ALGEBRAIC 3–FOLDS AND DIAGRAM METHOD. II

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Wegen der Stetigkeit der Abbildung von A auf E können wir eine positive Zahl $\varepsilon_n^x < \frac{1}{n}$ derart bestimmen daß die Menge $A \cdot S(x, \varepsilon_n^x)$

($S(x,\varepsilon)$ bedeutet die Menge der Punkte y für die $\rho(x,y) < \varepsilon$ ist)

eine Menge $B_n^x \subset E$ abgebildet werde, deren Breite $< \frac{1}{n}$ ist. Wir setzen dann

$$G_n = \sum_{x \subset A} S(x, \varepsilon_n^x)$$

(Vereinigungsmenge über alle x von A).

Offenbar ist $A \subset \prod_{n=1}^{\infty} G_n$

Es sei anderseits $\xi \subset \prod_{n=1}^{\infty} G_n$. Dann gibt es für jedes n ein x_n, so daß $\xi \subset S(x_n, \varepsilon_n)$; $(\varepsilon_n = \varepsilon_n^{x_n})$.

Da $x_n \subset A$ und $\varepsilon_n < \frac{1}{n}$, folgt hieraus $\xi \subset A_\alpha$.

Es seien nun n und m zwei verschiedene natürliche Zahlen; da ξ im Gebiete $S(x_n, \varepsilon_n) \cdot S(x_m, \varepsilon_m)$ enthalten ist, folgt aus $\xi \subset A_\alpha$ die Existenz eines Punktes $y_{n,m}$ aus A der in diesem Gebiete enthalten ist.

Die ~~Abbildungen~~ Bildpunkte von x_n bzw $y_{n,m}$ in E seien u_n, bzw $v_{n,m}$

Da $x_n + y_{n,m} \subset A \cdot S(x_n, \varepsilon_n)$, folgt aus der Bedingung für

ALGEBRAIC 3-FOLDS AND DIAGRAM METHOD.II

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0. Introduction.

Here we continue investigations of our paper [16], where we were developing the diagram method for non-singular algebraic 3-folds and 3-folds with simplest singularities. In this paper we transfer this results on 3-folds with terminal singularities.

In the introduction we will formulate the results only for Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities.

Let X be a Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Let R be an extremal ray of Mori polyhedron of X. We say that R has the *type (I)* (respectively *(II)*) if curves of R fill an irreducible divisor $D(R)$ of X and the contraction of the ray R contracts the divisor $D(R)$ in a point (respectively on a curve). An extremal ray R is called *small* if curves of this ray fill a curve on X.

A set $\mathcal{E}$ of extremal rays is called *extremal* if it is contained in a face γ of Mori polyhedron. The $\mathcal{E}$ has *Kodaira dimension 3* if the contraction of γ gives a morphism on a 3-fold. The first result of the paper gives a description for Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities of extremal sets $\mathcal{E}$ of Kodaira dimension 3 which contain only extremal rays of the type (I) or (II).

A set $\mathcal{L}$ of extremal rays is called *E-set* if $\mathcal{L}$ is not extremal but any proper subset of $\mathcal{L}$ is extremal. The second result of the paper gives for Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities a description of E-sets $\mathcal{L}$ of extremal rays such that any proper subset of $\mathcal{L}$ is extremal of Kodaira dimension 3 and $\mathcal{L}$ contains extremal rays of the type (I) or (II) only.

From this description of extremal sets and E-sets we get the following basic result of the paper.

Theorem. *Let X be Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Then for X one of the following statements holds:*

(1) $\dim N_1(X) \leq 8$.

(2) There exists a face of Mori polyhedron $\overline{NE}(X)$ such that the contraction of this face gives a morphism of X on a surface or a curve.

(3) There exists a small extremal ray R on X.

(4) There are $2t \geq 4$ different linear dependent extremal rays $R_{11}, R_{12}, R_{21}, R_{22}, ..., R_{t1}, R_{t2}$ of the type (II) such that the divisors $D(R_{i1}) = D(R_{i2})$ are coincided, $1 \leq i \leq t$, but the divisors $D(R_{i1}) = D(R_{i2})$ and $D(R_{j1}) = D(R_{j2})$ don't intersect one another for $1 \leq i < j \leq t$. (See figure 1, type $\mathfrak{B}_2$ below.)

(5) There are three different extremal rays S_1, R_1, R_2 of the type (II) such that the divisors $D(S_1) = D(R_2)$ are coincided, and $R_1 \cdot D(R_2) > 0, R_2 \cdot D(R_1) > 0$, and $S_1 \cdot D(R_2) = 0$. (See figure 2 below.)

We hope that later it will be possible to exclude the possibilities (5),(4),(2) and maybe (3) for some greater than 8 constant in (1). We should say that now it is not known that $\dim N_1(X)$ is bounded for Fano 3-folds X with $\mathbf{Q}$-factorial terminal singularities.

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CHAPTER 1. DIAGRAM METHOD.

Here we will give a simplest variant of the diagram method for multi-dimensional algebraic varieties. Precisely this method we shall use in the following chapter.

Let X be a projective algebraic variety with $\mathbf{Q}$-factorial singularities over an algebraically closed field. Let $\dim X \geq 2$. Let $N_1(X)$ be the $\mathbf{R}$-linear space generated by all algebraic curves on X by the numerical equivalence, and let $N^1(X)$ be the $\mathbf{R}$-linear space generated by all Cartier (or Weil) divisors on X by the numerical equivalence. Linear spaces $N_1(X)$ and $N^1(X)$ are dual one another by the intersection pairing. Let $NE(X)$ be a convex cone in $N_1(X)$ generated by all effective curves on X. Let $\overline{NE}(X)$ be the closer of the cone $NE(X)$ in $N_1(X)$. It is called *Mori cone (*or *polyhedron)* of X. A non-zero element $x \in N^1(X)$ is called *nef* if $x \cdot \overline{NE}(X) \geq 0$. Let $NEF(X)$ be the set of all nef elements of X and the zero. It is the convex cone in $N^1(X)$ dual to Mori cone $\overline{NE}(X)$. A ray $R \subset \overline{NE}(X)$ with origin 0 is called *extremal* if from $C_1 \in \overline{NE}(X)$, $C_2 \in \overline{NE}(X)$ and $C_1 + C_2 \in R$ it follows that $C_1 \in R$ and $C_2 \in R$.

We consider the following conditions (i), (ii) and (iii) for some set $\mathcal{R}$ of extremal rays on X.

(i)*If $R \in \mathcal{R}$ then all curves $C \in R$ fill out an irreducible divisor $D(R)$ on X.*

In this case we can correspond to $\mathcal{R}$ (and subsets of $\mathcal{R}$) an oriented graph $G(\mathcal{R})$ in the following way: Two different rays R_1 and R_2 are joined by an arrow R_1R_2 with the beginning in R_1 and the end in R_2 if $R_1 \cdot D(R_2) > 0$. Here and in what follows, for an extremal ray R and a divisor D we write $R \cdot D > 0$ if $r \cdot D > 0$ for $r \in R$ and $r \neq 0$. (The same for the symbols $\leq$, $\geq$ and $<$.)

A set $\mathcal{E}$ of extremal rays is called *extremal* if it is contained in a face of $\overline{NE}(X)$. Equivalenty, there exists a nef element $H \in N^1(X)$ such that $\mathcal{E} \cdot H = 0$. Evidently, a subset of an extremal subset is extremal too.

We consider the following condition (ii) for extremal subsets of $\mathcal{R}$.

(ii)*If $\mathcal{E} = \{R_1, ..., R_n\} \subset \mathcal{R}$ is extremal and $m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)$ is an effective divisor, then there exists a ray $R_j \in \mathcal{E}$ such that $R_j \cdot (m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)) < 0$. In particular, it follows that the divisor $m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)$ is not nef.*

A set $\mathcal{L}$ of extremal rays is called *E-subset* if the $\mathcal{L}$ is not extremal but every proper subset of $\mathcal{L}$ is extremal.

We consider the following condition (iii) for E-subsets of $\mathcal{R}$.

(iii) *If $\mathcal{L} = \{R_1, ..., R_n\} \subset \mathcal{R}$ is an E-subset, then there exists a non-zero effective nef divisor $D(\mathcal{L}) = m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)$.*

Lemma 1.1. *Suppose that for a set $\mathcal{R}$ of extremal rays the conditions (i), (ii) and (iii) above hold.*

Then any E-subset $\mathcal{L} \subset \mathcal{R}$ is connected in the following sense: For any decomposition $\mathcal{L} = \mathcal{L}_1 \coprod \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are not empty, there exists an arrow R_1R_2 such that $R_1 \in \mathcal{L}_1$ and $R_2 \in \mathcal{L}_2$. If $\mathcal{L} \subset \mathcal{R}$ and $\mathcal{M} \subset \mathcal{R}$ are two different E-subsets, then there exists an arrow LM where $L \in \mathcal{L}$ and $M \in \mathcal{M}$.

Proof. Let $\mathcal{L} = \{R_1, ..., R_n\}$. By (iii), there exists a nef divisor $D(\mathcal{L}) = m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)$. If one of the coefficients $m_1, ..., m_n$ is equal to zero, we get a contradiction with the conditions (ii) and (iii). It follows that all the coefficients $m_1, ..., m_n$ are positive. Let $\mathcal{L} = \mathcal{L}_1 \coprod \mathcal{L}_2$ where $L_1 = \{R_1, ..., R_k\}$ and $L_2 = \{R_{k+1}, ..., R_n\}$. The divisors $D_1 = m_1D(R_1) + ... + m_kD(R_k)$ and $D_2 = m_{k+1}D(R_{k+1}) + ... + m_nD(R_n)$ are non-zero. By (ii), there exists a ray $R_i, 1 \leq i \leq k$, such that $R_i \cdot D_1 < 0$. On the other hand, $R_i \cdot D(\mathcal{L}) = R_1 \cdot (D_1 + D_2) \geq 0$. It follows, that there exists j, $k+1 \leq j \leq n$, such that $R_i \cdot D(R_j) > 0$. It means that R_iR_j is an arrow.

Let us prove the second statement. By the condition (iii), for every ray $R \in \mathcal{L}$, we have the inequality $R \cdot D(\mathcal{M}) \geq 0$. If $R \cdot D(\mathcal{M}) = 0$ for any $R \in \mathcal{L}$, then the set $\mathcal{L}$ is extremal, and we get the contradiction. It follows that there exists a ray $R \in \mathcal{L}$ such that $R \cdot D(\mathcal{M}) > 0$. It follows the statement. ▷

The Theorem 1.2 below is an analog for algebraic varieties of arbitrary dimension of the Lemma 3.4 from [7] and the Lemma 1.4.1 from [10], which were devoted to surfaces.

Let $NEF(X) = \overline{NE}(X)^* \subset N^1(X)$ be the cone of nef elements of X and $\mathcal{M}(X) = NEF(X)/\mathbf{R}^+$ its projectivization. Let $\mathcal{R}(X)$ be the set of all extremal rays of X. If γ is a face of $\mathcal{M}(X)$, then

$$\mathcal{R}(\gamma) = \{R \in \mathcal{R}(X) \mid \exists \mathbf{R}^+ H \in \gamma : R \cdot H = 0\}.$$

A convex polyhedron is called *closed* if it is the convex hull of a finite set of points. A closed polyhedron is called *simplicial* if all its faces are simplexes. A closed polyhedron is called *simple* (equivalently, it has simplicial angles) if it is dual to a simlicial one. In other words, a polyhedron is simple if its face of codimension k is contained exactly in k its faces of the highest dimension. A polyhedron $\mathcal{M}$ is called *simple in a face* γ if every face $\gamma_1 \subset \gamma$ (including $\gamma_1 = \gamma$) is contained exactly in codim γ_1 (in $\mathcal{M}$) highest dimension faces of $\mathcal{M}$. In other words, for $\gamma_1 \subset \gamma$, the dual face γ_1^* is a simplex of the dual polyhedron $\mathcal{M}^*$.

Let A, B be two vertices of an oriented graph G. The *distance* $\rho(A, B)$ in G is a length (the number of links) of the shortest oriented path of the graph G with the beginning in A and the end in B. The distance is $+\infty$ if this path does not exist. The *diameter* diam G of the oriented graph G is the maximum distance between ordered pairs of its vertices. By the Lemma 1.1, the diameter of an E-subset is a finite number.

Theorem 1.2. *Let X be a projective algebraic variety with $\mathbf{Q}$-factorial singularities and dim $X \geq 2$. Let γ be a closed face of $\mathcal{M}(X)$ and $\mathcal{M}(X)$ is simple in the face γ. Suppose that the set $\mathcal{R}(\gamma)$ satisfies the conditions (i), (ii) and (iii) above. Suppose that there are some constants d, C_1, C_2 such that the conditions (a) and (b) below hold:*

(a) diam $\mathcal{L} \leq d$ *for every E-subset* $\mathcal{L} \subset \mathcal{R}(\gamma)$;

(b) $\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \sharp\mathcal{E}$;

and

$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid d+1 \leq \rho(R_1, R_2) \leq 2d+1\} \leq C_2 \sharp\mathcal{E}$ *for every extremal subset* $\mathcal{E} \subset \mathcal{R}$.

Then dim $\gamma < (16/3)C_1 + 4C_2 + 6$.

Proof.

We use the following Lemma 1.3 which was proved in [6]. The Lemma was used in [6] to get a bound (≤ 9) on the dimension of a hyperbolic (Lobachevsky) space admitting an action of an arithmetic reflection group with a field of definition of the degree $> N$. Here N is some constant.

Lemma 1.3. *Let $\mathcal{M}$ be a convex closed simple polyhedron of the dimension n, and $A_n^{i,k}$ the average number of i-dimensional faces of k-dimensional faces of $\mathcal{M}$.*

Then for $n \geq 2k - 1$

$$A_n^{i,k} < \frac{\binom{n-i}{n-k} \cdot \left(\binom{[n/2]}{i} + \binom{n-[n/2]}{i}\right)}{\binom{[n/2]}{k} + \binom{n-[n/2]}{k}}.$$

In particular, if $n \geq 3$

$$A_n^{0,2} < \begin{cases} \frac{4(n-1)}{n-2} & \text{if } n \text{ is even,} \\ \frac{4n}{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. See [6]. ▷

From the estimate of $A_n^{0,2}$ of the Lemma, it follows the following analog of Vinberg's Lemma from [13]. Vinberg's Lemma was used by him to obtain an estimate (dim < 30) for the dimension of a hyperbolic space admitting an action of a discrete reflection group with bounded fundamental polyhedron.

Lemma 1.4. *Let $\mathcal{M}$ be a convex simple polyhedron of the dimension n. Let C be a positive number and D a number. Suppose that oriented angles (2-dimensional plane) of $\mathcal{M}$ are supplied with weights and the following conditions (1) and (2) hold:*

(1) The sum of weights of all oriented angles at any vertex of M is not greater than $Cn + D$.

(2) The sum of weights of all oriented angles of any 2-dimensional face of $\mathcal{M}$ is at least $5 - k$ where k is the number of vertices of the 2-dimensional face.

Then

$$n < 8C + 2 + \begin{cases} (8d+8)/n & \text{if } n \text{ is even,} \\ (8d+7)/n & \text{if } n \text{ is odd} \end{cases} \leq 8C + 4D + 6.$$

Proof. We correspond to a non-oriented angle of $\mathcal{M}$ the weight which is equal to the sum of two corresponding oriented angles. Evidently, the conditions of the Lemma hold for the weights of non-oriented angles too if we forget about the word "oriented". Then we obtain Vinberg's lemma from [13] which we formulate a little bit more precisely here. Since the proof is very simle, we give the proof here.

Let Σ be the sum of all (non-oriented) angles of the polyhedron $\mathcal{M}$. Let α_0 be the number of vertices of $\mathcal{M}$ and α_2 the number of 2-dimensional faces of $\mathcal{M}$. Since $\mathcal{M}$ is simple,

$$\alpha_0 \frac{n(n-1)}{2} = \alpha_2 A_n^{0,2}.$$

From this inequality and the conditions of the Lemma, it follows

$$(Cn+D)\alpha_0 \geq \Sigma \geq \sum \alpha_{2,k}(5-k) = 5\alpha_2 - \alpha_0 = \alpha_0\left(\frac{n(n-1)}{2A_n^{0,2}} - 1\right),$$

where $\alpha_{2,k}$ is the number of 2-dimensional faces with k vertices of $\mathcal{M}$. From this inequalities and the bound for $A_n^{0,2}$ of the Lemma 1.3, we get the Lemma 1.4. $\rhd$

The proof of the Theorem 1.2. (Cf. [13].) Let Δ be an oriented angle of γ. Let $\mathcal{R}(\Delta) \subset \mathcal{R}(\gamma)$ be the set of all extremal rays of $\mathcal{M}(X)$ which are orthogonal (with respect to the intersection pairing) to the vertex of Δ. We can define the extremal rays $R_1(\Delta)$ and $R_2(\Delta)$ by the conditions that $\mathcal{R}(\Delta) - \{R_2(\Delta)\}$ and $\mathcal{R}(\Delta) - \{R_1(\Delta)\}$ are orthogonal to the first and second side of the oriented angle Δ respectively. Evidently, the set $\mathcal{R}(\Delta)$ and the ordered pair of rays (R_1, R_2) define the oriented angle Δ uniquely. We define the weight $\sigma(\Delta)$ by the formula:

$$\sigma(\Delta) = \begin{cases} 2/3, & \text{if } 1 \leq \rho(R_1(\Delta), R_2(\Delta)) \leq d, \\ 1/2, & \text{if } d+1 \leq \rho(R_1(\Delta), R_2(\Delta)) \leq 2d+1, \\ 0, & \text{if } 2d+2 \leq \rho(R_1(\Delta), R_2(\Delta)). \end{cases}$$

Here we take the distance in the subgraph $G(\mathcal{R}(\Delta))$. Let us prove the conditions of the Lemma 1.4 with the constants $C = (2/3)C_1 + C_2/2$ and $D = 0$.

The condition (1) is obvious.

Let us prove the condition (2). Let γ_3 be a triangle of γ. The set $\mathcal{R}(\gamma_3)$ is the union of the set $\overline{\mathcal{R}(\gamma_3)}$ of extemal rays, which are orthogonal to the plane of the triangle γ_3, and the rays R_1, R_2, R_3, which are orthogonal to the sides of the triangle γ_3. The union of the set $\overline{\mathcal{R}(\gamma_3)}$ with every two rays from R_1, R_2, R_3 is extremal, since it is orthogonal to a vertex of γ_3. On the other hand, the set $\mathcal{R}(\gamma_3) = \overline{\mathcal{R}(\gamma_3)} \cup \{R_1, R_2, R_3\}$ is not extremal, since it is not orthogonal to a point of $\mathcal{M}(X)$. Indeed, the sets of points of $\mathcal{M}(X)$, which are orthogonal to the sets $\overline{\mathcal{R}(\gamma_3)} \cup \{R_2, R_3\}$, $\overline{\mathcal{R}(\gamma_3)} \cup \{R_1, R_3\}$, and $\overline{\mathcal{R}(\gamma_3)} \cup \{R_1, R_2\}$ are the vertices A_1, A_2 and A_3 respectively of the triangle γ_3, and the intersection of the sets of vertices is empty. Thus, there exists an E-subset $\mathcal{L} \subset \mathcal{R}(\gamma_3)$, which contains the rays R_1, R_2, R_3. By the condition (a), the graph $G(\mathcal{L})$ contains an oriented path s of the length $\leq d$ which connects the rays R_1, R_3. If this path does not contain the ray R_2, then the oriented angle of γ_3 defined by the set $\overline{\mathcal{R}(\gamma_3)} \cup \{R_1, R_3\}$ and the pair (R_1, R_3) has the weight $2/3$. If this path contains the ray R_2, then the oriented angle of γ_3 defined by the set $\overline{\mathcal{R}(\gamma_3)} \cup \{R_1, R_2\}$ and the pair (R_1, R_2) has the weight $2/3$. Thus, we proved that the side A_2A_3 of the triangle γ_3 defines an oriented angle of the triangle with the weight $2/3$ and the first side A_2A_3 of the oriented angle. The triangle has three sides. It follows the condition (2) of the Lemma 1.4 for the triangle.

Let γ_4 be a quadrangle of γ. In this case,

$$\mathcal{R}(\gamma_4) = \overline{\mathcal{R}(\gamma_4)} \cup \{R_1, R_2, R_3, R_4\}$$

where $\overline{\mathcal{R}(\gamma_4)}$ is the set of all extremal rays which are orthogonal to the plane of the quadrangle and the rays R_1, R_2, R_3, R_4 are orthogonal to the consecutive sides of the quadrangle. As above, one can see that the sets $\overline{\mathcal{R}(\gamma_4)} \cup \{R_1, R_3\}$, $\overline{\mathcal{R}(\gamma_4)} \cup \{R_2, R_4\}$ are not extremal, but the sets $\overline{\mathcal{R}(\gamma_4)} \cup \{R_1, R_2\}$, $\overline{\mathcal{R}(\gamma_4)} \cup \{R_2, R_3\}$, $\overline{\mathcal{R}(\gamma_4)} \cup \{R_3, R_4\}$, and $\overline{\mathcal{R}(\gamma_4)} \cup \{R_4, R_1\}$ are extremal. It follows that there are E-sets $\mathcal{L}, \mathcal{N}$ such that $\{R_1, R_3\} \subset \mathcal{L} \subset \overline{\mathcal{R}(\gamma_4)} \cup \{R_1, R_3\}$ and $\{R_2, R_4\} \subset \mathcal{N} \subset \overline{\mathcal{R}(\gamma_4)} \cup \{R_2, R_4\}$. By the Lemma 1.1, there are rays

$R \in \mathcal{L}$ and $Q \in \mathcal{N}$ such that RQ is an arrow. By the condition (a) of the Theorem, one of the rays R_1, R_3 is joined by an oriented path s_1 of the length $\leq d$ with the ray R and this path does not contain another ray from R_1, R_3 (here R is the end of the path s_1). We can suppose that this ray is R_1 (otherwise, one should replace the ray R_1 by the ray R_3). As above, we can suppose that the ray Q is connected by the oriented path s_2 of the length $\leq d$ with the ray R_2 and this path does not contain the ray R_4. The path $s_1 RQ s_2$ is an oriented path of the length $\leq 2d+1$ in the oriented graph $\mathrm{G}(\overline{\mathcal{R}(\gamma_4)}) \cup \{R_1, R_2\})$. It follows that the angle of the quadrangle γ_4, such that the consecutive sides of this angle are orthogonal to the rays R_1 and R_2, has the weight $\geq 1/2$. Thus, we proved that for a pair of opposite sides of γ_4 there exists an oriented angle with weight $\geq 1/2$ such that the first side of this oriented angle is one of this opposite sides of the quadrangle. A quadrangle has two pairs of opposite sides. It follows that the sum of weights of oriented angles of γ_4 is ≥ 1. It proves the condition (2) of the Lemma 1.4 and the Theorem. $\triangleright$.

Below, we will apply the Theorem 1.2 to threefolds with singularities.

CHAPTER 2. THREEFOLDS.

1. Contractible extremal rays.

We consider normal projective 3-folds X with $\mathbf{Q}$-factorial singularities.

Let R be an extremal ray of Mori polyhedron $\overline{NE}(X)$ of X. A morphism $f : X \to Y$ on a normal projective variety Y is called the *contraction* of the ray R if for an irreducible curve C of X the image $f(C)$ is a point iff $C \in R$. The contraction f is defined by a linear system H on X (H gives the nef element of $N^1(X)$, which we denote by H also). It follows that an irreducible curve C is contracted iff $C \cdot H = 0$. We assume that the contraction f has properties: $f_*\mathcal{O}_X = \mathcal{O}_Y$ and the sequence

$$0 \to \mathbf{R}R \to N_1(X) \to N_1(Y) \to 0 \qquad (1.1)$$

is exact where the arrow $N_1(X) \to N_1(Y)$ is f_*. An extremal ray R is called *contractible* if there exists its contraction f with these properties.

The number $\kappa(R) = \dim Y$ is called *Kodaira dimension* of the contractible extremal ray R.

A face γ of $\overline{NE}(X)$ is called *contractible* if there exists a morphism $f : X \to Y$ on a normal projective variety Y such that $f_*\gamma = 0$, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f contracts curves lying in γ only. The $\kappa(\gamma) = \dim Y$ is called *Kodaira dimension of* γ.

Let H be a general nef element orthogonal to a face γ of Mori polyhedron. *Numerical Kodaira dimension of* γ is defined by the formula

$$\kappa_{num}(\gamma) = \begin{cases} 3, & \text{if } H^3 > 0; \\ 2, & \text{if } H^3 = 0 \text{ and } H^2 \not\equiv 0; \\ 1, & \text{if } H^2 \equiv 0. \end{cases}$$

It is obvious that for a contractible face γ we have $\kappa_{num}(\gamma) \geq \kappa(\gamma)$. In particular, $\kappa_{num}(\gamma) = \kappa(\gamma)$ for a contractible face γ of Kodaira dimension $\kappa(\gamma) = 3$.

2. Pairs of extremal rays of Kodaira dimension three lying in contractible faces of $\overline{NE}(X)$ of Kodaira dimension three.

Further X is a projective normal threefold with $\mathbf{Q}$-factorial singularities.

Lemma 2.2.1. *Let R be a contractible extremal ray of Kodaira dimension 3 and $f : X \to Y$ its contraction.*

Then there are three possibilities:

(I) All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ in a point and $R \cdot D(R) < 0$.

(II) All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ on an irreducible curve and $R \cdot D(R) < 0$.

(III) All curves $C \in R$ give a finite set of irreducible curves and the contraction f contracts these curves in points.

Proof. Assume that some curves of R fill an irreducible divisor D. Then $R \cdot D < 0$ (this inequality follows from the Proposition 2.2.6 below). Suppose that $C \in R$ and D does not contain C. It follows that $R \cdot D \geq 0$. We get a contradiction. It follows the Lemma. ▷

According to the Lemma 2.2.1, we say that an extremal ray R has the *type (I), (II) or (III)* if it is contractible of Kodaira dimension 3 and the statements (I), (II) or (III) respectively hold.

Lemma 2.2.2. *Let R_1 and R_2 are two different extremal rays of the type (I). Then divisors $D(R_1)$ and $D(R_2)$ do not intersect one another.*

Proof. Otherwise $D(R_1)$ and $D(R_2)$ have a common curve and the rays R_1 and R_2 are not different. ▷

For a divisor D on X let

$$\overline{NE}(X, D) = (\text{image } N_1(D)) \cap \overline{NE}(X).$$

Lemma 2.2.3. *Let R be an extremal ray of the type (II), and f its contraction.*
Then $\overline{NE}(X, D(R)) = \mathbf{R}^+F + \mathbf{R}^+S$, where F is a fiber of f and $\mathbf{R}^+f_*S = \mathbf{R}^+(f(D))$.

Proof. This follows at once from the exact sequence (1.1). $\triangleright$

Lemma 2.2.4. *Let R_1 and R_2 are two different extremal rays of the type (II) such that the divisors $D(R_1) = D(R_2)$.*
Then for $D = D(R_1) = D(R_2)$ we have:

$$\overline{NE}(X, D) = R_1 + R_2.$$

In particular, do not exist three different extremal rays of the type (II) such that their divisors are coincided.

Proof. This follows from the Lemma 2.2.3. $\triangleright$

Lemma 2.2.5. *Let R be an extremal ray of the type (II) and f its contraction.*

Then there does not exist more than one extremal ray Q of the type (I) such that $D(R) \cap D(Q)$ is not empty. If Q is this ray, then $D(R) \cap D(Q)$ is a curve and any irreducible component of this curve is not contained in fibers of f.

Proof. The last statement is obvious. Let us proof the first one. Suppose that Q_1 and Q_2 are two different extremal rays of the type (I) such that $D(Q_1) \cap D(R)$ and $D(Q_2) \cap D(R)$ are not empty. Then the plane angle $\overline{NE}(X, D(R))$ (see the Lemma 2.2.3) contains three different extremal rays: Q_1, Q_2 and R. It is impossible. $\triangleright$

The following key Proposition is very important.

Proposition 2.2.6. *Let X be a projective 3-fold with $\mathbf{Q}$-factorial singularities, $D_1, ..., D_m$ irreducible divisors on X and $f : X \to Y$ a surjective morphism such that $\dim X = \dim Y$ and $\dim f(D_i) < \dim D_i$. Let $y \in f(D_1) \cap ... \cap f(D_m)$.*

Then there are $a_1 > 0$, ..., $a_m > 0$ and an open U, $y \in U \subset f(D_1) \cup ... \cup f(D_m)$, such that

$$C \cdot (a_1 D_1 + ... + a_m D_m) < 0$$

if a curve $C \subset D_1 \cup ... \cup D_m$ belongs to a non-trivial algebraic family of curves on $f(D_1) \cup ... \cup f(D_m)$ and $f(C) =$point$\in U$.

Proof. It is the same as for the well-known case of surfaces (but, for surfaces, it is not necessary to suppose that C belongs to a nontrivial algebraic family). Let H be an irreducible ample divisor on X and $H' = f_*H$. Since $\dim f(D_i) < \dim D_i$, it follows that $f(D_1) \cup ... \cup f(D_m) \subset H'$. Let ϕ be a non-zero rational function on Y which is regular in a neighborhood U of y on Y and is equal to zero on the divisor H'. In the open set $f^{-1}(U)$ the divisor

$$(f^*\phi) = \sum_{i=1}^{m} a_i D_i + \sum_{j=1}^{n} b_j Z_j$$

where all $a_i > 0$ and all $b_j > 0$. Here every divisor Z_j is different from any divisor D_i. We have

$$0 = C \cdot \sum_{i=1}^{m} a_i D_i + C \cdot \sum_{j=1}^{n} b_j Z_j.$$

Here $C \cdot (\sum_{j=1}^{n} b_j Z_j) > 0$ since C belongs to a nontrivial algebraic family of curves on a surface $f(D_1) \cup ... \cup f(D_m)$ and one of the divisors Z_j is the hyperplane section H. $\triangleright$

Lemma 2.2.7. *Let R_1, R_2 are two extremal rays of the type (II), divisors $D(R_1), D(R_2)$ are different and $D(R_1) \cap D(R_2) \neq \emptyset$. Assume that R_1, R_2 belong to a contractible face of $\overline{NE}(X)$ of Kodaira dimension 3. Let $0 \neq F_1 \in R_1$ and $0 \neq F_2 \in R_2$.*
Then

$$(F_1 \cdot D(R_2))(F_2 \cdot D(R_1)) < (F_1 \cdot D(R_1))(F_2 \cdot D(R_2)).$$

Proof. Let f be the contraction of a face of Kodaira dimension 3, which contains both rays R_1, R_2. By the proposition 2.2.6, there are $a_1 > 0, a_2 > 0$ such that

$$a_1(F_1 \cdot D(R_1)) + a_2(F_1 \cdot D(R_2)) < 0 \quad \text{and} \quad a_1(F_2 \cdot D(R_1)) + a_2(F_2 \cdot D(R_2)) < 0.$$

Or

$$-a_1(F_1 \cdot D(R_1)) > a_2(F_1 \cdot D(R_2)) \quad \text{and} \quad -a_2(F_2 \cdot D(R_2)) > a_1(F_2 \cdot D(R_1))$$

where $F_1 \cdot D(R_1) < 0, F_2 \cdot D(R_2) < 0$ and $F_1 \cdot D(R_2) > 0, F_2 \cdot D(R_1) > 0$. Multiplying inequalities above, we obtain the Lemma. ▷

3. A classification of sets of simple extremal rays of the type (I) and (II).

As above, we assume that X is a projective normal 3-fold with $\mathbf{Q}$-factorial singularities.

Definition 2.3.1. An extremal ray R of the type (II) is called *simple* if

$$R \cdot (D(R) + D) \geq 0$$

for any irreducible divisor D such that $R \cdot D > 0$.

The following statement gives a simple sufficient condition for an extremal ray to be simple.

Proposition 2.3.2. *Let R be an extremal ray of the type (II) and $f : X \to Y$ the contraction of R. Suppose that the curve $f(D(R))$ is not contained in the set of singularities of Y.*

Then

(1) the ray R is simple;

(2) if the characteristic of the ground field is 0 and X has only isolated singularities then a general element C of the ray R (a general fiber of the morphism $f \mid D(R)$) is isomorphic to $\mathbf{P}^1$ and the divisor $D(R)$ is non-singular along C. If additionally $R \cdot K_X < 0$, then $C \cdot D(R) = C \cdot K_X = -1$.

In particular, both statements (1) and (2) are true if X has terminal singularities and $R \cdot K_X < 0$.

Proof. Let D be an irreducible divisor on X such that $R \cdot D > 0$. Since $R \cdot D(R) < 0$, the divisor D is different from $D(R)$ and the intersection $D \cap D(R)$ is a curve which does not belong to R. Then $D' = f_*(D)$ is an irreducible divisor on Y and $\Gamma = f(D(R))$ is a curve on D'. Let $y \in \Gamma$ be a non-singular point of Y. Then the divisor D' is defined by some local equation ϕ in a neighborhood U of y. Evidently, in the open set $f^{-1}(U)$ the divisor

$$(f^*\phi) = D + m(D(R))$$

where $m \geq 1$. Let a curve $C \in R$ and $f(C) = y \in U \cap f(D(R))$. Then $0 = C \cdot (D + m(D(R))) = C \cdot (D + D(R)) + C \cdot (m-1)(D(R))$. Since $m \geq 1$ and $C \cdot D(R) < 0$, it follows that $C \cdot (D + D(R)) \geq 0$.

Let us prove (2). Let us consider a linear system $|H|$ of hyperplane sections on Y and the corresponding linear systems on the resolutions of singularities of Y and X. Let us apply Bertini's theorem (see, for example, [14, ch.III, Corollary 10.9 and the Exercise 11.3]) to this linear systems. Singularities of X and Y are isolated. Then by Bertini theorem, for a general element H of $|H|$ we obtain that: (a) H and $f^{-1}(H)$ are irreducible and non-singular. (b) H intersects Γ transversely in non-singular points of Γ. Let us consider the corresponding birational morphism $f' = f|H' : H' \to H$ of the non-singular irreducible surfaces. It is a composition of blowing ups in non-singular points. Thus, fibers of f' over $H \cap \Gamma$ are trees of non-singular rational curves. The exceptional curve of the first of these blowing ups is identified with the fiber of the projectivization of the normal bundle $\mathbf{P}(\mathcal{N}_{\Gamma/Y})$. Thus, we obtain a rational map over the curve Γ

$$\phi : \mathbf{P}(\mathcal{N}_{\Gamma/Y}) \to D(R)$$

of the irreducible surfaces. Evidently, it is the injection in the general point of $\mathbf{P}(\mathcal{N}_{\Gamma/Y})$. It follows that ϕ is a birational isomorphism of the surface. Since ϕ is a birational map over the curve Γ, it follows that the general fibers of this maps are birationally isomorphic. It follows that a general fiber of f' is $C \simeq \mathbf{P}^1$. Since C is non-singular and is an intersection of the non-singular surface H' with the surface $D(R)$, and since X has only isolated singularities, it follows that $D(R)$ is non-singular along the general curve C.

The X and $D(R)$ are non-singular along $C \simeq \mathbf{P}^1$ and the curve C is non-singular. Then the canonical class $K_C = (K_X + D(R))|C$ where both divisors K_X and $D(R)$ are Cartier divisors on X along C. It follows that $-2 = \deg K_C = K_X \cdot C + D(R) \cdot C$, where the both numbers $K_X \cdot C$ and $D(R) \cdot C$ are negative integers. Then $D(R) \cdot C = K_X \cdot C = -1$.

If X has terminal singularities and $R \cdot K_X < 0$, then Y has terminal singularities too (see, for example, [2]). Moreover, terminal singularities are isolated. From (1), (2), it follows the last statement of the Proposition. ▷

Let R_1, R_2 are two extremal rays of the type (I) or (II). They *are joined* if $D(R_1) \cap D(R_2) \neq \emptyset$. It defines *connected components* of a set of extremal rays of the type (I) or (II).

We recall that a set $\mathcal{E}$ of extremal rays is called *extremal* if it is contained in a face of $\overline{NE}(X)$. We say that $\mathcal{E}$ is *extremal of Kodaira dimension 3* if it is contained in a face of numerical Kodaira dimension 3 of $\overline{NE}(X)$.

We prove the following classification theorem.

Theorem 2.3.3. *Let $\mathcal{E} = \{R_1, R_2, ..., R_n\}$ be an extremal set of extremal rays of the type (I) or (II). Suppose that every extremal ray of $\mathcal{E}$ of the type (II) is simple. Assume that $\mathcal{E}$ is contained in a contractible face with Kodaira dimension 3 of $\overline{NE}(X)$. (In particular, $\mathcal{E}$ is extremal of Kodaira dimension 3.)*

Then every connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$ below (see figure 1).

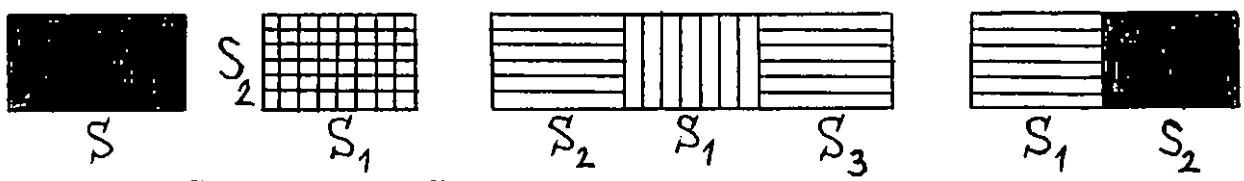
($\mathfrak{A}_1$) *One extremal ray of the type (I).*

($\mathfrak{B}_2$) *Two different extremal rays S_1, S_2 of the type (II) such that their divisors $D(S_1) = D(S_2)$ are coincided.*

($\mathfrak{C}_m$) *$m \geq 1$ extremal rays $S_1, S_2, ..., S_m$ of the type (II) such that the divisors $D(S_2), D(S_3), ..., D(S_m)$ do not intersect one another, and $S_1 \cdot D(S_i) = 0$, $i = 2, ..., m$, but $S_i \cdot D(S_1) > 0$, $i = 2, ..., m$.*

($\mathfrak{D}_2$) *Two extremal rays S_1, S_2, where S_1 is of the type (II) and S_2 of the type (I), $S_1 \cdot D(S_2) > 0$ and $S_2 \cdot D(S_1) > 0$. If $b_1 \geq 0, b_2 \geq 0$ and one of b_1, b_2 is not equal to zero then either $S_1 \cdot (b_1 D(S_1) + b_2 D(S_2)) < 0$ or $S_2 \cdot (b_1 D(S_1) + b_2 D(S_2)) < 0$.*

The following inverse statement is true: If $\mathcal{E} = \{R_1, R_2, ..., R_n\}$ is a connected set of extremal rays of the type (I) or (II) and $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$ above, then $\mathcal{E}$ generates a simplicial face $R_1 + ... + R_n$ of the dimension n and numerical Kodaira dimension 3 of $\overline{NE}(X)$. In particular, the set $\mathcal{E}$ is linearly independent.



Type $\mathfrak{A}_1$ *Type* $\mathfrak{B}_2$ *Type* $\mathfrak{C}_3$ *Type* $\mathfrak{D}_2$

Figure 1

Proof. Let us prove the first statement. We can assume that $\mathcal{E}$ is connected. We have to prove that then $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. If $n = 1$, it is obvious.

Let $n = 2$. From the Lemma 2.2.2, it follows that one of the rays R_1, R_2 has the type (II). Let R_1 has the type (II) and R_2 the type (I). Since $D(R_1) \cap D(R_2) \neq \emptyset$, then evidently $R_2 \cdot D(R_1) > 0$. If $R_1 \cdot D(R_2) = 0$, then evidently the curve $D(R_1) \cap D(R_2)$ belongs to the ray R_1. It follows, that the rays R_1 and R_2 contain the same curve. We get the contradiction. Thus, $R_1 \cdot D(R_2) > 0$. The rays R_1, R_2 belong to a contractible face of Kodaira dimension 3 of Mori polyhedron. Let f be a contraction of this face. By the Lemma 2.2.3, f contracts the divisors $D(R_1), D(R_2)$ in a same point. By the Proposition 2.2.6, there are positive a_1, a_2 such that $R_1 \cdot (a_1 D(R_1) + a_2 D(R_2)) < 0$ and $R_2 \cdot (a_1 D(R_1) + a_2 D(R_2)) < 0$. Now suppose that for some $b_1 > 0$ and $b_2 > 0$ the inequalities $R_1 \cdot (b_1 D(R_1) + b_2 D(R_2)) \geq 0$ and $R_2 \cdot (b_1 D(R_1) + b_2 D(R_2)) \geq 0$ hold.

There exsits $\lambda > 0$ such that $\lambda b_1 \leq a_1$, $\lambda b_2 \leq a_2$ and one of these inequalities is the equality. For example, let $\lambda b_1 = a_1$. Then

$$R_1 \cdot (a_1 D(R_1) + a_2 D(R_2)) = R_1 \cdot \lambda(b_1 D(R_1) + b_2 D(R_2)) + R_1 \cdot (a_2 - \lambda b_2) D(R_2) \geq 0.$$

We get the contradiction. It proves that in this case $\mathcal{E}$ has the type $\mathfrak{D}_2$.

Assume that both rays R_1, R_2 have the type (II). Since the rays R_1, R_2 are simple, from the Lemma 2.2.7, it follows that either $R_1 \cdot D(R_2) = 0$ or the $R_2 \cdot D(R_1) = 0$. If both these equalities hold, the rays R_1, R_2 have the common curve. We get the contradiction. Thus, in this case $\mathcal{E}$ has the type $\mathfrak{C}_2$.

Let $n = 3$. Every proper subset of $\mathcal{E}$ has connected components $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. Using the Lemmas 2.2.2—2.2.5, one can see very easy that then either $\mathcal{E}$ has the type $\mathfrak{C}_3$ or we have the following case:

The rays R_1, R_2, R_3 have the type (II), every two element subset of $\mathcal{E}$ has the type $\mathfrak{C}_2$ and we can change the numeration so that $R_1 \cdot D(R_2) > 0$, $R_2 \cdot D(R_3) > 0$ and $R_3 \cdot D(R_1) > 0$. Let $\mathcal{E} \subset \gamma$ where γ is a contractible face of $\overline{NE}(X)$ of the Kodaira dimension 3. Let f be a contraction of the face γ. By the Lemma 2.2.3, f contracts the divisors $D(R_1), D(R_2), D(R_3)$ in a one point. By the Proposition 2.2.6, there are positive a_1, a_2, a_3 such that

$$R_i \cdot (a_1 D(R_1) + a_2 D(R_2) + a_3 D(R_3)) < 0,$$

$i = 1, 2, 3$. On the other hand, from the simplicity of the rays R_1, R_2, R_3, it follows that

$$R_i \cdot (D(R_1) + D(R_2) + D(R_3)) \geq 0.$$

Let $a_1 = \min\{a_1, a_2, a_3\}$. From the last inequality, $R_1 \cdot (a_1 D(R_1) + a_2 D(R_2) + a_3 D(R_3)) = R_1 \cdot a_1 (D(R_1) + D(R_2) + D(R_3)) + R_1 \cdot ((a_2 - a_1) D(R_2) + (a_3 - a_1) D(R_3)) \geq 0$. We get the contradiction with the inequality above.

Let $n > 3$. We have proved that every two or three element subset of $\mathcal{E}$ has connected components $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. It follows very easy that then $\mathcal{E}$ has the type $\mathfrak{C}_n$.

Let us prove the inverse statement.

For the type $\mathfrak{A}_1$ it is obvious.

Let $\mathcal{E}$ has the type $\mathfrak{B}_2$. Since the rays S_1, S_2 are extremal of Kodaira dimension 3, there are nef elements H_1, H_2 such that $H_1 \cdot S_1 = H_2 \cdot S_2 = 0$, ${H_1}^3 > 0$, ${H_2}^3 > 0$. Let $0 \neq C_1 \in S_1$ and $0 \neq C_2 \in S_2$. Let D be the divisor of the rays S_1 and S_2. Let us consider the map

$$(H_1, H_2) \to H = (-D \cdot C_2)(H_2 \cdot C_1) H_1 + (-D \cdot C_1)(H_1 \cdot C_2) H_2 + (H_2 \cdot C_1)(H_1 \cdot C_2) D. \quad (3.1)$$

For the fixed H_1, we get the linear map $H_2 \to H$ of the set of nef elements H_2 orthogonal to S_2 into the set of nef elements H_2 orthogonal to S_1 and S_2. This map has a one dimensional kernel, generated by $(-D \cdot C_2) H_1 + (H_1 \cdot C_2) D$. It follows that $S_1 + S_2$ is the two dimensional face of $\overline{NE}(X)$.

For the general nef element $H = a_1 H_1 + a_2 H_2 + bD$ orthogonal to this face, where $a_1, a_2, b > 0$, we have: $H^3 = (a_1 H_1 + a_2 H_2 + bD)^3 \geq (a_1 H_1 + a_2 H_2 + bD)^2 (a_1 H_1 + a_2 H_2) = (a_1 H_1 + a_2 H_2 + bD)(a_1 H_1 + a_2 H_2 + bD)(a_1 H_1 + a_2 H_2) \geq (a_1 H_1 + a_2 H_2)^2 (a_1 H_1 + a_2 H_2 + bD) \geq (a_1 H_1 + a_2 H_2)^3 > 0$, since $a_1 H_1 + a_2 H_2 + bD$ and $a_1 H_1 + a_2 H_2$ are nef. It follows that the face $S_1 + S_2$ has Kodaira dimension 3.

Let $\mathcal{E}$ has the type $\mathfrak{C}_m$. Let H be a nef element orthogonal to the ray S_1. Let $0 \neq C_i \in S_i$. Let us consider the map

$$H \to H' = H + \sum_{i=2}^{m} (-(H \cdot C_i)/(C_i \cdot D(S_i))) D(S_i). \quad (3.2)$$

It is the linear map of the set of nef elements H orthogonal to S_1 into the set of nef elements H' orthogonal to the rays $S_1, S_2, ..., S_m$. The kernel of the map has the dimension $m-1$. It follows that the rays $S_1, S_2, ..., S_m$ belong to the face of $\overline{NE}(X)$ of the dimension $\leq m$. On the other hand, multiplying rays $S_1, ..., S_m$ on the divisors $D(S_1), ..., D(S_m)$, one can see very easy that the rays $S_1, ..., S_m$ are linearly independent. Thus, they generate the m-dimensional face of $\overline{NE}(X)$. Let us show that this face is $S_1 + S_2 + ... + S_m$. To prove it, we show that every $m-1$ ray subset of $\mathcal{E}$ is contained in a face of $\overline{NE}(X)$ of the dimension $\leq m-1$.

If this subset contains the ray S_1, this subset has the type $\mathfrak{C}_{m-1}$. We have proved that this subset belongs to the face of $\overline{NE}(X)$ of the dimension $m-1$. Let us consider the subset $\{S_2, S_3, ..., S_m\}$. Let H be an ample element on X. For the element H the map (3.2) gives the element H' which is orthogonal to the rays $S_2, S_3, ..., S_m$, but is not orthogonal to the ray S_1. It follows that the set $\{S_2, S_3, ..., S_m\}$ belongs to the face of the Mori polyhedron of the dimension $< m$. Like above, one can see that for the general H orthogonal to S_1 the element H' has $(H')^3 \geq H^3 > 0$.

Let $\mathcal{E}$ has the type $\mathfrak{D}_2$. Let H be a nef element orthogonal to the ray S_2. Let $0 \neq C_i \in S_i$. Let us consider the map

$$H \to H' = H + \frac{(H \cdot C_1)((-D(S_2) \cdot C_2)D(S_1) + (D(S_1) \cdot C_2)D(S_2))}{(D(S_2) \cdot C_2)(D(S_1) \cdot C_1) - (D(S_1) \cdot C_2)(D(S_2) \cdot C_1)}. \quad (3.3)$$

Evidently, $C_2 \cdot ((-D(S_2) \cdot C_2)D(S_1) + (D(S_1) \cdot C_2)D(S_2)) = 0$. From this equality and the inequality from the definition of the system $\mathfrak{D}_2$, it follows that $C_1 \cdot ((-D(S_2) \cdot C_2)D(S_1) + (D(S_1) \cdot C_2)D(S_2)) < 0$. Thus, the denominator from the formula (3.3) is positive. Then (3.3) is the linear map of the set of nef element H orthogonal to the ray S_2 into the set of nef elements H' orthogonal to the rays S_1, S_2. Evidently, the map has the one dimensional kernel. Thus, the rays S_1 and S_2 generate the two dimensional face $S_1 + S_2$ of Mori polyhedron. As above, for the general element H orthogonal to S_2 we have $(H')^3 \geq (H)^3 > 0$. ▷

Corollary 2.3.4. *Let $\mathcal{E} = \{R_1, R_2, ..., R_n\}$ be an extremal set of extremal rays of the type (I) or (II) and every extremal ray of $\mathcal{E}$ of the type (II) is simple. Assume that $\mathcal{E}$ is contained in a contractible face with Kodaira dimension 3 of the $\overline{NE}(X)$. Let $m_1 \geq 0, m_2 \geq 0, ..., m_n \geq 0$ and at least one of $m_1, ..., m_n$ is positive.*

Then there exists i, $1 \leq i \leq n$, such that

$$R_i \cdot (m_1 D(R_1) + ... + m_n D(R_n)) < 0.$$

Proof. It is sufficient to prove this statement for the connected $\mathcal{E}$. For every type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ and $\mathfrak{D}_2$ of the Theorem 2.3.3, one can prove it very easy. ▷

Unfortunately, in general, the inverse statement of the Theorem 2.3.3 holds only for a connected extremal set $\mathcal{E}$. We will give two cases when it is true for a non-connected $\mathcal{E}$.

Definition 2.3.5. A threefold X is called *strongly projective* (respectively *very strongly projective*) if the following statement holds: a set $\{Q_1, ..., Q_n\}$ of extremal rays of the type (II) is extremal of Kodaira dimension 3 (respectively generates the simplicial face $Q_1 + ... + Q_n$ of $\overline{NE}(X)$ of the dimension n) if its divisors $D(Q_1), ..., D(Q_n)$ do not intersect one another.

Theorem 2.3.6. *Let $\mathcal{E} = \{R_1, R_2, ..., R_n\}$ be a set of extremal rays of the type (I) or (II) such that every connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$.*

Then:

(1) $\mathcal{E}$ is extremal of Kodaira dimension 3 if and only if the same is true for any subset of $\mathcal{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another. In particular, it holds if X is strongly projective.

(2) $\mathcal{E}$ generates the simplicial face $R_1 + ... + R_n$ with Kodaira dimension 3 of the Mori polyhedron if and only if the same is true for any subset of $\mathcal{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another. In particular, it is true if X is very strongly projective.

Proof. Let us prove (1). Only the inverse statement is non-trivial. We prove it using an induction by n. For $n = 1$ the statement is obvious.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1$. Suppose that this conponent contains the ray R_1. By induction, there exists a nef element H such that $H^3 > 0$ and $H \cdot R_i = 0$ if $i > 1$. Then there exists $k \geq 0$, such that $H' = H + kD(R_1)$ is nef and $H' \cdot \mathcal{E} = 0$. As above, one can prove that $(H')^3 \geq H^3 > 0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{B}_2$. Suppose that this component contains the rays R_1, R_2 and $D(R_1) = D(R_2) = D$. Then, by induction, there are nef elements H_1 and H_2 such that ${H_1}^3 > 0, {H_2}^3 > 0$ and $H_1 \cdot \{R_1, R_3, ..., R_n\} = 0, H_2 \cdot \{R_2, R_3, ..., R_n\} = 0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathfrak{B}_2$, there are $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$ such that the element $H = k_1H_1 + k_2H_2 + k_3D$ is nef, $H \cdot \mathcal{E} = 0$ and $H^3 > 0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{C}_m, m > 1$. We use the notations of the Theorem 2.3.3 for this connected component. Let it is $\{S_1, S_2, ..., S_m\}$. By the induction, there exists a nef element H such that H is orthogonal to $\mathcal{E} - \{S_2, ..., S_m\}$ and $H^3 > 0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathfrak{C}_m$, there are $k_2 \geq 0, ..., k_m \geq 0$ such that $H' = H + k_2D(S_2) + ... + k_mD(S_m)$ is nef, $H' \cdot \mathcal{E} = 0$ and $(H')^3 \geq H^3 > 0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{D}_2$. We use notations of the Theorem 2.3.3 for this connected component. Let it is $\{S_1, S_2\}$. By the induction, there exists nef element H such that $H^3 > 0$ and H is orthogonal to $\mathcal{E} - \{S_1\}$. As for the theorem 2.3.3, there are $k_1 \geq 0, k_2 \geq 0$ such that $H' = H + k_1D(S_1) + k_2D(S_2)$ is nef, $H' \cdot \mathcal{E} = 0$ and $(H')^3 \geq H^3 > 0$.

If every connected component of $\mathcal{E}$ has the type $\mathfrak{C}_1$, then the statement holds by the condition of the Theorem.

Let us prove (2). Only the inverse statement is non-trivial. We prove it using an induction by n. For $n = 1$ the statement is true. It is sufficient to prove that $\mathcal{E}$ is contained in a face of a dimension $\leq n$ of Mori polyhedron because, by the induction, any its $n-1$ element subset generates a simplicial face of the dimension $n-1$ of Mori polyhedron.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1$. Suppose that the ray R_1 belongs to this component and $0 \neq C_1 \in R_1$. Let us consider the map

$$H \to H' = H + ((H \cdot C_1)/(-D(R_1) \cdot C_1))D(R_1).$$

of the set of nef elements H orthogonal to the set $\{R_2, ..., R_n\}$ into the set of nef elements H' orthogonal to the $\mathcal{E}$. It is the linear map with one dimensional kernel. Since, by the induction, the set $\{R_2, ..., R_n\}$ is contained in a face of Mori polyhedron of the dimension $n-1$, it follows that $\mathcal{E}$ is contained in a face of the dimension n.

If $\mathcal{E}$ has a connected component of the type $\mathfrak{B}_2, \mathfrak{C}_m, m > 1$, or $\mathfrak{D}_2$, the proof is the same if one uses the maps (3.1), (3.2) and (3.3) above.

If all connected components of $\mathcal{E}$ have the type $\mathfrak{C}_1$, the statement holds by the condition. ▷

Remark 2.3.7. Like the statement (1) of the Theorem 2.3.6, one can prove that a set $\mathcal{E}$ of extremal rays with connected components of the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$ is extremal if and only if the same is true for any subset of $\mathcal{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another. ▷

The following statement is useful also.

Proposition 2.3.8. *Assume that a set $\mathcal{E}$ of extremal rays has connected components of the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$.*

Then the following conditions are equivalent:

(i) The set $\mathcal{E}$ is linearly dependent.

(ii) The set $\mathcal{E}$ contains ≥ 2 connected components of the type $\mathfrak{B}_2$ such that their rays are linearly dependent. Let these components are $\mathfrak{B}^1, ..., \mathfrak{B}^t, t \geq 2$. Then we can choose the numeration so that $\mathfrak{B}^i = \{R_{i1}, R_{i2}\}$ and a linear dependence has a form

$$a_{11}R_{11} + a_{21}R_{21} + ... + a_{t1}R_{t1} = a_{12}R_{12} + a_{22}R_{22} + ... + a_{t2}R_{t2}.$$

where all $a_{ij} > 0$.

Proof. Let $\mathcal{E} = \{R_1, ..., R_m\}$ and a non-trivial linear dependence is $a_1R_1 + ... + a_mR_m = 0$. If we multiply this equality on divisors $D(R_1), ..., D(R_m)$, we get that $a_k = 0$ if the ray R_k belongs to a connected component of the type $\mathfrak{A}_1, \mathfrak{C}_m$ or $\mathfrak{D}_2$. Thus, there are connected components of $\mathcal{E}$

$$\mathfrak{B}^1 = \{R_{11}, R_{12}\}, \mathfrak{B}^2 = \{R_{21}, R_{22}\}, ..., \mathfrak{B}^t = \{R_{t1}, R_{t2}\},$$

$t \geq 1$, of the type $\mathfrak{B}_2$ such that we have the linear dependence

$$a_{11}R_{11} + a_{12}R_{12} + a_{21}R_{21} + a_{22}R_{22} + ... + a_{t1}R_{t1} + a_{t2}R_{t2} = 0,$$

where either $a_{i1} \neq 0$ or $a_{i2} \neq 0$ for any $1 \leq i \leq t$.

Here $t \geq 2$ since the rays R_{11} and R_{12} are different. If we multiply this equality on the divisor $D_i = D(R_{i1}) = D(R_{i2})$, we get for some $\alpha_i > 0, \beta_i > 0$ that $\alpha_i a_{i1} + \beta_i a_{i2} = 0$. Thus, a_{i1} and a_{i2} have opposite sings. It follows the Proposition. ▷

4. A classification of E-sets of extremal rays of the type (I) or (II).

As above, we suppose that X is a projective normal 3-fold with $\mathbf{Q}$-factorial singularities.

We recall that a set $\mathcal{L}$ of extremal rays is called *E-set* if it is not extremal but any proper subset of $\mathcal{L}$ is extremal (it is contained in a face of $\overline{NE}(X)$).

Theorem 2.4.1. *Let $\mathcal{L}$ be a E-set of extremal rays of the type (I) or (II). Suppose that every ray of the type (II) of $\mathcal{L}$ is simple and every proper subset of $\mathcal{L}$ is contained in a contractible face of Kodaira dimension 3 of Mori polyhedron.*

Then we have one of the following cases:

(a) $\mathcal{L}$ is connected and $L = \{R_1, R_2, R_3\}$, where any R_i has the type (II) and every of 2-element subsets $\{R_1, R_2\}, \{R_2, R_3\}, \{R_3, R_1\}$ of $\mathcal{L}$ has the type $\mathfrak{C}_2$. Here $R_1 \cdot D(R_2) > 0, R_2 \cdot D(R_3) > 0, R_3 \cdot D(R_1) > 0$ but $R_2 \cdot D(R_1) = R_3 \cdot D(R_2) = R_1 \cdot D(R_3) = 0$. The divisor $D(\mathcal{L}) = D(R_1) + D(R_2) + D(R_3)$ is nef.

(b) $\mathcal{L}$ is connected and $\mathcal{L} = \{R_1, R_2\}$, where the rays R_1, R_2 have the type (I) or (II). There are positive m_1, m_2 such that $R \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0$ for any extremal ray R of the type (I) or simple extremal ray of the type (II) on X.

(c) $\mathcal{L}$ is connected and $\mathcal{L} = \{R_1, R_2\}$ where both R_1 and R_2 have the type (II) and there exists the simple extremal ray S_1 of the type (II) such that the rays R_1, S_1 define the extremal set of the type $\mathfrak{B}_2$ (it means that $S_1 \neq R_1$ but the divisors $D(S_1) = D(R_1)$)and the rays S_1, R_2 define the extremal set of the type $\mathfrak{C}_2$, where $S_1 \cdot D(R_2) = 0$ but $R_2 \cdot D(S_1) > 0$. Here there do not exist positive m_1, m_2 such that the divisor $m_1 D(R_1) + m_2 D(R_2)$ is nef, since evidently $S_1 \cdot (m_1 D(R_1) + m_2 D(R_2)) < 0$. See figure 2 below.

(d) $\mathcal{L} = \{R_1, ..., R_k\}$ where $k \geq 2$, all rays $R_1, ..., R_k$ have the type (II) and the divisors $D(R_1), ..., D(R_k)$ do not intersect one another. Any proper subset of $\mathcal{L}$ is contained in a contractible face of Kodaira dimension 3 of Mori polyhedron but $\mathcal{L}$ is not contained in a face of Mori polyhedron.

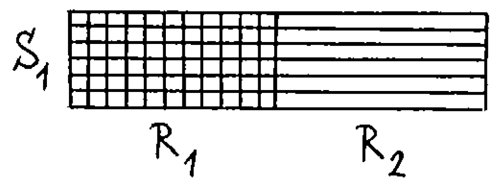


Figure 2

Proof. Let $\mathcal{L} = \{R_1, ..., R_n\}$ be a E-set of extremal rays satisfying to the conditions of the Theorem. Let us consider two cases.

The case 1. Let $\mathcal{L}$ is not connected. Then every connected component of $\mathcal{L}$ is extremal and, by the theorem 2.3.3, it has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. If some of these components has not the type $\mathfrak{C}_1$, then, by the statement (1) of the Theorem 2.3.6, $\mathcal{L}$ is extremal and we get the contradiction. Thus, we obtain the case (d) of the Theorem.

The case 2. Let $\mathcal{L} = \{R_1, ..., R_n\}$ is connected.

Let $n \geq 4$. By the Theorem 2.3.3, any proper subset of $\mathcal{L}$ has connected components of the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. Like for the proof of the Theorem 2.3.3, it follows that $\mathcal{L}$ has the type $\mathfrak{C}_n$. By the Theorem 2.3.3, then $\mathcal{L}$ is extremal. We get the contradiction.

Let $n = 3$. Then, like for the proof of the Theorem 2.3.3, we get that $\mathcal{L}$ has the type (a).

Let $n = 2$ and $\mathcal{L} = \{R_1, R_2\}$. If both rays R_1, R_2 have the type (I), then, by the Lemma 2.2.2, $\mathcal{L}$ is not connected and we get the contradiction.

Let R_1 has the type (I) and R_2 has the type (II). Since the system $\mathcal{L}$ is not extremal, by the Theorem 2.3.3, there are positive m_1, m_2 such that $R_1 \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0$ and $R_2 \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0$.

By the Lemma 2.2.3, it follows that $C \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0$ if the curve C is contained in the $D(R_1) \cup D(R_2)$. If C is not contained in $D(R_1) \cup D(R_2)$, then obviously $C \cdot (m_1 D(R_1) + m_2 D(R_2)) \geq 0$. It follows, that the divisor $m_1 D(R_1) + m_2 D(R_2)$ is nef. Thus, we obtain the case (b).

Let both rays R_1, R_2 have the type (II). If $D(R_1) = D(R_2)$, then we get an extremal set $\{R_1, R_2\}$ by the Theorem 2.3.3. Thus, the divisors $D(R_1)$ and $D(R_2)$ are different. By the Lemma 2.2.1, the curve $D(R_1) \cap D(R_2)$ has not an irreducible component which belongs to both rays R_1 and R_2. Since rays R_1, R_2 are simple, it follows that $R_1 \cdot (D(R_1) + D(R_2)) \geq 0$ and $R_2 \cdot (D(R_1) + D(R_2)) \geq 0$. Let R be an extremal ray of the type (I) or simple extremal ray of the type (II). If the divisor $D(R)$ does not coincide with the divisor $D(R_1)$ or $D(R_2)$, then obviously $R \cdot (D(R_1) + D(R_2)) \geq 0$. Thus, if there does not exist an extremal ray R which has the same divisor as the ray R_1 or R_2, we get the case (b).

Assume that $D(R) = D(R_1)$. Then, by the Lemma 2.2.5, the ray R has the type (II) too. If $R \cdot D(R_2) = 0$, we get the case (c) of the Theorem where $S_1 = R$. If $R \cdot D(R_2) > 0$, then $R \cdot (D(R_1) + D(R_2)) \geq 0$ since the ray R is simple. Then we get the case (b) of the Theorem. ▷

5. An application of the diagram method to the general threefolds.

Now we can apply the results of the Chapter 1 and of the Chapter 2 above to 3-folds.

Theorem 2.5.1. *Let X be a normal projective 3-fold with $\mathbf{Q}$-factorial singularities. Let γ be a face of the polyhedron $\mathcal{M}(X) = NEF(X)/\mathbf{R}^+$ such that γ is closed and the following conditions (α) and (β) hold:*

(α) The set

$$\mathcal{R}(\gamma) = \{extremal\ ray\ R \mid \exists \mathbf{R}^+ H \in \gamma\ :\ R \cdot H = 0\}$$

contains extremal rays of the type (I) or simple extremal rays of the type (II) only; any face of Mori polyhedron orthogonal to a point of γ is contractible and has Kodaira dimension 3.

(β) If Q is an extremal ray of the $\overline{NE}(X)$, which is not of the type (I) and is not simple of the type (II), then $Q \cdot D(R) \geq 0$ for any extremal ray $R \in \mathcal{R}(\gamma)$.

Then we have one of the following cases (1)—(4):

(1) $\dim \gamma \leq 7$.

(2) There are extremal rays $\{R_1, ..., R_t\} \subset \mathcal{R}(\gamma)$ such that the rays R_i have the type (II), their divisors $D(R_1), ..., D(R_t)$ do not intersect one another, the set $\{R_1, ..., R_t\}$ is contained in a face of Kodaira dimension 3 of $\overline{NE}(X)$ which is orthogonal to a vertex of γ, but the simplicial cone $R_1 + ... + R_t$ is not the face of $\overline{NE}(X)$.

(3) The set $\mathcal{R}(\gamma)$ contains extremal rays R_1, R_2 which together with some simple extremal ray S_1 of the type (II) on X give the configuration (c) of the Theorem 2.4.1.

(4) There are extremal rays $\{R_1, ..., R_t\} \subset \mathcal{R}(\gamma)$ such that the rays R_i have the type (II), their divisors $D(R_1), ..., D(R_t)$ do not intersect one another, but the set $\{R_1, ..., R_t\}$ is not extremal (it is not contained in a face of $\overline{NE}(X)$).

Proof. Let us suppose that the cases (2), (3) and (4) do not hold. Then let us apply the Theorem 1.2 to the face γ.

The conditions (i), (ii) and (iii) of the Theorem 1.2 follow from the Theorem 2.3.3, Corollary 2.3.4, Theorems 2.3.6 and 2.4.1 (to prove the condition (iii), one should use the condition (β) of the Theorem also). From the Theorems 2.3.3, 2.3.6 and 2.4.1 it follows that the conditions of the Theorem 1.2 hold with the constants $d = 2$, $C_1 = 1$ and $C_2 = 0$. From the Theorem 1.2 we get the inequality

$$\dim \gamma \leq 11.$$

To obtain more strong inequality (1) of the Theorem, we should analyze carefully the proof of the Theorem 1.2 in our concrete case.

Let an oriented angle Δ is defined by the triplet $\mathcal{R}(\Delta), R_1(\Delta), R_2(\Delta)$. We define the weight $\sigma(\Delta)$ by the rule: $\sigma(\Delta) = 2/3$ if simultaneously $\rho(R_1(\Delta), R_2(\Delta)) = 1$ and $\rho(R_2(\Delta), R_1(\Delta)) = 0$. It means that $R_1(\Delta)R_2(\Delta)$ is an arrow but $R_2(\Delta)R_1(\Delta)$ is not one. And $\sigma(\Delta) = 0$ otherwise. Let us prove that the conditions of the Lemma 1.4, which we used to prove the Theorem 1.2, hold with the constants $C = 2/3$ and $D = -2/3$.

From the Theorem 2.3.3, it follows the condition (1) of the Lemma 1.4.

In the case of the triangle γ_3, the extremal rays $\{R_1, R_2, R_3\}$ are contained in a E-subset $\mathcal{L}$ of $\mathcal{R}(\gamma)$. By the Theorem 2.4.1, this subset has 3 elements, and we have the case (a) of the Theorem 2.4.1. In this case the triangle γ_3 has three oriented angles of the weight 2/3 and their sum is 2. Thus, the condition (2) of the Lemma 1.4 holds for a triangle γ_3.

Let us consider the case of quadrangle γ_4. By the proof of the Theorem 1.2, we can suppose that there exists an oriented path from the extremal ray R_1 to the extremal ray R_2 in the set $\overline{\mathcal{R}(\gamma_4) \cup \{R_1, R_2\}}$ of extremal rays orthogonal to the vertex of γ_4. From the Theorem 2.3.3, it follows that the rays R_1, R_2 define the system $\mathfrak{D}_2$ or $\mathfrak{C}_2$. Let us consider these cases.

Assume that R_1, R_2 define the system $\mathfrak{D}_2$ where the ray R_1 has the type (II) and the ray R_2 has the type (I). The rays R_2 and R_4 belong to an E-subset $\mathcal{L}$ of $\mathcal{R}(\gamma)$. By the Theorem 2.4.1, this subset has two elements R_2, R_4 only, since the ray R_2 has the type (I). By the Lemma 1.1, R_2R_4 and R_4R_2 are the arrows and $D(R_2) \cap D(R_4)$ is a curve. From the Lemma 2.2.2, it follows that the ray R_4 has the type (II), since the ray R_2 has the type (I). The curve $C = D(R_1) \cap D(R_2)$ belongs to the ray R_2 of the type (I). Then $C \cdot D(R_4) > 0$. By the Theorem 2.3.3, then the rays R_1, R_4 define the system of the type $\mathfrak{C}_2$. From the Lemma 2.2.3, it follows that three different extremal rays R_1, R_2, R_4 generate a 2-dimensional subspace in $N_1(X)$. We get the contradiction.

Thus, we proved that the rays R_1R_2 define the system of the type $\mathfrak{C}_2$, where R_1R_2 is the arrow (i.e. $R_1 \cdot D(R_2) > 0$), but R_2R_1 is not (i.e. $R_2 \cdot D(R_1) = 0$). Thus, the weight of the oriented angle of γ_4 defined by the ordered set (R_1, R_2) of extremal rays is equal to 2/3.

From the proof of the Theorem 1.2, it follows that there exists at least one other oriented angle of γ_4 which has the same weight. Thus, the sum of weights of these angles is equal to $4/3 > 1$. It finishes the proof of the condition (2) of the Lemma 1.4. $\triangleright$

6. An application of the diagram method to Fano threefolds.

We recall that a 3-fold X with $\mathbf{Q}$-factorial singularities is called *Fano 3-fold* if the anticanonical class $-K_X$ is numerically ample, i.e. $-K_X \cdot C > 0$ for any effective curve C.

The following statement is interesting because it is true for 3-folds with just nef $-K_X$.

Theorem 2.6.1. *Let X be a 3-fold with isolated $\mathbf{Q}$-factorial singularities and $-K_X$ is nef. Assume that Mori polyhedron $\overline{NE}(X)$ is generated by a finite set of extremal rays of the type (I) or (II) and any face of $\overline{NE}(X)$ is contractible. Assume that for every extremal ray R of the type (II) $R \cdot K_X < 0$ and its contraction $f: X \to X'$ gives a 3-fold X' with isolated singularities (e.g., the last statement is true if X has $\mathbf{Q}$-factorial terminal singularities).*

Then:

If a set $\mathcal{E}$ of extremal rays on X (it may be empty) is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3, then every connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. The following inverse statement is true: If every connected component of a set $\mathcal{E}$ of extremal rays has a type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$, then $\mathcal{E}$ is contained in a face of $\overline{NE}(X)$, and this face has Kodaira dimension 3 if $(-K_X)^3 > 0$. Moreover, for a set $\mathcal{E}$ of extremal rays with connected components of the types $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$ one of the statements (1)—(4) below holds:

(1) $\mathcal{E}$ generates a linear subspace of $N_1(X)$ of the codimension ≤ 3.

(2) $\mathcal{E}$ is contained in a face with Kodaira dimension 1 or 2 of $\overline{NE}(X)$.

(3) $\mathcal{E}$ is contained in a set $\mathcal{E}'$ of extremal rays such that $\mathcal{E}'$ is contained in a face of Kodaira dimension 3 of $\overline{NE}(X)$ and $\mathcal{E}'$ has $t \geq 2$ connected components $\mathfrak{B}^1 = \{R_{11}, R_{12}\}, .., \mathfrak{B}^t = \{R_{t1}, R_{t2}\}$ of the type $\mathfrak{B}_2$ with a linear dependence

$$a_{11}R_{11} + a_{21}R_{21} + \ldots + a_{t1}R_{t1} = a_{12}R_{12} + a_{22}R_{22} + \ldots + a_{t2}R_{t2}$$

for some $a_{ij} > 0$.

(4) There are extremal rays S_1, R_1, R_2 of the type (II) which define the configuration (c) of the Theorem 2.4.1 and every set $\mathcal{E} \cup \{R_1\}, \mathcal{E} \cup \{R_2\}$ is contained in a face (its own) of Kodaira dimension 3 of $\overline{NE}(X)$.

Proof. The first direct statement follows from the Proposition 2.3.2 and the Theorem 2.3.3.

Let us prove the first inverse statement. Let $\mathcal{E}$ has connected components of the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. Let $R_1, ..., R_n$, $n > 1$, are extremal rays of the type (II) of $\mathcal{E}$ and the divisors $D(R_1), ..., D(R_n)$ do not intersect one another. By the Proposition 2.3.2, $R_i \cdot K_X = R_i \cdot D(R_i)$ for $1 \leq i \leq n$. Since all extremal rays of X have the type (I) or (II), it follows that the divisor $T = -K_X + D(R_1) + ... + D(R_n)$ is nef and $R_i \cdot T = 0$. It follows that $R_1, ..., R_n$ belong to the face of $\overline{NE}(X)$ orthogonal to T. From the Theorem 2.3.6 and the Remark 2.3.7, it follows that the set $\mathcal{E}$ is extremal.

Let $(-K_X)^3 > 0$. Then $T^3 \geq T^2 \cdot (-K_X) \geq T \cdot (-K_X)^2 \geq (-K_X)^3 > 0$, since $-K_X$ and T are nef.

Suppose that the statement (2) of the Theorem is not true. Then we can apply the Theorem 2.5.1 to the face γ which is orthogonal to $\mathcal{E}$. From the first statement which we have proved, it follows that the statement (4) of the Theorem 2.5.1 does not hold.

The case (3) of the Theorem 2.5.1 gives the case (4) of the theorem 2.6.1.

Let us consider the case (2) of the theorem 2.5.1. Let v be a vertex of γ and $R_1, ..., R_t$ are the corresponding to this case extremal rays orthogonal to v. Thus, $R_1 + ... R_t$ is not a face of the $\overline{NE}(X)$. Let $\mathcal{E}'$ be the set of all extremal rays orthogonal to the vertex v. Evidently, $\mathcal{E} \subset \mathcal{E}'$ and $\{R_1, ..., R_n\} \subset \mathcal{E}$. If all rays of $\mathcal{E}'$ are linearly independent, then $\mathcal{E}'$ generates the simplicial face of the $\overline{NE}(X)$. Then $R_1 + ... + R_t$ is a simplicial face of $\overline{NE}(X)$ too, and we get the contradiction. Thus, the the set $\mathcal{E}'$ of extremal rays is linearly dependent. From the Proposition 2.3.8, it follows the case (3) of the theorem 2.6.1.

The case (1) of the Theorem 2.5.1 gives the case (2) of the Theorem 2.6.1. ▷

From the Theorem 2.6.1, it follows the following basic statement of the paper.

Theorem 2.6.2. *Let X be a Fano 3-fold with isolated $\mathbf{Q}$-factorial log-terminal singularities. Assume that all extremal rays of $\overline{NE}(X)$ have the type (I) or (II). Assume that for every extremal ray R of the type (II) its contraction $f: X \to X'$ gives a 3-fold X' with isolated singularities (e.g., this statement is true if X has $\mathbf{Q}$-factorial terminal singularities).*

Then a set $\mathcal{E}$ (it may be empty) is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3 if and only if every connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. Moreover, for the set $\mathcal{E}$ of extremal rays one of the statements (1)—(4) below holds:

(1) $\mathcal{E}$ generates a linear subspace of $N_1(X)$ of the codimension ≤ 8.

(2) $\mathcal{E}$ is contained in a face with Kodaira dimension 1 or 2 of $\overline{NE}(X)$.

(3) $\mathcal{E}$ is contained in a set $\mathcal{E}'$ of extremal rays such that $\mathcal{E}'$ is contained in a face of Kodaira dimension 3 of $\overline{NE}(X)$ and $\mathcal{E}'$ has $t \geq 2$ connected components $\mathfrak{B}^1 = \{R_{11}, R_{12}\}, .., \mathfrak{B}^t = \{R_{t1}, R_{t2}\}$ of the type $\mathfrak{B}_2$ with a linear dependence

$$a_{11}R_{11} + a_{21}R_{21} + ... + a_{t1}R_{t1} = a_{12}R_{12} + a_{22}R_{22} + ... + a_{t2}R_{t2}$$

for some $a_{ij} > 0$.

(4) There are extremal rays S_1, R_1, R_2 of the type (II) which define the configuration (c) of the Theorem 2.4.1 and every set $\mathcal{E} \cup \{R_1\}, \mathcal{E} \cup \{R_2\}$ is contained in a face of Kodaira dimension 3 of $\overline{NE}(X)$.

Proof. From the results of [2] and [12], it follows that any face of $\overline{NE}(X)$ is contractible and $\overline{NE}(X)$ is generated by a finite set of extremal rays. Moreover, $(-K_X)^3 > 0$ since X is Fano 3-fold. From the Theorem 2.6.1 we get the statement. ▷

From this Theorem, we get

Corollary 2.6.3. *Let X be a Fano 3-fold with isolated $\mathbf{Q}$-factorial log-terminal singularities. Assume that all extremal rays of $\overline{NE}(X)$ have the type (I) or (II). Assume that for every extremal ray R of the type (II) its contraction $f: X \to X'$ gives a 3-fold X' with isolated singularities (e.g., this statement is true if X has $\mathbf{Q}$-factorial terminal singularities).*

Then one of the statements (1)—(4) below holds:

(1) $N_1(X) \leq 8$.

(2) There exists a face of $\overline{NE}(X)$ of the Kodaira dimension 1 or 2.

(3) There exists a set $\mathcal{E}$ of extremal rays with $t \geq 2$ connected components $\mathfrak{B}^1 = \{R_{11}, R_{12}\}, .., \mathfrak{B}^t = \{R_{t1}, R_{t2}\}$ of the type $\mathfrak{B}_2$ such that

$$a_{11}R_{11} + a_{21}R_{21} + ... + a_{t1}R_{t1} = a_{12}R_{12} + a_{22}R_{22} + ... + a_{t2}R_{t2}$$

for some $a_{ij} > 0$.

(4) There are extremal rays S_1, R_1, R_2 of the type (II) which define the configuration (c) of the Theorem 2.4.1.

Proof. This follows from the Theorem 2.6.2 for $\mathcal{E} = \emptyset$. ▷

Now we want to apply the Theorem 2.5.1 to a suitable resolution of singularities of Fano 3-fold with log-terminal singularities.

Definition 2.6.4. Let Y be a 3-dimensional variety with log-terminal singularities. A birational morphism $\sigma : X \to Y$ is called ***minimal terminal resolution*** of singularities of Y if the following conditions hold: X has $\mathbf{Q}$-factorial terminal singularities; the exceptional set of σ is a union of irreducible divisors F_i and in the formula

$$K_X = \sigma^* K_Y + \sum \alpha_i F_i \qquad (6.1)$$

the pair $(X, \sum(-\alpha_i)F_i)$ has log-terminal singularities in the sense of [2]. In particular, $-1 < \alpha_i \leq 0$. ▷

We can apply the Theorem 2.5.1 to a case when all extremal rays on X have the type (I) or (II) and any extremal ray of the type (II) is simple. Thus, in our case it is natural to suppose that the following condition takes place:

Condition 2.6.5. The morphism σ is the contraction of a simplicial face $R_1 + ... + R_n$ of $\overline{NE}(X)$ which is generated by extremal rays $R_1, ..., R_n$ of the type (I) or (II) and every extremal ray R_i, $1 \leq i \leq n$, of the type (II) is simple.

We want to note that by the Proposition 2.3.2, it is sufficient to suppose that the contraction of any R_i of the type (II) gives a 3-fold with isolated singularities. Also, by the Theorem 2.3.3, the set $R_1, ..., R_n$ of extremal rays has connected components of the type $\mathfrak{A}_1, \mathfrak{B}_2, \mathfrak{C}_m$ or $\mathfrak{D}_2$. The Theorem 2.3.6 gives the inverse statement. Thus, the condition 2.6.5 is mostly the condition on the singularities of Y. ▷

We say that a 3-fold Y with log-terminal singularities has ***simplest singularities*** if there exists a minimal terminal resolution of singularities of Y with the condition 2.6.5. We should mention that in [16] we considered much more narrow class of simplest singularities.

Lemma 2.6.6. *Let Y be Fano 3-fold with simplest log-terminal singularities and $\sigma : X \to Y$ a minimal terminal resolution of singularities of Y.*

Then $\overline{NE}(X)$ is generated by a finite set of contractible extremal rays and all its faces are contractible. Every extremal ray R of $\overline{NE}(X)$ belongs to one of the following cases below:

(i) R is one of the rays $R_1, ..., R_n$ which are contracted by the morphism σ;

(ii) $R \cdot D(R_i) < 0$ for one of divisors $D(R_i)$ of the extremal rays $R_1, ..., R_n$ above. Thus, any curve of R is contained in $D(R_i)$.

(iii) $R \cdot K_X < 0$.

Proof. Compare with the proof of the Lemma 1.3.2 in [17] or the Lemma 2.1 in [7].

By the Theorem 2.3.3, there are rational $\epsilon_i > 0$ such that for $\Delta = \sum \epsilon_i D(R_i)$ we have the inequality $R_i \cdot \Delta < 0$ for any R_i from (i). Since $-K_Y$ is ample on Y and the morphism σ is the contraction of the face $R_1 + ... + R_n$, it follows that for sufficiently small $\epsilon > 0$ we have the inequality $-(\sigma^* K_Y + \epsilon\Delta) \cdot T > 0$ for any non-zero $T \in \overline{NE}(X)$. It follows that the element $H = -(\sigma^* K_Y + \epsilon\Delta)$ is ample for sufficiently small $\epsilon > 0$. By the formula (6.1), we have

$$K_X = \sigma^* K_Y + \sum \alpha_i D(R_i), \text{ where } -1 < \alpha_i \leq 0. \qquad (6.2)$$

It follows that $H = -(K_X + \sum(-\alpha_i + \epsilon\epsilon_i)D(R_i))$, where for sufficiently small $\epsilon > 0$ we have the inequalities $0 \leq -\alpha_i + \epsilon\epsilon_i < 1$, since $0 \leq -\alpha_i < 1$. Since the pair $(X, \sum(-\alpha_i)D(R_i))$ is log-terminal (in the sense [2]), for sufficiently small $\epsilon > 0$ the pair $(X, \sum(-\alpha_i + \epsilon\epsilon_i)D(R_i))$ is evidently log-terminal too. Since the element $-(K_X + \sum(-\alpha_i)D(R_i))$ is ample on X, from [2, Theorem 4.5], it follows that $\overline{NE}(X)$ is generated by a finite set of extremal rays and every its face is contractible.

Let R be an extremal ray on X and R is different from the rays R_i. Suppose that $-R \cdot K_X \geq 0$. Since the ray R is contractile and $-\sigma^* K_Y$ is ample on Y, we have $R \cdot \sigma^* K_Y < 0$. From the formula (6.1), it follows that $R \cdot D(R_i) < 0$ for some i. Since the ray R is contractible, any curve of R belongs to the divisor $D(R_i)$. Otherwise, $R \cdot D(R_i) \geq 0$. ▷

From the Lemma 2.6.6 and the Theorem 2.5.1, we get

Theorem 2.6.7. *Let Y be Fano 3-fold with simplest log-terminal singularities and $\sigma : X \to Y$ a minimal terminal resolution of singularities of Y. Suppose that any extremal ray on X is an extremal ray of the type (I) or simple extremal ray of the type (II).*

Then we have one of the cases (1), (2) or (3) below:

(1) $dim N_1(X) \leq 8$.

(2) One of the faces of $\overline{NE}(X)$ is contractible and has Kodaira dimension 1 or 2.

(3) There are extremal rays $S_1, S_2, ..., S_t, t \geq 2$, of the type (II) such that the divisors $D(S_1), D(S_2), ..., D(S_t)$ do not intersect one another, but $S_1 + ... + S_t$ is not a face of $\overline{NE}(X)$ of the Kodaira dimension 3.

(4) There are extremal rays S_1, R_1, R_2 of $\overline{NE}(X)$ which define the configuration (c) of the theorem 2.4.1.

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