HOMOGENEOUS FOLIATIONS: TOPOLOGY OF LEAVES AND APPLICATIONS TO POLYGONAL BILLIARDS

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ABSTRACT. The aim of this paper is to describe the topology of the leaves of a homogeneous foliation on $\mathbf{C}^2$ with isolated singularity and reduced tangent cone. This description leads to a complete topological classification of translation surfaces associated to a polygonal billiard.

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1. INTRODUCTION

Let $\overline{P} \subset \mathbf{R}^2$ be a polygon and $Q(\overline{P})$ the set of its vertices. Through this article we consider polygons without vertices $P := \overline{P} \setminus Q(\overline{P})$. Let $S(P)$ denote the translation real surface arising by unfolding billiard trajectories inside P. In general, there exist a "non dicritical" holomorphic homogeneous foliation $\mathcal{F} = \mathcal{F}(P)$ on $\mathbf{C}^2$ whose generic leaves are isomorphic, *as translation surfaces*, to the surface $S(P)$ [7]. Before precising what genericity and generality mean, (see §2 and §4 respectively) we recall what is known about the topology of the orientable surface $S(P)$:

Lemma 1. [4] *Let the angles of P be $\pi m_i/n_i$, $i = 1, \ldots, k$, where m_i and n_i are coprime, and N be the last common multiple of n_i's. Then*

$$\text{genus } S(P) = 1 + \frac{N}{2}\left(k - 2 - \Sigma\frac{1}{n_i}\right)$$

In this article we give complete *topological* description of the orientable surfaces $S(P)$ when the angles of P are not necessarily rational. We achieve this by prooving the following

Theorem 1. *Let $\mathcal{F}$ be an homogeneous holomorphic foliation on $\mathbf{C}^2$ with isolated singularity and reduced tangent cone. Then, every non singular leaf which is not a separatrix is homeomorphic to either:*

1. $\mathbf{C} \setminus \mathbf{Z}[i]$, *endowed with the standart topology.*
2. $S \setminus E$, *where S is a compact* RIEMANN *surface and* $E \subset S$ *is a finite set of points.*

(3) $M\backslash E'$, *where* M *is the* LOCH NESS *monster, i.e. the plane to which we stick a countable set of handles, and* $E' \subseteq M$ *is a countable and discrete set of points.*
(4) M, *the* LOCH NESS *monster.*

The foregoing theorem completes a result presented in [7]. From the same reference, we deduce that any translation surface $S(P)$ is homeomorphic to either (2), (3) or (4). It is relevant to remark that, up to punctures, all the topological surfaces in our list are contained in the milestone

Theorem 2. [3] *Let $\mathcal{G}$ be an orientable dimension 2 lamination on an arbitrary compact space. Then, if $\mathcal{G}$ has no compact leaf, a non countable set of its leaves are diffeomorphic to either:*
(1) *The plane* $\mathbf{R}^2$*,*
(2) *The* LOCH NESS *monster,*
(3) *The cylinder* $\mathbf{R} \times \mathbf{S}^1$*,*
(4) JACOB'S STAIR*, i.e. the cylinder to which we stick a countable set of handles in both directions,*
(5) CANTOR'S TREE*, i.e. the sphere* $\mathbf{S}^2$ *to which we remove a* CANTOR *set,*
(6) CANTOR'S FLORID TREE*, i.e., a* CANTOR'S TREE *to which we stick a countable number of handles in all directions.*

2. GENERALITIES

In this section we precise our language and notation. A holomorphic foliation on $\mathbf{C}^2$ is said to be *homogeneous* if it is invariant under the natural action of the homothetic transformation group $\{T_k(z_1, z_2) := k(z_1, z_2) \mid k \in \mathbf{C}^*\}$. In this article we consider homogeneous holomorphic foliations on $\mathbf{C}^2$ with isolated singularity and having $\nu + 1$, $\nu \geq 2$, complex lines through the singularity as leaves. We suppose that this set of invariant lines, usually called the *tangent cone* of the foliation, is given by the zeros of the irreducible polynomial $P_{\nu+1} = \prod_{j=1}^{\nu+1}(z_2 - a_j z_1)$, $a_j \in \mathbf{C}$, $a_i \neq a_j$, $\forall i, j$, and set $a = (a_1, \ldots, a_{\nu+1})$. From [2], this foliation is given by the holomorphic 1-form

$$\frac{\omega_{a,\lambda}}{P_{\nu+1}} = \sum_{j=1}^{\nu+1} \lambda_j \frac{d(z_2 - a_j z_1)}{z_2 - a_j z_1}, \tag{1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_{\nu+1})$ is a point of the affine hyperplane $\{(\lambda_1, \ldots, \lambda_{\nu+1}) \in \mathbf{C}^n \mid \sum_{j=1}^{\nu+1} \lambda_j = K, \ \lambda_j \neq 0 \, \forall j\}$ and $K \in \mathbf{Z} \setminus 0$ is an arbitrary normalization constant whose choice does not affect the foliation defined by (1). We denote this foliation $\mathcal{F}_{a,\lambda}$. We remark that

$$F_{a,\lambda}(z_1, z_2) = \prod_{j=1}^{\nu+1} (z_2 - a_j z_1)^{\lambda_j} \tag{2}$$

is a first integral for $\mathcal{F}_{a,\lambda}$. This expression is usually multivaluated. Let $\sim_\lambda$ be the equivalence relation induced by the $\mathbf{Z}^3$ action on $\mathbf{C}^*$ defined by $((n_1, n_2, n_3), z) \longrightarrow e^{2\pi i \Sigma_j n_j \lambda_j} z$. The connected components of the fibers $F_{a,\lambda} : \mathbf{C}^2 \longrightarrow \mathbf{C}^* / \sim_\lambda$ are the leaves of $\mathcal{F}_{a,\lambda}$ in the complement of the tangent cone. Abusing notation we will write $F_{a,\lambda}(z_1, z_2) = k$, $k \in \mathbf{C} / \sim_\lambda$, for such a fiber. A straightfoward computation shows that if $k, k' \in \mathbf{C}^*$, the homothecy $T_{k/k'}$ defines a diffeomorphism between $F_{a,\lambda}(z_1, z_2) = k$ and $F_{a,\lambda}(z_1, z_2) = k'$. In particular, each leaf in $\mathcal{F}_{a,\lambda}$ is invariant by any homothecy $T_{e^{2\pi i \sum_j n_j \lambda_j}}$, $n_j \in \mathbf{Z}$, $\forall j$.

We will call *generic leaf* any leave in the complement of the tangent cone of $\mathcal{F}_{a,\lambda}$. We denote it by $\mathcal{L} \in \mathcal{F}_{a,\lambda}$. The foliation $\mathcal{F}_{a,\lambda}$ presents, up to diffeomorphism, three kinds of leaves: *a (singular) point,* $\mathbf{C}^*$ *(separatrices) and a generic leaf* $\mathcal{L}$*.*

3. PROOF, THEOREM 1

The topological type of the generic leaf of $\mathcal{F}_{a,\lambda}$ depends only in the parameters $\lambda_1, \ldots, \lambda_{\nu+1}$. Indeed, let's consider $\widetilde{\mathcal{F}_{a,\lambda}}$, the extension of the foliation $\mathcal{F}_{a,\lambda}$ to HIRZEBRUCH'S first surface. This complex ruled surface, which we denote $\widetilde{\mathbf{CP}(2)}$, is obtained by blowing up the origin of the affine chart $\mathbf{C} = \{(z_1, z_2)\}$ of $\mathbf{CP}(2)$. A leaf in $\widetilde{\mathcal{F}_{a,\lambda}}$ is called generic if it comes from a generic leaf in $\mathcal{F}_{a,\lambda}$. Let $\pi : \widetilde{\mathbf{CP}(2)} \longrightarrow \mathbf{P}^1(\mathbf{C})$ be the natural projection onto the exceptional divisor. For each $j = 1, \ldots, \nu + 1$, we denote as well a_j the intersection of the complex line $z_2 = a_j z_1$ with the exceptional divisor. Let $B_a := \mathbf{P}^1(\mathbf{C}) \setminus \{a_1, \ldots, a_{\nu+1}\}$. We remark that $\widetilde{\mathcal{F}_{a,\lambda}}$ is transversal to the fibration defined by π on B_a. Furthermore, for every generic leaf $\mathcal{L} \in \widetilde{\mathcal{F}_{a,\lambda}}$, the restriction $\pi_| : \mathcal{L} \longrightarrow B_a$ is a covering. The monodromy of this covering is given by the image in $Diff(\mathbf{C}, 0)$ of the holonomy representation of $\widetilde{\mathcal{F}_{a,\lambda}}$ with respect to the fibration defined by π on B_a. This holonomy representation can be explicitly calculated: let $\{\gamma_j\}_{j=1}^{\nu+1}$ be generators for $\pi_1(B_a, *)$, having index δ_i^j with respect to each a_j, $j = 1, \ldots, \nu+1$, and such that $\gamma_i \cap \gamma_j = *$ whenever $i \neq j$. The holonomy representation morphism $hol : \pi_1(B_a, *) \longrightarrow Diff(\mathbf{C}, 0)$ of $\widetilde{\mathcal{F}_{a,\lambda}}$ is given by:

$$hol(\gamma_j) = z \rightsquigarrow e^{2\pi i \lambda_j / K} z. \tag{3}$$

This is clear from (2). In particular, the monodromy action of the covering $\pi_|$ is abelian.

Definition 1. *We will say that the set* $\lambda = (\lambda_1, \ldots, \lambda_{\nu+1})$ *is:*

- Not rational, *when* $\lambda_j \in \mathbf{C}^* \setminus \mathbf{Q}$ *for all* $j = 1, \ldots, \nu + 1$.
- Semi-rational, *when there exists a pair* $i \neq j$ *such that* $\lambda_i \in \mathbf{C}^* \setminus \mathbf{Q}$ *and* $\lambda_j \in \mathbf{Q}$.
- Rational, *when* $\lambda_j \in \mathbf{Q}$ *for all* $j = 1, \ldots, \nu + 1$.

Definition 2. [6] *A real surface* S *is of* infinite genus *if there is no bounded subset* S' *such that* $S \setminus S'$ *is of genus zero.*

Following [*Ibid*], the *ideal boundary* of a real surface S (not necessarily compact) is a nested triple of topological spaces $B''(S) \subset B'(S) \subset B(S)$. This triple codes the topological type of the surface, that is:

Theorem 3 (Kerékjártó). [Ibid] *Let* S *and* $\widehat{S}$ *be two separable real surfaces of the same genus and orientability class. Then* S *and* $\widehat{S}$ *are homeomorphic if and only if their ideal boundaries (considered as triples of spaces) are topologically equivalent.*

The topological spaces $B'(S)$ and $B''(S)$ are the not planar and not orientable parts of the *whole* ideal boundary $B(S)$. In the context of this article, $B''(\mathcal{L}) = \emptyset$, for the generic leaf of the foliation $\mathcal{F}_{a,\lambda}$ is oriented by its natural complex structure. Up to homeomorphism, the LOCH NESS monster is the only infinite genus real surface whose ideal boundary is just a point.

Lemma 2. *If* λ *is not rational, then the generic leaf* $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ *is homeomorphic to a* LOCH NESS *monster.*

Proof. First we claim that, for every compact $K \subset \mathcal{L}$ there exists a compact subset $K \subset K' \subset \mathcal{L}$ such that $\mathcal{L} \setminus K'$ is connected by arcs. From the definition of *boundary component*, this implies that the ideal boundary of $\mathcal{L}$ is just a point. Indeed, let $t_0 \in \mathbf{P}^1(\mathbf{C}) \setminus \{a_1, \ldots, a_{\nu+1}, *\}$. Let $C_1, \ldots, C_{\nu+1}$ be line segments joining a_j to t_0 for every $j = 1, \ldots, \nu + 1$ and such that $C_i \cap C_j = t_0$, whenever $i \neq j$. Without loss of generality, we suppose as well that the cardinality of $C_i \cap \gamma_j$ is given by KRONECKER's delta δ_{ij}. The open set

$$U := \mathbf{P}^1(\mathbf{C}) \setminus \cup_{j=1}^{\nu+1} C_j \tag{4}$$

is simply connected. Let $\mathbf{P}^1(\mathbf{C}) \simeq \mathbf{S}^2$ be the standard identification of the complex proyective line with RIEMANN's sphere and $\mathbf{S}^2 \hookrightarrow \mathbf{R}^3$ a fixed smooth embeding. Then, the restriction of the standard metric to the image of this embeding induces a complete metric on the exceptional divisor that we denote δ. For every positive real number $\tau < inf_{i\neq j, i,j=1,\dots\nu+1}\delta(a_i, a_j)$, the complement in U of the set $\cup_{j=1}^{\nu+1}\{t \in \mathbf{P}^1(\mathbf{C}) \mid \delta(t, a_j) < \tau\}$ is closed and simply connected. We denote this set by U_τ.

For every point ξ in a fiber $\pi_|^{-1}(t)$, $t \neq a_1, \dots, a_{\nu+1}, t_0$, let $U_{\tau,\xi}$ be the lifting of U_τ to the leaf $\mathcal{L}$ satisfying $U_{\tau,\xi} \cap \xi = \xi$. We define $K_{\tau,\,\xi}$ to be the closure of $U_{\tau,\xi}$ in $\mathcal{L}$. This set is compact in $\mathcal{L}$.

Let

$$\tau(K) := \frac{1}{2} inf_{j=1,\dots,\nu+1}\delta(\pi(K), a_j). \tag{5}$$

Since K is compact, there exists a finite set of points $P = \{p_1, \dots, p_s\} \subset B_a$ such that

$$K \subset K' := \cup_{\xi\in K\cap\pi_|^{-1}(P)} K_{\tau(K),\xi}. \tag{6}$$

The set $K \cap \pi_|^{-1}(P)$ is finite and therefore K' is compact. We claim that, for every positive real number $\tau < inf_{i\neq j, i,j=1,\dots,\nu+1}\delta(a_i, a_j)$ and finite subset $Z \subset \pi_|^{-1}(t)$, $t \neq a_1, \dots, a_{\nu+1}, t_0$, the set $\mathcal{L} \setminus \cup_{\xi\in Z} K_{\tau,\xi}$ is arcwise connected. This implies that $\mathcal{L} \setminus K'$ is arcwise connected as well.

Indeed, for every point η in a fiber $\pi_|^{-1}(t)$, $t \neq a_1, \dots, a_{\nu+1}, t_0$, let U_η be the lifting of U (4) satisfying $U_\eta \cap \eta = \eta$. We denote $\overline{U_\eta}$ the closure in $\mathcal{L}$ of U_η. Clearly, $\mathcal{L} = \cup_{\eta\in\pi_|^{-1}(t)} \overline{U_\eta}$. Consider two disctint points $q \in \overline{U_\eta}$ and $q' \in \overline{U_{\eta'}}$. If $\eta = \eta'$ and $\cup_{\xi\in Z} K_{\tau,\xi} \cap \overline{U}_\eta$ has empty interior, then the points q and q' can be joined by an arc in $\mathcal{L}$. When $\eta \neq \eta'$, the point q can be joined by an arc to a point $\tilde{q}$ such that $\pi(\tilde{q}) \cap \pi(K_{\tau,\xi}) = \emptyset$. Without loss of generality, we assume that $\pi(\tilde{q})$ is contained in a small neighborhood in exceptional divisor of a point a_i, for some $i = 1, \dots, \nu + 1$. Let be $\gamma(\tilde{q})$ be a simple loop passing through $\pi(\tilde{q})$ and contained in this small neighborhood. There exist a lift of $\gamma(\tilde{q})$ to the generic leaf $\mathcal{L}$ joining $\tilde{q}$ to a point $\hat{q} \in \overline{U_{\eta''}}$ such that $\cup_{\xi\in Z} K_{\tau,\xi} \cap \overline{U_{\eta''}}$ has empty interior. This is true since $\lambda_i \in \mathbf{C}^* \setminus \mathbf{Q}$, for all i, and therefore the cyclic subgroup $\{hol(\gamma_i)^n\}_{n\in\mathbf{Z}}$ of $Diff(\mathbf{C}, 0)$ is torsion-free. Proceeding analogously, the point q' can be joined by an arc to a point $\hat{q'} \in \overline{U_{\eta''}}$. This leads to the previous case in which $\eta = \eta'$. We conclude that a path between the points $q \neq q'$ always exist.

Lemma 3. *If there exist a pair $i \neq j$, $1 \leq i, j \leq \nu + 1$, such that γ_i, γ_j and $\gamma_i\gamma_j$ are not contained in $hol^{-1}(Id_{\mathbf{C}})$ (3), then $\mathcal{L}$ has genus different from zero.*

Proof, lemma 3. Let $m \in \pi_1(B_a, *)$ and $\widetilde{m}$ be a lifting of this loop to the generic leaf $\mathcal{L}$ via de projection $\pi_|^{-1}$. We denote by $U_m \subset \mathcal{L}$ the analytic continuation of this lifting to the simply connected domain U (4). Let $[\gamma_i, \gamma_j] := \gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}$ and consider the following cases:

(1) $\gamma_i\gamma_j^{-1} \in hol^{-1}(Id_{\mathbf{C}})$. Then $U_{[\gamma_i,\gamma_j]}$ is homeomorphic to a torus minus a disc.

(2) $\gamma_i\gamma_j^{-1} \notin hol^{-1}(Id_{\mathbf{C}})$. Then $U_{[\gamma_i,\gamma_j]}$ is homeomorphic to the annular domain $\{z \in \mathbf{C} \mid 1 <\mid z \mid< 2\}$. The domain $U_{[\gamma_i^{-1},\gamma_j][\gamma_i,\gamma_j]}$ is homeomorphic to torus minus a disc, for the conditions $\gamma_i^2 \in hol^{-1}(Id_{\mathbf{C}})$ and $\gamma_j^2 \in hol^{-1}(Id_{\mathbf{C}})$ are mutually exclusive. □

Such a pair of generators γ_i, γ_j always exists whenever $\lambda_j \in \mathbf{C}^* \setminus \mathbf{Q}$, for all j. Suppose the opposite, that is, for every $i \neq j$, $1 \leq i, j \leq \nu+1$, we have $\gamma_i\gamma_j \in hol^{-1}(Id_{\mathbf{C}})$. Then, for every such a pair, $\lambda_i + \lambda_j = q_{ij} \in \mathbf{Z}$. Since $\sum_{j=1}^{\nu+1} \lambda_j \in \mathbf{Z}$, such integral relations imply that $\lambda_{j_0} \in \mathbf{Z}$, for some $1 \leq j_0 \leq \nu + 1$, which is a contradiction.

Whenever a pair $\{\lambda_i, \lambda_j\}$ satisfying the hypothesis of lemma 3 coexists with $\lambda_k \in \mathbf{C}^* \setminus \mathbf{Q}$, k not necessarily different form i or j, the action of the free-torsion group $\{hol(\gamma_k)^n\}_{n\in\mathbf{Z}}$ on the fibers of the covering $\pi_| : \mathcal{L} \longrightarrow B_a$ generates a countable family of handles. In particular, every generic leaf $\mathcal{L} \in \mathcal{F}_{a,\lambda}$, for which λ is not rational, has infinite genus. This concludes the proof of lemma 2. □

Semi-rational case, the set E'. Without loss of generality we assume that $\lambda = (\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_{\nu+1})$, where $\lambda_j \in \mathbf{Q}$ for $1 \leq j \leq s$ and $\lambda_j \in \mathbf{C}^* \setminus \mathbf{Q}$ for $s + 1 \leq j$. In the chart $z_2 = tz_1$ of $\widetilde{\mathbf{CP}}(2)$, the first integral (2) has the form $F_{a,\lambda}(z_1, t) := z_1^K \prod_{j=1}^{\nu+1}(t - a_j)^{\lambda_j}$. Let $i \leq s$ and $\lambda_i = \frac{p_i}{q_i} \in \mathbf{Q}$. Then, any generic leaf $\mathcal{L} \in \widetilde{\mathcal{F}_{a,\lambda}}$, is given by a connected component of

$$z_1^K (t - a_i)^{p_i/q_i} \prod_{j \neq i} (t - a_j)^{\lambda_j} = z_0, \tag{7}$$

where $z_0 \in \mathbf{C}^*$ is constant. For $t \sim a_i$, $\prod_{j\neq i}(t - a_j)^{\lambda_j}$ is a unity. Then, up to a coordinate change, the generic leaf $\mathcal{L}$ defined by (7) is given, in a neighborhood $t \sim a_i$, by a countable set of algebraic branches of the form

$$z_1^K (t - a_i)^{p_i/q_i} = \text{constant}. \tag{8}$$

For every such a branch, we compactify the abstract surface $\mathcal{L}$ by adding the points corresponding to the limit $t \to a_i$ in (8). We denote $\overline{\mathcal{L}}$ topological surface obtained from this local compactification process when the index i varies in $\{1, \ldots, s\}$. We define $E' := \overline{\mathcal{L}} \setminus \mathcal{L}$.

For example, if λ is rational, then $\overline{\mathcal{L}}$ is a compact orientable surface whose genus can be calculated from (3) using the RIEMANN-HURWITZ formula. For instance, if $\lambda \in \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})\}$, then $\overline{\mathcal{L}}$ is homeomorphic to the standard torus $\mathbf{S}^1 \times \mathbf{S}^1$.

Lemma 4. *If λ is semi-rational, then the ideal boundary of $\overline{\mathcal{L}}$ is just a point.*

Proof. As before, we claim that every compact subset $K \subset \overline{\mathcal{L}}$ can be covered by a compact subset $K' \subset \overline{\mathcal{L}}$ such that $\overline{\mathcal{L}} \setminus K'$ is connected by arcs. In deed, for positive real number $\tau < inf_{i \neq j, i,j=s+1,\ldots\nu+1} \delta(a_i, a_j)$, we define V_τ to be the complement in U of the set $\cup_{j=s+1}^{\nu+1} \{t \in \mathbf{P}^1(\mathbf{C}) \mid \delta(t, a_j) < \tau\}$. As in the proof of lemma 2, we define $V_{\tau,\xi}$ to be the lifting of V_τ to a point ξ in a generic fiber of $\pi_|$.

We consider $\tau(K)$ as in (5), but taking the infimum over the set $j = s+1, \ldots, \nu+1$. To define K' it is sufficient to consider a the closure in $\overline{\mathcal{L}}$ of a set

$$\cup_{\xi \in Z} V_{\tau,\xi}, \tag{9}$$

where $Z \subset \pi_|^{-1}(t)$, $t \neq a_1, \ldots, a_{\nu+1}, t_0$, is a "sufficiently large" finite subset. The rest of the proof is analog to the proof of lemma 2. Indeed, considering different cases, any two points $q \neq q'$ in $\overline{\mathcal{L}}$ can be joined by an arc, for the action of any free-torsion group $\{hol(\gamma_j)^n\}_{n \in \mathbf{Z}}$, $j = s+1, \ldots, \nu+1$, on the covering $\pi_| : \mathcal{L} \longrightarrow B_a$ permits to scape to any of compact subsets K' in $\overline{\mathcal{L}}$ previously constructed. $\square$

Corollary 1. *Let λ be a semi-rational. Then, the ideal boundary of the generic leaf $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ is homeomorphic to $0 \cup \{1/n\}_{n \in \mathbf{N}} \subset \mathbf{C}$ endowed with the subspace topology.*

If two coordinates λ_i, λ_j of a semi-rational point λ meet the hypothesis of lemma 3, then the ideal boundary of the generic leaf of $\mathcal{F}_{a,\lambda}$ presents a distinguished point $B'(\mathcal{L})$, corresponding to 0 in terms of the preceding corollary, representing the not planar end of $\mathcal{L}$. In such a case, $\mathcal{L}$ is homeomorphic to the LOCH NESS monster punctured in a countable set of points. In a semi-rational point λ, such two coordinates always exist in the complement of the set:

$$\lambda_j \in \mathbf{Z},\ \forall j \leq s \quad \text{and} \quad \lambda_j + \lambda_k \in K\mathbf{Z},\ \forall s+1 \leq j, k. \tag{10}$$

Lemma 5. *If a semirational point $\lambda = (\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_{\nu+1})$ satisfies (10), then the generic leaf of $\mathcal{F}_{a,\lambda}$ is homeomorphic to a* LOCH NESS *monster puntured in a countable set of points.*

Proof. First we prove the lemma in the illustrative case $\nu = 2$. We assume that $(a_1, a_2, a_3) = (0, 1, \infty)$. Let $W := \mathbf{P}^1(\mathbf{C}) \setminus ([\infty, 0] \cup [1, \infty])$. From condition $\lambda_1 + \lambda_2 + \lambda_3 = K$, we deduce that it is sufficient to consider the action of the holonomy generators $z \rightsquigarrow e^{2\pi i \lambda_j / K} z$, $j = 2, 3$ to reconstruc the generic leaf $\mathcal{L}$ from W. We identify W with figure 1

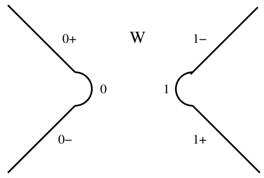

Figure 1.

Then, the generic leaf $\mathcal{L}$ is homeomorphic to

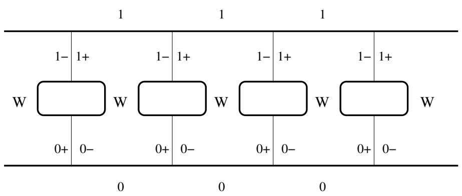

Figure 2.

which clearly shows that $\mathcal{L}$ is homeomorphic to $\mathbf{C} \setminus \mathbf{Z}[i]$. When ν takes values greater than 2 but $\nu - s$ remains equal to 2, a new discrete set of punctures appears in the leaf $\mathcal{L}$, for the image on every generator γ_j, $j \leq s$, in $Diff(\mathbf{C}, 0)$ is trivial.

We remark that, if $\nu - s > 1$, there is no semirational point λ satisfying (10). Indeed, in such a case set $\lambda_j = p_j - \lambda_{s+1}$, $p_j \in \mathbf{Z}$, for all $s + 1 < j$. Then $\sum_{j=1}^{\nu+1} = K$ becomes

$$(1 - (\nu - s))\lambda_{s+1} = K',$$

for a certain $K' \in \mathbf{Z}$. This is a contradiction, for λ is semirational. $\square$

4. Applications to polygonal billiards

Let $\lambda_1 \pi, \ldots, \lambda_{\nu+1} \pi$ be the angles of the non degenerated $\nu + 1$-side polygon P, $\nu \geq 2$. That is, $0 < \lambda_j < 2$, for all $j = 1, \ldots, \nu + 1$ and $\sum_{j=1}^{\nu+1} \lambda_j = \nu - 1$. Let $\hat{n} := (n_1, \ldots, n_{\nu+1}) \in \mathbf{Z}^{\nu+1}$ and $\psi(\hat{n}) := \sum_{j=1}^{\nu+1} n_j \lambda_j$. We define $G := \psi^{-1}(\mathbf{Z})$ and $\widetilde{\psi}(g) := e^{\frac{2\pi i}{\nu - 1}\phi(g)}$, for every

$g \in G$. Then $\widetilde{G} := \widetilde{\psi}(G)$ is a subgroup of $G_{\nu-1} := \{z \in \mathbf{C} \mid z^{\nu-1} = 1\}$. Polygons for which $\widetilde{G} = \{1\}$ are called *reasonable*. For example, all triangles are reasonable. Rectangles are *not reasonable*. This nomenclature comes from the same reference as the following

Theorem 4. [7] *Let P be a reasonable polygon. Then, there exist $a \in \mathbf{C}^{\nu+1}$ such that the translation surface $S(P)$ and the generic leaf of $\mathcal{F}_{a,\lambda}$ are isomorphic* as translation surfaces.

The determination of the parameter $a \in \mathbf{C}^{\nu+1}$ is a SCHWARZ-CHRISTOFFEL parameter problem [3]. Above, the translation surface structure of a leaf $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ is defined by the restriction of a holomorphic vector field in $\mathbf{C}^2$ generating $Ker(\omega_{a,\lambda}(p))$, for every $p \in \mathbf{C}^2$. The action:

$$\mathbf{C}^2 \times \widetilde{G} \ni ((z_1, z_2), \tilde{g}) \rightsquigarrow \tilde{g}(z_1, z_2) \tag{11}$$

has no fixed points except for the origin. It leaves invariant each connected component in the "fibers" of the first integral $F_{a,\lambda}$. We denote $\mathbf{C}^2_\lambda$ the quotient of $\mathbf{C}^2 \setminus \{z_1 = z_2 = 0\}$ by the preceding action. Let $\pi_\lambda : \mathbf{C}^2 \longrightarrow \mathbf{C}^2_\lambda$ be the natural projection. Then, $\{\pi_\lambda(\mathcal{L}) \mid \mathcal{L} \in \mathcal{F}_{a,\lambda}\}$ is a non singular foliation on $\mathbf{C}^2_\lambda$. We denote it $\widehat{\mathcal{F}_{a,\lambda}}$. The image by π_λ of a generic leaf in $\mathcal{F}_{a,\lambda}$ is called generic as well.

Theorem 5. [7] *Let P be a not reasonable. Then, there exist $a \in \mathbf{C}^{\nu+1}$ such that the translation surface $S(P)$ and the generic leaf of the foliation $\widehat{\mathcal{F}_{a,\lambda}}$ are isomorphic* as translation surfaces.

Here the translation surface structure any generic leaf in $\widehat{\mathcal{F}_{a,\lambda}}$ is inherited from any generic leaf in $\mathcal{F}_{a,\lambda}$ contained in its preimage via the projection π_λ. By definition, if $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ is generic, then $\pi_{\lambda\ |\mathcal{L}}$ is a finite covering. The monodromy group of this covering is isomorphic to $\widetilde{G}$.

Corollary 2. *Let P be a non degenerated polygon of angles $\lambda_1\pi, \ldots \lambda_{\nu+1}\pi$* without vertices.

(1) *If λ is algebraic, then $S(P)$ is homeomorphic to a compact* RIEMANN *surface punctured in a finite set of points.*
(2) *If λ is semi-rational, then $S(P)$ is homeomorphic to a* LOCH NESS *monster punctured in a countable set of points.*
(3) *If λ is not rational, then $S(P)$ is homeomorphic to a* LOCH NESS *monster.*

Certanly, since we are dealing with non-degenerated polygons, there is no λ in this case satisfying the conditions (10). For (2)-(3)

above, it is sufficient to remark that, if M is a LOCH NESS monster and $M \longrightarrow N$ is a covering map whose fibers are finite, then N is also a LOCH NESS monster.

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