Construction of Symbols on Adelic Matrix Groups

Richard Hill

University College London
Department of Mathematics
Gower Street
London WC1E 6BT

England

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

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Abstract

This article is a reinterpretation of the work of Kubota [4] and Hill [2], in which the reciprocity law for Kummer extensions of algebraic number fields are proved using geometric arguments. Here we show that the same methods can be used to describe higher metaplectic cocycles. We also describe an analogy of the Hilbert Reciprocity law for a skew field.

1 Introduction

1.1 Main Results

Let μ be a finite group and denote by μ^{ab} the Abelianization of μ, ie. $\mu^{\mathrm{ab}} = \mu/\mu'$ where μ' is the subgroup of μ generated by the commutators. Let $L = \mathbb{Z}^n$, $V = \mathbb{Q}^n$, $L_p = \mathbb{Z}_p^n$, $V_p = \mathbb{Q}_p^n$, $V_{\mathbb{A}} = \mathbb{A}^n$ and $X = V_\infty/L$. Suppose we have an action of μ on L satisfying:

- For all $l \in L \setminus \{0\}$ and $\zeta \in \mu \setminus \{1\}$, one has $\zeta \cdot l \neq l$.
- There is a singular n-chain $\mathcal{P}$ in $C_n(X)$ which is a linear combination of parallelotopes, such that
$$\sum_{\zeta\in\mu} \zeta\mathcal{P}$$
is a cycle and generates the homology group $H_n(X, \mathbb{Z})$. Furthermore, ther vertices of $\mathcal{P}$ generate a finite subgroup of X.

The chain referred to above can be thought of as a generalized fundamental domain. We shall refer to is as a fundamental chain.

Let $G_{\mathbb{Z}} = \mathrm{Aut}_\mu(V) \cap \mathrm{End}_\mu(L)$ and $G_{\mathbb{Q}} = \mathrm{Aut}_\mu(V)$. We think of this as an algebraic group and use the standard notation. Let

$$\begin{aligned}
\Omega_{\mathbb{Z}} &:= \left\{(\alpha,\beta) \in G_{\mathbb{Z}}^2 : \begin{array}{c} \alpha\beta = \beta\alpha, \text{ and} \\ \alpha L + \beta L = \alpha L + \#\mu \cdot L = \beta L + \#\mu \cdot L = L \end{array}\right\}, \\
\Omega_{\mathbb{Q}} &:= \{(\alpha,\beta) \in G_{\mathbb{Q}}^2 : \alpha\beta = \beta\alpha\}. \\
\Omega_{\mathbb{A}} &:= \{(\alpha,\beta) \in G_{\mathbb{A}}^2 : \alpha\beta = \beta\alpha\}.
\end{aligned}$$

We shall construct a map

$$\mathrm{Hilb} : \Omega_{\mathbb{A}} \to \mu^{\mathrm{ab}}$$

with the following properties:

- Hilb is bimultiplicative: if $(\alpha, \beta), (\alpha', \beta) \in \Omega_{\mathbb{A}}$ then

$$\mathrm{Hilb}(\alpha\alpha', \beta) = \mathrm{Hilb}(\alpha, \beta)\mathrm{Hilb}(\alpha', \beta).$$

- Hilb is continuous with respect to the adele topology.

- Hilb is skew-symmetric:-

$$\mathrm{Hilb}(\alpha, \beta)\mathrm{Hilb}(\beta, \alpha) = 1, \quad \mathrm{Hilb}(\alpha, \alpha) = 1.$$

- Hilb has the property:-

$$\mathrm{Hilb}(\alpha, 1 - \alpha) = 1.$$

- Hilb splits over $\mathbb{Q}$: if $(\alpha, \beta) \in \Omega_{\mathbb{Q}}$ then

$$\mathrm{Hilb}(\alpha, \beta) = 1.$$

- Hilb is a product of local factors:

$$\mathrm{Hilb}(\alpha, \beta) = \prod_v \mathrm{Hilb}_v(\alpha_v, \beta_v),$$

  where $\alpha_v, \beta_v \in G(\mathbb{Q}_v)$ are the v-th components of α and β and $\mathrm{Hilb}_v : G(\mathbb{Q}_v) \times G(\mathbb{Q}_v) \to \mu^{\mathrm{ab}}$ is a continuous bicharacter.

- Hilb commutes with direct sums. Suppose $L = L^1 \oplus L^2$ as μ-modules and $\alpha V_{\mathbb{A}}^1 = V_{\mathbb{A}}^1$, $\beta V_{\mathbb{A}}^1 = V_{\mathbb{A}}^1$, $\alpha V_{\mathbb{A}}^2 = V_{\mathbb{A}}^2$, $\beta V_{\mathbb{A}}^2 = V_{\mathbb{A}}^2$. Then

$$\mathrm{Hilb}(\alpha, \beta) = \mathrm{Hilb}(\alpha|_{L^1}, \beta|_{L^1}) \cdot \mathrm{Hilb}(\alpha|_{L^2}, \beta|_{L^2}).$$

- If $\mu_0 \subset \mu$ then one has

$$\mathrm{Hilb}_{\mu_0} = \mathrm{Verl}_{\mu_0}^{\mu} \mathrm{Hilb}_{\mu},$$

  where Verl is the transfer homomorphism.

Having given several properties of Hilb it remains to say how it is defined. This is achieved is several stages. Let S be a finite set of places of $\mathbb{Q}$, including all places v for which $|\#\mu|_v \neq 1$. We shall use the standard notation

$$\mathbb{Q}_S = \bigoplus_{v \in S} \mathbb{Q}_v \quad , \mathbb{Z}^S = \{x \in \mathbb{Q} : \forall v \notin S, \ |x|_v \leq 1\}.$$

We begin by constructing a bicharacter Hilb_S on $\Omega_{\mathbb{Q}_S}$ which is trivial on $\Omega_{\mathbb{Z}^S}$. This will be compatibe with the injections $G(\mathbb{Q}_S) \to G(\mathbb{Q}_{S'})$ for $S \subset S'$. Thus taking the limit over S we obtain a bicharacter on $\Omega_{\mathbb{A}}$.

We now give a brief description of Hilb_S, which should be avoided on first reading but which may be useful for reference purposes. To define Hilb_S we let X_S be the abelian group

$$X_S = V(\mathbb{Q}_S)/V(\mathbb{Z}^S).$$

Let $\Omega_{\mathbb{Z}^S}$ be the set of pairs $(\alpha, \beta) \in G(\mathbb{Z}^S) \times G(\mathbb{Z}^S)$ satisfying $\alpha\beta = \beta\alpha$, $\alpha V(\mathbb{Z}^S) + \beta V(\mathbb{Z}^S) = V(\mathbb{Z}^S)$. Let $f : X_S \to \mathbb{Z}$ be any function satisfying

$$\sum_{\zeta \in \mu} f(\zeta x) = \#\mathrm{Stab}_\mu(x).$$

Then for $(\alpha, \beta) \in \Omega_{\mathbb{Z}^S}$ we define

$$\mathrm{Hilb}_S(\alpha, \beta) := \prod_{x \in X^*[\beta], \zeta \in \mu} \zeta^{f(\zeta x) f(\alpha x)} \prod_{x \in X^*[\alpha], \zeta \in \mu} \zeta^{-f(\zeta x) f(\beta x)}$$

where

$$X^*[\alpha] := \{x \in X_S : \alpha x = 0, \ x \neq 0\}.$$

Most of the hard work in the paper goes into proving that Hilb_S can be extended uniquely to a continuous function on $\Omega_{\mathbb{Q}_S}$. The properties of Hilb follow naturally from the construction. Our techniques will be geometric. Most of the arguments can be found in a slightly different context in [2], which in turn is based on ideas from [4] and [1]. The main object of study is the decident, which in some form at least dates back to Gauss.

1.2 Group Extensions

Let $H/\mathbb{Q}$ be an algebraic group and let $\rho : H \oplus H \oplus \mu \to \mathrm{Aut}\, V$ be a $\mathbb{Q}$-rational representation. Then one may pull Hilb back to a continuous bicharacter $\mathrm{Hilb}_\rho : H(\mathbb{A}) \times H(\mathbb{A}) \to \mu^{\mathrm{ab}}$. Any bicharacter is automatically a 2-cocycle in the group theoretical sense, since one has

$$\begin{aligned}
(\partial \mathrm{Hilb}_\rho)(\alpha, \beta, \gamma) &:= \mathrm{Hilb}_\rho(\alpha\beta, \gamma)\mathrm{Hilb}_\rho(\alpha, \beta)\mathrm{Hilb}_\rho(\alpha, \beta\gamma)^{-1}\mathrm{Hilb}_\rho(\beta, \gamma)^{-1} \\
&= \mathrm{Hilb}_\rho(\alpha, \gamma)\mathrm{Hilb}_\rho(\beta, \gamma)\mathrm{Hilb}_\rho(\alpha, \beta) \\
&\quad\ \mathrm{Hilb}_\rho(\alpha, \beta)^{-1}\mathrm{Hilb}_\rho(\alpha, \gamma)^{-1}\mathrm{Hilb}_\rho(\beta, \gamma)^{-1} \\
&= 1.
\end{aligned}$$

Thus Hilb_ρ represents a 2-cohomology class with values in μ^{ab}. Corresponding to this there is a central group extension

$$1 \to \mu^{\mathrm{ab}} \to \widetilde{H(\mathbb{A})} \to H(\mathbb{A}) \to 1.$$

Set theoretically $\widetilde{H(\mathbb{A})}$ consistes of all pairs (α, ζ) with $\alpha \in H(\mathbb{A}), \zeta \in \mu^{\mathrm{ab}}$. The group law is then given by

$$(\alpha, \zeta) \cdot (\alpha', \zeta') = (\alpha\alpha', \zeta\zeta' \mathrm{Hilb}_\rho(\alpha, \alpha')).$$

Since Hilb_ρ is trivial on $H(\mathbb{Q}) \times H(\mathbb{Q})$ it follows that the extension splits over $H(\mathbb{Q})$.

1.3 Example 1 : The Hilbert Symbol

We now describe the connections between our construction and the real world. Suppose that V is a number field and μ is its group of roots of unity, acting on V be scalar multiplication. Then for any two ideles α, β of V we may define $\mathrm{Hilb}(\alpha, \beta)$. It turns out that this coincides with the global Hilbert symbol.

1.4 Example 2 : Metaplectic Extensions

Let K be a number field. Suppose that H is the subgroup of diagonal matrices in GL_n/K. Let N be the K-vector space of upper triangular matrices whose diagonal enties are all zero. As with the previous example let μ be a group of roots of unity in K. We define a representation ρ of $H \oplus H \oplus \mu$ as follows:-

$$\rho(\alpha, \beta, \zeta)n := \zeta \alpha n \beta^{-1},$$

where the multiplication by ζ is scalar multiplication and the left and right multiplication by α and β is matrix multiplication. Then the corresponding extension $\widetilde{H(\mathbb{A}_K)}$ of $H(\mathbb{A}_K)$ is the restriction to H of the standard metaplectic extension of $\mathrm{GL}_n(\mathbb{A}_K)$ (see [3] §0). To obtain the twisted metaplectic extensions one adds to N copies of the vector space of diagonal matrices, acted on by α, β and ζ in the obvious way.

We now mention another connection with metaplectic extensions. Let σ be the cocycle of the metaplectic extension, ie. $\sigma : \mathrm{SL}_n(\mathbb{A}_K) \times \mathrm{SL}_n(\mathbb{A}_K) \to \mu(K)$. Let $V = K^n$. This is acted on by μ and also by SL_n/K. The action of μ on K^n satisfies our conditions. Furthermore, elements of SL_n/K commute with the action of $\mu(K)$ on K^n. We may therefore think of $\mathrm{Rest}_{\mathbb{Q}}^K(\mathrm{GL}_n/K)$ as a subgroup of $G/\mathbb{Q} := \mathrm{Aut}_\mu(V)/_{\mathbb{Q}}$. With this notation we have

Theorem. *Let $\alpha, \beta \in \mathrm{SL}_n(\mathbb{A}_K)$ such that $\alpha\beta = \beta\alpha$. Suppose α and β are regular. Then*

$$\sigma(\alpha, \beta)\sigma(\beta, \alpha)^{-1} = \mathrm{Hilb}(\alpha, \beta).$$

1.5 Another Example

We now give another example which shows that the conditions of the theorem are not only satisfied when μ is a group of roots of unity of a number field acting on a vector space over the number field (in which case Hilb is quite closely related to the Hilbert symbol). Now let V be the skew field

$$V = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Q}\},$$

where

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j.$$

Let μ be the (non-abelian) group $\{\pm 1, \pm i, \pm j, \pm k\}$. This acts on V by left scalar multiplication. The action has the required properties: clearly the action is free because V is a skew field, and the other property can be checked by constructing a fundamental chain explicitly. The fundamental chain will be a sum of parallelotopes of the form

$$[0, \pm 1/2] \times [0, \pm i/2] \times [0, \pm j/2] \times [0, \pm k/2].$$

Note that μ^{ab} is a Klein 4-group. For any two commuting elements α, β of $V_{\mathbb{A}}^{\times}$, their right actions on V commute with the left action of μ. We may therefore define $\mathrm{Hilb}(\alpha, \beta)$ in terms of the right actions of α and β on V. This is a kind of analogy of the global Hilbert symbol for the skew field V. It is surjective, taking values in the Klein 4-group μ^{ab}, and it is trivial on $V^{\times} \times V^{\times}$.

2 Combinatorial Methods

Some of the results of this section have been described in a different notation by Smirnov [8]; the basic ideas are originally from [2], but there they were not described in the generality which we require.

2.1 The Sheaf of Fundamental Functions

Let X be a set with a chosen element $0 \in X$. Let μ be a finite group acting on X and suppose that every element of μ fixes 0. Denote by μ^{ab} the Abelianization of μ. A function $f : X \to \mathbb{Z}$ will be called *fundamental* at $x \in X$ iff it satisfies

$$\sum_{\zeta \in \mu} f(\zeta x) = \#\mathrm{Stab}_{\mu}(x).$$

The function will be called fundamental iff it is fundamental at all $x \in X$.

Let $Y \subset X$ be a finite μ-invariant subset and suppose that μ acts freely on Y. Let $\mathcal{F}(Y)$ be the $\mathbb{Z}$-span of the functions $f : X \to \mathbb{Z}$ which are fundamental on Y. There is a map $\deg : \mathcal{F}(Y) \to \mathbb{Z}$ given by

$$\deg(f) = \frac{\sum_{\zeta \in \mu} f(\zeta x)}{\#\mathrm{Stab}_{\mu}(x)}$$

for any $x \in Y$. We also introduce a skew product on $\mathcal{F}(Y)$ given by

$$< f, g >_Y := \prod_{x \in Y, \zeta \in \mu} \zeta^{f(\zeta x) g(x)} \in \mu^{\mathrm{ab}}.$$

The properties of this inner product can be summed up as follows:-

Proposition 1 *If $f, g, h \in \mathcal{F}(Y)$ then one has*

$$< f + g, h >_Y = < f, h >_Y < g, h >_Y,$$

$$< f, g >_Y < g, f >_Y = 1, \quad < f, f >_Y = 1.$$

If $\deg f = \deg g = \deg h$ *then one has*

$$< f, g >_Y < g, h >_Y = < f, h >_Y .$$

If Y is the disjoint union of $Y1$ and $Y2$ and if $Y1$ and $Y2$ are μ-invariant then

$$< f, g >_Y = < f, g >_{Y1} < f, g >_{Y2} .$$

Proof. The first two properties are easily seen from the definition. To prove that $< f, f >_Y = 1$ we express f as a linear combination of functions $g \in \mathcal{F}(Y)$ which take only the values 0 and 1 on Y. It is easily seen that one has $< g, g >_Y = 1$ for such functions and the general case follows from this. The final relation is also trivial. It remains to show that $< f, g >_Y < g, h >_Y = < f, h >_Y$. We shall prove this in two steps.

(i) We first show that for all $f, g \in \mathcal{F}(Y)$ with $\deg f = \deg g = 0$ one has $< f, g >_Y = 1$. Let f be such a function and define

$$\theta(x) := \prod_{\zeta \in \mu} \zeta^{f(\zeta x)}.$$

Note that for $\xi \in \mu$ we have

$$\begin{aligned} \theta(\xi x) &= \prod_{\zeta \in \mu} \zeta^{f(\zeta \xi x)} \\ &= \prod_{\zeta \in \mu} (\zeta \xi^{-1})^{f(\zeta x)} \\ &= \prod_{\zeta \in \mu} \zeta^{f(\zeta x)} \times \xi^{-\sum_{\zeta \in \mu} f(\zeta x)} \\ &= \theta(x) \xi^{-\deg f} = \theta(x). \end{aligned}$$

Thus θ is constant on μ-orbits. We shall write $[x]$ for the μ-orbit of a point $x \in Y$. Then we have

$$\begin{aligned} < f, g >_Y &= \prod_{x \in Y} \theta(x)^{g(x)} \\ &= \prod_{[x] \subset Y} \prod_{y \in [x]} \theta(x)^{g(y)} \\ &= \prod_{[x] \subset Y} \theta(x)^{\sum_{y \in [x]} g(y)} \\ &= \prod_{[x] \subset Y} \theta(x)^{\deg g} = 1. \end{aligned}$$

(ii) Now let $f, g, h \in \mathcal{F}(Y)$ satisfy $\deg f = \deg g = \deg h$. By the previous parts of the proposition we have

$$\begin{aligned} < f, g >_Y < g, h >_Y &= < f, g >_Y < h, g >_Y^{-1} \\ &= < f - h, g >_Y \\ &= < f - h, g - h >_Y < f - h, h >_Y \\ &= < f - h, g - h >_Y < f, h >_Y . \end{aligned}$$

However, using the fact that $\deg(g-f) = \deg(h-f) = 0$ we know by (i) that $< f-h, g-h >_Y = 1$. This proves the proposition.

2.2 Decidents

Let G be the set of μ-covariant self maps $\alpha : X \to X$ such that $\alpha(0) = 0$ and $\alpha^{-1}(\{0\})$ is finite and the action of μ on $\alpha^{-1}(\{0\}) \setminus \{0\}$ is free. For $\alpha \in G$ we shall use the notation

$$X[\alpha] := \{x \in X : \alpha(x) = 0\},$$

$$X^*[\alpha] := \{x \in X : \alpha(x) = 0,\ x \neq 0\}.$$

Let

$$\Omega = \left\{ (\alpha,\beta) \in G^2 : \begin{array}{c} \alpha\beta = \beta\alpha, \alpha \text{ permutes } X[\beta] \\ \text{and } \beta \text{ permutes } X[\alpha] \end{array} \right\}.$$

For $(\alpha,\beta) \in \Omega$ we define the *Decident*

$$\mathrm{Dec}(\alpha,\beta) := \mathrm{Dec}(\alpha,\beta,X,\mu) := < f, f\circ\alpha >_\beta,$$

where f is any fundamental function and we are using the notation

$$< -,- >_\beta := < -,- >_{X^*[\beta]}.$$

Proposition 2 *The decident is independent of the choice of f and satisfies*

$$\mathrm{Dec}(\alpha\alpha',\beta) = \mathrm{Dec}(\alpha,\beta) \cdot \mathrm{Dec}(\alpha',\beta).$$

(Here it is not necessary that $\alpha\circ\alpha' = \alpha'\circ\alpha$.)

Proof. Suppose that f and g are fundamental. Then by the previous proposition one has

$$< f, f\circ\alpha >_\beta < f\circ\alpha, g\circ\alpha >_\beta < g\circ\alpha, g >_\beta < g, f >_\beta = 1.$$

This implies

$$< f, f\circ\alpha >_\beta < g, g\circ\alpha >_\beta^{-1} = < f\circ\alpha, g\circ\alpha >_\beta^{-1} < f, g >_\beta .$$

However since α permutes $X^*[\beta]$ we have

$$< f\circ\alpha, g\circ\alpha >_\beta = \prod_{x,\zeta} \zeta^{f(\zeta\alpha x)g(\alpha x)} = \prod_{x,\zeta} \zeta^{f(\zeta x)g(x)} = < f, g >_\beta .$$

Therefore $< f, f\circ\alpha >_\beta = < g, g\circ\alpha >_\beta$ and so the decident is independent of f.

Now let $g = f\circ\alpha$ for some fundamental function f. It follows that g is fundamental on $X^*[\beta]$. We therefore have

$$\begin{aligned} \mathrm{Dec}(\alpha\circ\alpha',\beta) &= < f, f\circ\alpha\circ\alpha' >_\beta \\ &= < f, f\circ\alpha >_\beta < f\circ\alpha, f\circ\alpha\circ\alpha' >_\beta \\ &= < f, f\circ\alpha >_\beta < g, g\circ\alpha' >_\beta \\ &= \mathrm{Dec}(\alpha,\beta) \cdot \mathrm{Dec}(\alpha',\beta). \end{aligned}$$

From now on, in addition to our other assumptions we shall suppose that X is an Abelian group and that 0 is the identity element of X and that μ and α and β act on X by surjective group homomorphisms. We shall write the group law of X additively. This gives rise to a group law on $\mathrm{End}_\mu(X)$, which we shall also write additively. We then have

Proposition 3 *The decident is multiplicative in the second factor, ie.*

$$\mathrm{Dec}(\alpha, \beta \circ \beta') = \mathrm{Dec}(\alpha, \beta)\mathrm{Dec}(\alpha, \beta').$$

(Here it is not necessary that $\beta \circ \beta' = \beta' \circ \beta$.)

Proof. We define a special fundamental function g which will simplify the situation. Let f be any fundamental function, and define g by

$$g(x) = \begin{cases} f(\beta x) & x \notin X^*[\beta], \\ f(x) & x \in X^*[\beta]. \end{cases}$$

Then it follows that g is fundamental on $X^*[\beta \circ \beta']$. We have

$$\mathrm{Dec}(\alpha, \beta \circ \beta') = < g, g \circ \alpha >_{\beta \circ \beta'} = < g, g \circ \alpha >_{X[\beta\beta'] \setminus X[\beta]} < g, g \circ \alpha >_\beta .$$

The second term on the right is $\mathrm{Dec}(\alpha, \beta)$. We shall examine the first term.

$$< g, g \circ \alpha >_{X^*[\beta\beta'] \setminus X^*[\beta]} = \prod_{\beta\beta' x = 0, \beta x \neq 0, \zeta \in \mu} \zeta^{f(\zeta\beta x) f(\beta\alpha x)}.$$

We shall write $N(\beta)$ for the number of elements of $X[\beta]$. Since β is a surjective group homomorphism, this is the number of preimages of any element of X. We therefore have (substituting $y = \beta x$)

$$< g, g \circ \alpha >_{X^*[\beta\beta'] \setminus X^*[\beta]} = \prod_{\beta' y = 0, y \neq 0, \zeta \in \mu} \zeta^{f(\zeta y) f(\alpha y) N(\beta)} = \mathrm{Dec}(\alpha, \beta')^{N(\beta)}.$$

However since μ acts freely on $X^*[\beta]$ we have $N(\beta) \equiv 1 \bmod \#\mu$ and therefore

$$< g, g \circ \alpha >_{X^*[\beta\beta'] \setminus X^*[\beta]} = \mathrm{Dec}(\alpha, \beta').$$

This proves the proposition.

Lemma 1 *If* $\alpha \equiv I \bmod \beta$ *in* $\mathrm{End}_\mu(X)$ *then* $\mathrm{Dec}(\alpha, \beta) = 1$.

Proof. The action of α on $X[\beta]$ is trivial.

The following result is the first step in proving a kind of reciprocity law for decidents.

Proposition 4 *Let α and β be as above and let f be fundamental. Then*

$$\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} = < f, f\circ\alpha >_{\alpha\beta\backslash\alpha} < f, f\circ\beta >^{-1}_{\alpha\beta\backslash\beta} \times \prod_{\zeta\in\mu} \zeta^{\frac{(N(\alpha)-1)(N(\beta)-1)}{\#\mu^2}}$$

Here we are using the notation

$$< -,- >_{\alpha\beta\backslash\alpha} := < -,- >_{X^*[\alpha\beta]\backslash X^*[\alpha]} .$$

Proof. Note that $N(\alpha\beta) = N(\alpha)N(\beta)$. Also note that since α permutes $X[\beta]$, we have $X[\alpha]\cap X[\beta] = \{0\}$. From this follows $\#(X[\alpha]+X[\beta]) = N(\alpha\beta)$. On the other hand, since α and β commute we clearly have $X[\alpha]+X[\beta]\subset X[\alpha\beta]$. We therefore have

$$X[\alpha\beta] = X[\alpha]\oplus X[\beta].$$

This implies the following expression for $X^*[\alpha\beta]$ as a disjoint union:

$$X^*[\alpha\beta] = X^*[\alpha]\dot{\cup} X^*[\beta]\dot{\cup} Y$$

where $Y = \{x\in X|\alpha\beta x = 0 \text{ but } \alpha x\neq 0,\ \beta x\neq 0\}$. This leads to an identity of the products over these sets:

$$\begin{aligned} < f, f\circ\alpha >_{\alpha\beta\backslash\alpha} < f, f\circ\beta >^{-1}_{\alpha\beta\backslash\beta} &= < f, f\circ\alpha >_{\beta} < f, f\circ\alpha >_{Y} \\ &\quad < f, f\circ\beta >^{-1}_{\alpha} < f, f\circ\beta >^{-1}_{Y} \\ &= \mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} < f, f\circ\alpha >_{Y} < f\circ\beta, f >_{Y} \\ &= \mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} < f\circ\beta, f\circ\alpha >_{Y} . \end{aligned}$$

It remains to calculate the inner product $< f\circ\beta, f\circ\alpha >_Y$. To do this we also use the direct sum decomposition of $X[\alpha\beta]$. This gives us

$$Y = \{x_1 + x_2 : x_1\in X^*[\alpha],\ x_2\in X^*[\beta]\}.$$

This implies

$$\begin{aligned} < f\circ\beta, f\circ\alpha >_{Y} . &= \prod_{\zeta\in\mu}\prod_{x_1\in X^*[\alpha]}\prod_{x_2\in X^*[\beta]} \zeta^{f(\zeta\beta(x_1+x_2))f(\alpha(x_1+x_2))} \\ &= \prod_{\zeta\in\mu}\prod_{x_1\in X^*[\alpha]}\prod_{x_2\in X^*[\beta]} \zeta^{f(\zeta\beta x_1)f(\alpha x_2)} \\ &= \prod_{\zeta\in\mu} \zeta^{\left(\sum_{x_1\in X^*[\alpha]} f(\zeta\beta x_1)\right)\left(\sum_{x_2\in X^*[\beta]} f(\alpha x_2)\right)} \\ &= \prod_{\zeta\in\mu} \zeta^{\frac{(N(\alpha)-1)(N(\beta)-1)}{\#\mu^2}} \end{aligned}$$

2.3 The Plan

Let $L = \mathbb{Z}^n$, and suppose we have an action of a finite group μ on L. We shall also use the following notation

$$\text{for a prime } p,\ L_p := L \otimes \mathbb{Z}_p; \qquad V := L \otimes \mathbb{Q};$$

$$\text{for a prime } p,\ V_p := L \otimes_{\mathbf{Z}} \mathbb{Q}_p; \qquad V_\infty := L \otimes \mathbb{R}; \qquad V_{\mathbb{A}} := L \otimes \mathbb{A}.$$

The tensor products are all over $\mathbb{Z}$. Define

$$G_{\mathbf{Q}} := \mathrm{Aut}_\mu V, \qquad G_{\mathbf{Z}} := \mathrm{End}_\mu L \cap \mathrm{Aut}_\mu V,$$

$$G_{\mathbb{R}} := \mathrm{Aut}_\mu V_\infty, \qquad G_{\mathbb{A}} := \mathrm{Aut}_\mu V_{\mathbb{A}}.$$

Also write $G^0_{\mathbb{R}}$ for the connected component of I in $G_{\mathbb{R}}$. Furthermore define

$$\Omega_{\mathbf{Z}} := \{(\alpha,\beta) \in G_{\mathbf{Z}} : \alpha \circ \beta = \beta \circ \alpha \text{ and } \alpha L + \beta L = L\},$$

$$\Omega_{\mathbf{Q}} := \{(\alpha,\beta) \in G^2_{\mathbf{Q}} : \alpha \circ \beta = \beta \circ \alpha\},$$

$$\Omega_{\mathbb{A}} := \{(\alpha,\beta) \in G^2_{\mathbb{A}} : \alpha \circ \beta = \beta \circ \alpha\},$$

$$G^\alpha_{\mathbf{Z}} := \{\beta \in G_{\mathbf{Z}} : (\alpha,\beta) \in \Omega_{\mathbf{Z}}\}. \qquad G^\alpha_{\mathbb{R}} := \{\beta \in G_{\mathbb{R}} : (\alpha,\beta) \in \Omega_{\mathbb{R}}\}.$$

$$G^\alpha_{\mathbb{A}} := \{\beta \in G_{\mathbb{A}} : (\alpha,\beta) \in \Omega_{\mathbb{A}}\}.$$

Let $X := V_\infty / L$. Then μ acts on X, as does any element of $G_{\mathbf{Z}}$. For any pair $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ we define $\mathrm{Dec}(\alpha,\beta)$ in terms of the actions of α,β,μ on X. In this article we shall prove

Theorem 1 *There is an open subgroup $U \subset G_{\mathbb{A}^f}$ such that if $(\alpha,\beta) \in \Omega_{\mathbf{Z}} \cap U^f \times U^f$, and if $\alpha \in G^0_{\mathbb{R}}$ and if $\beta \in G^{\alpha 0}_{\mathbb{R}}$ then one has*

$$\mathrm{Dec}(\alpha,\beta) = \mathrm{Dec}(\beta,\alpha).$$

Here $\mathbb{A}^f$ are the finite adeles.

This implies the following

Theorem 2 *There is an open subgroup $U' \subset G_{\mathbb{A}^f}$ such that if $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ and $\alpha \in U'$ and if (α,β) is in the connected component of* 1 *in $\Omega_{\mathbb{R}}$ then one has*

$$\mathrm{Dec}(\alpha,\beta) = \mathrm{Dec}(\beta,\alpha).$$

The arguments used to show that Theorem 1 implies Theorem 2 are contained in the introduction to [2]. Theorem 2 in turn will imply the following

Theorem 3 *There is a unique continuous map* Hilb : $\Omega_{\mathbb{A}} \to \mu^{\mathrm{ab}}$ *which is trivial on* $\Omega_{\mathbb{Q}}$ *and such that* Hilb *is bimultiplicative and if* $(\alpha, \beta) \in \Omega_{\mathbb{Z}}$ *then one has*

$$\mathrm{Hilb}(\alpha_f, \beta_f) = \mathrm{Dec}(\alpha, \beta)\mathrm{Dec}(\beta, \alpha)^{-1}.$$

Here α_f *and* β_f *are adeles which are equal to* α *and* β *at finite places and are otherwise the identity.*

The ideas used in proving Theorem 3 from Theorem 2 are contained in the discussion of symbols in [7]. In fact, Hilb will be independent of L, and will only depend on V:

Proposition 5 *Let* $L' \subset L$ *be another* μ*-invariant lattice. If* $(\alpha, \beta) \in \Omega \cap \Omega'$ *satisfies* $(\det \beta, [L : L']) = 1$ *then*

$$\mathrm{Dec}(\alpha, \beta) = \mathrm{Dec}'(\alpha, \beta).$$

Thus one has

$$\mathrm{Hilb} = \mathrm{Hilb}'.$$

3 Geometric Methods

Again the ideas here can be found in [2] but in insufficient generality. Many of the ideas are also contained in either [4] or [1] (see [2] for a description of where each idea comes from). Some technical improvments and simplifications have been made here.

Let $L = \mathbb{Z}^n$, and suppose we have an action of a finite group μ on L.

We shall think of V_∞ as a real vector space. Our next aim is to define the singular homology groups of a topological space. For our purposes, the definitions given in [6] are most convenient. Later in this section we shall construct using the homology groups a class of fundamental functions. At the end of the section we shall find a formula for the skew product $< f^1, f^2 >$, where f^1 and f^2 are from the class of fundamental functions which we shall construct.

The connection with the homology groups is the following: we define $f^1(x)$ to be the degree of a map $\mathcal{P}^1 : I^n \to X$ at the point $x \in X$, where I^n is a hypercube.

3.1 Singular Homology

1. Let I be the closed interval $[0, 1]$ in $\mathbb{R}$. We shall write I^r for the cartesian product of r copies of I. I^0 will be a topological space with exactly one point.

2. Let X be a topological space. A continuous map $\mathcal{T} : I^r \longrightarrow X$ will be called a singular r-cube in X. We shall write $Q_r(X)$ for the $\mathbb{Z}$-module generated by the set of singular r-cube in X, and with relations

$$\mathcal{T} + \mathcal{T} \circ (i\, j) = 0 \quad 1 \leq i < j \leq r,$$

where

$$(i\ j)(x_1, \ldots, x_r) := (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_r).$$

Therefore one always has in $Q_r(X)$ the identity $\mathcal{T} \circ \Phi = \mathrm{sign}(\Phi).\mathcal{T}$, where Φ is an element of the symmetry group S_r, acting on I^r by permutation of the coordinates.

3. A singular r-cube $\mathcal{T}$ is called *degenerate*, if the function $\mathcal{T}(x_1, \ldots, x_r)$ is independent of at least one of the coordinates x_i. We shall write $D_r(X)$ for the submodule of $Q_r(X)$ generated by the degenerate r-cubes. The quotient $C_r(X) := Q_r(X)/D_r(X)$ will be called the group of r-chains in X.

4. Let $\mathcal{T}$ be a singular r-cube. We now define the i^{th} *front face of* $\mathcal{T}$,
$$\mathrm{A}_i\mathcal{T} : I^{r-1} \longrightarrow X$$
$$(x_1, \ldots, x_{r-1}) \longmapsto \mathcal{T}(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{r-1})$$
and the i^{th} *back face of* $\mathcal{T}$,
$$\mathrm{B}_i\mathcal{T} : I^{r-1} \longrightarrow X$$
$$(x_1, \ldots, x_{r-1}) \longmapsto \mathcal{T}(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{r-1}).$$
The faces of an r-cubes are $(r-1)$-cubes.

5. The *boundary* of an r-cube $\mathcal{T}$ is defined to be the element of $Q_{r-1}(X)$ given by the following formula
$$\partial_r\mathcal{T} := \sum_{i=1}^{r}(-1)^i\ (\mathrm{A}_i\mathcal{T} - \mathrm{B}_i\mathcal{T}).$$
This definition can be extended by $\mathbb{Z}$-linearity to $Q_r(X)$.
$$\partial_r : Q_r(X) \longrightarrow Q_{r-1}(X).$$
This induces a homomorphism of the chain modules
$$\partial_r : C_r(X) \longrightarrow C_{r-1}(X).$$
We define the *r-cycles* to be the kernel of the boundary map
$$Z_r(X) := \mathrm{Ker}\ (\partial_r) \subset C_r(X),$$
and the *r-boundaries* to be its image
$$B_r(X) := \mathrm{Im}\ (\partial_{r+1}) \subset C_r(X).$$
One can check that every boundary is a cycle
$$B_r(X) \subset Z_r(X).$$
We can thus define the r^{th} *singular homology group of* X to be the quotient of the cycles by the boundaries:
$$H_r(X) := Z_r(X)/B_r(X).$$

6. Now let Y be a subspace of X. Clearly there is an inclusion

$$Q_r(Y) \subset Q_r(X).$$

This induces an inclusion of chain modules

$$C_r(Y) \subset C_r(X),$$

and we define the relative chain modules of X with respect to Y to be the quotient:

$$C_r(X,Y) := C_r(X)/C_r(Y).$$

The boundary map induces a homomorphism of relative chain modules

$$\partial_r : C_r(X,Y) \longrightarrow C_{r-1}(X,Y),$$

and we define as before the relative cyles to be the kernel; the relative boundaries to be the image; and the relative homology groups to be the quotient of the relative cycles by the relative boundaries.

$$Z_r(X,Y) := \text{Ker } \partial_r \subset C_r(X,Y),$$

$$B_r(X,Y) := \text{Im } \partial_{r+1} \subset C_r(X,Y),$$

$$H_r(X,Y) := Z_r(X,Y)/B_r(X,Y).$$

7. The base set $|\mathcal{T}|$ of a singular r-cube $\mathcal{T}$ is defined to be the image of $\mathcal{T}$, if $\mathcal{T}$ is non-degenerate, and the empty set, if $\mathcal{T}$ is degenerate. The base set of an element of $C_r(X)$ is defined to be the union of all base-sets of singular r-cubes in its support.

8. Let X be an Abelian topological group (whose group law we shall write additively) and let $\mathcal{T}$ be a singular r-cube and $\mathcal{U}$ a singular s-cube in X. We can define a product $(r+s)$-cube:

$$\mathcal{T} \times \mathcal{U} : I^{r+s} \longrightarrow X$$

$$(x_1, ..., x_r, y_1, ..., y_s) \longmapsto \mathcal{T}(x_1, ..., x_r) + \mathcal{U}(y_1, ..., y_s).$$

This product operation can be extended by bilinearity

$$Q_r(X) \times Q_s(X) \longrightarrow Q_{r+s}(X),$$

and this induces a product operation on the chain modules:

$$C_r(X) \times C_s(X) \longrightarrow C_{r+s}(X).$$

9. Let X be a manifold. If $x \in X$ then

$$H_n(X, X \setminus \{x\}) \cong \mathbb{Z}.$$

(This is a non-canonical isomorphism.) The manifold X is called orientable, if one can associate to each point $x \in X$ an isomorphism

$$\mathrm{Iso}_x : H_n(X, X \setminus \{x\}) \longrightarrow \mathbb{Z}$$

with the property that for every $x \in X$ there is a neighbourhood U of x, such that for every $y \in U$ the diagram commutes

$$\begin{tikzcd} & H_n(X, X \setminus U) \arrow[dl] \arrow[dr] & \\ H_n(X, X \setminus \{x\}) \arrow[dr, "\mathrm{Iso}_x"'] & & H_n(X, X \setminus \{y\}) \arrow[dl, "\mathrm{Iso}_y"] \\ & \mathbb{Z} & \end{tikzcd}$$

Such a set of isomorphisms is called an orientation. An n-dimensional, differentiable manifold, which posesses a global, non-vanishing differential n-form, is orientable.

Assume that X is orientable, and fix an orientation Iso. Let $\mathcal{T} \in C_n(X)$. Then $\partial\mathcal{T}$ is a singular $n-1$ chain. Suppose that $x \in X$ does not lie in the base set $|\partial\mathcal{T}|$ of $\partial\mathcal{T}$. Then $\mathcal{T}$ represents a homology class in $H_n(X, X \setminus \{x\})$. We define the degree of $\mathcal{T}$ at the point x to be

$$\mathbb{I}_{\mathcal{T}}(x) \ := \ \mathrm{Iso}_x(\mathcal{T}).$$

From our condition on Iso, we have a locally constant function

$$\mathbb{I}_{\mathcal{T}} \ : \ X \setminus |\partial\mathcal{T}| \ \longrightarrow \mathbb{Z}.$$

10. The space $X := V_\infty / L$ is orientable (it is a torus).

11. If S is a discrete subset of an n-dimensional orientable manifold X, and if $\mathcal{T}$ is an n-chain in X with $S \cap |\partial\mathcal{T}| = \emptyset$, then we define

$$\{\{S|\mathcal{T}\}\} := \sum_{x \in S} \mathbb{I}_{\mathcal{T}}(x).$$

Since $|\mathcal{T}|$ is compact, the sum has finite support. If $\sigma : X \longrightarrow X$ is a homeomorphism then

$$\{\{\sigma S|\sigma\mathcal{T}\}\} = \mathrm{sign}(\sigma)\{\{S|\mathcal{T}\}\},$$

where $\mathrm{sign}\sigma = \pm 1$ depending on whether σ is orientation preserving or orientation reversing. If X is a real vector space or a torus and $\sigma : X \to X$ is a composition of a (real-)linear bijection and a translation, then $\mathrm{sign}(\sigma)$ is the sign of the determinant of the linear bijection.

12. The singular 0-cubes in a topological space X correspond to the points x of X. We shall write $[x]$ for the singular 0-cube corresponding to x. The singular 1-cubes in X are paths between points x and y in X. Let X be an Abelian topological group and let $\mathcal{T}$ be a singular r-cube in X. Then

$$[x] \times \mathcal{T}$$

is a translation of $\mathcal{T}$ by x, and one has (because 0 is an even number)

$$[x] \times \mathcal{T} = \mathcal{T} \times [x].$$

This equality is at the level of singular r-cubes. We shall use the notation

$$\mathrm{Transl}(x)T := [x] \times \mathcal{T}.$$

3.2 Remarks

1. Let $\mathcal{T} : I^n \longrightarrow X$ be a singular n-cube in X, and let Y be a subspace of the space X with

$$|\partial\mathcal{T}| \subset Y.$$

Then $\mathcal{T}$ represents a homology class in $H_n(X, Y)$. We cut $\mathcal{T}$ into two pieces:

$$\mathcal{T}_1 : I^n \longrightarrow X$$
$$(x_1, \ldots, x_n) \longmapsto \mathcal{T}\left(\frac{x_1}{2}, x_2, \ldots, x_n\right)$$

and

$$\mathcal{T}_2 : I^n \longrightarrow X$$
$$(x_1, \ldots, x_n) \longmapsto \mathcal{T}\left(\frac{x_1 + 1}{2}, x_2, \ldots, x_n\right).$$

If in addition

$$|\partial\mathcal{T}_1| \subset Y \text{ and } |\partial\mathcal{T}_2| \subset Y,$$

then we have in $H_n(X, Y)$ the equation

$$\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}.$$

2. Let $\mathcal{T}$ be a singular n-cube in an oriented manifold X. Then we may define as in §3.1.9 a function $\mathbb{I}_{\mathcal{T}}$. Let $x \in X$ lie outside the base set of $\mathcal{T}$. Then we always have

$$\mathbb{I}_{\mathcal{T}}(z) = 0.$$

We shall now use the homology groups to construct fundamental functions. We shall define the function $f : X \rightarrow \mathbb{Z}$ to be $\mathbb{I}_{\mathcal{P}}$, where $\mathcal{P}$ is a sum of singular n-cubes.

3.3 The Fundamental Chain

As before, let μ be a finite group and V a vector space over $\mathbb{Q}$ on which μ acts. We have a μ-invariant lattice L in V. Let $X = V_\infty/L$. We assume that μ acts freely on $V \setminus \{0\}$.

An *edge* $e : I \to V_\infty$ is a 1-chain of the form

$$e(z) = z \cdot a,$$

where $a \neq 0$. A parallelotope in V_∞ is an n-cube in V_∞ of the form

$$\mathrm{Transl}(v)\left(\prod_{i=1}^{n} e_i\right),$$

where each e_i is an edge and the vectors $e_i(1)$ are linearly independent over $\mathbb{R}$. A parallelotope in X is a projection of a parallelotope in V_∞. The vertices of a parallelotope are the points $\{v + \sum_{i\in S} e_i(1) \ : \ S \subset \{1,\dots,n\}\}$. The following is essentially due to Kubota [4].

Theorem 4 *Suppose that μ is cyclic, and suppose μ acts on $V/\mathbb{Q}$ and its action is free on $V \setminus \{0\}$. Then there is a chain $\mathcal{P}$ in X such that*

$$\sum_{\zeta\in\mu} \det(\zeta)\cdot \zeta\mathcal{P}$$

is a cycle in X, whose homology class generates $H_n(X)$. The chain $\mathcal{P}$ is a finite linear combination over $\mathbb{Z}$ of parallelotopes. Furthermore, $\mathcal{P}$ is a finite linear combination over $\mathbb{Z}$ of parallelotopes whose vertices generate a finite subgroup of X.

In fact Kubota proved this in the case that μ is the group of roots of unity of a number field and V is the number field. The general case can be reduced to this as follows. Since μ acts freely on $V\setminus\{0\}$, if $1 \neq H$ is a subgroup of μ then $\sum_{\zeta\in H} \zeta v = 0$ for all $v \in V$. This implies that V is a vector space over the cyclotomic field whose roots of unity are μ. The chain $\mathcal{P}$ can be constructed from chains for the action of μ on the cyclotomic field.

We shall call a chain $\mathcal{P}$ with the properties guaranteed by the above theorem a fundamental chain. We shall call a representation of μ 'Kubotan' if it has a fundamental chain (this depends on the $\mathbb{Q}$-structure but is independent of the lattice L). From now on, we shall assume that we have a Kubotan representation on μ and we shall fix a fundamental chain $\mathcal{P}$. We shall lay down some notation for later use. There is an expression for $\mathcal{P}$ as a linear combination of parallelotopes $\mathcal{P}_i$:

$$\mathcal{P} = \sum_i \mathrm{wt}(i)\cdot \mathcal{P}_i. \tag{1}$$

Each parallelotope can be written as a product of edges:

$$\mathcal{P}_i = \mathrm{Transl}(x_i)\prod_{j=1}^{n} e_{i,j}. \tag{2}$$

We shall use the notation $\mathcal{F} < \mathcal{P}_i$ to say that $\mathcal{F}$ is a face of $\mathcal{P}_i$. Then we have an expression for the boundary of $\mathcal{P}_i$:

$$\partial \mathcal{P}_i = \sum_{\mathcal{F} < \mathcal{P}_i} \mathrm{wt}(\mathcal{F}, i) \cdot \mathcal{F}. \tag{3}$$

For any face $\mathcal{F}$ we have an expression

$$\mathcal{F} = \mathrm{Transl}(v_{\mathcal{F}}) \prod_{j=1}^{n-1} e_{\mathcal{F},j}.$$

Since $\partial \left(\sum_{\zeta \in \mu} \det(\zeta) \cdot \zeta \mathcal{P} \right) = 0$, we have for any face $\mathcal{F}$,

$$\sum_i \sum_{\zeta \in \mu} \det(\zeta) \cdot \mathrm{wt}(i) \cdot \mathrm{wt}(\zeta \mathcal{F}, i) = 0. \tag{4}$$

We shall write $\delta \in \mathbb{N}$ for an annihilator of the subgroup of X generated by ther vertices of the fundamental chain. We shall sometimes use the notation

$$\tilde{X}^*[\alpha] = \{ v \in V_\infty : \mathrm{pr}(v) \in X^*[\alpha] \}.$$

3.4 Modifications

Let $\mathcal{E}$ be the set of edges of $\mathcal{P}$. We introduce an equivalence relation on $\mathcal{E}$ as follows: two edges are said to be equivalent if one is ζ times the other for some (unique) element $\zeta \in \mu$. Let $\mathcal{E}_o$ be a set of representatives of eqivalence classes of edges of $\mathcal{P}$.

Suppose m assigns to each element e of $\mathcal{E}_o$ a path $m(e)$ in V_∞ with the same start and end points as e. We call such an m a "modification".

Let m be a modification and e any edge of $\mathcal{P}$. Then there is a unique expression $e = \zeta e_o$ for $e_o \in \mathcal{E}_o$. We define

$$m(e) := \zeta m(e_o).$$

We have an expression for $\mathcal{P}$ of the form

$$\mathcal{P} = \sum_i \mathrm{Transl}(v_i) \prod_{j=1}^{n} e_{i,j}.$$

We define the modified chain

$$m(\mathcal{P}) := \sum_i \mathrm{Transl}(v_i) \prod_{j=1}^{n} m(e_{i,j}).$$

We also define a function

$$f^m := \mathbb{I}_{m(\mathcal{P})}.$$

3.5 Homotopies Between Modifications

Let m^0 and m^1 be two modifications. We shall now introduce the idea of a homotopy from m^0 to m^1. Let $\mathcal{E}_o$ be as above, and let

$$M := \overset{\circ}{\bigcup_{e \in \mathcal{E}_o}} I_e,$$

where each I_e is a copy of I. Then a modification m can be thought of as a map

$$m : M \to V_\infty,$$

such that $m(0_e)$ and $m(1_e)$ are the start and end points of e.

If m^0 and m^1 are two modifications then a homotopy from m^0 to m^1 is a continuous map

$$h : M \times I \to X$$

such that $h(-,0) = m^0$, $h(-,1) = m^1$ and such that for any $t \in I$, $h(-,t)$ is a modification.

If e is any edge of $\mathcal{P}$ then one has $e = \zeta e_o$ for some $e_o \in \mathcal{E}_o$. We then define for $y, t \in I$,

$$h^{(e)}(y,t) := \zeta h^{(e_o)}(y,t).$$

Suppose $\mathcal{P}_i$ is a parallelotope in $\mathcal{P}$. Then we have

$$\mathcal{P}_i = \prod_{j=1}^{n} e_{i,j}.$$

Define

$$\tilde{h}_i(y_1, \ldots, y_n, t) := \sum_{j=1}^{n} h^{(e_{i,j})}(y_j, t).$$

The following is essentially due to Kubota.

Theorem 5 *Let $\mathcal{P}$ be a fundamental chain and let m be a modification. Then*

$$\sum_{\zeta \in \mu} \det(\zeta) \cdot \zeta m(\mathcal{P})$$

is a cycle in X, whose homology class generates $H_n(X)$.

Corollary 1 *Let $\mathcal{P}$ satisfy Theorem 4 and let m be a modification. Then for every $x \in X$ whose μ-orbit does not intersect $\partial m(\mathcal{P})$, f^m is fundamental at X.*

Proof. Choose homotopies from e to $m(e)$ for each $e \in \mathcal{E}_o$. Use these to construct a homotopy from $\sum_\zeta \det(\zeta) \cdot \zeta\mathcal{P}$ to its modification. Thus $\sum_{\zeta\in\mu} \det(\zeta) \cdot \zeta m(\mathcal{P})$ is homologous to $\sum_{\zeta\in\mu} \det(\zeta) \cdot \zeta\mathcal{P}$.

3.6 A Formula for the Skew Product

What follows is based on a technique used in Habicht's paper [1]. We investigate the difference $f^0 - f^1$ of two of the functions, which we constructed in §3.8. The basic idea is to express $m^0(\mathcal{P}) - m^1(\mathcal{P})$ as a sum of pieces, each piece being associated to a face $\mathcal{F}$ of a parallelotope of $\mathcal{P}$. Actually these pieces will be homotopies from $m^0(\mathcal{F})$ to $m^1(\mathcal{F})$. This method will lead to a formula for the skew product $< f^1, f^2 >$.

Notation Let h be a homotopy from m^0 to m^1. Then for any parallelotope $\mathcal{P}_i = \mathrm{Transl}(v_i)\prod_{j=1}^n e_{i,j}$ we define

$$\tilde{h}_i(x_1,\ldots,x_n,t) := v_i + \sum_{j=1}^n h^{(e_{i,j})}(x_j,t).$$

Similarly for any face $\mathcal{F} = \mathrm{Transl}(v_\mathcal{F})\prod_{j=1}^{n-1} e_{\mathcal{F},j}$ we define

$$\tilde{h}_\mathcal{F}(x_1,\ldots,x_{n-1},t) := v_\mathcal{F} + \sum_{j=1}^{n-1} h^{(e_{\mathcal{F},j})}(x_j,t).$$

Then $\tilde{h}_i$ is a homotopy from $m^0(\mathcal{P}_i)$ to $m^1(\mathcal{P}_i)$ and $\tilde{h}_\mathcal{F}$ is a homotopy from $m^0(\mathcal{F})$ to $m^1(\mathcal{F})$, and one has in $C_n(X)$:-

$$\begin{aligned}\partial\tilde{h}_i &= \sum_{j=1}^{n+1}(-1)^j\left(\mathrm{A}_j(\tilde{h}_i) - \mathrm{B}_j(\tilde{h}_i)\right)\\ &= (-1)^{n+1}\cdot\left(m^0(\mathcal{P}_i) - m^1(\mathcal{P}_i)\right) + \sum_{\mathcal{F}<\mathcal{P}_i}\mathrm{wt}(\mathcal{F},i)\cdot\tilde{h}_\mathcal{F},\end{aligned}$$

Letting $\tilde{h} = \sum \tilde{h}_i$ in $C_{n+1}(X)$ one then has

$$m^0(\mathcal{P}) - m^1(\mathcal{P}) = (-1)^n\sum_{\mathcal{F}<\mathcal{P}}\mathrm{wt}(\mathcal{F})\,\tilde{h}_\mathcal{F} - (-1)^n\partial\tilde{h}.$$

If x is a point of X which is not contained in $|\partial\tilde{h}_\mathcal{F}|$ for any $\mathcal{F}$ then one has in $H_n(X, X\setminus x)$

$$m^0(\mathcal{P}) - m^1(\mathcal{P}) = (-1)^n\sum_{\mathcal{F}<\mathcal{P}}\mathrm{wt}(\mathcal{F})\,\tilde{h}_\mathcal{F},$$

and therefore (in the notation of §3.8)

$$f^0(x) - f^1(x) = (-1)^n\sum_{\mathcal{F}<\mathcal{P}}\mathrm{wt}(\mathcal{F})\mathbb{I}_{\tilde{h}_\mathcal{F}}(x).$$

We now consider the equivalence classes of faces of $\mathcal{P}$ under the action of μ. We shall refer to the class of $\mathcal{F}$ as $[\mathcal{F}]$. We split the above sum over $\mathcal{F} < \mathcal{P}$ into sums over the classes:-

$$\begin{aligned}f^0(x) - f^1(x) &= \frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\sum_{[\mathcal{F}]}\sum_{\xi\in\mu}\mathrm{wt}(\xi\mathcal{F})\cdot\mathbb{I}_{\tilde{h}_{\xi\mathcal{F}}}(x)\\ &= \frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\sum_{[\mathcal{F}]}\sum_{\xi\in\mu}\mathrm{wt}(\xi\mathcal{F})\cdot\det(\xi)\cdot\mathbb{I}_{\tilde{h}_\mathcal{F}}(\xi^{-1}x). \qquad (5)\end{aligned}$$

Note that since $\mathcal{P}$ is a fundamental chain, we have

$$\sum_{\zeta\in\mu} \det(\zeta)\mathrm{wt}(\zeta\mathcal{F}) = 0. \tag{6}$$

Let $Y \subset X$ be a finite set on which μ acts freely, so for $\xi \in \mu$ and $\mathcal{T} \in H_n(X, X \setminus Y)$, one has

$$\sum_{x\in Y} \mathbb{I}_{\mathcal{T}}(z) \quad = \sum_{x\in Y} \det(\xi)\mathbb{I}_{\xi\mathcal{T}}(z).$$

We therefore have by Proposition 1,

$$\begin{aligned}
< f^0, f^1 >_Y &= < f^0, f >_Y < f, f^1 >_Y \\
&= < f^0, f >_Y < f^1, f >_Y^{-1} \\
&= < f^0 - f^1, f >_Y,
\end{aligned}$$

and by definition of $< -, - >$,

$$= \prod_{\zeta\in\mu}\prod_{x\in Y} \zeta^{(f^0(\zeta x)-f^1(\zeta x))f(x)}.$$

Formula (5) now gives

$$= \prod_{\zeta\in\mu}\prod_{[\mathcal{F}]}\prod_{\xi\in\mu}\prod_{x\in Y} \zeta^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\mathbf{I}_{\widetilde{h}_{\mathcal{F}}}(\zeta\xi^{-1}x)f(x)}.$$

We now reparametrize the product:

$$= \prod_{\zeta\in\mu}\prod_{[\mathcal{F}]}\prod_{\xi\in\mu}\prod_{x\in Y} (\zeta\xi)^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\mathbf{I}_{\widetilde{h}_{\mathcal{F}}}(\zeta x)f(x)}.$$

Formula (6) now gives

$$= \prod_{\zeta\in\mu}\prod_{[\mathcal{F}]}\prod_{\xi\in\mu}\prod_{x\in Y} \xi^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\mathbf{I}_{\widetilde{h}_{\mathcal{F}}}(\zeta x)f(x)}.$$

Reparametrizing again we get

$$= \prod_{\zeta\in\mu}\prod_{[\mathcal{F}]}\prod_{\xi\in\mu}\prod_{x\in Y} \xi^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\mathbf{I}_{\widetilde{h}_{\mathcal{F}}}(x)f(\zeta^{-1}x)}.$$

Since f is fundamental we have

$$\begin{aligned}
&= \prod_{[\mathcal{F}]}\prod_{\xi\in\mu}\prod_{x\in Y} \xi^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\mathbf{I}_{\widetilde{h}_{\mathcal{F}}}(x)} \\
&= \prod_{[\mathcal{F}]}\prod_{\xi\in\mu} \xi^{\frac{(-1)^n}{\#\mathrm{Stab}_\mu\mathcal{F}}\mathrm{wt}(\xi\mathcal{F})\det(\xi)\{\{\widetilde{h}_{\mathcal{F}}|Y\}\}}.
\end{aligned}$$

We have therefore proved:

Theorem 6 (Formula for the Skew Product) *Let m^0 and m^1 be two modifications from 0 to 1 in V_∞, and let h be a homotopy from m^0 to m^1. For a face*

$$\mathcal{F} = \mathrm{Transl}(v_{\mathcal{F}})\prod_{j=1}^{n-1} e_{\mathcal{F},j}$$

we define an n-cube

$$\tilde{h}_{\mathcal{F}}(x_1,\ldots,x_{n-1},t) = v_{\mathcal{F}} + \sum_{j=1}^{n-1} h^{(e_{\mathcal{F}},j)}(x_j,t), \qquad x_j,t \in I.$$

If for every face $\mathcal{F} < \mathcal{P}$ *we have* $Y \cap \left|\partial \tilde{h}_{\mathcal{F}}\right| = \emptyset$ *then*

$$< f^1, f^2 >_Y = \prod_{[\mathcal{F}]} \zeta_{[\mathcal{F}]}^{\{\{\tilde{h}_{\mathcal{F}}|Y\}\}},$$

where the constants $\zeta_{[\mathcal{F}]} \in \mu^{\mathrm{ab}}$ *depend only on the fundamental chain* $\mathcal{P}$.

4 Admissible Paths and Admissible Homotopies

Let $\mathcal{T} : I^n \to X$ be a singular n-cube. If $x \in |\partial\mathcal{T}|$ then $\mathbb{I}_{\mathcal{T}}(x)$ is not defined. Thus if we write formulae involving $\mathbb{I}_{\mathcal{T}}(x)$, we must be certain that $x \notin |\partial\mathcal{T}|$. This is the purpose of this section. The proofs here are quite technical. We first state a technical lemma, which we shall need for the other proofs in the section.

Lemma 2 *Let* Φ *be a Hausdorff, real, topological vector space and* V_∞ *an n-dimensional, real vector space. Let* Z *be a compact polyhedron of dimension less than* n. *Let* $B : \Phi \times Z \to V$ *be a map with the following properties:*

- *B is continuous and piecewise differentiable.*
- $\forall z \in Z$, $B(-,z) : \Phi \to V_\infty$ *is an affine map.*
- $\forall z \in Z$ *where* $B(-,z) : \Phi \to V_\infty$ *is not surjective, one has* $0 \notin B(\Phi, z)$. *We shall call such points z 'degenerate'.*

We define a subset

$$\Psi := \{\phi \in \Phi \mid \forall z \in Z \text{ one has } B(\phi, z) \neq 0\}.$$

Then Ψ *is a dense, open subset of* Φ.

Proof. See [2].

4.1 Admissible Paths

We are interested in the values of our various fundamental functions on the subset $X^*[\alpha\beta]$ of X. This will be refered to as the *critical set*. It is important that our functions are fundamental on this set. The functions which we constructed in §3 are fundamental outside the boundaries of the modified parallelotopes used in their construction. We shall therefore try to modify our parallelotopes in such a way that their boundaries avoid the critical set.

We call a modification m *admissible*, iff

$$|\partial m(\mathcal{P})| \cap X^*[\alpha\beta] = \emptyset.$$

This means that the function f^m is fundamental on the critical set $X^*[\alpha\beta]$.

Lemma 3 (Existence of Admissible Paths) *For every neighbourhood U of zero in V_∞ there is a differentiable, admissible modification m^1 with*

$$\frac{d}{dz}\left(m^1(e)(z) - e(z)\right) \in U \quad \text{and} \quad m^1(e)(z) - e(z) \in U.$$

And if we define

$$m^\alpha(e)(z) := \begin{cases} \alpha^{-1}m^1(e)(2z) & z \leq \frac{1}{2} \\ \alpha^{-1}e(1) + (1-\alpha^{-1})m^1(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^\beta(e)(z) := \begin{cases} \beta^{-1}m^1(e)(2z) & z \leq \frac{1}{2} \\ \beta^{-1}e(1) + (1-\beta^{-1})m^1(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\alpha,\beta}(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1}m^1(e)(4z) & z \leq \frac{1}{4} \\ \alpha^{-1}\beta^{-1}e(1) + (\alpha^{-1} - \alpha^{-1}\beta^{-1})m^1(e)(4z-1) & \frac{1}{4} \leq z \leq \frac{1}{2} \\ \alpha^{-1}e(1) + (1-\alpha^{-1})m^1(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\beta,\alpha}(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1}m^1(e)(4z) & z \leq \frac{1}{4} \\ \alpha^{-1}\beta^{-1}e(1) + (\beta^{-1} - \alpha^{-1}\beta^{-1})m^1(e)(4z-1) & \frac{1}{4} \leq z \leq \frac{1}{2} \\ \beta^{-1}e(1) + (1-\beta^{-1})m^1(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^\$(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1}m^1(e)(2z) & z \leq \frac{1}{2} \\ \alpha^{-1}\beta^{-1}e(1) + (1-\alpha^{-1}\beta^{-1})m^1(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

then the modifications $m^\alpha, m^\beta, m^{\alpha,\beta}, m^{\beta,\alpha}, m^\$$ are also all admissible.

Proof. (i) Let Φ be the real vector space of functions $\phi : M \to V_\infty$ satisfying the conditions for all $e \in \mathcal{E}0$

$$\phi^{(e)}(0) = \frac{d\phi^{(e)}}{dz}(0) = 0,$$

$$\forall\, 0 \leq z \leq \frac{\#\mu^{\mathrm{ab2}} - 1}{\#\mu^{\mathrm{ab2}}} \text{ one has } \phi^{(e)}\left(z + \frac{1}{\#\mu^{\mathrm{ab2}}}\right) = \phi^{(e)}(z).$$

Φ becomes a topological vector space with the following norm:

$$\|\phi\| := \sup_{z\in I}\sup_{e\in\mathcal{E}0}\left\{\left|\phi^{(e)}(z)\right|, \left|\frac{d\phi^{(e)}}{dz}(z)\right|\right\}.$$

With this topology Φ is Hausdorff.

To each face $\mathcal{F} < \mathcal{P}$ we define $Z_{\mathcal{F}}$ to be the disjoint union of six copies of I^{n-1}. We call these copies $Z^1_{\mathcal{F}}$, $Z^\alpha_{\mathcal{F}}$, $Z^\beta_{\mathcal{F}}$, $Z^{\alpha,\beta}_{\mathcal{F}}$, $Z^{\beta,\alpha}_{\mathcal{F}}$ and $Z^\$_{\mathcal{F}}$. Let X be the disjoint union over all $\mathcal{F} < \mathcal{P}$ of the sets $Z_{\mathcal{F}}$. Then X is a compact, $(n-1)$-dimensional polyhedron.

To each $x \in X^*[\alpha\beta]$ we shall define a map $B_x : X \times \Phi \to V_\infty$ to which we shall apply the technical lemma §4.1.

(ii) Let $\phi \in \Phi$. We define modifications.

$$m^{1,\phi}(e)(z) := e(z) + \phi(z),$$

$$m^{\alpha,\phi}(e)(z) := \begin{cases} \alpha^{-1} m^{1,\phi}(e)(2z) & z \leq \frac{1}{2} \\ \alpha^{-1} e(1) + (1-\alpha^{-1}) m^{1,\phi}(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\beta,\phi}(e)(z) := \begin{cases} \beta^{-1} m^{1,\phi}(e)(2z) & z \leq \frac{1}{2} \\ \beta^{-1} e(1) + (1-\beta^{-1}) m^{1,\phi}(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\alpha,\beta,\phi}(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1} m^{1,\phi}(e)(4z) & z \leq \frac{1}{4} \\ \alpha^{-1}\beta^{-1} e(1) + (\alpha^{-1} - \alpha^{-1}\beta^{-1}) m^{1,\phi}(e)(4z-1) & \frac{1}{4} \leq z \leq \frac{1}{2} \\ \alpha^{-1} e(1) + (1-\alpha^{-1}) m^{1,\phi}(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\beta,\alpha,\phi}(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1} m^{1,\phi}(e)(4z) & z \leq \frac{1}{4} \\ \alpha^{-1}\beta^{-1} e(1) + (\beta^{-1} - \alpha^{-1}\beta^{-1}) m^{1,\phi}(e)(4z-1) & \frac{1}{4} \leq z \leq \frac{1}{2} \\ \beta^{-1} e(1) + (1-\beta^{-1}) m^{1,\phi}(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

$$m^{\$,\phi}(e)(z) := \begin{cases} \alpha^{-1}\beta^{-1} m^{1,\phi}(e)(2z) & z \leq \frac{1}{2} \\ \alpha^{-1}\beta^{-1} e(1) + (1-\alpha^{-1}\beta^{-1}) m^{1,\phi}(e)(2z-1) & z \geq \frac{1}{2} \end{cases}$$

(iii) Let $\underline{z} = (z_1, \ldots, z_{n-1}) \in Z^1_{\mathcal{F}}$, where $\mathcal{F} = \mathrm{Transl}(v_{\mathcal{F}}) \prod_{j=1}^{n-1} e_{\mathcal{F},j}$. We define for $x \in \tilde{X}^*[\alpha\beta]$,

$$B_x(\underline{z}, \phi) := m^{1,\phi}(\mathcal{F})(\underline{z}) - x = v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{1,\phi}(e_{\mathcal{F},j})(z_j) - x.$$

The point $\underline{z}$ is degenerate (in the sense of §4.1) precisely when for all $j = 1, \ldots, n-1$, $z_j \in \{0, \frac{1}{\#\mu^{\mathrm{ab2}}}, \ldots, 1\}$. If that is the case, then $v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{1,\phi}(e_{\mathcal{F},j})(z_j)$ is a $\delta\#\mu^{\mathrm{ab2}}$-division point of L, and is therefore not in $\tilde{X}^*[\alpha\beta]$. Therefore, if $\underline{z} \in Z^1_{\mathcal{F}}$ is degenerate, then $B_x(\underline{z}, \phi) \neq 0$.

(iv) Let $\underline{z} = (z_1, \ldots, z_{n-1}) \in Z^\alpha_{\mathcal{F}}$. We define

$$B_x(\underline{z}, \phi) := m^{\alpha,\phi}(\mathcal{F})(\underline{z}) - x = v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j) - x.$$

The point $\underline{z}$ is degenerate precisely when for all $j = 1, \ldots, n-1$, $z_j \in \{0, \frac{1}{2\#\mu^{\mathrm{ab2}}}, \ldots, 1\}$. We want to show that in that case, $B_x(\underline{z}, \phi) \neq 0$. Let $\underline{z}$ be degenerate. We shall compute $\alpha(v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j))$ modulo L. We shall often use the fact that $\alpha \equiv I \bmod \delta\#\mu^{\mathrm{ab}2}$. One has

$$\alpha\left(v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j)\right) \equiv v_{\mathcal{F}} + \sum_{j=1}^{n-1} \alpha m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j) \quad \bmod L$$

Suppose $z_j > \frac{1}{2}$. Then

$$\begin{aligned} \alpha m^{\alpha,\phi} e_{\mathcal{F},j}(z_j) &= e_{\mathcal{F},j}(1) + (\alpha - 1) e_{\mathcal{F},j}(2z_j - 1) \\ &\equiv e_{\mathcal{F},j}(1) \quad \bmod L \\ &\equiv \alpha m^{\alpha,\phi}(e_{\mathcal{F},j})\left(\frac{1}{2}\right). \end{aligned}$$

We can thus assume $z_j \leq \frac{1}{2}$. We then have

$$\alpha\left(v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j)\right) \equiv v_{\mathcal{F}} + \sum_{j=1}^{n-1} e_{\mathcal{F},j}(2z_j) \bmod L.$$

This is a $\delta\#\mu^{\mathrm{ab}2}$-division point of L and can only be in L if for all j, $z_j = 0$. Thus $v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j)$ can only be an $\alpha\beta$-division point, if for all j, $z_j = 0$. If that is the case, then $v_{\mathcal{F}} + \sum_{j=1}^{n-1} m^{\alpha,\phi}(e_{\mathcal{F},j})(z_j) = v_{\mathcal{F}} \notin X^*[\alpha\beta]$. It follows that for degenerate $\underline{z} \in Z_{\mathcal{F}}^{\alpha}$, $B_x(\underline{z}, \phi) \neq 0$.

(v) We define further for $\underline{z} \in Z_{\mathcal{F}}^{\beta}$, $x \in \tilde{X}^*[\alpha\beta]$:

$$B_x(\underline{z}, \phi) := m^{\beta,\phi}(\mathcal{F})(\underline{z}) - x,$$

and so on. As in (iv), we show that if $\underline{z}$ is degenerate, then for all $\phi \in \Phi$, $B_x(\underline{z}, \phi) \neq 0$. We can now apply Lemma 2.

(vi) We define for $x \in \tilde{X}^*[\alpha\beta]$,

$$\Psi_x := \{\phi \in \Phi \mid \forall z \in Z \text{ one has } B_x(z, \phi) \neq 0\}.$$

From §4.1, Ψ_x is a dense, open subset of Φ. If $\phi \in \Psi_x$, then by definition of B_x,

$$x \notin |\partial\mathcal{P}^{1,\phi}| \cup |\partial\mathcal{P}^{\alpha,\phi}| \cup \ldots \cup |\partial\mathcal{P}^{\$,\phi}|.$$

(vii) Let $S_{\alpha\beta}^{finite}$ be the set of all elements of $\tilde{X}^*[\alpha\beta]$, in a large compact subset of V_∞. This is a finite set, but if $x \in \tilde{X}^*[\alpha\beta] \setminus S_{\alpha\beta}^{finite}$, then Ψ_x contains a neighbourhood U' of 0 in Φ, which is independent of x. Let $\Psi := \bigcap_{x \in X^*[\alpha\beta]} \Psi_x$. Then $\Psi \cap U' = \bigcap_{x \in S_{\alpha\beta}^{finite}} \Psi_x$. This is also dense and open in U'. We can therefore choose a $\phi \in \Psi$ arbitrarily close to 0. Let $m^1(e)(z) := e(z) + \phi(z)$. Then m^1 satisfies the conditions of the lemma.

4.2 Admissible Homotopies

In Theorem 6 we obtained a formula for the skew product $< f^0, f^1 >_{\alpha\beta}$ where m^0 and m^1 are two admissible modifications. Our formula depends on the choice of a homotopy h from m^0 to m^1, where h satisfies the following condition:

$$X^*[\alpha\beta] \cap \left(\bigcup_{\mathcal{F}<\mathcal{P}} \left|\partial \widetilde{h}_{\mathcal{F}}\right| \right) = \emptyset,$$

with $\widetilde{h}_{\mathcal{F}}$ as is §3.10. We shall call a homotopy which satisfies this condition *admissible*. To be able to apply Theorem 6 we must show that admissible homotopies exist. The following statements are easily proved (so we won't prove them).

- If m^0, m^1 and m^2 are three admissible modifications, and h' and h'' are admissible homotopies from m^0 to m^1 and from m^1 to m^2, then the composition (in the category of modifications),

$$\left(h' \circ h''\right)^{(e)}(x,t) := \begin{cases} h'^{(e)}(x, 2t) & t \leq \frac{1}{2} \\ h''^{(e)}(x, 2t-1) & t \geq \frac{1}{2} \end{cases}$$

  is an admissible homotopy from m^0 to m^2.

- If h is an admissible homotopy from m^0 to m^1 and of h' is pointwise close to h and also a homotopy from m^0 to m^1, then h' is also admissible.

We now show that close to any homotopy, there is always an admissible homotopy.

Lemma 4 (Existence of Admissible Homotopies) *Let m^0 and m^1 be two admissible modifications, and let h_0 be any homotopy from m^0 to m^1. Then for any neighbourhood of zero $0 \in U \subset V_\infty$, there is an admissible homotopy $h : I^2 \to V_\infty$ from m^0 to m^1, with the property that for all $(x,t) \in I^2$, $e \in \mathcal{E}0$, one has*

$$h^{(e)}(x,t) - h_0^{(e)}(x,t) \in U.$$

If the functions m^0, m^1 and h_0 are differentiable, then we may also require that h is differentiable, and in addition that

$$\frac{\partial}{\partial x}\left(h^{(e)}(x,t) - h_0^{(e)}(x,t)\right) \in U \quad \textit{and} \quad \frac{\partial}{\partial t}\left(h^{(e)}(x,t) - h_0^{(e)}(x,t)\right) \in U.$$

Proof. (i) We first prove the lemma in the case that m^0, m^1 and h_0 are differentiable.

For each $\mathcal{F} < \mathcal{P}$ let $Z_{\mathcal{F}}$ be the set ∂I^n, and let Z be the disjoint union of all the $Z_{\mathcal{F}}$. Then Z is a compact, $(n-1)$-dimensional polyhedron. We shall write points of $Z_{\mathcal{F}}$ as $(\underline{z}, t)$, where $\underline{z} = (z_1, \ldots, z_{n-1}) \in I^{n-1}$ and $t \in I$.

(ii) Let Φ be the real vector space of differentiable functions

$$\phi : M \times I \longrightarrow V_\infty$$

$$\partial(M \times I) \longrightarrow 0,$$

whose restrictions to $\partial M \times I$ are zero. We give Φ the topology induced by the following norm:

$$\|\phi\| := \sup_{e \in \mathcal{E}0} \sup_{(z,t)\in M\times I} \left\{ \left|\phi^{(e)}(z,t)\right|, \left|\frac{\partial \phi^{(e)}}{\partial z}(z,t)\right|, \left|\frac{\partial \phi^{(e)}}{\partial t}(z,t)\right| \right\}.$$

For every $x \in \tilde{X}^*[\alpha\beta]$ we define a function $B_x : Z \times \Phi \to V_\infty$. If $(\underline{z}, t) \in Z_{\mathcal{F}}$ then we define

$$B_x((\underline{z}, t), \phi) := v_{\mathcal{F}} + \sum_{j=1}^{n-1} \left(h_0^{(e_{\mathcal{F},j})}(z_j, t) + \phi^{(e_{\mathcal{F},j})}(z_j, t) \right) - x.$$

Since h_0 and ϕ are differentiable, B_x is also differentiable.

(iii) A point $(\underline{z}, t) \in Z_{\mathcal{F}}$ is degenerate precisely when either $t \in \{0, 1\}$, or $\underline{z}$ is a vertex of I^{n-1}.

- If $t = 0$ then $B_x((\underline{z}, t), \phi) = m^0(\mathcal{F})(\underline{z}) - x$. Since m^0 is admissible, $B_x((\underline{z}, t), \phi) \neq 0$.
- If $t = 1$ then since m^1 is admissible, $B_x((\underline{z}, t), \phi) \neq 0$.
- If $\underline{z}$ is a vertex of I^{n-1}, then $B_x((\underline{z}, t), \phi) + x$ is a vertex of a parallelotope $\mathcal{P}_i$. Therefore $B_x((\underline{z}, t), \phi) \neq 0$.

The function B_x therefore satisfies the conditions of Lemma 2.

(iv) Let $\Psi_x := \{\phi \in \Phi \mid \forall z \in Z \text{ one has } B_x(z, \phi) \neq 0\}$. By Lemma 2, Ψ_x is dense and open in Φ. Let $\Psi := \bigcap_{x \in X^*[\alpha\beta]} \Psi_x$. As in the proof of the previous lemma, Ψ is also a dense, open subset of Φ. We choose $\phi \in \Psi$ close to 0. Since $\phi \in \Psi$, one has for all $x \in X^*[\alpha\beta]$ and all $\mathcal{F} < \mathcal{P}$,

$$x \notin \left|\partial \tilde{h}^{\phi}_{\mathcal{F}}\right|,$$

where $h^{\phi(e)}(x, t) := h_0^{(e)}(x, t) + \phi^{(e)}(x, t)$. The homotopy h^ϕ is therefore admissible. Since ϕ close to 0, h^ϕ is close to h_0. The proof in the differentiable case is finished.

The non-differentiable case

Now consider continuous functions

$$m^0(e) : I \longrightarrow V_\infty, \qquad m^0(e) : I \longrightarrow V_\infty, \qquad h_0^{(e)} : I^2 \longrightarrow V_\infty$$

with the conditions

$$m^0(e)(0) = m^1(e)(0) = h_0^{(e)}(0, t) = 0,$$
$$m^0(e)(1) = m^1(e)(1) = h_0^{(e)}(1, t) = e(1),$$
$$h_0^{(e)}(z, 0) = m^0(e)(z),$$
$$h_0^{(e)}(z, 1) = m^1(e)(z),$$

and with m^0 and m^1 admissible. There are differentiable functions

$$m^{d0}(e) : I \longrightarrow V_\infty, \qquad m^{d1}(e) : I \longrightarrow V_\infty, \qquad h_{d0}^{(e)} : I^2 \longrightarrow V_\infty,$$

which are pointwise close to $m^0(e)$, $m^1(e)$ and $h_0^{(e)}$, and which satisfy the same conditions. Since m^0 and m^1 are admissible, and m^{d0} and m^{d1} are close to them, m^{d0} and m^{d1} are also admissible. From what we have already proved, there is an admissible homotopy h_d close to h_{d0}. We define

$$\begin{aligned} h'^{(e)}(z,t) &= (1-t)m^0(e)(z) + tm^{d0}(e)(z) \\ \text{and} \quad h''^{(e)}(z,t) &= (1-t)m^1(e)(z) + tm^{d1}(e)(z) \end{aligned}$$

The two homotopies h' and h'' are admissible. Now let

$$h^{(e)}(z,t) = \begin{cases} h'^{(e)}\left(z, \frac{t}{\epsilon}\right) & t \leq \epsilon \\ h_{d0}^{(e)}\left(z, \frac{1}{1-2\epsilon}(t-\epsilon)\right) & \epsilon \leq t \leq 1-\epsilon \\ h''^{(e)}\left(z, \frac{1-t}{\epsilon}\right) & t \geq 1-\epsilon. \end{cases}$$

The function h is an admissible homotopy from m^0 to m^1, and for small ϵ, h is close to h_0.

5 Proof of Theorem 1

In this section we prove the following:

Let $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ with the following conditions:

- $\alpha \equiv \beta \equiv I \bmod 2\delta\#\mu^{\mathrm{ab}2}$;
- *α and β are in the connected component of I in $G_{\mathbb{R}}$.*

Then

$$\mathrm{Dec}(\alpha,\beta) = \mathrm{Dec}(\beta,\alpha).$$

5.1 Proportional Equivalence Classes

We can embed the multiplicative group $\mathbb{R}^{>0}$ in $G_{\mathbb{R}}$ by the map $r \longmapsto r \cdot I$. We write the quotient group $G_{\mathbb{R}}/\mathbb{R}^{>0}$ as $G_{\mathbb{R}}\!:$. We call the cosets of $\mathbb{R}^{>0}$ (the elements of $G_{\mathbb{R}}\!:$) *proportional equivalence classes*. We write $\alpha\!:$ for the proportional equivalence class of α. If $\alpha\!: = \beta\!:$ then we say, that α and β are *proportionally equivalent*.

We shall assume that $\alpha\!:$ and $\beta\!:$ are both in a small neighbourhood of $I\!:$ in $G_{\mathbb{R}}\!:$. This implies in particular that α and β are in the connected component of I in $G_{\mathbb{R}}$. It also implies that the modifications m^α, m^β, $m^{\alpha,\beta}$ and $m^{\beta,\alpha}$ consist of nearly straight paths, and that the maps $m^\alpha(\mathcal{P}_i)$, $m^\beta(\mathcal{P}_i)$, $m^{\alpha,\beta}(\mathcal{P}_i)$ and $m^{\beta,\alpha}(\mathcal{P}_i) : I^n \to X$ are injective.

5.2 Remark

Let $\gamma \in G_{\mathbf{Z}}$, $v \in X$ and $e \in \mathcal{E}$. If $\mathrm{Transl}(v)\gamma m^1(e) \times \mathcal{T}$ is an element of $Z_n(X, X \setminus X[\alpha\beta])$ such that $\gamma \equiv 0 \bmod \alpha^{-1}\beta^{-1}\delta\#\mu^{\mathrm{ab}}$, then one has:

$$\left\{\left\{X[\alpha\beta]\middle|\mathrm{Transl}(v)\gamma m^1(e) \times \mathcal{T}\right\}\right\} \equiv 0 \bmod \#\mu^{\mathrm{ab}}.$$

We will often use this fact.

The next lemmas are similar to lemmas due to Habicht in the case where μ is the group of cube roots of 1 acting on the number field which they generate.

Lemma 5 *If* $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ *with* $\alpha \equiv 1 \bmod \delta\#\mu^{\mathrm{ab}}$ *then*

$$< f^\alpha, f^1 \circ \alpha >_{\alpha\beta\setminus\alpha} = 1.$$

Proof. We have $m^\alpha(\mathcal{P}) = \sum_i m^\alpha(\mathcal{P}_i)$ for some set of parallelotopes $\mathcal{P}_i$. Similarly we have $f^\alpha = \sum_i \mathbb{I}_{m^\alpha(\mathcal{P}_i)}$. For each parallelotope there is an expression

$$m^\alpha(\mathcal{P}_i) = \mathrm{Transl}(v_i)\prod_{j=1}^{n} m^\alpha(e_{i,j}),$$

where vertices of $\mathcal{P}_i$ are in $X[\delta]$. By definition (§4.3) of $m^\alpha(e_{i,j})$, this is the equivalent to

$$\mathrm{Transl}(v_i)\prod_{j=1}^{n}\left(\alpha^{-1}m^1(e_{i,j}) + \mathrm{Transl}\left(\alpha^{-1}e_{i,j}(1)\right)\left(1-\alpha^{-1}\right)m^1(e_{i,j})\right).$$

Expanding the brackets we obtain:

$$\mathrm{Transl}(v_i)\prod_{j=1}^{n}\alpha^{-1}m^1(e_{i,j})$$

$$+ \text{ parallelotopes, at least one of whose edges is a vector in } \alpha^{-1}\#\mu^{\mathrm{ab}}L.$$

The first term is equal to $\mathrm{Transl}((1-\alpha^{-1})v_i)\alpha^{-1}m^1(\mathcal{P}_i)$. Summing over the set of parallelotopes $\mathcal{P}_i$ we obtain

$$\begin{aligned} m^\alpha(\mathcal{P}_i) &= \mathrm{Transl}((1-\alpha^{-1})v_i)\alpha^{-1}m^1(\mathcal{P}) \\ &\quad + \text{ parallelotopes, at least one of whose edges is a vector in } \alpha^{-1}\#\mu^{\mathrm{ab}}L. \end{aligned}$$

The function $x \mapsto f^1(\zeta^{-1}\alpha x)$ is periodic with respect to $X[\alpha]$. The set $X[\alpha\beta] \setminus X[\alpha]$ is also invariant under translations by elements of $X[\alpha]$. Therefore the sum of $f^1(\zeta^{-1}\alpha x)$ over points of $X[\alpha\beta] \setminus X[\alpha]$ in a parallelotope with at least one edge in $\alpha^{-1}\#\mu^{\mathrm{ab}}L$ vanishes modulo $\#\mu^{\mathrm{ab}}$. We therefore have:

$$
\begin{aligned}
< f^\alpha, f^1 \circ \alpha >_{\alpha\beta\backslash\alpha} &= \prod_{\zeta\in\mu} \prod_{x\in X[\alpha\beta]\backslash X[\alpha]} \zeta^{\mathbf{I}_{m^\alpha(\mathcal{P})}(x)\cdot f(\zeta^{-1}\alpha x)} \\
&= \prod_i \prod_{\zeta\in\mu} \prod_{x\in X[\alpha\beta]\backslash X[\alpha]} \zeta^{\mathbf{I}_{m^\alpha(\mathcal{P}_i)}(x)\cdot f(\zeta^{-1}\alpha x)} \\
&= \prod_i \prod_{\zeta\in\mu} \prod_{x\in X[\alpha\beta]\backslash X[\alpha]} \zeta^{\mathbf{I}_{\mathrm{Transl}((1-\alpha^{-1})v_i)\alpha^{-1}m^1(\mathcal{P}_i)}(x)\cdot f(\zeta^{-1}\alpha x)} \\
&= \prod_i \prod_{\zeta\in\mu} \prod_{\tilde{x}\in \tilde{X}[\alpha\beta]\backslash \tilde{X}[\alpha]} \zeta^{\mathbf{I}_{\mathrm{Transl}((1-\alpha^{-1})v_i)\alpha^{-1}m^1(\mathcal{P}_i)}(\tilde{x})\cdot f(\zeta^{-1}\alpha \tilde{x})} \\
&= \prod_{\zeta\in\mu} \prod_{\tilde{x}\in \tilde{X}[\alpha\beta]\backslash \tilde{X}[\alpha]} \zeta^{\mathrm{sign}(\alpha)\cdot\mathbf{I}_{m^1(\mathcal{P}_i)}(\alpha\tilde{x}-(\alpha-1)v_i)\cdot f(\zeta^{-1}\alpha \tilde{x})} \\
&= \prod_{\zeta\in\mu} \prod_{\tilde{x}\in \tilde{X}[\beta]\backslash L} \zeta^{\mathrm{sign}(\alpha)\cdot\mathbf{I}_{m^1(\mathcal{P})}(\tilde{x})\cdot f(\zeta^{-1}\tilde{x})} \\
&= < f^1, f^1 >_\beta^{\mathrm{sign}(\alpha)} = 1.
\end{aligned}
$$

Here $\mathrm{sign}(\alpha)$ is the sign of the determinant of α. Note that the $(\alpha-1)v_i$ has disappeared because it is in L.

Lemma 6 *There is a neighbourhood* Nbd *of* I: *in* $G_{\mathbf{R}}$: *with the following property. If* $(\alpha,\beta)\in\Omega_{\mathbf{Z}}$ *with* $\alpha\equiv\beta\equiv 1 \bmod \delta\#\mu^{\mathrm{ab}2}$ *and* $\alpha:,\beta:\in U$, *then*

$$< f^\alpha, f^1 >_\alpha = < f^{\beta,\alpha}, f^\beta >_{\alpha\beta}.$$

Proof. (i) The proof is quite long but the idea is simple. In the one dimensional case this is all trivial because the fundamental functions are independent of the modifications. The lemma can be easily understood for fields of degree 2. In higher dimensions some new phenomena arise and the two-dimensional picture becomes inaccurate. A full impression of the proof can be gained by considering three-dimensional cases, in which everything goes wrong that can go wrong.

Our calculations will be mainly in the homology group $H_n(X, X\setminus X^*[\alpha\beta])$. The lemma follows because the difference between $m^{\beta,\alpha}$ and m^β is essentially β^{-1} of the difference between m^α and m^1 (this can be seen by drawing a picture). On the other hand the product on the right is over $\alpha\beta$-division points, whereas that on the left is only over α-division points. The proof will use the skew product formula (Theorem 6).

(ii) To apply Theorem 6, we need admissible homotopies from m^α to m^1 and from $m^{\beta,\alpha}$ to m^β. We now construct these homotopies. Let U be a neighbourhood of 0 in V_∞. We define for an edge e,

$$h_0^{(e)}(x,t) := t\cdot m^1(e)(x) + (1-t)\cdot m^\alpha(e)(x).$$

Then h_0 is a homotopy from m^1 to m^α. We choose using Lemma 4 an admissible homotopy h_1 from m^1 to m^α, which is uniformly close to h_0 (it is not so important which admissible homotopy we choose). We compress this by β^{-1}, and obtain a homotopy from $\beta^{-1}m^1$ to $\beta^{-1}m^\alpha$, which we shall denote h_2:

$$h_2^{(e)}(x,t) \ := \ \beta^{-1}h_1^{(e)}(x,t),$$

Finally we extend h_2 by a constant homotopy from $\beta^{-1}e(1)+(1-\beta^{-1})m^1(e)$ to itself, thus obtaining a homotopy h_3 from m^β to $m^{\beta,\alpha}$:

$$h_3^{(e)}(x,t) := \begin{cases} h_2^{(e)}(2x,t) & \text{if } x \le \frac{1}{2}, \\ \beta^{-1}e(1)+(1-\beta^{-1})m^1(e)(2x-1) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

The admissibility of h_3 follows from that of h_1. We construct as described in §3.5 the homotopies $\widetilde{h_1}$, $\widetilde{h_2}$ and $\widetilde{h_3}$.

(iii) From Theorem 6 we have

$$< f^\alpha, f^1 >_\alpha = \prod_{[\mathcal{F}]} \zeta_{[\mathcal{F}]}^{\left\{\left\{X^*[\alpha] \middle| \widetilde{h_1}_{\mathcal{F}}\right\}\right\}},$$

$$< f^{\beta\alpha}, f^\beta >_{\alpha\beta} = \prod_{[\mathcal{F}]} \zeta_{[\mathcal{F}]}^{\left\{\left\{X^*[\alpha\beta] \middle| \widetilde{h_3}_{\mathcal{F}}\right\}\right\}},$$

To show that $< f^\alpha, f^1 >_\alpha = < f^{\beta\alpha}, f^\beta >_{\alpha\beta}$, it is clearly sufficient to prove for each face $\mathcal{F} < \mathcal{P}$ that

$$\left\{\left\{X^*[\alpha] \middle| \widetilde{h_{1\mathcal{F}}}\right\}\right\} \equiv \left\{\left\{X^*[\alpha\beta] \middle| \widetilde{h_{3\mathcal{F}}}\right\}\right\} \bmod \#\mu^{\mathrm{ab}}.$$

We shall prove this.

(iv) Let $\mathcal{F}$ be a general face of a parallelotope of $\mathcal{P}$. The there is an expression of the form

$$\mathcal{F} = \mathrm{Transl}(v_{\mathcal{F}}) \prod_{i=1}^{n-1} e_{\mathcal{F},i},$$

where $v_{\mathcal{F}}, e_{\mathcal{F},i}(1) \in X[\delta]$. We cut the n-cube $\widetilde{h_{3\mathcal{F}}}$ into 2^{n-1} pieces. This cutting process corresponds to cutting $\mathcal{F}$ into 2^{n-1} pieces, each half as big as $\mathcal{F}$. We thus have in $H_n(X, X \setminus X^*[\alpha\beta])$:

$$\widetilde{h_{3\mathcal{F}}} = \sum_{T \subset \{1,2,\ldots,n-1\}} \widetilde{h_{3\mathcal{F}}^T},$$

where

$$\widetilde{h_{3\mathcal{F}}^T}(x_1, x_2, \ldots, x_{n-1}, t) = v_{\mathcal{F}} + \sum_{j \in \{1,2,\ldots,n-1\} \setminus T} h_3^{(e_{\mathcal{F},j})}\left(\frac{x_j}{2}, t\right) + \sum_{j \in T} h_3^{(e_{\mathcal{F},j})}\left(\frac{x_j+1}{2}, t\right).$$

Note that the boundary of $\widetilde{h_{3\mathcal{F}}^T}$ has no intersection with $X^*[\alpha\beta]$. This follows because the boundary can be covered by translations of the boundary of $\widetilde{h_{3\mathcal{F}}}$ by $\alpha\beta$-division points, and h_3 is admissible. We therefore have

$$\left\{\left\{X^*[\alpha\beta] \middle| \widetilde{h_{3\mathcal{F}}}\right\}\right\} = \sum_{T \subset \{1,2,\ldots,n-1\}} \left\{\left\{X^*[\alpha\beta] \middle| \widetilde{h_{3\mathcal{F}}^T}\right\}\right\}.$$

We now compute the terms in this sum.

(v) First suppose T is empty. We then have

$$\begin{aligned}
\widetilde{h_{3\mathcal{F}}^{\emptyset}}(x_1, x_2, \ldots, x_{n-1}, t) &= v_{\mathcal{F}} + \sum_{j=1}^{n-1} h_3^{(e_{\mathcal{F},j})}\left(\frac{x_j}{2}, t\right) \\
&= v_{\mathcal{F}} + \sum_{j=1}^{n-1} h_2^{(e_{\mathcal{F},j})}(x_j, t) \\
&= v_{\mathcal{F}} + \beta^{-1} \sum_{j=1}^{n-1} h_1^{(e_{\mathcal{F},j})}(x_j, t) \\
&= \beta^{-1}\left(v_{\mathcal{F}} + \sum_{j=1}^{n-1} h_1^{(e_{\mathcal{F},j})}(x_j, t)\right) + (1-\beta^{-1})v_{\mathcal{F}}.
\end{aligned}$$

Thus

$$\widetilde{h_{3\mathcal{F}}^{\emptyset}} = \mathrm{Transl}((1-\beta^{-1})v_{\mathcal{F}})\beta^{-1}\widetilde{h_{1\mathcal{F}}},$$

and therefore

$$\begin{aligned}
\left\{\left\{X^*[\alpha\beta] \,\middle|\, \widetilde{h_{3\mathcal{F}}^{\emptyset}}\right\}\right\} &= \left\{\left\{\alpha^{-1}\beta^{-1}L \setminus L \,\middle|\, \mathrm{Transl}((1-\beta^{-1})v_{\mathcal{F}})\beta^{-1}\widetilde{h_{1\mathcal{F}}},\right\}\right\} \\
&= \left\{\left\{\alpha^{-1}\beta^{-1}L \setminus \{0\} \,\middle|\, \mathrm{Transl}((1-\beta^{-1})v_{\mathcal{F}})\beta^{-1}\widetilde{h_{1\mathcal{F}}},\right\}\right\} \\
&= \left\{\left\{\frac{1}{\alpha}L \setminus \{0\} \,\middle|\, \mathrm{Transl}((\beta-1)v_{\mathcal{F}})\widetilde{h_{1\mathcal{F}}}\right\}\right\}.
\end{aligned}$$

We now distinguish two cases. First suppose $v_{\mathcal{F}} = 0$. We then have immediately

$$\left\{\left\{X^*[\alpha\beta] \,\middle|\, \widetilde{h_{3\mathcal{F}}^{\emptyset}}\right\}\right\} = \left\{\left\{\frac{1}{\alpha}L \setminus \{0\} \,\middle|\, \widetilde{h_{1\mathcal{F}}}\right\}\right\} = \left\{\left\{X^*[\alpha] \,\middle|\, \widetilde{h_{1\mathcal{F}}}\right\}\right\}.$$

In the other case $v_{\mathcal{F}} \neq 0$. Then there is a neighbourhood of $|\mathcal{F}|$, which is disjoint from L. We therefore have for α: and β: sufficiently close to I: and $m^1(e)$ sufficiently close to e and h_1 sufficiently close to h_0, $L \cap |\widetilde{h_{1\mathcal{F}}}| = \emptyset$. Therefore (since $(\beta-1)v_{\mathcal{F}} \in L$) $\left|\mathrm{Transl}((\beta-1)v_{\mathcal{F}})\widetilde{h_{1\mathcal{F}}}\right| \cap L = \emptyset$. We thus have as in the first case

$$\left\{\left\{X^*[\alpha\beta] \,\middle|\, \widetilde{h_{3\mathcal{F}}^{\emptyset}}\right\}\right\} = \left\{\left\{X^*[\alpha] \,\middle|\, \widetilde{h_{1\mathcal{F}}}\right\}\right\}.$$

(vi) Now suppose T is non-empty. Without loss of generality, assume $1 \in T$. Then

$$\widetilde{h_{3\mathcal{F}}^{T}}(x_1, x_2, \ldots, x_{n-1}, t) = h_3^{(e_{\mathcal{F},1})}\left(\frac{x_1+1}{2}, t\right) + g(x_2, \ldots, x_{n-1}, t)$$

with a suitable function g. However $h_3^{(e_{\mathcal{F},1})}\left(\frac{x_1+1}{2}, t\right) = \beta^{-1}e_{\mathcal{F},1}(1) + (1-\beta^{-1})m^1(e_{\mathcal{F},1})(x_1)$. We therefore have

$$\widetilde{h_{3\mathcal{F}}^{T}} = (1-\beta^{-1})m^1(e_{\mathcal{F},1}) \times g'.$$

Since T is non-empty and α: is close to I:, it follows that $|\widetilde{h_{3\mathcal{F}}^{T}}|$ contains no point of L. Therefore

$$\left\{\left\{X^{*}[\alpha\beta]\middle|\widetilde{h_{3\mathcal{F}}^{T}}\right\}\right\} = \left\{\left\{X[\alpha\beta]\middle|\widetilde{h_{3\mathcal{F}}^{T}}\right\}\right\},$$

so by remark §5.2, we have

$$\left\{\left\{X^{*}[\alpha\beta]\middle|\widetilde{h_{3\mathcal{F}}^{T}}\right\}\right\} \equiv 0 \bmod \#\mu^{\mathrm{ab}}.$$

(vii) We have shown in (iv), (v) and (vi) that for every face $\mathcal{F} < \mathcal{P}$,

$$\left\{\left\{X^{*}[\alpha\beta]\middle|\widetilde{h_{3\mathcal{F}}}\right\}\right\} \equiv \left\{\left\{X^{*}[\alpha]\middle|\widetilde{h_{1\mathcal{F}}}\right\}\right\} \bmod \#\mu^{\mathrm{ab}}.$$

Therefore by Theorem 6,

$$< f^{\alpha}, f^{1} >_{\alpha} = < f^{\beta,\alpha}, f^{\beta} >_{\alpha\beta} .$$

Lemma 7 *There is a neighbourhood* Nbd$'$ *of* I: *in* $G_{\mathbf{R}}$: *with the follwoing property. If* $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ *satisfies* $\alpha \equiv \beta \equiv I \bmod 2\delta\#\mu^{\mathrm{ab}2}$ *and* α:, β:$\in$ Nbd$'$, *then*

$$< f^{\alpha,\beta}, f^{\$} >_{\alpha\beta} = 1.$$

Proof. (i) We first consider the case that $\#\mu^{\mathrm{ab}}$ is odd. The minor changes required for the case that $\#\mu^{\mathrm{ab}}$ is even will be described at the end of the proof.

(ii) We recall that the functions $f^{\$}$ and $f^{\alpha,\beta}$ are defined using the modifications $m^{\$}$ and $m^{\alpha,\beta}$, where

$$m^{\$}(e)(x) = \begin{cases} \alpha^{-1}\beta^{-1}m^{1}(e)(2x) & x \le \frac{1}{2}, \\ \alpha^{-1}\beta^{-1}e(1) + (1-\alpha^{-1}\beta^{-1})m^{1}(e)(2x-1) & x \ge \frac{1}{2}, \end{cases}$$

$$m^{\alpha,\beta}(e)(x) = \begin{cases} \alpha^{-1}\beta^{-1}m^{1}(e)(2x) & x \le \frac{1}{2}, \\ \alpha^{-1}\beta^{-1}e(1) + (\alpha^{-1}-\alpha^{-1}\beta^{-1})m^{1}(e)(4x-2) & \frac{1}{2} \le x \le \frac{3}{4}, \\ \alpha^{-1}e(1) + (1-\alpha^{-1})m^{1}(e)(4x-3) & x \ge \frac{3}{4}. \end{cases}$$

From this we see that the difference between $m^{\$}(e)$ and $m^{\alpha,\beta}(e)$ is essentially a triangle whose vertices ($\alpha^{-1}\beta^{-1}e(1)$, $\alpha^{-1}e(1)$ and $e(1)$) are congruent modulo $\alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}2}L$. We shall exploit this congruence to show that $< f^{\alpha,\beta}, f^{\$} >_{\alpha\beta} = 1$.

(iii) We shall construct a special admissible homotopy h von $[0,1]^{\$}$ to $[0,1]^{\alpha,\beta}$. Then by the skew product formula (Theorem 6),

$$< f^{\alpha,\beta}, f^{\$} >_{\alpha\beta} \equiv \prod_{[\mathcal{F}]} \zeta_{[\mathcal{F}]}^{\left\{\left\{X^{*}[\alpha\beta]\middle|\tilde{h}_{\mathcal{F}}\right\}\right\}}.$$

For every face $\mathcal{F} < \mathcal{P}$ we shall show that

$$\left\{\left\{X^{*}[\alpha\beta]\middle|\tilde{h}_{\mathcal{F}}\right\}\right\} \equiv 0 \bmod \#\mu^{\mathrm{ab}}.$$

From this it follows that $< f^{\alpha,\beta}, f^{\$} >_{\alpha\beta} \equiv 0$. The difficult thing is to find the right homotopy h.

(iv) We now begin to construct the homotopy h. The two paths $m^{\$}(e)$ and $m^{\beta,\alpha}(e)$ are the same from 0 to $\alpha^{-1}\beta^{-1}e(1)$. We call this part of the paths the singular part. In the singular part, whose preimage in I is $\left[0, \frac{1}{2}\right]$, we define $h^{(e)}(x,t)$ to be independent of t. Thus for $x \leq \frac{1}{2}$ we have

$$h(x,t) \;=\; \alpha^{-1}\beta^{-1}m^1(e)(2x).$$

The rest of h depends on t, since $m^{\$}(e)$ and $m^{\beta,\alpha}(e)$ are not the same between $\alpha^{-1}\beta^{-1}e(1)$ and $e(1)$. We call this part of h the non-singular part. If the face $\mathcal{F}$ is given by the product

$$\mathrm{Transl}(v_{\mathcal{F}}) \prod_{i=1}^{n-1} e_{\mathcal{F},i},$$

then we have

$$\tilde{h}_{\mathcal{F}}(x_1,\ldots,x_{n-1},t) \;=\; v_{\mathcal{F}} \;+\; \sum_{x_i\in[0,\frac{1}{2}]} \alpha^{-1}\beta^{-1}m^1(e_{\mathcal{F},i})(2x_i) + \sum_{x_i\notin[0,\frac{1}{2}]} h^{(e_{\mathcal{F},i})}(x_i,t).$$

To make this more readable, we define for every subset $T \subset \{1,2,\ldots,n-1\}$ a function

$$g_T\left(x_1,\ldots,x_{n-1},t\right) = v_{\mathcal{F}} + \sum_{j\in T} \alpha^{-1}\beta^{-1}m^1(e_{\mathcal{F},j})(x_j) + \sum_{j\notin T} h^{(e_{\mathcal{F},j})}\left(\frac{x_j+1}{2}, t\right).$$

The function g_T is a singular n-cube in X. We have an equivalence in $H_n(X, X\setminus X^*[\alpha\beta])$:

$$\tilde{h}_{\mathcal{F}} \simeq \sum_{T\subset\{1,2,\ldots,n-1\}} g_T.$$

We shall construct the non-singular part of h such that for every T one has

$$\left\{\left\{X^*[\alpha\beta]\middle| g_T\right\}\right\} \equiv 0 \bmod \#\mu^{\mathrm{ab}}.$$

If T is empty then g_T is degenerate and the equation follows immediately. Thus the totally singular part of $\tilde{h}_{\mathcal{F}}$ vanishes. Now suppose T is non-empty. Since α: and β: are close to I:, we can (and shall) choose h is such a way that for non-empty T the sets $|g_T|$ and L are disjoint. It is then sufficient to show that

$$\left\{\left\{X[\alpha\beta]\middle| g_T\right\}\right\} \equiv 0 \bmod \#\mu^{\mathrm{ab}}.$$

(v) We would now like to construct the non-singular part of h. For this purpose we define

$$h^{ns(e)} \;:\; I^2 \;\longrightarrow\; V_\infty$$

$$(x,t) \longmapsto h^{(e)}\Big(\frac{x+1}{2}, t\Big).$$

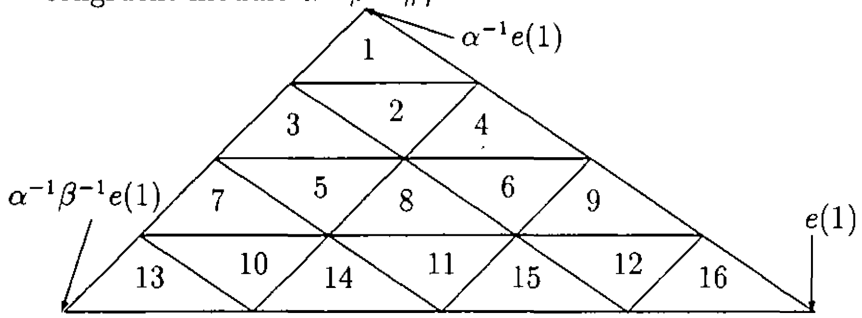
This function is a homotopy between the non-singular parts of $m^{\alpha,\beta}$ and $m^{\$}$. We can now express g_T more easily:

$$g_T(x_1,\ldots,x_{n-1},t) = v_{\mathcal{F}} + \sum_{j\in T} \alpha^{-1}\beta^{-1}m^1(e_{\mathcal{F},j})(x_j) + \sum_{j\notin T} h^{ns(e_{\mathcal{F},j})}(x_j,t).$$

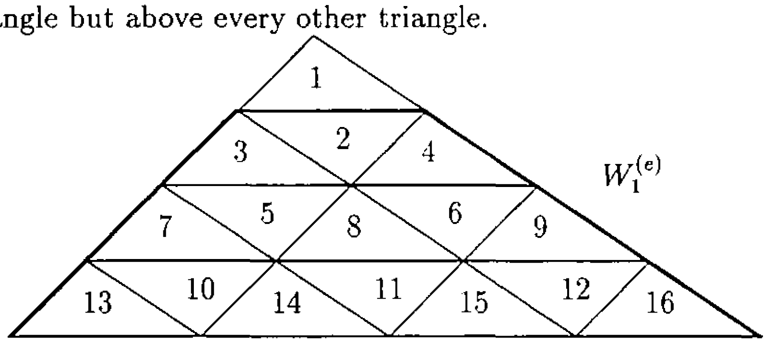
(vi) We now construct a sequence of paths between $\alpha^{-1}\beta^{-1}e(1)$ and $e(1)$:

$$W_0^{(e)} \ , \ W_1^{(e)} \ , \ W_2^{(e)} \ , \ \ldots \ , \ W_{\#\mu^{\mathrm{ab}2}}^{(e)}$$

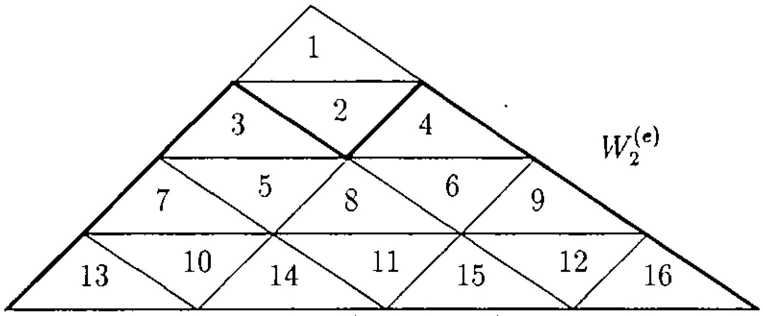
where $W_0^{(e)}$ is the non-singular part of $m^{\alpha,\beta}(e)$ and $W_{\#\mu^{\mathrm{ab}2}}^{(e)}$ is the non-singular part of $m^{\$}(e)$. Between $W_0^{(e)}$ and $W_{\#\mu^{\mathrm{ab}2}}^{(e)}$ there is a modified triangle, whose vertices are $\alpha^{-1}\beta^{-1}e(1)$, $\alpha^{-1}e(1)$ and $e(1)$. These vertices are congruent modulo $\alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}2}L$. We cut this triangle into $\#\mu^{\mathrm{ab}2}$ smaller, similar triangles. The vertices of the smaller triangles are congruent modulo $\alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}}L$.



We number the triangles as shown in the diagram. Thus the path $W_0^{(e)}$ runs above all triangles in the diagram. We now construct the path $W_1^{(e)}$ to run below of the first triangle but above every other triangle.



Similarly $W_2^{(e)}$ runs below the first two triangles, and so on.



We choose the paths $W_{i-1}^{(e)}$ and $W_i^{(e)}$ so that they are equal outside the subinterval $[c_i, d_i]$ of I, which is mapped to the boundary of the i^{th} triangle. Thus $W_{i-1}^{(e)}(x) = W_i^{(e)}(x)$ for $x \notin (c_i, d_i)$. We shall choose homotopies $h_i^{ns(e)}(x,t)$ from $W_{i-1}^{(e)}$ to $W_i^{(e)}$ in such a way that they are independent of t for $x \notin (c_i, d_i)$. Thus for $x \notin (c_i, d_i)$,

$$h_i^{ns(e)}(x,t) \;=\; W_i^{(e)}(x) \;=\; W_{i-1}^{(e)}(x).$$

We now choose by Lemma 4 $h_1^{ns(e)}$ and $h_2^{ns(e)}$ for x in (c_1, d_1) and (c_2, d_2) such that h_1 and h_2 are admissible. If $i > 2$ then the i^{th} triangle is a translation either of the first or of the second triangle by an element $t_i^{(e)}$ of $\alpha^{-1}\beta^{-1}\#\mu^{\text{ab}}L$. We can now construct $h_i^{ns(e)}(x,t)$ for $x \in (c_i, d_i)$ as follows:

$$h_i^{(e)}(c_i + (d_i - c_i)x, t) = \; t_i^{(e)} \;+\; \begin{cases} h_1^{(e)}(c_1 + (d_1 - c_1)x, t) \\ h_2^{(e)}(c_2 + (d_2 - c_2)x, t) \end{cases}.$$

We define the non-singular part, h^{ns} of h:

$$h^{ns(e)}(x,t) := h_i^{ns(e)}(x, \#\mu^{\text{ab2}}t - i + 1), \quad \text{for } \frac{i-1}{\#\mu^{\text{ab2}}} \le t \le \frac{i}{\#\mu^{\text{ab2}}}.$$

We also define

$$g_T^i(x_1, \ldots, x_{n-1}, t) := v_{\mathcal{F}} + \sum_{j\in T} \alpha^{-1}\beta^{-1}m^1(e_{\mathcal{F},j})(x_j) + \sum_{j\notin T} h_i^{ns(e_{\mathcal{F},j})}(x_j, t).$$

There is an equivalence in $H_n(X, X \setminus X[\alpha\beta])$:

$$g_T \simeq \sum_{i=1}^{\#\mu^{\text{ab2}}} g_T^i.$$

(vii) We now consider the functions h_i^{ns} and g_T^i in more detail. We have

$$\begin{aligned} g_T^i(x_1, \ldots, x_{n-1}, t) \;&=\; v_{\mathcal{F}} + \sum_{j\in T} \alpha^{-1}\beta^{-1}m^1(e_{\mathcal{F},j})(x_j) + \sum_{j\notin T,\; x_j \notin (c_i,d_i)} W_i^{(e_{\mathcal{F},j})}(x_j) \\ &\quad + \sum_{j\notin T,\; x_j\in(c_i,d_i)} h_i^{(e_{\mathcal{F},j})}(x_j, t). \end{aligned}$$

If T^1 and T^2 are two subsets of $\{1, 2, \dots, n-1\}$, such that T, T^1 and T^2 are pairwise disjoint, then we define

$$\begin{aligned} l^i_{T^1,T^2}(x_1, \dots, x_{n-1}, t) &= v_{\mathcal{F}} + \sum_{j \in T} \alpha^{-1}\beta^{-1} m^1(e_{\mathcal{F},j})(x_j) + \sum_{j \in T^1} W_i^{(e_{\mathcal{F},j})}(c_i x_j) \\ &\quad + \sum_{j \in T^2} W_i^{(e_{\mathcal{F},j})}((1-d_i)x_j + d_i) \\ &\quad + \sum_{j \notin T \cup T^1 \cup T^2} h_i^{(e_{\mathcal{F},j})}(c_i + (d_i - c_i)x_j, t). \end{aligned}$$

We then have an equivalence in $H_n(X, X \setminus X[\alpha\beta])$:

$$g^i_T \simeq \sum_{T^1, T^2 \subset \{1,2,\dots,n-1\} \setminus T,\ T^1 \cap T^2 = \emptyset} l^i_{T^1,T^2}$$

We shall compute the terms of this sum.

(viii) If T^1 is non-empty, then $l^i_{T^1,T^2}$ is a product of $W_i^{(e_{\mathcal{F},j})}|_{[0,c_i]}$ with other things. We know however that $W_i^{(e_{\mathcal{F},j})}|_{[0,c_i]}$ is a sum of modified line segments, whose lengths are in $\alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}}L$. Therefore by remark §5.2,

$$\{\{X[\alpha\beta]|\text{These terms}\}\} \equiv 0 \quad \text{modulo } \#\mu^{\mathrm{ab}}.$$

The terms in which T^2 is non-empty vanish in the same way. We are therefore only interested in the term, for which T^1 and T^2 are empty. First suppose the i^{th} triangle is a translation by $t_i^{(e)}$ of the first triangle . We then have

$$\begin{aligned} l^i_{\emptyset,\emptyset}(x_1, \dots, x_{n-1}, t) &= v_{\mathcal{F}} + \sum_{j \in T} \alpha^{-1}\beta^{-1} m^1(e_{\mathcal{F},j})(x_j) + \sum_{j \notin T} h_i^{(e_{\mathcal{F},j})}(c_i + (d_i - c_i)x_j, t) \\ &= v_{\mathcal{F}} + \sum_{j \in T} \alpha^{-1}\beta^{-1} m^1(e_{\mathcal{F},j})(x_j) \\ &\quad + \sum_{j \notin T} \left(t_i^{(e_{\mathcal{F},j})} + h_1^{ns(e_{\mathcal{F},j})}(c_1 + (d_1 - c_1)x_j, t) \right) \\ &= \sum_{j \notin T} t_i^{(e_{\mathcal{F},j})} + l^1_{\emptyset,\emptyset}(x_1, \dots, x_{n-1}, t). \end{aligned}$$

Since $t_i^{(e_{\mathcal{F},j})} \in \alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}}L$, we must have $\sum_{j \notin T} t_i^{(e_{\mathcal{F},j})} \in \alpha^{-1}\beta^{-1}\#\mu^{\mathrm{ab}}L$. In particular this translation is in $\alpha^{-1}\beta^{-1}L$. Therefore

$$\left\{\left\{X[\alpha\beta]|l^i_{\emptyset,\emptyset}\right\}\right\} = \left\{\left\{X[\alpha\beta]|l^1_{\emptyset,\emptyset}\right\}\right\}.$$

Analogously, if the i^{th} triangle is a translation of the second triangle,

$$\left\{\left\{X[\alpha\beta]|l^i_{\emptyset,\emptyset}\right\}\right\} = \left\{\left\{X[\alpha\beta]|l^2_{\emptyset,\emptyset}\right\}\right\}.$$

The number the triangles which are translations of the first triangle is $\frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}+1)}{2}$. The number the triangles which are translations of the second triangle is $\frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}-1)}{2}$. We therefore have

$$\{\{X[\alpha\beta]|g_T\}\} \equiv \frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}+1)}{2}\left\{\left\{X[\alpha\beta]|l^1_{\emptyset,\emptyset}\right\}\right\} + \frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}-1)}{2}\left\{\left\{X[\alpha\beta]|l^2_{\emptyset,\emptyset}\right\}\right\}.$$

Since both these numbers are divisible by $\#\mu^{\mathrm{ab}}$, the lemma is proved in the case that $\#\mu^{\mathrm{ab}}$ is odd.

(ix) We now consider the case that $\#\mu^{\mathrm{ab}}$ is even. The whole proof would be the same, but at the end one doesn't have the result that $\frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}+1)}{2}$ and $\frac{\#\mu^{\mathrm{ab}}(\#\mu^{\mathrm{ab}}-1)}{2}$ are divisible by $\#\mu^{\mathrm{ab}}$. Instead we require at the beginning that $\alpha, \beta \equiv 1 \bmod 2\delta\#\mu^{\mathrm{ab}2}$. We cut the large triangle into $4\#\mu^{\mathrm{ab}2}$ instead of $\#\mu^{\mathrm{ab}2}$ pieces. At the end we have for the two numbers $\frac{2\#\mu^{\mathrm{ab}}(2\#\mu^{\mathrm{ab}}+1)}{2}$ and $\frac{2\#\mu^{\mathrm{ab}}(2\#\mu^{\mathrm{ab}}-1)}{2}$, which are divisible by $\#\mu^{\mathrm{ab}}$.

Lemma 8 *There is a neighbourhood* Nbd_2 *of* I: *in* $G_{\mathbf{R}}$: *with the following property. Let* $(\alpha, \beta) \in \Omega_{\mathbf{Z}}$ *satisfy* $\alpha \equiv \beta \equiv 1 \bmod 2\delta\#\mu^{\mathrm{ab}2}$ *and* $\alpha:, \beta: \in \mathrm{Nbd}_2$. *Then*

$$\mathrm{Dec}(\alpha, \beta) = \mathrm{Dec}(\beta, \alpha).$$

Proof. This will follow from the previous three lemmas together with the combinatorial properties of decidents and skew products. Let Nbd_2 be the intersection of the two neighbourhoods constructed in the previous 2 lemmas. By Proposition 4 we have

$$\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} = < f^1, f^1\circ\alpha >_{\alpha\beta\backslash\alpha} < f^1\circ\beta, f^1 >_{\alpha\beta\backslash\beta}.$$

By Lemma 5 we have

$$\begin{aligned}\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} &= < f^1, f^1\circ\alpha >_{\alpha\beta\backslash\alpha} < f^1\circ\alpha, f^\alpha >_{\alpha\beta\backslash\alpha} \\ &\quad < f^\beta, f^1\circ\beta >_{\alpha\beta\backslash\beta} < f^1\circ\beta, f^1 >_{\alpha\beta\backslash\beta}.\end{aligned}$$

By Proposition 1 we have

$$\begin{aligned}\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} &= < f^1, f^\alpha >_{\alpha\beta\backslash\alpha} < f^\beta, f^1 >_{\alpha\beta\backslash\beta} \\ &= < f^1, f^\alpha >_{\alpha\beta} < f^1, f^\alpha >_\alpha^{-1} < f^\beta, f^1 >_{\alpha\beta} < f^\beta, f^1 >_\beta^{-1}.\end{aligned}$$

Now Lemma 6 implies

$$\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} = < f^1, f^\alpha >_{\alpha\beta} < f^\beta, f^{\beta,\alpha} >_{\alpha\beta}^{-1} < f^\beta, f^1 >_{\alpha\beta} < f^{\alpha,\beta}, f^\alpha >_{\alpha\beta}^{-1}.$$

By Proposition 1 we have

$$\begin{aligned}\mathrm{Dec}(\alpha,\beta)\mathrm{Dec}(\beta,\alpha)^{-1} &= < f^1, f^\alpha >_{\alpha\beta} < f^{\beta,\alpha}, f^\beta >_{\alpha\beta} < f^\beta, f^1 >_{\alpha\beta} < f^\alpha, f^{\alpha,\beta} >_{\alpha\beta} \\ &= < f^{\beta,\alpha}, f^\beta >_{\alpha\beta} < f^\beta, f^1 >_{\alpha\beta} < f^1, f^\alpha >_{\alpha\beta} < f^\alpha, f^{\alpha,\beta} >_{\alpha\beta} \\ &= < f^{\beta,\alpha}, f^{\alpha,\beta} >_{\alpha\beta} \\ &= < f^{\beta,\alpha}, f^\$ >_{\alpha\beta} < f^\$, f^{\alpha,\beta} >_{\alpha\beta}.\end{aligned}$$

Lemma 7 now implies $\mathrm{Dec}(\alpha, \beta) = \mathrm{Dec}(\beta, \alpha)$.

We now prove the result stated at the beginning of the chapter:

Corollary 2 (Theorem 1) *Let* $(\alpha,\beta) \in \Omega_{\mathbf{Z}}$ *satisfy* $\alpha \equiv \beta \equiv 1 \bmod 2\delta\#\mu^{\mathrm{ab}2}$ *and* $\alpha \in G^0_{\mathbb{R}}$ *and* $\beta \in G^{\alpha 0}_{\mathbb{R}}$. *Then*

$$\mathrm{Dec}(\alpha,\beta) = \mathrm{Dec}(\beta,\alpha).$$

Proof. First let $\alpha{:}\in \mathrm{Nbd}_2$, and let $\alpha,\beta \equiv 1 \bmod 2\delta\#\mu^{\mathrm{ab}2}$ and $\beta \in G^{\alpha 0}_{\mathbb{R}}$. The set $\{b^{-1}\beta' \mid b \in \mathbb{N} \text{ and } \beta' \in G^{\alpha}_{\mathbf{Z}}, \beta' \equiv 1 \bmod 2\delta\#\mu^{\mathrm{ab}2}\}$ is dense in $G^{\alpha}_{\mathbb{R}}$. Since $\beta \in G^{\alpha 0}_{\mathbb{R}}$, we can find a β' such that $(\beta'^{\#\mu^{\mathrm{ab}}}\beta){:}\in \mathrm{Nbd}_2$ and $\beta' \equiv I \bmod 2\delta\#\mu^{\mathrm{ab}2}$. We have from the previous lemma

$$\mathrm{Dec}(\alpha,\beta'^{\#\mu^{\mathrm{ab}}}\beta) = \mathrm{Dec}(\beta'^{\#\mu^{\mathrm{ab}}}\beta,\alpha).$$

By Proposition 3 we have $\mathrm{Dec}(\alpha,\beta'^{\#\mu^{\mathrm{ab}}}\beta) = \mathrm{Dec}(\alpha,\beta)$. Similarly by Proposition 2, $\mathrm{Dec}(\beta'^{\#\mu^{\mathrm{ab}}}\beta,\alpha) = \mathrm{Dec}(\alpha,\beta)$. Therefore

$$\mathrm{Dec}(\alpha,\beta) = \mathrm{Dec}(\beta,\alpha).$$

With the same trick we can remove the condition that $\alpha{:}\in \mathrm{Nbd}_2$.

References

[1] Habicht, Ein elementarer Beweis des kubischen Reziprozitätsgesetzes, Math. Annalen 139 (1959-60) 343-365.

[2] R.Hill : A geometric proof of a reciprocity law, to appear in Nagoya Math.J. March 1995.

[3] D.Kazhdan, S.J.Patterson : Metaplectic Forms, Publ.IHES.

[4] T. Kubota, Geometry of Numbers and Class Field Theory : Japan. J. Math. Vol. 13, No 2, (1987) 235-275.

[5] T. Kubota, Geometric Foundation of Class Field Theory, (in Japanese) Sugaka 44 1 (1992) 1-12.

[6] Massey : Singular Homology. Springer GTM.

[7] Serre : Algebraic Groups and Class Fields. Springer GTM.

[8] Smirnov : preprint 1994.

[9] A. Weil, Basic Number Theory, Grundlehren der Math. Wissenschaften in Einzeldarstellung Band 144, Springer-Verlag (1967).