with $H_1 \cap H_2 = \{0\}$. For $s \geq 1$, consider the group of invertible bounded operators defined by

$$\mathrm{GL}_s = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \} \ , \tag{4.1.2}$$

where $a \in \mathrm{Fred}(H_1)$, $b \in \mathcal{L}_{2s}(H_2, H_1)$, $c \in \mathcal{L}_{2s}(H_1, H_2)$, $d \in \mathrm{Fred}(H_2)$. We define GL_0 to consist of those elements for which b and c are finite rank operators. With suitable metric topologies, each GL_s is a complex analytic Banach Lie group modeled on the appropriate Banach space, and there is a chain of embeddings $\mathrm{GL}_0 \subset \mathrm{GL}_1 \subset \cdots \subset \mathrm{GL}_\infty$, where for $s' \leq s$, each $\mathrm{GL}_{s'}$ is dense in GL_s . There is also a corresponding chain of embeddings of Banach analytic Lie groups given by

$\mathrm{GL}^0 \subset \mathrm{GL}^1 \subset \cdots \subset \mathrm{GL}^\infty$, where for $s \geq 1$, GL^s consists of all invertible linear operators on H of the form $T - 1 \in \mathcal{L}_s(H_1, H_2)$.

Let Δ_s be the subgroup of block triangular operators in GL_s where $c = 0$. This leads to defining the Grassmannian $\mathrm{Gr}_s(H_1, H) = \mathrm{GL}_s/\Delta_s$ for which Δ_s is the stabiliser of the subspace H_1 . Elements of $\mathrm{Gr}_s(H_1, H)$ may be viewed as closed subspaces $W = g \cdot H_1$, for some $g \in \mathrm{GL}_s$ as in (4.1.2), which split H and which are similar to H_1 . In this way, $\mathrm{Gr}_s(H_1, H)$ can be seen to be a Banach manifold modeled on the Banach space $\mathcal{L}_{2s}(H_1, H_2)$, and is thus an example of the Banach manifold $\mathrm{Gr}(F, E)$.

4.2. **Admissible bases.** For elements $W \in \mathrm{Gr}_s(H_1, H)$, there are certain bases called *admissible bases* which loosely speaking, allow us to make sense of a determinant *det* W, in part thanks to the group GL^s . The relevant space $V(p, A)$ is the Stiefel manifold of admissible bases denoted by $V_s(H_1, H)$, for which there is a principal GL^s–bundle

$$\mathrm{GL}^s \hookrightarrow V_s(H_1, H) \longrightarrow \mathrm{Gr}_s(H_1, H) \ , \tag{4.2.1}$$

being the restriction to such bases of the locally trivial principal $\mathrm{GL}(H)$–bundle given by (3.3.1).

4.3. **The Banach algebra $A = \mathcal{L}_J(H)$.** Let H be a separable complex Hilbert space admitting an orthogonal direct sum decomposition $H = H_+ \oplus H_-$, where the $H_\pm$ are closed subspaces, for which the decomposition is specified by a unitary operator $J : H \to H$ such that $J|H_\pm = \pm 1$. In the context of the preceding example, we now consider (without loss of generality) the case $s = 1$ and set $\mathrm{Gr}_1(H_+, H) = \widehat{\mathrm{Gr}}(H_+, H)$. Relative to $\widehat{\mathrm{Gr}}(H_+, H)$, we consider closed splitting subspaces W that are *commensurable with* H_+ (that is, for which $W \cap H_+$ has finite codimension in both W and H_+). Specifically, elements of $\widehat{\mathrm{Gr}}(H_+, H)$ consist of closed subspaces $W \subset H$ such that :

(1) the orthogonal projection $p_+ : W \longrightarrow H_+$ is Fredholm, and

(2) the orthogonal projection $p_- : W \longrightarrow H_-$ is Hilbert–Schmidt .

The subspaces $W \in \widehat{\mathrm{Gr}}(H_+, H)$ on which p_+ is an isomorphism, form a dense open subset of the latter which we call the *big cell*.

The relevant algebra to consider for these Grassmannians is $A = \mathcal{L}_J(H)$, the Banach algebra of bounded linear operators $T : H \to H$ such that $[J, T]$ is a Hilbert–Schmidt operator. The relevant norm $\| \ \|_J$ is defined by

$$\|T\|_J = \|T\| \ + \ \|[J, T]\|_2 \ . \tag{4.3.1}$$

Together with the topology induced by $\| \ \|_J$, the group of units $G(A)$ is a complex Banach Lie group. Now the subgroup of unitaries $U(A) \subset G(A)$ acts transitively on $\widehat{\mathrm{Gr}}(H_+, H)$, and consequently $G(A)$ is seen to be identifiable with GL_1 following (4.1.2) (see [25] §6.2, 7.1).

For our purposes we will often deal directly with the Banach algebra A . For the appropriate choice of $p \in P(A) \subset A$ such that $p \in \mathrm{Sim}(p_+, A)$, we will identify $\mathrm{Gr}(p, A)$ with $\widehat{\mathrm{Gr}}(H_+, H)$ via the spatial correspondence. Further, the Banach Lie group $G(p) = G(pAp)$ is identified with GL^1, and elements $w = \{w_i\} \in V(p, A)$, are viewed as admissible bases for W as above, and are preserved under $G(p) = \mathrm{GL}^1$. In view of (4.2.1) for the case $s = 1$, we have

$$\mathrm{Gr}(p, A) = V(p, A)/G(p) = G(A)/G(\Pi(p)) \ , \tag{4.3.2}$$

whereby $\mathrm{Gr}(p, A)$ is a complex Hilbert manifold modeled on $\mathcal{L}_2(H_+, H_-)$. We also have the universal bundle $\mathcal{E}(p, A) \longrightarrow \mathrm{Gr}(p, A)$ which is the holomorphic Hilbert bundle associated to the principal bundle $V(p,A) \longrightarrow \mathrm{Gr}(p,A)$ (a more general situation is considered in [9]). For tangent spaces (pointwise), we have the complex type decomposition

$$T^{\mathbb{C}}\mathrm{Gr}(p,A) = T^{(1,0)}\mathrm{Gr}(p,A) \oplus T^{(0,1)}\mathrm{Gr}(p,A) \ , \tag{4.3.3}$$

where for $W = p(H) \in \mathrm{Gr}(p,A)$ we deduce from (3.3.2) :

$$\begin{aligned} T^{(1,0)}_W\mathrm{Gr}(p,A) &\cong \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}) \cong pA\hat{p} \ , \\ T^{(0,1)}_W\mathrm{Gr}(p,A) &\cong \mathrm{Hom}_{\mathbb{C}}(W^{\perp}, W) \cong \hat{p}Ap \ . \end{aligned} \tag{4.3.4}$$

4.4. **The Baker function ψ_W.** Starting from the Hilbert space $H = L^2(S^1, \mathbb{C}^n)$, we recall the Banach algebra $A = \mathcal{L}_J(H)$ and the Fredholm Grassmanian $\widehat{\mathrm{Gr}}(H_+,H) = \mathrm{Gr}(p,A)$. We next consider a specific class of functions obtained from the development of [27] §5. Let D be the closed unit disc and denote by Γ_+ the group of holomorphic maps $g : D \longrightarrow \mathbb{C}^*$, such that $g(0) = 1$. The group Γ_+ acts on $\widehat{\mathrm{Gr}}(H_+,H)$ via multiplication operators on H, and for each $W \in \widehat{\mathrm{Gr}}(H_+,H)$, we define a dense open subset of Γ_+ by

$$\Gamma_+^W = \{g \in \Gamma_+ : g^{-1}W \text{ is transverse to } H_-\} \ . \tag{4.4.1}$$

We also define Γ_- as the group of holomorphic maps of the form $g(\frac{1}{z})$ where $g \in \Gamma_+$. It can be seen (cf. [27], Proposition 3.4) that Γ_- acts freely and transitively on $\widehat{\mathrm{Gr}}(H_+,H) = \mathrm{Gr}(p,A)$ (recall that $p \in \mathrm{Sim}(p_+,A))$.

Now for a given $g \in \Gamma_+^W$ there exists a unique $h_g \in \Gamma_-$ determined by W, satisfying the relation $g^{-1}W = h_g H_+$, that is,

$$W = g h_g H_+ \ . \tag{4.4.2}$$

Accordingly, there exists a function $\psi_W = \psi_W(x,z)$ given by

$$\psi_W = \psi_W(x,z) = g(z)(1 + \sum_{i=1}^{\infty} a_i(g)\ z^{-i}) \ , \tag{4.4.3}$$

where the a_i are analytic functions on Γ_+^W extending to meromorphic functions on all of Γ_+ . The function ψ_W is the unique function of that type lying in $g^{-1}W$, and via orthogonal projection $p_+^g : g^{-1}W \longrightarrow H_+$, we have $\psi_W = (p_+^g)^{-1}(\mathbf{1})$. Further, the $g \in \Gamma_+$ can be expressed uniquely as

$$g(z) = \exp(xz + t_2 z^2 + t_3 z^3 + \cdots) \ , \tag{4.4.4}$$

so that $\psi_W = \psi_W(x,z;t_2,t_3,\ldots)$.

We call ψ_W the *Baker function of the subspace* $W \in \widehat{\mathrm{Gr}}(H_+,H)$.

4.5. **The τ–function.** In the analysis and transformations of integrable systems the Plücker embedding equations explicitly describe the orbit of the vacuum state (see e.g. [11],[26]). From the embedding equations an essential feature known as the *τ–function* is derived. We recall now its definition. Relative to an admissible basis represented by $w \in V(p,A)$, the embedding $\mathrm{Gr}(p,h)$ leads to a function

$$\mathrm{Gr}_w(p,h)(w') = \ det\ \langle w, w' \rangle \ , \tag{4.5.1}$$

where $\langle w, w' \rangle$ denotes the matrix whose (i,j)–entry is $\langle w_i, w'_j \rangle$. Denoting the vacuum state (the canonical section of DET*) by $\mathrm{Gr}_{\mathbf{1}}(p,h)$, the *$\tau$–function* is defined as

$$\tau_w(g) := \ \langle \mathrm{Gr}_{\mathbf{1}}(p,h),\ g\ \mathrm{Gr}_w(p,h) \rangle \ , \qquad g \in \Gamma_+ \ . \tag{4.5.2}$$

We recall that the Baker function ψ_W has the property of extending to an analytic function of z in $|z| > 1$ (fixing $g \in \Gamma_+^W$). For $\zeta \in \mathbb{C}$ satisfying $|\zeta| > 1$, the map defined by $q_\zeta(z) = 1 - z\zeta^{-1}$, leads to the notable relationship between the Baker and τ–functions [27] Proposition 5.14 :

$$\psi_W(g, \zeta) = \tau_W(g \cdot q_\zeta)(\tau_W(g))^{-1} \ . \tag{4.5.3}$$

Remark 4.5.1. In Sato's formulation [26], the relevant algebra A consists of semi-infinite matrices parametrizing graphs of linear maps between subspaces of $H = H_+ \oplus H_-$ commensurable with H_+. Now $H_\pm$ consist of formal power series, $H_+ = \{\sum_{j\geq 0} c_j z^j | c_j \in \mathbb{C}\}$ and $H_- = \{\sum_{j<0} c_j z^j | c_j \in \mathbb{C}\}$; two subspaces V, W are commensurable when $\dim(V+W)/V \cap W$ is finite. The attendant formal Grassmannian, Baker and τ functions, have the advantage of encapsuling all the possible solutions to various differential equations, as well as the obvious disadvantage of not yielding numerical values in general. A significant example of an element of the Sato Grassmannian that does not converge is given by the τ function of the 2D topological gravity identified by Kac and Schwarz in [12], namely the one corresponding to the subspace V that satisfies $LV \subset V$, $L = \frac{3}{2}z + \frac{1}{2z}\frac{d}{dz} - \frac{1}{4}z^{-2}$.

5. Pseudodifferential operators as subrings of A

In this section we report on Sato's theory, to the effect of identifying each element of the Grassmannian (strictly speaking, the open dense "big cell") with a formal pseudodifferential operator L. One of the goals of this paper is to extend the coefficients of L.

5.1. **Rings of formal pseudodifferential operators.** Let $\mathbb{B}$ be the algebra of analytic functions $U \longrightarrow \mathbb{C}$ where U is a connected open neighbourhood of the origin in $\mathbb{C}$. The (noncommutative) algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in $\mathbb{B}$, consists of expressions

$$\sum_{i=0}^{N} a_i \ \partial^i \ , \ (a_i \in \mathbb{A} \ , \ \text{some } N \in \mathbb{Z}) \ . \tag{5.1.1}$$

Here $\partial \equiv \partial/\partial x$ and the a_i can be regarded as multiplication operators for which multiplication is defined by

$$[\partial, a] = \partial a - a\partial = d(a) \equiv \partial a/\partial x \ . \tag{5.1.2}$$

Let $\mathbb{A}$ denote a commutative subalgebra of $\mathbb{B}[\partial]$. The *rank* of $\mathbb{A}$ is defined to be the greatest common divisor of all orders of the operators in $\mathbb{A}$. For instance, given an operator of the form

$$L_0 = \partial^r + u_{r-2} \ \partial^{r-2} + \ldots u_1 \ \partial + u_0 \ , \tag{5.1.3}$$

the algebra $\mathbb{C}[L_0]$ has rank equal to r .

Next we proceed to the algebra $\mathbb{B}[\partial^{-1}]$ of formal pseudodifferetial operators with coefficients in $\mathbb{B}$. This algebra is obtained from $\mathbb{B}[\partial]$ by formally inverting the operator ∂ . Thus $P \in \mathbb{B}[\partial^{-1}]$ may be written as

$$P = \sum_{i>-\infty}^{N} a_i \ \partial^i \ . \tag{5.1.4}$$

The operator P admits a decomposition $P = P_+ + P_-$, where

$$P_+ = \sum_{i=0}^{N} a_i \ \partial^i \ , \ P_- = \sum_{i>-\infty}^{-1} a_i \ \partial^{-i} \ . \tag{5.1.5}$$

5.2. **The Baker function ψ_W as eigenfunction.** Recall that the n–th generalized KdV–hierarchy consists of all evolution operators for $n-1$ unknown functions $u_0(x,t),\ldots,u_{n-2}(x,t)$ that can be expressed as

$$\frac{\partial L}{\partial t} = [P, L] \ , \tag{5.2.1}$$

where $L \in \mathbb{B}[\partial^{-1}]$ is an n–th order differential operator

$$L = \partial^n + u_{n-2}\ \partial^{n-2} + \ldots u_1\ \partial + u_0\ ,\ (\partial = \frac{\partial}{\partial x})\ , \tag{5.2.2}$$

and P is some differential operator. In order to study the evolution of eigenfunctions of L via comparison with the constant operators ∂^n, we want to find an operator K conjugating L such that $K(L)K^{-1} = \partial^n$. So if ψ_0 is an eigenfunction of ∂^n, then $\psi = K\psi_0$ will be an eigenfunction of L . Following [27] §4, there exists such an operator $K \in \mathbb{B}[\partial^{-1}]$ of the form

$$K = 1 + \sum_{i=1}^{\infty} a_i(x)\ \partial^{-i}\ , \tag{5.2.3}$$

which is determined up to right multiplication by a constant coefficient operator of the form $1 + c_1\ \partial^{-1} + \ldots$ (noting that only constant coefficient operators commute with ∂^n).

If $W \in \widehat{\mathrm{Gr}}(H_+, H)$, then there exists a unique differential operator P_n such that

$$P_n\psi_W = z^n\psi_W\ . \tag{5.2.4}$$

Furthermore, the operator P_n admits the conjugation property $P_n = K(\partial^n)K^{-1}$ (see [27] §5).

In order to simplify the exposition, we will henceforth relax the dependence on the variables $(t_2, t_3, \ldots)$, and so $\psi_W = \psi_W(x,z)$ unless otherwise stated.

5.3. **The Burchnall-Chaundy ring.** In order to simplify the exposition, we will relax temporarily the dependence on the variables $(t_2, t_3, \ldots)$, and so $\psi_W = \psi_W(x,z)$ unless otherwise stated.

We start by considering the eigen–equation $L\psi = z^n\psi$ which admits a formal power series solution as determined by the Baker function

$$\psi = e^{xz}(1 + \sum_{i=1}^{\infty} a_i(x)\ z^{-i})\ , \tag{5.3.1}$$

which is unique up to multiplication with constant coefficients of the form $1 + c_1\ z^{-1} + \ldots$.

It will be sufficient to deal with the case $n = 1$, and take an operator L of order 1 in $\mathbb{B}[\partial^{-1}]$, such that L is given by

$$L = \partial + \sum_{i>-\infty}^{-1} u_i\ \partial^i\ . \tag{5.3.2}$$

Observe that conjugation by K turns elements of $\mathbb{B}[\partial^{-1}]$ into constant-coefficient operators. The set of ordinary differential operators that commute with a given L turns out to be a maximal-commutative subalgebra of $\mathbb{B}[\partial]$, which could be called "Burchnall-Chaundy ring". Burchnall and Chaundy in the 1920s proved essentially that any rank-1 commutative subalgebra of $\mathbb{B}[\partial]$ can be conjugated into a constant-coefficient subring of $\mathbb{B}[\partial^{-1}]$, by reversing the conjugation above.

Finally, using the elegant idea of [26], the correspondence '$(\frac{\partial}{\partial x})^{-1} \leftrightarrow$ multiplication by z', here realizes commutative subrings $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$ as subrings of $\mathbb{C}[[z]][z^{-1}]$. Thus for $H = L^2(S^1, \mathbb{C})$ and the Banach algebra $A = \mathcal{L}_J(H)$, we have the following preliminary result which may be seen as an interpretation of [26] in terms of the relevant algebras.

Theorem 5.3.1. *Provided its Baker function is associated to a subspace* $W \in \widehat{\mathrm{Gr}}(H_+, H)$, *the ring* $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$ *conjugates as a subring into the Banach algebra* $A = \mathcal{L}_J(H)$ *up to constant coefficient operators.*

Proof. We start with the subspace $W \in \widehat{\mathrm{Gr}}(H_+, H) = \mathrm{Gr}(p, A)$ given by (4.4.2) , and consider the corresponding Baker function $\psi_W = \psi_W(x, z)$ as before. We can assume that W belongs to the big cell, so under the orthogonal projection $p_+^g : W \longrightarrow H_+$, we have $\psi_W = (p_+^g)^{-1}(\mathbf{1})$. For $P \in \widehat{\mathbb{A}}$, $\partial = \frac{\partial}{\partial x}$, we assign $P(\frac{\partial}{\partial x})$ to $f(z^{-1})$ via

$$P(\frac{\partial}{\partial x})\psi_W = f(z^{-1})\psi_W \ . \tag{5.3.3}$$

The form of W given by (4.4.2) and the definition of ψ_W lead to $\psi_W = We^{zx}$. With L as in (5.3.2), the relevant eigenvalue problem of (5.2.4) $L\psi_W = z\psi_W$, thus corresponds to $\partial(e^{zx}) = ze^{zx}$.

Recall from (5.2.3), we have $K = 1 + \sum_{i=1}^{\infty} a_i(x)\ \partial^{-i}$, such that each $L \in \widehat{\mathbb{A}}$ satisfies the conjugation property $L = K(\partial)K^{-1}$. Comparing this with (4.4.2) and (4.4.3), we can write $K = h_g(\partial)$, for $h_g \in \Gamma_-$. Thus via the assignment $\partial \mapsto z^{-1}$, we obtain

$$L \mapsto h_g(z^{-1})\ (z^{-1})\ (h_g(z^{-1}))^{-1} \ . \tag{5.3.4}$$

Since the h_g was uniquely determined by the $W \in \mathrm{Gr}(p, A)$, the latter therefore determines the conjugation of $\widehat{\mathbb{A}}$ into a commutative subring of A . □

Remark 5.3.1. The condition of Theorem 5.3.1 for the Baker fuction to correspond to an element of the Grassmannian could be removed if we were working with Sato's formal Grassmannian. For the case we are considering, namely $H = L^2(S^1, \mathbb{C}^n)$, a requirement on ψ to ensure membership is not known, cf. the comment on the class $\mathcal{C}^{(n)}$ in [27], Section 5.

Theorem 5.3.1 is a theme on which we shall perform certain variations, but in stages of generality. Implicit in the proof of Theorem 5.3.1 is the fact that the KP–hierarchy is manifestly a sequence of deformations for $L = K(\partial)K^{-1}$, $K \in \widehat{\mathbb{A}}$, whereby the Baker (eigen)function $\psi_W = We^{zx}$ converts differentiation into multiplication. As shown explicitly in [2], the solution flow of $L\psi_W = z\psi_W$, arises on subspaces W for which the vectors

$$(\psi_W,\ \partial\psi_W, \ldots,\ \partial^{k-1}\psi_W, \ldots)|_{x=0}\ ,\ (\partial \equiv \partial/\partial x)\ , \tag{5.3.5}$$

comprise a basis for W, the so–called 'admissible bases'.

We mention briefly that Sato's theory achieves the construction of all solutions to the KP hierarchy, namely the τ functions associated to elements of the formal Grassmannian, regarded as functions of $(t_1 = x, t_2, ..., t_k, ...)$: the KP hierarchy turns out to give the Plücker relations, namely the condition for a formal power series in $(t_1, t_2, ..., t_k, ...)$ to be the τ function associated to an element of the Grassmannian. The set-up we covered is part of a more general construction: for a more general Kac-Moody algebra (in our case, $\widehat{\mathrm{gl}}(\infty)$) and a given Heisenberg algebra in it, its basic representation can be given in terms of vertex operators, in such a way that the defining equations of the orbits of the associated Kac-Moody group through the highest weight vector are a generalized KP hierarchy, cf. [11]. In view of this important role of the Grassmannian, it is worth underscoring two properties achieved by the framework based on the Banach algebra $A = \mathcal{L}_J(H)$:

Proposition 5.3.1. *The tautological bundle over the Grassmannian is the vector bundle associated to the principal fibration 3.3.1, which was seen to be locally trivial. The τ function in the (essentially unique) globally holomorphic section of the dual of its determinant bundle (*[25], *§2.9,* [27], *Proposition 3.3).*

Notably, when a Jacobian $\mathrm{Jac}(X)$ is embedded into the Grassmannian via the Krichever map ([27], §6), the pull-back of the determinant bundle is the Poincaré line bundle; it would be interesting to extend such an interpretation to the generalization given in [3] of the Krichever map to a general abelian variety (roughly speaking, the Grassmannian of $\widehat{\mathrm{gl}}(\infty)$ is replaced by a homogeneous space of a symplectic loop group). If a group action analogous to Γ_+'s could be defined, the corresponding hierarchy of partial differential equations should be related to the one identified in [4].

Proposition 5.3.2. *The tangent space to the Grassmannian at an element $W = p(H)$, which was seen to be isomorphic to $T\mathrm{Gr}(p, A) = pA\hat{p} + \hat{p}Ap \cong \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}) \oplus \mathrm{Hom}_{\mathbb{C}}(W^{\perp}, W)$ 4.3.4, is isomorphic to the dual of the non-commutative stabilizer algebra S_W of W (*[1], *§3.2 and Lemma 3.3.4), the transformations that map W to itself. In this identification, the KP flows are defined by Marsden-Weinstein reduction as Hamiltonians of Ad^* invariant functions (ibid, §8).*

6. Grassmannians over Hilbert C*–modules

6.1. **Standard operators on Hilbert C*–modules.** Let us start by describing some basic properties of Hilbert modules over C*–algebras. We shall skip some of the usual elementary definitions since we will be restricting our discussion to a particular class; thus the reader is referred to e.g. [6] [15] [16] for comprehensive details of the subject.

Consider a unital C*–algebra $\boldsymbol{\mathcal{A}}$ and the standard (free countable dimensional) Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ over $\boldsymbol{\mathcal{A}}$ as given by

$$H_{\boldsymbol{\mathcal{A}}} = \{\{\zeta_i\}\ ,\ \zeta_i \in \boldsymbol{\mathcal{A}}\ ,\ i \geq 1\ :\sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \boldsymbol{\mathcal{A}}\ \} \cong \oplus \boldsymbol{\mathcal{A}}_i \tag{6.1.1}$$

where each $\boldsymbol{\mathcal{A}}_i$ represents a copy of $\boldsymbol{\mathcal{A}}$.

Example 6.1.1. Suppose H is a separable Hilbert space. We can form the algebraic tensor product $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ on which there is an $\boldsymbol{\mathcal{A}}$–valued inner product

$$\langle x \otimes \zeta\ ,\ y \otimes \eta \rangle\ =\ \langle x, y \rangle\ \zeta^* \eta \qquad x, y \in H\ ,\ \zeta, \eta \in \boldsymbol{\mathcal{A}}\ .$$

In this way, $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ becomes an inner product $\boldsymbol{\mathcal{A}}$–module whose completion is denoted by $H \otimes \boldsymbol{\mathcal{A}}$. Given an orthonormal basis for H, then there is the identification (unitary equivalence) $H \otimes \boldsymbol{\mathcal{A}} \approx H_{\boldsymbol{\mathcal{A}}}$ (see e.g. [15]).

There is a natural $\boldsymbol{\mathcal{A}}$–valued scalar product on $H_{\boldsymbol{\mathcal{A}}}$ leading to classes of bounded linear operators on $H_{\boldsymbol{\mathcal{A}}}$ (as studied in e.g. [22]). Let $A = \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ denote the space of $\boldsymbol{\mathcal{A}}$–linear bounded operators with $\boldsymbol{\mathcal{A}}$–linear bounded adjoints. Then A is a Banach algebra whose group of units $G(A)$ retracts onto the subgroup of unitaries $U(A) = U(H_{\boldsymbol{\mathcal{A}}})$. We have

$$P(A) = \{p \in A : p = p^2 = p^* \text{ and } p \sim 1 \sim 1 - p\}\ , \tag{6.1.2}$$

where "$\sim$" denotes the Murray–von Neumann equivalence of projections (see e.g. [6]).

Compact linear operators on $H_{\boldsymbol{\mathcal{A}}}$ are defined as $\boldsymbol{\mathcal{A}}$–norm limits of finite rank operators. We denote the space of such compact operators by $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$. Finite–rank $\boldsymbol{\mathcal{A}}$–submodules of $H_{\boldsymbol{\mathcal{A}}}$ are well–defined and a Fredholm operator on $H_{\boldsymbol{\mathcal{A}}}$ is one whose kernel and image are finite–rank $\boldsymbol{\mathcal{A}}$–submodules. We denote the space of such Fredholm operators by $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$. There exists a canonical index homomorphism $\mathrm{Ind}_{\mathcal{A}} : \mathrm{Fred}(H_{\mathcal{A}}) \longrightarrow K_0(\mathcal{A})$, where $K_0(\mathcal{A})$ denotes the topological K–group of $\mathcal{A}$ (see e.g. [6]).

6.2. **The Grassmannian $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$.** Since $A = \mathcal{L}(H_{\mathcal{A}})$ is a Banach algebra, we can consider the Grassmannians $\mathrm{Gr}(p, A)$ for suitable $p \in P(A)$. A particularly important example is the Grassmannian $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ as studied in [13]. It is a Banach manifold modeled on the Banach space $\mathcal{K}(H_{\mathcal{A}})$, and its unitary and topological structures can be determined in a way similar to that of $\widehat{\mathrm{Gr}}(H_+, H)$ (in the case of $\mathcal{A} = \mathbb{C}$) as in [25]. Specifically, we fix a direct sum decomposition of Hilbert $\mathcal{A}$–modules $H_{\mathcal{A}} = H_+ \oplus H_-$, where $H_\pm$ are isomorphic to $H_{\mathcal{A}}$ as $\mathcal{A}$–modules. Then elements of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ consist of $\mathcal{A}$–submodules W of $H_{\mathcal{A}}$ such that :

(1) the orthogonal projection $p_+ \ : \ W \longrightarrow H_+$ is in $\mathrm{Fred}(H_{\mathcal{A}})$, and

(2) the orthogonal projection $p_- \ : \ W \longrightarrow H_-$ is in $\mathcal{K}(H_{\mathcal{A}})$.

We define the *big cell* of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ as the collection of all $\mathcal{A}$–submodules W of $H_{\mathcal{A}}$ such that p_+ is an isomorphism.

6.3. **Generalized Birkhoff stratification.** Suppose now we fix a path component $\widehat{\mathrm{Gr}}(\gamma)$ of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ corresponding to an element $\gamma \in K_0(\mathcal{A})$ where γ equals the index of p_+ restricted to an element V of this component. Consider then pairs $(\alpha, \beta) \in K_0(\mathcal{A}) \times K_0(\mathcal{A})$ for which $\gamma = \alpha - \beta$. Given any such pair, we denote by $B_{\alpha,\beta} \subset V$ the subset for which

$$[\ker\ p_+|V] = \alpha\ ,\ [\mathrm{coker}\ p_+|V] = \beta\ . \tag{6.3.1}$$

Then following [13] we have :

$$\widehat{\mathrm{Gr}}(\gamma) = \bigcup_{\alpha,\beta} B_{\alpha,\beta}\ . \tag{6.3.2}$$

Furthermore, the $B_{\alpha,\beta}$ are Banach analytic subspaces of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ and (6.3.2) describes a complex analytic stratification of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$.

7. The ring $\mathbb{B}[\partial^{-1}]$ and the Banach algebra $A = \mathcal{L}(H_{\mathcal{A}})$

7.1. **Generalization of the $n = 1$ case.** Let $\mathcal{A}$ be a (unital) C*–algebra. Given a separable Hilbert space H, we will adopt the Hilbert C*–module of Example 6.1.1 where we have the unitary equivalence $H \otimes \mathcal{A} \approx H_{\mathcal{A}}$ together with the direct sum decomposition of Hilbert $\mathcal{A}$–modules $H_{\mathcal{A}} = H_+ \oplus H_-$ for which $H_\pm \cong H_{\mathcal{A}}$. In the following we set

$$\widetilde{A} = A \otimes \mathcal{A}\ ,\ \text{where } A = \mathcal{L}_J(H)\ , \tag{7.1.1}$$

Without too much loss of generality, we keep with the separable Hilbert space $H = L^2(S^1, \mathbb{C})$. We will also use the identification $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}}) = \mathrm{Gr}(\tilde{p}, \widetilde{A})$ with $\tilde{p} \in \mathrm{Sim}(\tilde{p}_+, \widetilde{A})$ where $\tilde{p}_+ \in \mathrm{Fred}(H_{\mathcal{A}})$ is the orthogonal projection as previously.

Here we take an operator L of order 1 in the ring $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}] \otimes \mathcal{A}$ with coefficients $u_i \in \mathbb{B} \otimes \mathcal{A}$, such that L is given by

$$L = \partial + \sum_{i>-\infty}^{-1} u_i\, \partial^i\ ,\ (\partial \equiv \partial/\partial x)\ . \tag{7.1.2}$$

Proposition 7.1.1. *Provided its Baker function is associate to an element of the Grassmannian $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$, the commutative ring $\widehat{A}$ conjugates into the Banach algebra $A = \mathcal{L}(H_{\mathcal{A}})$ as a subring up to constant coefficient operators.*

Proof. We use essentially the same argument in proving Theorem 5.3.1, except that the coefficient variables are taken to be in $\boldsymbol{\mathcal{A}}$. Once more we consider $K \in \mathbb{B}[\partial^{-1}]$ given by

$$K = 1 + \sum_{i=1}^{\infty} a_i(x)\ \partial^{-i}\ , \tag{7.1.3}$$

with the (commuting) variables $x \in \boldsymbol{\mathcal{A}}$, so that effectively $K \in \mathbb{B}[\partial^{-1}] \otimes \boldsymbol{\mathcal{A}}$. Essentially the same argument as before implies the conjugation property, namely $L = K(\partial)K^{-1}$.

Let $W \in \widehat{\mathrm{Gr}}(H_+, H_{\boldsymbol{\mathcal{A}}}) = \mathrm{Gr}(p, A)$ belong to the big cell, so that $p_+ : W \longrightarrow H_+$ in (6.2) is an isomorphism. As an element of $H \otimes \boldsymbol{\mathcal{A}}$, let us consider W to be of the form

$$W = 1 + \sum_{i=1}^{\infty} a_i(x)\ z^{-i}\ ,\ x \in \boldsymbol{\mathcal{A}}\ . \tag{7.1.4}$$

We consider the group Γ_+ of holomorphic maps as before, but now extending Γ_+ via tensoring with $\boldsymbol{\mathcal{A}}$, to act as multiplication operators on $H \otimes \boldsymbol{\mathcal{A}}$. For $g \in \Gamma_+$, we consider the orthogonal projection

$$p_+^g\ :\ g^{-1}(W) \longrightarrow H_+\ ,\ p_+^g \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})\ , \tag{7.1.5}$$

and define the Baker function associated to W by $\psi_W = (p_+^g)^{-1}(\mathbf{1}) = We^{xz}$, such that as before we have $L\psi_W = zW$. The assignment $\partial^{-1} \mapsto z$ induces $L \mapsto W(\partial)W^{-1}$, and thus $\widehat{A}$ is conjugated as a subring of $A = \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$. □

8. Towards an operator–valued Baker function

In this section we outline an abstraction at the level of operator–valued functions. An astute–minded reader will see that the idea requires replacing the variable z by a single operator.

8.1. **Laurent series generator.**

Definition 8.1.1. [5] Let A be a unital Fréchet algebra. An invertible element $\zeta \in G(A) \subset A$ is said to be a *Laurent series generator for* A if each $a \in A$ is expressible as

$$a = \sum_{i=-\infty}^{\infty} a_i\ \zeta^i\ , \tag{8.1.1}$$

for scalars a_i, and the series converges absolutely with respect to each continuous seminorm on A . We say that A has the *unique expression property* if such a representation is always unique.

As shown in [5], such algebras A with a Laurent series generator include algebras which are isomorphic to various types of function algebras defined on S^1 or on the annulus.

8.2. **Fredholm operators on Banach spaces.** The next step involves introducing from [28] the notion of compact and Fredholm operators between complex Banach spaces E and E' . Let $\mathcal{K}(E, E')$ denote the compact operators. The more general meaning of a Fredholm operator is here based on the notion of 'right and left aggregation'; we refer to [28] for details. An operator $T \in \mathrm{Fred}(E)$ is stable under compact perturbations and admits a well–defined *index* given by $\mathrm{Ind}(T) = \dim \mathrm{Ker}\ T - \mathrm{codim}\ \mathrm{Im}\ T$. The index $\mathrm{Ind}(T)$ is constant on connected components, is invariant under compact perturbations, and satisfies $\mathrm{Ind}(T_1T_2) = \mathrm{Ind}(T_1) + \mathrm{Ind}(T_2)$. Moreover, there is an induced homomorphism $\mathrm{Ind} : \mathrm{Fred}(E) \to \mathbb{Z}$.

Let E be a complex Banach space admitting a decomposition of the type (3.4.1). Let

$$\widehat{G} \subset \{ \begin{bmatrix} T_1 & *_1 \\ *_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F) \ , \ T_2 \in \mathrm{Fred}(F^c) \} \ , \tag{8.2.1}$$

be a Banach Lie group that generates a Banach algebra A acting on E, but with possibly a different norm. Here, $*_i$ for $i = 1, 2$ tentatively represent operators on E of some specified class.

Suppose that $\widehat{G}$ acts analytically on $\mathrm{Gr}(A)$ with a typical orbit denoted by $\widehat{\mathrm{Gr}}(A)$. Fixing $p \in P(A)$, let $\widehat{\mathrm{Gr}}(p, A) = \widehat{\mathrm{Gr}}(A) \cap \mathrm{Gr}(p, A)$, and let $\widehat{\mathrm{Gr}}_\alpha(A)$ denote a connected component of $\widehat{\mathrm{Gr}}(A)$ for which $\mathrm{Ind}\ T_1 = \alpha$. Accordingly, we define $\widehat{\mathrm{Gr}}_\alpha(p, A) = \widehat{\mathrm{Gr}}_\alpha(A) \cap \mathrm{Gr}(p, A)$. The restriction $V_\alpha(p, A) = V(p, A)|\ \widehat{\mathrm{Gr}}_\alpha(p, A)$, thus provides a framing for elements of $\widehat{\mathrm{Gr}}_\alpha(p, A)$ [9].

8.3. **Conjugation into the Banach algebra A.** Suppose now that A is a unital Banach algebra with Laurent series generator ζ (with the unique expression property) acting cyclically on E . We observe that the idempotents (projections) of A under the action form certain subspaces of the complex Banach space E . We shall assume a decomposition as in (3.4.1), $E = F \oplus F^c$ where the closed (splitting) subspaces F and F^c are specified as follows.

Let ϕ be a cyclic vector for this action and take F to be the closed linear span of all vectors $\zeta^i \phi$ for $i \geq 0$, and F^c the closed linear span of $\zeta^i \phi$ for $i < 0$. With regards to (8.2.1), we now take the operators $*_1 \in \mathcal{K}(F^c, F)$ and $*_2 \in \mathcal{K}(F, F^c)$.

Here we assume that a fixed $p \in P(A)$ acts as the projection of E on F along F^c . We take $\widehat{\mathrm{Gr}}(p, A)$ to be the Grassmannian consisting of subspaces $W \in \mathrm{Gr}(F, E)$ where $W = q(E)$ for $q \in P(A)$ such that :

(1) the projection $p_1 = pq \ : \ W \longrightarrow F$ is in $\mathrm{Fred}(E)$, and

(2) the projection $p_2 = (1 - p)q \ : \ W \longrightarrow F^c$ is in $\mathcal{K}(E)$.

We define the *big cell* of $\widehat{\mathrm{Gr}}(p, A)$ as the collection of all subspaces W of E such that p_1 is an isomorphism.

Let D be the closed unit disc centered at the origin in $\mathbb{C}$ which contains the spectrum of the generator ζ . We define $\widetilde{\Gamma}_+$ to be the group of (invertible) holomorphic maps $g : D \longrightarrow G(A)$, such that $g(0) = \mathbf{1}$, and define an action of $\widetilde{\Gamma}_+$ on F by $g \cdot v = g(\zeta)v$ where $g(\zeta)$ is given by the holomorphic functional calculus. We also define $\widetilde{\Gamma}_-$ as the group of holomorphic maps of the form $g(\frac{1}{z})$ where $g \in \widetilde{\Gamma}_+$, and it is assumed that $\widetilde{\Gamma}_-$ acts freely and transitively on $\widehat{\mathrm{Gr}}(F, E) = \widehat{\mathrm{Gr}}(p, A)$, where $p \in \mathrm{Sim}(p_1, A)$.

As previously, we consider subspaces $W \in \widehat{\mathrm{Gr}}(p, A)$ of the form

$$W = g h_g F \ , \tag{8.3.1}$$

with $g \in \widetilde{\Gamma}_+$ and $h_g \in \widetilde{\Gamma}_-$. Also for $g \in \widetilde{\Gamma}_+$, we consider projections

$$p_1^g \ : \ g^{-1}(W) \longrightarrow F \ , \ p_1^g \in \mathrm{Fred}(E) \ , \tag{8.3.2}$$

and define

$$\widetilde{\Gamma}_+^W = \{ g \in \widetilde{\Gamma}_+ : p_1^g \text{ is an isomorphism} \} \ . \tag{8.3.3}$$

The operator–valued Baker function associated to W is defined as :

$$\begin{aligned}\psi_W &= (p_1^g)^{-1}(\mathbf{1}) = We^{x\zeta}\ ,\\ &= g(z)(1+\sum_{i=1}^{\infty} a_i(g)\zeta^{-i})\ ,\end{aligned} \tag{8.3.4}$$

where $g \in \widetilde{\Gamma}_+^W$ and the a_i are analytic functions on $\widetilde{\Gamma}_+^W$ extending to meromorphic functions on all of $\widetilde{\Gamma}_+$. By the same principles of §5, we obtain the equation $L\psi_W = \zeta\psi_W$. Let $\alpha = \mathrm{Ind}\ p_+^g$ and recall $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$. Adopting the same strategies used in establishing Theorems 5.3.1 and 7.1.1, leads to :

Proposition 8.3.1. *Let A be a unital Banach algebra with Laurent series generator ζ (with the unique expression property) acting cyclically on E . Relative to subspaces $W \in \widehat{\mathrm{Gr}}_\alpha(p, A)$ given by* (8.3.1)*, the assignment $\partial^{-1} \mapsto \zeta$ induces a conjugation of $\widehat{\mathbb{A}}$ into a subring of A up to constant coefficient operators.*

9. KP–hierarchy in several variables

There does not yet exist a satisfactory generalization of Sato's theory to a Baker function of several variables, though several attempts have been made. One which is close in spirit to the Burchnall-Chaundy theory, namely Parshin's ring, will be outlined here, because our construction of an operator-valued Baker function seems suitable to extending Sato's theory to this situation.

9.1. **Parshin's ring of formal pseudodifferential operators.** In this section we will start by extracting several ideas from [23] [24] to which we refer for complete details.

Let $\mathcal{A}_0$ be an associative ring with derivation $d : \mathcal{A}_0 \longrightarrow \mathcal{A}_0$. As a part generalization of an earlier section, we consider the ring $\mathcal{A}_0[\partial^{-1}]$ of formal pseudodifferential operators in $\mathcal{A}_0$ as a left $\mathcal{A}_0$–module of all formal expressions.

$$L = \sum_{i>-\infty}^{N} a_i\ \partial^i\ ,\ a_i \in \mathcal{A}_0\ . \tag{9.1.1}$$

For a given $a \in \mathcal{A}_0$, we have

$$\begin{aligned}[][\partial, a] &= \partial a - a\partial = d(a)\\ [\partial^{-1}, a] &= \partial^{-1}a - a\partial^{-1} = -d(a)\partial^{-2} + d^2(a)\partial^{-3} - \dots\end{aligned} \tag{9.1.2}$$

Following [23], $\mathcal{A}_0[\partial^{-1}]$ is an associative ring and moreover, the above construction can be iterated. Given N variables $x_1, \dots x_N$, let $\mathbb{C}((x_1))\dots((x_N))$ be the N–dimensional local field of iterated Laurent series where $\mathbb{C}((x_1))\dots((x_i))$ is defined to be the quotient field of $\mathbb{C}((x_1))\dots((x_{i-1}))[[x_i]]$.

On setting $\partial_i = \partial/\partial x_i$, for $1 \leq i \leq N$, we let

$$\begin{aligned}\mathcal{Q} &= \mathbb{C}((x_1))\dots((x_N))((\partial_1^{-1}))\dots((\partial_N^{-1}))\ ,\\ \mathcal{E} &= \mathbb{C}[[x_1, \dots, x_N]]((\partial_1^{-1}))\dots((\partial_N^{-1}))\ .\end{aligned} \tag{9.1.3}$$

Then $\mathcal{E} \subset \mathcal{Q}$ is a subring and there is the following decomposition into non–negative (+) and negative (−) orders of operators as given by :

$$\mathcal{Q} = \mathcal{Q}_+ + \mathcal{Q}_-\ ,\ \mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-\ ,\ \mathcal{E}_\pm = \mathcal{E} \cap \mathcal{Q}_\pm\ . \tag{9.1.4}$$

From the conjugacy theorem of [23] (Th.1), we have :

Theorem 9.1.1. [23] *Let $L_1 \in \partial_1 + \mathcal{E}_-, \ldots,\ L_N \in \partial_N + \mathcal{E}_-$. There exists an operator $K \in 1 + \mathcal{E}_-$, such that*

$$L_1 = K(\partial_1)K^{-1}\ , \ldots ,\ L_N = K(\partial_N)K^{-1} \tag{9.1.5}$$

if and only if $L_1, \ldots, L_N$ pairwise commute.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi–index such that $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$. We will proceed to the $\mathbb{C}[[x]]$–module of partial differential operators

$$\mathcal{P} = \{ \sum_\alpha a_\alpha\ \partial^\alpha\ :\ a_\alpha \in \mathcal{A}_{00}\ ,\ \alpha \in \mathbb{Z}\ \} \subseteq \mathcal{E}\ . \tag{9.1.6}$$

Since $[\partial_i, \partial_j] = 0$, it is a consequence of the Leibnitz rule that $\mathcal{P}$ is endowed with a $\mathbb{C}$–algebra structure. The *residue* of the operator $L \in \mathcal{P}$ is defined as

$$\mathrm{res}_{\mathcal{P}}(L) = a_{-1\ \ldots\ -1} \in \mathbb{C}\ , \tag{9.1.7}$$

and for $L, M \in \mathcal{P}$, the pairing

$$\langle L, M\rangle_{\mathcal{P}} = \mathrm{res}_{\mathcal{P}}(LM)\ , \tag{9.1.8}$$

defines a nondegenerate bilinear form on $\mathcal{P}$ [23].

Remark 9.1.1. Although the orthogonal splitting determined by this pairing plays a major role in the representation theory of the loop group, it has nothing to do with the holomorphic splitting of 4.3.4.

In the following, $\mathbb{C}[[t]]$ denotes $\mathbb{C}[[\{t_\alpha\}]]$ the ring of Taylor series in $\{t_\alpha\}$, $\partial_\alpha = \frac{\partial}{\partial t_\alpha}$, and we take L^α to denote $L_1^{\alpha_1} \cdots L_N^{\alpha_N}$.

The KP–hierarchy for N variables, henceforth KP(N), consists of the following Lax system

$$\begin{gathered} \partial_\alpha L_i = [(L^\alpha)_+\ ,\ L_i]\ , \\ [L_i, L_j] = 0 \text{ (commutativity relations)}\ , \end{gathered} \tag{9.1.9}$$

for $1 \leq i, j \leq N\ ,\ 0 \subset \alpha$, and for which

$$L_i = \partial_i + \sum_\alpha u_{i\alpha}(t)\ \partial^\alpha \in \mathcal{P} \otimes \mathbb{C}[[t]]\ . \tag{9.1.10}$$

Let V denote the vector space $\mathbb{C}((x_1))\ldots((x_N))$ of the iterated Laurent series, with its filtration $V_n = x_N^n\ \mathbb{C}((x_1))\ldots((x_{N-1}))[[x_N]]$, for $n \in \mathbb{Z}$.

Suppose now $\boldsymbol{\mathcal{A}}$ is a (unital) commutative $C*$–algebra. Then $A = V \otimes \boldsymbol{\mathcal{A}}$ becomes an $\boldsymbol{\mathcal{A}}$–module, and via the spatial correspondence, we have the Grassmannian $\mathrm{Gr}(\mathcal{L}(V \otimes \boldsymbol{\mathcal{A}})) \cong \mathrm{Gr}(A)$.

Now we observe the following data (**) relative to the system (9.1.9) (9.1.10) :

1. Let $W_0 = \mathbb{C}[x_1^{-1}, \ldots, x_N^{-1}] \otimes \boldsymbol{\mathcal{A}}$, and let $W_0^c \subset V \otimes \boldsymbol{\mathcal{A}}$ be a complementary subspace such that $W_0 \oplus W_0^c \cong V \otimes \boldsymbol{\mathcal{A}}$.
2. Let $F_0 = \mathrm{Hom}(W_0, \widehat{W}_0^c) \subset \mathrm{Gr}(V \otimes \boldsymbol{\mathcal{A}})$, where $\widehat{W}_0^c$ is a commensurable subspace formed by taking a certain inverse limit just as in [24] (note that F_0 here plays the role of the 'big cell').

3. We have a nondegenerate bilinear $\boldsymbol{\mathcal{A}}$–valued form on $\mathcal{P} \otimes \boldsymbol{\mathcal{A}}$ defined by :

$$\langle L \otimes a \ , \ M \otimes b \rangle \ = \ \langle L, M \rangle_{\mathcal{P}} \ (a^* b) \ ,$$

for $L, M \in \mathcal{P}$ and $a, b \in \boldsymbol{\mathcal{A}}$, that induces the same $\langle \ , \ \rangle$ on $V \otimes \boldsymbol{\mathcal{A}}$, via $\partial^{\alpha} \mapsto x_{\alpha}$.

Theorem 9.1.2. *Let $\boldsymbol{\mathcal{A}}$ be a (unital) commutative $C*$–algebra. Given the data (**) as above, there exists for a given $W \in F_0 \subset \mathrm{Gr}(A)$, an eigenfunction ψ_W belonging to a class of wave functions which induces a conjugation of the ring $\mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ as a subring of the algebra $A = \mathcal{L}(V \otimes \boldsymbol{\mathcal{A}})$.*

Proof. The proof is a straightforward modification of the techniques of [24]. Firstly, we establish a preliminary conjugation result. Consider the operators

$$L_i = \partial_i + \sum_{\alpha} u_{i\alpha}(t,a) \ \partial^{\alpha} \in \mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}} \ , \ u_{i\alpha}(t,a) \in \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}} \ . \tag{9.1.11}$$

Then there exists $K \in 1 + \mathcal{P}_{-} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ such that for $1 \leq i \leq N$,

$$\begin{aligned} L_i &= K(\partial_i) K^{-1} \ , \\ \partial_{\alpha} K &= -(K(\partial^{\alpha})K^{-1})_{-} \ K \ . \end{aligned} \tag{9.1.12}$$

Equivalently, the operators $\{L_1, \ldots, L_N\}$ admit a wave function $\psi(x,t,a)$ satisfying :

$$\begin{aligned} L_i \psi &= x_i \cdot \psi \ , \qquad 1 \leq i \leq N \ , \\ \partial_{\alpha} \psi &= (L^{\alpha})_{+} \psi \ , \qquad 0 \subset \alpha \ . \end{aligned} \tag{9.1.13}$$

The next step is to show that there exists a Baker function corresponding to a point in $\mathrm{Gr}(A)$ that is indeed such a wave function, and one that can be implemented so as to achieve the conjugation into the algebra A, just as for the one–variable case. In order to see this, we introduce formal oscillating functions

$$\phi(t,x,a) = (1 + \sum_{0 \subset \alpha} u_{\alpha}(t,a) \ x^{\alpha}) \ e^{\xi(t,x)} \ , \tag{9.1.14}$$

where

$$e^{\xi(t,x)} := \exp(\sum_{0 \subset \beta} t_{\beta} \ x^{-\beta}) \ , \qquad t = \{t_{\beta} : 0 \subset \beta\} \ , \tag{9.1.15}$$

and $u_{\alpha}(t,a) \in \mathbb{C}[[\{t_{\alpha}\}_{0 \subset \alpha}]] \otimes \boldsymbol{\mathcal{A}}$.

Relative to $W \in \mathrm{Gr}(A)$ we define the function ψ_W and a basis $w = \{w_i\}_{i \in I} \in V(p, A)$, such that

$$\psi_W(t,x,a) = \sum_{i \in I} t_{v(w_i)} w_i \ , \tag{9.1.16}$$

where $v : A \longrightarrow \mathbb{Z}^N$ maps $\sum_{\alpha} u_{\alpha} \ x^{\alpha}$ to the multi–index α such that $u_{\alpha} = 0 \ , \ \forall \beta < \alpha$.

Using the bilinear form on $V \otimes \boldsymbol{\mathcal{A}}$ in the data (**) 3., there is the following 'orthogonality' relation deduced from [24]. Relative to $W \in F_0 \subset \mathrm{Gr}(A)$ in (**) 2., we have for all s, t :

$$\langle \psi_W(t,x,a) \ , \ \phi_W(s, x_N, a) \rangle \ = 0 \ . \tag{9.1.17}$$

In particular, there exists a natural map from the set of such wave functions for the $\mathrm{KP}(N)$ hierarchy to $F_0 \subset \mathrm{Gr}(V)$, and conversely, if $W \in F_0$, then the Baker function corresponding to W is a wave function for this hierarchy (these facts follow essentially from [24] Theorems 3.5 and 3.6). Thus we have established the desired properties of the function ψ_W which implements the conjugation of $\mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ into a subring of A . □

10. Concluding remarks

In this paper, an operator-theory version of Sato's universal Grassmann manifold is constructed. There are obvious advantages in endowing with a (Banach) topology the algebra of linear maps on which the Grassmannian is modeled, as well as allowing non-commutative coefficients. However, the main application we have in mind is a construction still in the making, namely a multivariable version of the Boson-Fermion correspondence, which would produce KP-like flows on a Grassmannian.

For that purpose, we focus our analysis on the Baker and τ functions associated to an element of the Grassmannian, and their role in conjugating the commutative ring of differential operators that stabilize that element, into one with constant coefficients (Theorem 5.3.1, Proposition 8.3.1 and Theorem 9.1.2). We give a version of operator-valued Baker function; we can also work out a version of operator-valued Plücker embedding and (consequently, since it is the global section of the embedding line bundle) τ function; we will write that part in the full-fledged version of this report.

We conclude by sketching Parshin's beautiful version of the KP hierarchy for a suitable non-commutative ring, in fact a higher local field. He gave no Grassmannian or Baker function model. An adaptation of Sato's theory was given in [24], but a more complete translation into the language of formal Grassmannian and formal Baker function was given more recently in [17], [18], [19]. It is to the latter that our programme would be best suited, since [19]'s Grassmannian is indeed obtained by singling out one variable, and allowing non-commutative coefficients (a Parshin ring in the remaining variables).

Another prospective application of our topologized programme would be to use the operator norm to describe the (co)tangent bundle to the Grassmannian (Proposition 5.3.2). This would be then suitable to describe the Hitchin system (cf. [20]) and, more importantly, Hitchin's projective connection, in the vein of [3], where a Grassmannian version of the heat equation for line bundles is given.

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Department of Mathematics, Tulane University, New Orleans, LA 70118 USA
E-mail address, M. J. Dupré: `mdupre@tulane.edu`

(Primary inst.) Department of Mathematics and Computer Science, Eastern Illinois University, Charleston IL 61920 USA, and

Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana IL 61801 USA
E-mail address, J. F. Glazebrook: `cfjfg@eiu.edu` , `glazebro@math.uiuc.edu`

Department of Mathematics and Statistics, Boston University, 111 Cummington Street, Boston MA 02215-2411 USA
E-mail address, E. Previato: `ep@bu.edu`