Global smoothing of Calabi-Yau threefolds

Yoshinori Namikawa,
J.H.M. Steenbrink

Department of Mathematics
Sophia University
Kioi-cho, Tokyo, 102
JAPAN

Mathematical Institute
University of Nijmegen
Toernooiveld
6525 ED Nijmegen
THE NETHERLANDS

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI/95-81

Global smoothing of Calabi-Yau threefolds

Yoshinori Namikawa and J.H.M. Steenbrink

Introduction

Friedman [Fr] has studied the relationship between local and global deformations of a threefold Z with isolated hypersurface singularities which admits small resolutions. One of his main results is as follows. Let Z be a Moishezon threefold with only ordinary double points $\{p_1, ..., p_n\}$. Assume that the canonical line bundle K_Z of Z is trivial. Let $\pi : \tilde{Z} \longrightarrow Z$ be a small resolution and let $C_i := \pi^{-1}(p_i) \cong \mathbf{P}^1$ be the exceptional curves. Then he showed that if there is a relation $\Sigma_{1 \leq i \leq n} \alpha_i [C_i] = 0$ with $\alpha_i \neq 0$ for all i in $H_2(\tilde{Z}, \mathbf{C})$ and if the Kuranishi space Def(Z) of Z is smooth, then there is a global smoothing of Z by a flat deformation, that is, there is a proper flat map $f : \mathcal{Z} \longrightarrow \Delta^1$ from an analytic space $\mathcal{Z}$ to a 1-dimensional disc Δ^1 such that $f^{-1}(0) = Z$ and that $f^{-1}(t)$ is a smooth threefold for $t \neq 0$. On the other hand, Clemens has compared the topology of $\tilde{Z}$ with that of $Z_t = f^{-1}(t)$ in [Cl]. We have a simple relation $e(\tilde{Z}) = e(Z_t) + 2n$ for the Euler numbers. However, the relations between Betti numbers are not so simple; there is a phenomenon called the *defect of singularities*. (See also [W], [Di].)

One can observe from these results that local deformations of singularities (Z, p_i) are not independent in global deformations of Z. The purpose of this paper is to generalize the above results to the case where Z has more general isolated hypersurface singularities which do not necessarily have small resolutions, and to clarify the mechanism of the dependence and the defect of singularities. We can recognize a special importance in studying such things for Calabi-Yau threefolds in the works of several people (cf. [H], [G-H], [W], [Wi], [Re 2]). We shall explain our results in more detail. Let Z be a complete algebraic variety with only isolated rational singularities. Let Weil(Z) (resp. Cart(Z)) be the group of Weil divisors of Z (resp. Cartier divisors of Z). Then the abelian group Weil(Z)/Cart(Z) is finitely generated (cf. [Ka 1, Lemma (1.1)]). We denote by $\sigma(Z)$ the rank of this group. When $\sigma(Z) = 0$, Z is called $\mathbf{Q}$-factorial. In this paper, by a Calabi-Yau threefold, we mean a projective threefold Z with only rational singularities, and with $K_Z \sim 0$. Note that there is an example of a Calabi-Yau threefold Z with one ordinary point where Z remains singular under any flat deformation ([Na, (5.8)]). This example suggests that some global condition is needed for Z to be smoothable. The notion of $\mathbf{Q}$-factoriality is nothing but this global condition, and it also has a deep connection with the defect of singularities in a smoothing. Our main results are the following.

Theorem(1.3) *Let Z be a* **Q**-*factorial Calabi-Yau threefold which admits only isolated rational hypersurface singularities. Then Z can be deformed to a smooth Calabi-Yau threefold.*

Theorem(2.4) *Let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities. Then Z can be deformed to a Calabi-Yau threefold with only ordinary double points.*

Theorem(3.2) *Let Z be a normal projective threefold with only isolated rational hypersurface singularities such that $H^2(Z, \mathcal{O}_Z) = 0$. Let $b_i(Z)$ denote the i-th Betti number for the singular cohomology of Z. Then $\sigma(Z) = b_4(Z) - b_2(Z)$. Moreover, if Z has a smoothing $f : \mathcal{Z} \longrightarrow \Delta^1$, then we have*

$$\sigma(Z) = b_3(Z) + \sum_{p \in Sing(Z)} m(p) - b_3(\mathcal{Z}_t)$$

for $t \in \Delta^1 - \{0\}$, where $m(p)$ denotes the Milnor number of the singularity (Z, p).

Theorem(1.3) is closely related to the classification theory of algebraic threefolds. In fact, let Y be a smooth projective threefold with Kodaira dimension zero. By the theory of minimal models in dimension 3 (cf. [Mo], [Ka 3]), Y is birationally equivalent to a normal projective threefold W with only terminal singularities such that $mK_W \sim 0$ for a positive integer m. We take the index 1-cover $\tau : Z' \longrightarrow W$ (cf. [K-M-M, 0-2-5]). Here Z' is a Calabi-Yau threefold with only terminal singularities, and τ is a finite morphism which is etale outside $Sing(W)$. We can take a **Q**-factorial Calabi-Yau threefold Z in such a way that Z is birational equivalent to Z' (cf. [Ka 1, (4.5)]). By Theorem(1.3), Z can be deformed to a smooth Calabi-Yau threefold Z_t. Then Y inherits some nice properties from Z_t through this construction. For example, as pointed out by Kollar in the preprint version of [Ko], we can prove that $\pi_1(Y)$ has a finite index Abelian subgroup by using the Bogomolov decomposition of Z_t (cf. [Be]). As a consequence, we have a generalization of the Bogomolov decomposition to a smooth projective threefold with Kodaira dimension zero:

Corollary(1.4)(Kollár) *Let Y be a smooth projective threefold with Kodaira dimension 0. Assume that $\pi_1(Y)$ has infinite order. Then Y has a finite etale cover $\pi : V \longrightarrow Y$ such that V is birationally equivalent to an abelian threefold or the product of a K3 surface and an elliptic curve.*

The proofs of (1.3) and (2.4) are both based on the fact that the Kuranishi space Def(Z) of Z is smooth (cf. [Na, Theorem A], [Ra], [Ka 4]). In this paper, we shall introduce two different approaches to the smoothing problem; one of them uses the vanishing theorem of Guillén, Navarro Aznar and Puerta (cf. [St 3]) which is a generalization of Kodaira-Akizuki-Nakano vanishing theorem, and another one uses the invariant μ introduced in [Na §5] for an isolated rational singularity. A merit of the first approach is that we can find a smoothing direction in Def(Z) in one step. But this approach cannot be applied to a non-**Q**-factorial Calabi-Yau threefold. On the other hand, if we employ the second approach, then we need some induction steps with respect to the invariant μ to find out a suitable smoothing direction in Def(Z). However, we can prove both theorems (1.3) and (2.4) by this method.

In §1 we shall prove Theorem(1.3) by the first method. The key result is the following theorem which is proved by using the vanishing theorem of Guillén, Navarro Aznar and Puerta:

Theorem(1.1) *Let (X, x) be an isolated singularity of a complex space, and let $\pi : Y \longrightarrow X$ be a resolution of X such that its exceptional divisor E has only normal crossings. Let $U = X \setminus \{x\}$. Then we have a natural map $\tau : H^1(U, \Omega^2) \to H^2_E(Y, \Omega^2_Y)$ as a coboundary map of the exact sequence of local cohomology.*

Suppose that (X, x) is a 3-dimensional isolated Gorenstein Du Bois singularity for which τ is the zero map. Then (X, x) is rigid.

Note here that $H^1(U, \Omega^2_U) \cong H^0(X, T^1_X)$ by Schlessinger [Sch, Theorem 2] if (X, x) is an isolated hypersurface singularity of dimension ≥ 3. Going back to the original situation, we let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities, and let Y be a resolution of Z. For each singularity $x \in Z$, we have the same map $\tau_x : T^1_{Z,x} \to H^2_{E_x}(Y, \Omega^2_Y)$ as above, where E_x is the exceptional divisor over $x \in Z$. Take an arbitrary smoothing direction $\zeta \in H^0(Z, T^1_Z)$. Then, by using the $\mathbf{Q}$-factoriality of Z, one can find an element $\eta \in \bigoplus_{x \in Sing(Z)} \ker(\tau_x)$ such that $\zeta + \eta$ comes from $Ext^1(\Omega^1_Z, \mathcal{O}_Z)$. By combining Theorem(1.1) and some results concerning the discriminant of the semi-universal deformation space of a hypersurface singularity, we see that $\eta_x \in T^1_{Z,x}$ is contained in the tangent cone of the discriminant locus for every $x \in Sing(Z)$. Thus, we have been able to find a smoothing direction in $Ext^1(\Omega^1_Z, \mathcal{O}_Z)$.

In §2 we shall prove Theorem(2.4) by the second method. Let (X, x) be a rational isolated singularity, and let $\pi : Y \to X$ be its resolution. Then $\mu(X, x)$ is defined to be the dimension of the cokernel of the map $(2\pi i)^{-1} d\log : H^1(Y, \mathcal{O}^*_Y) \otimes_{\mathbf{Z}} \mathbf{C} \to H^1(Y, \Omega^1_Y)$. We shall prove that, for a 3-dimensional isolated rational hypersurface singularity (X, x), $\mu(X, x) = 0$ if and only if (X, x) is a smooth point or an ordinary double point (cf. Theorem(2.2)). The proof uses the theory of spectrum of a hypersurface singularity developed by Arnold, Steenbrink, Varchenko, Morihiko Saito and others. The proof of Theorem(2.4) goes as follows. Assume that there is a singularity with $\mu > 0$ on a given Calabi-Yau threefold. Then one can find a small deformation of Z so that the lying singularity becomes better in the following sense (Proposition(2.3)): for any resolution Y of Z, this small deformation is outside the image of the map Def$(Y) \to$ Def(Z). By some inductive process, Z is eventually deformed to a Calabi-Yau threefold whose singularities all have $\mu = 0$. This implies that this Calabi-Yau actually has only ordinary double points.

In the final section, Theorem(3.2) is proved, and at the same time, we consider the Hodge theoretic meaning of a smoothing in dimension 3. For example, the following theorem is proved.

Corollary(3.13) *Let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities. Then Z can be deformed to a Calabi-Yau threefold Y with only ordinary double points whose cohomology groups $H^i(Y)$ $(0 \leq i \leq 6)$ have a pure Hodge* *structure* .

The arguments here are more or less standard. In particular, Theorem(3.2) follows immediately from a theorem of Goresky-MacPherson. The assumption that $\dim Z = 3$ is essential.

§1.

Let (X, x) be an isolated singularity of a complex space. Let X be a good representative for this germ and let $U = X \setminus \{x\}$. Let $\pi : Y \to X$ be a resolution of X such that $\pi^{-1}(U) \cong U$ and its exceptional divisor E has simple normal crossings. Identifying $\pi^{-1}(U)$ with U, we have a natural map $\tau : H^1(U, \Omega^2_U) \to H^2_E(\Omega^2_Y)$ as the coboundary map of the exact sequence of local cohomology. We claim the following

Theorem(1.1). *Suppose that (X, x) is a 3-dimensional isolated Gorenstein Du Bois singularity for which τ is the zero map. Then (X, x) is rigid.*

Proof. First observe that, as X is Gorenstein, (X, x) is rigid if and only if $H^1(U, \Omega^2_U) = 0$ by Schlessinger [Sch, Theorem 2]. We can factorize τ via $H^2_E(\Omega^2_Y(\log E)(-E))$. As $H^2(\Omega^2_Y(\log E)(-E)) = 0$ by the vanishing theorem of Guillén, Navarro Aznar and Puerta (cf.[St 3]), the map $H^1(U, \Omega^2_U) \to H^2_E(\Omega^2_Y(\log E)(-E))$ is surjective. Define

$$\omega^p_E := \Omega^p_E \text{ mod torsion } \simeq \Omega^p_Y / \Omega^p_Y(\log E)(-E).$$

Then we have the exact sequence

$$H^1(E, \omega^2_E) \xrightarrow{\alpha} H^2_E(\Omega^2_Y(\log E)(-E)) \to H^2_E(\Omega^2_Y)$$

and α factorizes via

$$H^1(E, \omega^2_E) \xrightarrow{\alpha'} H^1(E, \Omega^2_Y(\log E) \otimes \mathcal{O}_E)$$

which is to be interpreted as the natural map $Gr^2_F H^3(E, \mathbf{C}) \to Gr^2_F H^4_{\{x\}}(X, \mathbf{C})$; by semipurity α' is the zero map (see [St 2, Theorem 1.11]). Hence α is the zero map, and $H^2_E(\Omega^2_Y(\log E)(-E)) \to H^2_E(\Omega^2_Y)$ is injective. So we have proved that

$$\mathrm{im}(\tau) \cong H^2_E(\Omega^2_Y(\log E)(-E)).$$

As (X, x) is a Gorenstein Du Bois singularity, we have that $H^i(Y, \mathcal{O}_Y(-E)) = 0$ for $i = 1, 2$. We consider the spectral sequence of hypercohomology

$$E^{pq}_1 = H^q(Y, \Omega^p_Y(\log E)(-E)_{|E}) \Rightarrow 0$$

where the abutment 0 follows from the fact that for each point $y \in E$ the complex $\Omega^\bullet_Y(\log E)(-E)$ is acyclic. By the vanishing theorem quoted above, the only possibly nonzero terms in E_1 are E^{p0}_1 for all p and $E^{1,1}_1$, $E^{1,2}_1$ and $E^{2,1}_1$. As the sequence converges to 0, we have $E^{1,2}_1 = H^2(Y, \Omega^1_Y(\log E)(-E)) = 0$ and the map $d_1 : E^{1,1}_1 \to E^{2,1}_1$, i.e.

$$H^1(Y, \Omega^1_Y(\log E)(-E)) \to H^1(Y, \Omega^2_Y(\log E)(-E))$$

is surjective.

So suppose that τ is the zero map. Then $H^2_E(\Omega^2_Y(\log E)(-E)) = 0$. We have the surjection

$$H^2_E(\Omega^2_Y(\log E)(-E)) \to H^2_E(\Omega^2_Y(\log E))$$

as $H^2(E, \Omega^2_Y(\log E) \otimes \mathcal{O}_E) = Gr^2_F H^5_{\{x\}}(X, \mathbf{C}) = 0$. Hence also $H^2_E(\Omega^2_Y(\log E)) = 0$. By duality we get $H^1(Y, \Omega^1_Y(\log E)(-E)) = 0$ and hence by the remark above $H^1(Y, \Omega^2_Y(\log E)(-E)) = 0$. We have the exact sequence

$$H^1(Y, \Omega^2_Y(\log E)(-E)) \to H^1(U, \Omega^2_U) \to H^2_E(\Omega^2_Y(\log E)(-E))$$

hence $H^1(U, \Omega^2_U) = 0$. This means that (X, x) is rigid.

Remark τ is a homomorphism of $\mathcal{O}_{X,x}$-modules, so in general $\ker(\tau)$ is an $\mathcal{O}_{X,x}$-submodule of $H^1(U, \Omega^2_U)$. By the proof above, $\dim \ker(\tau) \leq \dim \mathrm{im}(\tau)$.

Let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities. We assume that Z has at least one singular point. If $H^1(Z, \mathcal{O}_Z) \neq 0$, then the Albanese map $Z \to Alb(Z)$ is a fiber bundle by Kawamata [Ka 2]. Since Z has only isolated singularities, this implies that Z is smooth. Thus, we can assume from the start that $H^1(Z, \mathcal{O}_Z) = 0$. Let $\pi : Y \to Z$ be a good resolution of Z. Then there is an injection $T^0_Z \to \pi_* \Omega^2_Y$. Let $x \in Z$ be a singular point, and let E_x be the exceptional divisor of π over x. We denote by ϕ_x the composition of the maps $H^2_{\{x\}}(Z, T^0_Z) \to H^2_{\{x\}}(Z, \pi_*\Omega^2_Y) \to H^2_{E_x}(Y, \Omega^2_Y)$. Let $\iota : H^2_{E_x}(Y, \Omega^2_Y) \to H^2(Y, \Omega^2_Y)$ be the natural map.

Proposition(1.2) *Assume that Z is $\mathbf{Q}$-factorial. Then $\iota \circ \phi_x : H^2_{\{x\}}(Z, T^0_Z) \to H^2(Y, \Omega^2_Y)$ is the zero map.*

Proof. Take a sufficiently small open neighborhood Z_x of $x \in Z$, and put $U_x = Z_x \setminus \{x\}$ and $Y_x = \pi^{-1}(Z_x)$. Since $H^1(U_x, \Omega^2_{U_x}) \cong H^1(U_x, \Theta_U) \cong H^2_{\{x\}}(Z_x, T^0_{Z_x})$ by Schlessinger [Sch Theorem 2], ϕ_x is identified with the coboundary map $H^1(U_x, \Omega^2_{U_x}) \to H^2_{E_x}(Y_x, \Omega^2_{Y_x})$ of the exact sequence of local cohomology. Here U_x and $\pi^{-1}(U_x)$ are identified. Thus, the map $\iota \circ \phi_x$ is identified with the composition of the following maps:

$$H^1(U_x, \Omega^2_{U_x}) \longrightarrow H^2_{E_x}(Y_x, \Omega^2_{Y_x}) \longrightarrow H^2(Y, \Omega^2_Y)$$

Since the natural map $H^1(Y, \Omega^1_Y) \to H^1(Y_x, \Omega^1_{Y_x})$ is the dual map of ι, it suffices to show that the composition of the maps:

$$H^1(Y, \Omega^1_Y) \longrightarrow H^1(Y_x, \Omega^1_{Y_x}) \longrightarrow H^1(U_x, \Omega^1_{U_x})$$

is the zero map. Consider the commutative diagram

$$\begin{array}{ccc} H^1(Y, \mathcal{O}^*_Y) \otimes_{\mathbf{Z}} \mathbf{C} & \longrightarrow & H^1(U_x, \mathcal{O}^*_{U_x}) \otimes_{\mathbf{Z}} \mathbf{C} \\ (2\pi i)^{-1} d\log \downarrow & & \downarrow (2\pi i)^{-1} d\log \\ H^1(Y, \Omega^1_Y) & \longrightarrow & H^1(U_x, \Omega^1_{U_x}) \end{array}$$

The vertical map on the left-hand side is an isomorphism by Hodge theory because $H^1(Z,\mathcal{O}_Z) = H^2(Z,\mathcal{O}_Z) = 0$. The top horizontal map is the zero map by [K-M, 12.1.6] since Z is $\mathbf{Q}$-factorial. Hence the map $H^1(Y,\Omega^1_Y) \to H^1(U_x,\Omega^1_{U_x})$ is the zero map. Q.E.D.

Theorem (1.3). *Let Z be a Calabi-Yau threefold which is $\mathbf{Q}$-factorial and whose singular points are all isolated hypersurface singularities. Then Z is smoothable.*

Proof. Let Σ denote the set of singular points of Z. Let U denote the regular locus of Z and let $\pi : Y \to Z$ denote a good resolution of Z. Choose contractible mutually disjoint neighborhoods Z_x for all points $x \in \Sigma$, put $Y_x = \pi^{-1}(Z_x)$ and $U_x = Z_x \setminus \{x\}$. Finally let $E = \pi^{-1}(\Sigma)$. One has the map

$$\tau = \oplus_{x\in\Sigma}\tau_x : \oplus_{x\in\Sigma}H^1(U_x,\Omega^2_{U_x}) \to H^2_E(Y,\Omega^2_Y)$$

whose composition with $H^2_E(Y,\Omega^2_Y) \to H^2(Y,\Omega^2_Y)$ is the zero map by Proposition (1.2). Since Z_x is a Stein open neighborhood of an isolated hypersurface singularity, we have

$$H^1(U_x,\Omega^2_{U_x}) \cong H^1(U_x,\Theta_{U_x}) \cong T^1_{Z,x}$$

by Schlessinger [Sch, Theorem 2].

As all singularities of Z are non-rigid, all maps τ_x are non-zero. Hence their kernels are proper submodules of the cyclic modules $T^1_{Z,x}$. The tangent cone to the discriminant in the semi-universal deformation of each singular point of Z is the linear space which corresponds to the maximal submodule of $T^1_{Z,x}$ (see [Te], p. 653), hence it contains $\ker(\tau_x)$ for each $x \in Z$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(U,\Omega^2_U) & \xrightarrow{\gamma} & H^2_E(Y,\Omega^2_Y) & \longrightarrow & H^2(Y,\Omega^2_Y) \\ \| & & \uparrow \oplus\phi_x & & \\ H^1(U,\Theta_U) & \xrightarrow{\alpha} & \oplus_{x\in\Sigma} H^2_x(Z,T^0_Z) \cong H^0(Z,T^1_Z) & & \end{array}$$

Note here that the composition of the maps: $\bigoplus_{x\in\Sigma}H^2_{\{x\}}(Z,T^0_Z) \cong \bigoplus_{x\in\Sigma}H^1(U_x,\Theta_{U_x}) \cong \bigoplus_{x\in\Sigma}H^2(U_x,\Omega^2_{U_x}) \xrightarrow{\tau} H^2_E(Y,\Omega^2_Y)$ coincides with $\bigoplus\phi_x$. Choose a smoothing direction $\zeta \in \oplus_{x\in\Sigma}H^2_{\{x\}}(Z,T^0_Z)$ and let ζ' denote its image in $H^2_E(Y,\Omega^2_Y)$; this maps to 0 in $H^2(Y,\Omega^2_Y)$, hence ζ' is of the form $\gamma(\eta)$ for some $\eta \in H^1(U,\Omega^2_U)$. Then the image $\alpha(\eta)$ of η in $\oplus_{x\in\Sigma}H^1(U_x,\Omega^2_{U_x})$ is a smoothing direction at every point. In fact, by definition, $\alpha(\eta) - \zeta \in \bigoplus_{x\in\Sigma}\ker(\phi_x)$. By the above observations, every element of $\ker(\phi_x)$ is contained in the tangent cone of the discriminant locus of $\mathrm{Def}(Z_x)$. This implies that $\alpha(\eta)$ is a smoothing direction. Q.E.D.

Corollary (1.4). *Let Y be a smooth projective threefold with Kodaira dimension $\kappa(Y) = 0$. Assume that $\pi_1(Y)$ has an infinite order. Then Y has a finite etale cover $\pi : V \to Y$ such that V is birationally equivalent to an abelian threefold or the product of a K3 surface and an elliptic curve.*

Proof. We first prove that $\pi_1(Y)$ has a finite index abelian subgroup. By the theory of minimal models ([Mo], [Ka 3]), Y is birationally equivalent to a normal projective threefold W with only terminal singularities such that $mK_W \sim 0$ for some positive integer m. Take the index 1-cover $\tau : Z' \to W$. Here τ is a finite morphism which is an etale morphism outside $Sing(W)$, and Z' is a Calabi-Yau threefold with only terminal singularities (cf. [K-M-M, 0-2-5]). By [Ka 1, 4.5] there are a $\mathbf{Q}$-factorial Calabi-Yau threefold Z with only terminal singularities, and a birational morphism $g : Z \to Z'$ which is an isomorphism in codimension 1. Note that a 3-dimensional Gorenstein terminal singularity is an isolated cDV point, and hence, it is an isolated rational hypersurface singularity. Thus, we have a smoothing $f : \mathcal{Z} \to \Delta^1$ of Z by Theorem (1.3). Let Z_t be a general fiber of f. Then Z_t is a smooth Calabi-Yau threefold. By the Bogomolov decomposition theorem (cf. [Be]), Z_t is a finite etale quotient of one of the following three types: an abelian threefold; the product of a K3 surface and an elliptic curve; a simply connected threefold. This implies that $\pi_1(Z_t)$ has a finite index abelian subgroup. There is a sequence of homomorphisms of fundamental groups:

$$\pi_1(Y) \cong \pi_1(W) \overset{\tau_*}{\leftarrow} \pi_1(Z') \overset{g_*}{\leftarrow} \pi_1(Z).$$

We have the first isomorphism by using a smooth threefold which birationally dominates both Y and W. Since τ is a finite morphism, the image of τ_* is a finite index subgroup of $\pi_1(W)$ by [Ko, 2.9]. By [Ko, 7.8], g_* is an isomorphism because both Z and Z' have only terminal singularities. There is a surjection $\pi_1(Z_t) \to \pi_1(Z)$ (cf. [Ko, 5.2.2]). Since $\pi_1(Z_t)$ has a finite index abelian subgroup, we see that $\pi_1(Y)$ also has a finite index abelian subgroup by the above observation.

We take an etale cover $\pi : V \to Y$ corresponding to the finite index abelian subgroup of $\pi_1(Y)$. Then $\pi_1(V)$ is an infinite abelian group by the assumption. Since π is an etale cover and $\kappa(Y) = 0$, we have $\kappa(V) = 0$. Now the result follows from the classification theory of threefolds with $\kappa = 0$ and $q > 0$ (cf. [Ka 2]).

§2

Let (X, x) be an isolated rational singularity of a complex space. Let $\pi : Y \to X$ be a resolution of X. Then the invariant $\mu(X, x)$ is defined as the codimension in $H^1(Y, \Omega^1_Y)$ of the image of the map

$$(2\pi i)^{-1} d\log : H^1(Y, \mathcal{O}^*_Y) \bigotimes_{\mathbf{Z}} \mathbf{C} \to H^1(Y, \Omega^1_Y).$$

Note that $\mu(X, x)$ is independent of the choice of the resolution by [Na, §5].

Proposition (2.1). *Let (X, x) be a rational isolated singularity and let $(Y, E) \to (X, x)$ be a good resolution. Then*

$$\mu(X, x) = \dim H^1(Y, \Omega^1_Y(\log E)(-E)).$$

Proof. As X has a rational singularity, $H^1(Y, O_Y) = 0 = H^2(Y, O_Y)$. Therefore $H^1(Y, O_Y^*) \simeq H^2(Y, \mathbf{Z}) \simeq H^2(E, \mathbf{Z})$. Also, $H^2(E, O_E) = 0$ hence $H^2(E, \mathbf{Z})$ is a pure Hodge structure of type $(1,1)$, with $Gr_F^1 \simeq H^1(E, \Omega_Y^1/\Omega_Y^1(\log E)(-E))$. As also $H^1(E, O_E) = 0$ we have $H^1(E, \mathbf{C}) = 0$ so the sequence

$$0 \to H^1(Y, \Omega_Y^1(\log E)(-E)) \to H^1(Y, \Omega_Y^1) \to H^2(E, \mathbf{C}) \to 0$$

is exact. The composition

$$H^1(Y, O_Y^*) \to H^1(Y, \Omega_Y^1) \to H^2(E, \mathbf{C})$$

is just the natural map $H^2(E, \mathbf{Z}) \to H^2(E, \mathbf{C})$. This proves the claim.

Theorem (2.2). *Let (X, x) be an isolated hypersurface singularity of dimension three which is rational and not an ordinary double point. Then $\mu(X, x) > 0$.*

Proof. Suppose that $\mu(X, x) = 0$. Then $H^1(Y, \Omega_Y^1(\log E)(-E)) = 0$ by Proposition (2.1). In the proof of Theorem (1.1), we have shown that $d : H^1(Y, \Omega_Y^1(\log E)(-E)) \to H^1(Y, \Omega_Y^2(\log E)(-E))$ is surjective. This implies that $H^1(Y, \Omega_Y^2(\log E)(-E)) = 0$. Hence the map $\tau : H^1(U, \Omega_U^2) \to H_E^2(Y, \Omega_Y^2(\log E)(-E))$ is an isomorphism by the exact sequence of local cohomology because $H^2(Y, \Omega_Y^2(\log E)(-E)) = 0$ by the vanishing theorem of Guillén, Navarro Aznar and Puerta (cf. [St 3]). Consider the exact sequence

$$0 = H^1(Y, \Omega_Y^1(\log E)(-E)) \to H^1(Y, \Omega_Y^1(\log E)) \to H^1(E, \Omega_Y^1(\log E) \otimes \mathcal{O}_E).$$

By duality, $h^1(Y, \Omega_Y^1(\log E)) = h_E^2(Y, \Omega_Y^2(\log E)(-E)) = \dim_{\mathbf{C}} T_X^1$. Since

$$H^1(E, \Omega_Y^1(\log E) \otimes \mathcal{O}_E) = Gr_F^1 H_{\{x\}}^3(X, \mathbf{C}),$$

we have $\dim_{\mathbf{C}} T_X^1 \leq \dim_{\mathbf{C}} H_{\{x\}}^3(X, \mathbf{C})$ by the exact sequence.

Now let $f = 0$ be a defining equation for X in $\mathbf{C}^4$. Let X_f denote the Milnor fibre of f and let T be the monodromy transformation of $H^3(X_f, \mathbf{C})$. Let T_s be the semi-simple part of T and define $H^3(X_f, \mathbf{C})_1 = \ker(T_s - I)$.

Claim 1. *All Jordan blocks of T for eigen-value 1 have size 1. Moreover, $dim_{\mathbf{C}} H^3(X_f, \mathbf{C}_1 = dim_{\mathbf{C}} Gr_F^2 H^3(X_f, \mathbf{C})_1$.*

Proof. It suffices to show that $Gr_i^W H^3(X_f, \mathbf{C})_1 = 0$ if $i \neq 4$. In fact, W is the weight filtration of $N = \log(T)$ on $H^3(X_f, \mathbf{C})_1$, by [St1] Cor. (4.9), hence triviality of W on $H^3(X_f, \mathbf{C})_1$ implies that $T = I$ on $H^3(X_f, \mathbf{C})_1$.

We shall use the following facts (cf. [St 1,2]):

(1) $N = \log T : H^3(X_f, \mathbf{C}) \to H^3(X_f, \mathbf{C})$ is a morphism of the mixed Hodge structure of type $(-1, -1)$;

(2) $N^r : Gr_{4+r}^W H^3(X_f, \mathbf{C})_1 \cong Gr_{4-r}^W H^3(X_f, \mathbf{C})_1$ for $r \geq 0$;

(3) $\dim_{\mathbf{C}} Gr_F^i Gr_r^W H^3(X_f, \mathbf{C})_1 = \dim_{\mathbf{C}} Gr_F^{r-i} Gr_r^W H^3(X_f, \mathbf{C})_1$ for $r \geq 0$ (Hodge symmetry);

(4) Assume that (X, x) is a rational singularity. Then $Gr_F^i H^3(X_f, \mathbf{C}) \neq 0$ only if $i = 1, 2$.

For simplicity, we shall write $h_1^{i,j}$ for $\dim_{\mathbf{C}} Gr_F^i Gr_{i+j}^W H^3(X_f, \mathbf{C})_1$. By (2), we only have to show that $Gr_r^W H^3(X_f, \mathbf{C})_1 = 0$ for r = 5, 6, 7. By (4), it suffices to show that $h_1^{1,4} = h_1^{2,3} = h_1^{1,5} = h_1^{2,4} = h_1^{1,6} = h_1^{2,5} = 0$. By (2) and (4), $h_1^{1,4} = h_1^{0,3} = 0$. By (2), (3) and (4), we have $h_1^{2,3} = h_1^{1,2} = h_1^{2,1} = h_1^{3,2} = 0$. Similarly, $h_1^{1,5} = h_1^{-1,3} = 0$, $h_1^{2,4} = h_1^{0,2} = 0$, $h_1^{1,6} = h_1^{-2,3} = 0$ and $h_1^{2,5} = h_1^{-1,2} = 0$. Thus, $\dim_{\mathbf{C}} H^3(X_f, \mathbf{C})_1 = \dim_{\mathbf{C}} Gr_4^W H^3(X_f, \mathbf{C})_1$. Finally, note that $h_1^{1,3} = h_1^{3,1} = 0$. From this it follows that $\dim_{\mathbf{C}} H^3(X_f, \mathbf{C})_1 = \dim_{\mathbf{C}} Gr_F^2 H^3(X_f, \mathbf{C})_1$.

We next consider the spectrum $Sp(f)$ of f. Let m be the Milnor numer of f. Then $Sp(f)$ is a non-decreasing sequence of m rational numbers $(\alpha_1, ..., \alpha_m)$ such that the frequency n_α of $\alpha \in \mathbf{Q}$ in this set is given by the dimension of $\mathbf{C}$-vector space $Gr_F^{[3-\alpha]} H^3(X_f, \mathbf{C})_\alpha$, where $H^3(X_f, \mathbf{C})_\alpha = \{x \in H^3(X_f, \mathbf{C}); T_s(x) = \exp(-2\pi i \alpha)x\}$. As f has a rational singularity, $n_\alpha = 0$ unless $0 < \alpha < 2$ and by the claim above $n_1 = \dim \ker(T - id) = \dim \ker(j) = \dim H^3_{\{x\}}(X, \mathbf{C})$ where j is the intersection form on $H_3(X_f, \mathbf{C})$. (As for the last equality, see [St-2, (2.3)]). In the above, we have shown that $\dim_{\mathbf{C}} T_X^1 \leq n_1$. On the other hand, we have the following

Claim 2. $\dim T_X^1 \geq \sum_{\alpha \leq 1} n_\alpha$.

Proof. Let Q_f be the Jacobian ring of f. Then we have an isomorphism $T_X^1 \cong Q_f / f Q_f$. By [S-S, §7., p.656], we have the filtration V on Q_f indexed by rational numbers such that $\dim_{\mathbf{C}} V_\alpha / V_{>\alpha} = n_\alpha$. By the proof of Theorem (7.1) in [S-S], the multiplication by f on Q_f maps V_α to $V_{\alpha+1}$. For an isolated rational hypersurface f, $n_\alpha = 0$ for $\alpha \leq 0$. Hence $f Q_f \subset V_\beta Q_f$, where β is the minimal spectrum number greater than 1. Thus, we have the inequality $\dim_{\mathbf{C}} T_X^1 \geq \dim_{\mathbf{C}}(Q_f / V_\beta Q_f) = \sum_{\alpha \leq 1} n_\alpha$.

Combining Claim 2 with the above observation, one has $n_\alpha = 0$ for $\alpha \neq 1$, i.e. T is the identity. This implies that X is an ordinary double point by A'Campo [AC]. Q.E.D.

Proposition (2.3) *Let Z be a Calabi-Yau threefold with $H^1(Z, \mathcal{O}_Z) = 0$ which admits only isolated rational hypersurface singularities. Let $\pi : Y \to Z$ be a resolution of Z. Let p_i $(1 \leq i \leq n)$ be the singular points on Z which are not ordinary double points, and let E_i be the exceptional divisor over p_i. Let Z_i be mutually disjoint, contractible Stein open neighborhoods of $p_i \in Z$. Set $Y_i = \pi^{-1}(Z_i)$. Consider the diagram*

$$Ext^1(\Omega_Z^1, \mathcal{O}_Z) \stackrel{\alpha}{\to} \bigoplus_{1 \leq i \leq n} H^0(Z_i, T_{Z_i}^1) \stackrel{\oplus_{1 \leq i \leq n} \beta_i}{\longleftarrow} \bigoplus_{1 \leq i \leq n} H^1(Y_i, \Theta_{Y_i}).$$

Then there is an element $\eta \in Ext^1(\Omega_Z^1, \mathcal{O}_Z)$ such that $\alpha(\eta)_i \notin \mathrm{im}(\beta_i)$ for all i. Moreover, when Z is $\mathbf{Q}$-factorial, the same as above holds even if we set $Sing(Z) = \{p_1, p_2, ..., p_n\}$.

Proof. Let $Sing(Z) = \{p_1, ..., p_n, p_{n+1}, ..., p_m\}$ and let $U = Z \backslash \{p_1, ..., p_m\}$. Consider the following commutative diagram similar to that in the proof of Theorem (1.3):

$$H^1(U,\Omega^2_U) \stackrel{\gamma}{\to} H^2_E(Y,\Omega^2_Y) \longrightarrow H^2(Y,\Omega^2_Y)$$

$$\| \qquad\qquad \uparrow \bigoplus \phi_i$$

$$H^1(U,\Theta_U) \stackrel{\alpha'}{\to} \bigoplus_{1\le i\le m} H^2_{p_i}(Z,T^0_Z) \cong H^0(Z,T^1_Z)$$

Denote by ι_i the natural map $H^2_{E_i}(Y,\Omega^2_Y) \to H^2(Y,\Omega^2_Y)$. In the above diagram, ϕ_i is factorized as follows:

$$H^2_{p_i}(Z,T^1_Z) \stackrel{\phi_i'}{\to} H^2_{E_i}(Y,\Theta_Y) \to H^2_{E_i}(Y,\Omega^2_Y).$$

We shall prove that the map

$$\iota_i : H^2_{E_i}(Y,\Omega^2_Y) \to H^2(Y,\Omega^2_Y)$$

is not an injection for each $i \le n$. If this is proved, then we take a non-zero element $\zeta_i \in Ker(\iota_i)$ for each $i \le n$. By the above diagram, there is an element $\eta \in Ext^1(\Omega^1_Z,\mathcal{O}_Z)$ such that $\phi_i \circ \alpha'(\eta)_i = \zeta_i \neq 0$. In particular, we have $\phi_i' \circ \alpha'(\eta)_i \neq 0$. We then see that $\alpha(\eta)_i \notin \text{image}(\beta_i)$ by the exact sequence

$$H^1(Y_i,\Theta_{Y_i}) \stackrel{\beta_i}{\to} H^2_{p_i}(Z,T^0_Z) \stackrel{\phi_i'}{\to} H^2_{E_i}(Y,\Theta_Y).$$

We shall finish the proof by showing the following claim.

Claim *The map ι_i is not an injection for $i \le n$.*

Proof. (CASE 1: $p_i \in Z$ is not an ordinary double point)

Since $H^1(Y,\mathcal{O}_Y) = H^2(Y,\mathcal{O}_Y) = 0$, there are isomorphisms

$$H^2(Y,\Omega^2_Y) = H^1(Y,\Omega^1_Y)^* \cong (H^1(Y,\mathcal{O}^*_Y)\bigotimes_{\mathbf{Z}}\mathbf{C})^*.$$

Hence ι_i is factored as follows:

(*)

$$H^2_{E_i}(Y,\Omega^2_Y) \to (H^1(Y_i,\mathcal{O}^*_{Y_i})\bigotimes_{\mathbf{Z}}\mathbf{C})^* \to H^2(Y,\Omega^2_Y)$$

.

The first map is the dual map of $(1/2\pi i)^{-1} d\log : H^1(Y_i,\mathcal{O}^*_{Y_i})\otimes_{\mathbf{Z}}\mathbf{C} \to H^1(Y_i,\Omega^1_{Y_i})$, which is not a surjection because $\mu(Z_i,p_i) > 0$ by Theorem (2.2). Thus, ι_i is not an injection. Q.E.D.

(CASE 2: $p_i \in Z$ is an ordinary double point, and Z is $\mathbf{Q}$-factorial.)

Since Z is $\mathbf{Q}$-factorial, and Z_i is not $\mathbf{Q}$-factorial, the second map in (*) is not an injection. The map $(1/2\pi i)^{-1} d\log : H^1(Y_i,\mathcal{O}^*_{Y_i})\otimes_{\mathbf{Z}}\mathbf{C} \to H^1(Y_i,\Omega^1_{Y_i})$ is an injection by [Na, §2. CLAIM]. The first map in (*) is nothing but the dual of this map. Thus, ι_i is not an injection. Q.E.D.

Theorem (2.4). *Let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities. Then Z can be deformed to a Calabi-Yau threefold with only ordinary double points.*

Proof. Let p_i $(1 \leq i \leq n)$ be the singular points on Z which are not ordinary double points. We shall use the same notation as Proposition (2.3). Let $\mathrm{Def}(Z_i)$ be the semi-universal deformation space of Z_i and let $\mathcal{Z}_i$ be the semi-universal family over $\mathrm{Def}(Z_i)$. $\mathrm{Def}(Z_i)$ has a stratification into Zariski locally closed, smooth subsets S_i^k $(k \geq 0)$ with the following properties:

1. $\mathrm{Def}(Z_i) = \coprod_{k \geq 0} S_i^k$;
2. S_i^0 is a non-empty Zariski open subset of $\mathrm{Def}(Z_i)$, and $\mathcal{Z}_i$ is smooth over S_i^0;
3. S_i^k are of pure codimension in $\mathrm{Def}(Z_i)$ for all $k \geq 0$, and $\mathrm{codim}_{\mathrm{Def}(Z_i)} S_i^k < \mathrm{codim}_{\mathrm{Def}(Z_i)} S_i^{k+1}$;
4. If $k > l$, then $\bar{S}_i^k \cap S_i^l = \emptyset$;
5. $\mathcal{Z}_i$ has a simultaneous resolution on each S_i^k, that is, there is a resolution $\mathcal{Z}_i^k$ of $\mathcal{Z}_i \times_{\mathrm{Def}(Z_i)} S_i^k$ such that $\mathcal{Z}_i^k$ is smooth over S_i^k.

For example, we can construct such a stratification as follows. Denote by f_i the projection from $\mathcal{Z}_i$ to $\mathrm{Def}(Z_i)$. Since Z_i has an isolated singularity, the locus of $\mathcal{Z}_i$ where f_i is not smooth is finite over $\mathrm{Def}(Z_i)$. Thus, by the theorem of Sard, we can find a non-empty Zariski open subset S_i^0 of $\mathrm{Def}(Z_i)$ on which f_i is a smooth morphism. Set $F_i^0 = \mathrm{Def}(Z_i) \setminus S_i^0$. If we replace $\mathrm{Def}(Z_i)$ by a small open neighborhood of the origin, we may assume that all irreducible components contain the origin. Let $F_i^{0,j}$ be its irreducible components of maximal dimension. Take their resolutions $\hat{F}_i^{0,j}$. Then we have a flat family of isolated hypersurface singularities over $\hat{F}_i^{0,j}$ by pulling back $\mathcal{Z}_i$. The total space of this flat family admits a resolution, and it is clear by the theorem of Sard that this resolution gives a simultaneous resolution of the flat family over a non-empty Zariski open subset of $\hat{F}_i^{0,j}$. We may assume that this Zariski open subset does not have any intersection with the exceptional locus of the resolution. Take the complement of this Zariski open subset in $\hat{F}_i^{0,j}$. Then its image on $F_i^{0,j}$ becomes a Zariski closed subset because the resolution is proper. Define S_i^1 to be the complement of the union of these Zariski closed subsets and the non-maximal irreducible components in F_i^0. By definition, $\mathcal{Z}_i$ has a simultaneous resolution on S_i^1, and S_i^1 is smooth of pure codimension. Next we set $F_i^1 = F_i^0 \setminus S_i^1$, and continue the same process. Then, we eventually obtain a desired stratification.

Let us fix such a stratification for each $\mathrm{Def}(Z_i)$. The origin of $\mathrm{Def}(Z_i)$ is contained in the minimal stratum S_i^k. By definition, the flat family $\mathcal{Z}_i \times_{\mathrm{Def}(Z_i)} S_i^k \to S_i^k$ admits a simultaneous resolution. This simultaneous resolution induces a resolution $\pi_i : Y_i \to Z_i$. Since π_i is an isomorphism over smooth points of Z_i, these fit together into a global resolution $\pi : Y \to X$. We here apply Proposition (2.3). Let $g : \mathcal{Z} \to \Delta$ be a small deformation of Z determined by $\eta \in Ext^1(\Omega_Z^1, \mathcal{O}_Z)$. It determines for each i a holomorphic map $\varphi_i : \Delta \to \mathrm{Def}(Z_i)$ with $\varphi_i(0) = 0$. If $p_i \in Z$ is not an ordinary double point, then the image of φ_i is not contained in S_i^k. Moreover, if we take a general point $t \in \Delta \setminus 0$, then $\varphi_i(t) \in S_i^{k'}$ for some $k' < k$ by the property (4) of the stratification. Since $\mathrm{Def}(Z_i)$ is a versal deformation space for the singular point of $\mathcal{Z}_t$ at $\varphi_i(t)$ (cf. [Lo (6.15)]), we can continue the same process as above for $\mathcal{Z}_t$ by using $\mathrm{Def}(Z_i)$. Finally, we reach a smooth Calabi-Yau threefold or a Calabi-Yau threefold whose singular points all have $\mu = 0$. In the first case, we have finished, and in the second case, the resulting Calabi-Yau threefold has only ordinary double points by Theorem (2.2). Q.E.D.

Remark Since $\mathbf{Q}$-factoriality is preserved by a small deformation by Kollár-Mori [K-M, 12.1.10], it follows from the above argument that any $\mathbf{Q}$-factorial Calabi-Yau threefold has a flat deformation to a smooth Calabi-Yau threefold.

§3.

Let Z be a normal projective variety with only isolated rational singularities. Denote by $\mathrm{Weil}(Z)$ (resp. $\mathrm{Cart}(Z)$) the group of Weil divisors of Z (resp. the group of Cartier divisors of Z). Set $Sing(Z) = \{p_1, ..., p_n\}$ and take a resolution $\pi : Y \to Z$ of the singularities such that the π-exceptional locus is a divisor with simple normal crossings. Put $E_i = \pi^{-1}(p_i)$ and $E = \Sigma_{1 \le i \le n} E_i$. Let $E_i = \Sigma_j E_{i,j}$ be the irreducible decomposition of E_i. Take a sufficiently small open neighborhood Y_i of E_i in Y. We then have the following isomorphism of abelian groups:

(3.1)

$$\mathrm{Weil}(Z)/\mathrm{Cart}(Z) \cong \mathrm{im}[H^1(Y, \mathcal{O}_Y^*) \to \bigoplus_{1 \le i \le n} (H^1(Y_i, \mathcal{O}_{Y_i}^*)/\Sigma_j \mathbf{Z}[E_{i,j}]$$

.

Since $p_i \in Z$ is a rational singularity, we have

$$H^1(Y_i, \mathcal{O}_{Y_i}^*) \cong H^2(Y_i, \mathbf{Z}) \cong H^2(E_i, \mathbf{Z}).$$

Hence $\mathrm{Weil}(Z)/\mathrm{Cart}(Z)$ is a finitely generated Abelian group. We let $\sigma(Z)$ denote its rank.

Theorem(3.2) *Let Z be a normal projective threefold with only isolated rational hypersurface singularities such that $H^2(Z, \mathcal{O}_Z) = 0$. Define* $\mathrm{def}(Z) = b_4(Z) - b_2(Z)$, *where $b_i(Z)$ denote the i-th Betti number for singular cohomology of Z. Then* $\mathrm{def}(Z) = \sigma(Z)$. *Moreover, if Z has a smoothing $f : \mathcal{Z} \longrightarrow \Delta^1$, then we have*

$$\mathrm{def}(Z) = b_3(Z) + \Sigma_{p \in Sing(Z)} m(p) - b_3(\mathcal{Z}_t)$$

for $t \in \Delta^1 - \{0\}$.

Set $\Sigma = Sing(Z)$ and $U = Z \setminus \Sigma$. First we need the following lemma.

Lemma (3.3).(cf. [Di])

$$\mathrm{def}(Z) = \dim_{\mathbf{C}} \mathrm{coker}[H^3(U, \mathbf{C}) \to H^4_\Sigma(Z, \mathbf{C})].$$

Proof. Consider the exact sequence of local cohomology:

$$H^3(U) \to H^4_\Sigma(Z) \to H^4(Z) \to H^4(U) \to H^5_\Sigma(Z)$$

Since Z has only isolated rational singularities, $H^5_\Sigma(Z) = 0$ by [St 2, (1.12)]. Thus, we have $\dim_{\mathbf{C}} \mathrm{coker}[H^3(U) \to H^4_\Sigma(Z, \mathbf{C})] = b_4(Z) - b_4(U)$ by the exact sequence. On the other hand, by duality, $\dim_{\mathbf{C}} H^4(U) = \dim_{\mathbf{C}} H^2_c(U)$. There is an isomorphism $H^2_c(U) \cong H^2(Z, \Sigma)$, where $H^2(Z, \Sigma)$ is the 2-nd relative cohomology of the pair (Z, Σ). Since Σ is isolated, we have $H^2(Z, \Sigma) \cong H^2(Z)$. Q.E.D.

Lemma (3.4). *Let Z be a normal projective threefold with only isolated hypersurface singularities. Suppose that Z has a smoothing $f : \mathcal{Z} \to \Delta^1$ by a 1-parameter flat deformation, i.e. $Z = f^{-1}(0)$ and $\mathcal{Z}_t$ is a smooth variety for $t \neq 0$. Denote by $m(p_i)$ the Milnor number of (Z, p_i). Then we have*

$$\mathrm{def}(Z) = b_3(Z) + \Sigma m(p_i) - b_3(\mathcal{Z}_t)$$

.

Proof. Let B_i be the Milnor fiber of (Z, p_i). Then we have an exact sequence

$$0 \to H^3(Z) \to H^3(\mathcal{Z}_t) \to \bigoplus H^3(B_i) \to H^4(Z) \to H^4(\mathcal{Z}_t) \to 0.$$

By the exact sequence we have

$$b_3(Z) + \Sigma m(p_i) - b_3(\mathcal{Z}_t) = b_4(Z) - b_4(\mathcal{Z}_t).$$

By Poincaré duality, $b_4(\mathcal{Z}_t) = b_2(\mathcal{Z}_t)$. Since $b_2(\mathcal{Z}_t) = b_2(Z)$, the result follows. Q.E.D.

The final step is to prove the following.

Lemma (3.5). *Let Z be a normal projective threefold with only isolated rational hypersurface singularities. Assume that $H^2(Z, \mathcal{O}_Z) = 0$. Then we have*

$$\sigma(Z) = \dim_{\mathbf{C}} \mathrm{coker}[H^3(U) \to H^4_\Sigma(Z)].$$

Proof. We shall use the same notation as above. Consider the commutative diagram (3.6)

$$
\begin{array}{ccccc}
H^3(U) & \stackrel{\psi}{\to} & H^4_E(Y) & \longrightarrow & H^4(Y) \\
\| & & \uparrow \phi & & \uparrow \\
H^3(U) & \stackrel{\varphi}{\to} & H^4_\Sigma(Z) & \longrightarrow & H^4(Z)
\end{array}
$$

where the horizontal sequences are exact, and the vertical maps are edge homomorphisms of the spectral sequence of Leray. By a theorem of Goresky-MacPherson (cf. [St3, (1.11), (1.12)], the map ϕ fits into the exact sequence (3.7)

$$0 \to H^4_\Sigma(Z) \to H^4_E(Y) \to H^4(E) \to 0.$$

Taking the dual of (3.6) and (3.7) we have

(3.6)'

$$
\begin{array}{ccccccc}
0 & \longleftarrow & \mathrm{im}(\psi)^* & \longleftarrow & H^2(E) & \longleftarrow & H^2(Y) \\
& & \| & & \downarrow & & \downarrow \\
0 & \longleftarrow & \mathrm{im}(\varphi)^* & \longleftarrow & H^4_\Sigma(Z)^* & \hookleftarrow & \mathrm{coker}(\varphi)^*
\end{array}
$$

and

(3.7)'

$$0 \longleftarrow H^4_\Sigma(Z)^* \longleftarrow H^2(E) \longleftarrow \bigoplus_{i,j} \mathbf{C}_{[E_{i,j}]} \longleftarrow 0.$$

By (3.6)' and (3.7)' we have

$$\mathrm{coker}(\varphi)^* = \mathrm{im}[H^2(Y) \to H^2(E)/\Sigma \mathbf{C}_{[E_{i,j}]}].$$

Since $H^2(Z, \mathcal{O}_Z) = 0$ and Z has only rational singularities, we have $H^2(Y, \mathcal{O}_Y) = 0$. From this it follows that

$$\mathrm{coker}(\varphi)^* = \mathrm{im}[H^1(Y, \mathcal{O}_Y^*) \bigotimes \mathbf{C} \to H^2(E)/\Sigma \mathbf{C}_{[E_{i,j}]}].$$

Comparing this with (3.1), we have the result. Q.E.D.

Example (3.8). Let Y be a smooth Calabi-Yau threefold with $H^1(Y, \mathcal{O}_Y) = 0$. Assume that there is a birational contraction $\pi : Y \to Z$ of rational curves on Y. Then Z has only Gorenstein terminal singularities because π is a small birational contraction. Thus, Z is a Calabi-Yau threefold with only isolated rational singularities (cf. [Re 1]). Let $Sing(Z) = \{p_1, ..., p_n\}$, and let $C_i = \pi^{-1}(p_i)$. Then C_i is a tree of smooth rational curves. Assume that Z is smoothable by a flat deformation. Since $H^2(Y, \mathcal{O}_Y) = 0$ by Serre duality, we can apply Theorem (3.2). Let n_i be the number of irreducible components of C_i and let $L \subset H_2(Y, \mathbf{C})$ be the subspace spanned by the 2-cycles associated with the exceptional curves of π. Put $l = \dim_{\mathbf{C}} L$. Then we have

$$b_2(\mathcal{Z}_t) = b_2(Y) - l$$

$$b_3(\mathcal{Z}_t) = b_3(Y) + \Sigma_i n_i + \Sigma_i m(p_i) - 2l$$

$$b_4(\mathcal{Z}_t) = b_4(Y) - l$$

We can also give a geometric description of the mixed Hodge structure on $H^3(Z)$ when Z is a normal projective threefold with only isolated rational hypersurface singularities and with $H^2(Z, \mathcal{O}_Z) = 0$. Let Z_i be a contractible Stein open neighborhood of p_i in Z. Denote by Weil(Z_i) (resp. Cart(Z_i)) the group of Weil divisors of Z_i (resp. the group of Cartier divisors of Z_i). Then we have

(3.9)

$$\mathrm{Weil}(Z_i)/\mathrm{Cart}(Z_i) \cong H^1(Y_i, \mathcal{O}^*_{Y_i})/\Sigma_j \mathbf{Z}[E_{i,j}]$$

in the same way as (3.1). We denote by $\sigma(p_i)$ the rank of this group.

Proposition (3.10). *Let Z be a normal complete algebraic variety of dimension 3 which admits only isolated rational singularities. Assume that $H^2(Z, \mathcal{O}_Z) = 0$. Then the weight filtration of the mixed Hodge structure on $H^3(Z)$ has the following description:*

$$Gr_k^W H^3(Z) = 0 \text{ for } k \neq 2, 3;$$

$$\dim_{\mathbf{C}} W_2(H^3(Z)) = \Sigma_i \sigma(p_i) - \sigma(Z).$$

Proof. It follows from the fact that Z is a complete algebraic variety that $Gr_k^W H^3(Z) = 0$ for $k > 3$. We shall prove the second statement. Consider the long exact sequence of mixed Hodge structures

(3.11)

$$\ldots \to H^2(U) \xrightarrow{\alpha} H^3_\Sigma(Z) \to H^3(Z) \to H^3(U) \to \ldots$$

Let $\pi : (Y, E) \to (Z, \Sigma)$ be a good resolution. By a theorem of MacPherson (cf. [St 3, (1.11), (1.12)], we have a surjection of mixed Hodge structures $H^2(Y) \to H^2(U)$ and an exact sequence of mixed Hodge structures

$$0 \to H^2_E(Y) \to H^2(E) \to H^3_\Sigma(Z) \to 0.$$

Therefore, $H^3_\Sigma(Z) = H^2(E)/\Sigma_{i,j}\mathbf{C}[E_{i,j}]$, $\sigma(p_i) = \dim H^3_{\{p_i\}}(Z)$ and

(3.12)

$$\mathrm{im}(\alpha) = \mathrm{im}[H^2(Y) \to H^2(U) \to H^2(E)/\Sigma_{i,j}\mathbf{C}[E_{i,j}]$$

Since U is a smooth open variety, we have $Gr_k^W H^3(U) = 0$ if $k < 3$. Hence by (3.11) and (3.12) we obtain

$$W_2(H^3(Z)) = H^3_\Sigma(Z)/\mathrm{im}(\alpha) = \mathrm{coker}[H^2(Y) \to H^2(E)/\Sigma_{i,j}\mathbf{C}[D_{i,j}]].$$

Since $H^2(Y, \mathcal{O}_Y) = 0$ by the assumption, we see that $\dim_{\mathbf{C}} W_2(H^3(Z)) = \Sigma_i \sigma(p_i) - \sigma(Z)$. The fact that $H^3_\Sigma(Z)$ is purely of weight two has been proved in the course of the proof of Theorem (2.2) Q.E.D.

Corollary (3.13). *Let Z be a Calabi-Yau threefold with only isolated rational hypersurface singularities. Then Z can be deformed to a Calabi-Yau threefold Y with only ordinary double points whose cohomologies $H^i(Y)$ $(0 \le i \le 6)$ have pure Hodge structures.*

Proof. Z is deformed to a Calabi-Yau threefold Y with only ordinary double points $\{p_1, ..., p_n\}$ by Theorem (2.4). By [St 2, 1.12] $H^i(Y)$ always has the pure Hodge structure for $i \ge 4$. It is clear that $H^i(Y)$ has the pure Hodge structure for $i \le 2$. Hence we only have to prove that $H^3(Y)$ has the pure Hodge structure. Let $\tilde{Y}$ be a small resolution of Y, i.e. its exceptional locus are disjoint union of $(-1,-1)$-smooth rational curves C_i $(1 \le i \le n)$. By Proposition (3.10) we have $\dim_{\mathbf{C}} W_2(H^3(Y)) = \Sigma_{1 \le i \le n} \sigma(p_i) - \sigma(Y)$. Suppose that the right-hand side is not zero. Then it follows that there is a non-trivial relation between $[C_i]$ in $H_2(\tilde{Y}, \mathbf{C})$. We then have a small deformation of Y in which some ordinary double points on Y are smoothed by [Fr, §4., (b)]. This implies that we may assume that $\Sigma_{1 \le i \le n} \sigma(p_i) - \sigma(Y) = 0$. Q.E.D.

References

[AC] A'Campo, N.: Le nombre de Lefschetz d'une monodromie, Indag. Math. (N.S.) **8**, 113-118 (1973)

[Be] Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. **18**, 755-782 (1983)

[Cl] Clemens, H.: Double solids, Adv. Math. **47**, 107-230 (1983)

[Di] Dimca, A.: Betti numbers of hypersurfaces and defects of linear systems, Duke Math.J. **60**, 285-298 (1990)

[Fr] Friedman, R.: Simultaneous resolution of threefold double points, Math. Ann. **274**, 671-689 (1986)

[G-H] Green, P.S., Hübsh, T.: Connecting moduli spaces of Calabi-Yau threefolds, Comm. Math. Phys. **119**, 431-441 (1988)

[H] Hirzebruch, F.: Some examples of threefolds with trivial canonical bundle, M.P.I. preprint, no. 85-58, Bonn (1985)

[Ka 1] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application of degenerations of surfaces, Ann. of Math. (2) **127**, 93-163 (1988)

[Ka 2] Kawamata, Y.: Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. **363**, 1-46 (1986)

[Ka 3] Kawamata, Y.: Abundance theorem for minimal threefolds, preprint (1991)

[Ka 4] Kawamata, Y.: Unobstructed deformations, a remark on a paper of Z. Ran, J. Algebraic Geom. **1**, 183-190 (1992)

[K-M-M] Kawamata, Y., Matsuda, K., Matsuki, K.: Introduction to the minimal model problem, Adv. Stu. Pure Math. **10**, 283-360 (1987), Kinokuniya and North-Holland

[K-M] Kollár, J., Mori, S.: Classification of three-dimensional flips, J. Amer. Math. Soc. **5**,(3) 533-703

[Ko] Kollár, J.: Shafarevich maps and plurigenera of algebraic varieties, Invent. Math. **113**, 177-215 (1993)

[Lo] Looijenga, E.J.N.: *Isolated singular points on complete intersections*, London Math. Soc. Lect. Note. Ser. **77**, Cambridge University Press 1984

[Mo] Mori, S.: Flip theorem and the existence of minimal threefolds, J. Amer. Math. Soc. **1**, 117-253 (1988)

[Na] Namikawa, Y.: On deformations of Calabi-Yau threefolds with terminal singularities, Topology **33**(3), 429-446 (1994)

[Ra] Ran, Z.: Deformations of Calabi-Yau Kleinfolds, in *Essays on Mirror manifolds*

[Re 1] Reid, M.: Minimal models of canonical threefolds, Adv. Stud. Pure Math. **1**, 131-180 (1983) Kinokuniya, North-Holland

[Re 2] Reid, M.: The moduli space of threefolds with $K = 0$ may nevertheless be irreducible, Math. Ann. **278**, 329-334 (1987)

[S-S] Scherk, J., Steenbrink, J.H.M.: On the mixed Hodge structure on the cohomology of the Milnor fibre, Math. Ann. **271**, 641-665 (1985)

[Sch] Schlessinger, M.: Rigidity of quotient singularities, Invent. Math. **14**, 17-26 (1971)

[St 1] Steenbrink, J.H.M.: Mixed Hodge structure on the vanishing cohomology, In: *Real and complex singularities*, Oslo 1976. P. Holm ed. pp. 525-563, Sijthoff–Noordhoff, Alphen a/d Rijn 1977

[St 2] Steenbrink, J.H.M.: Mixed Hodge structures associated with isolated singularities. Proc. Symp. Pure Math. **40** Part 2, 513–536 (1983)

[St 3] Steenbrink, J.H.M.: Vanishing theorems on singular spaces, Astérisque **130**, 330–341 (1985)

[Te] Teissier, B.: The hunting of invariants in the geometry of discriminants, In: Real and complex singularities, Oslo 1976. P. Holm ed. pp. 565-677, Sijthoff-Noordhoff, Alphen a/d Rijn 1977

[W] Werner, J.: Kleine Auflösungen spezieller dreidimensionaler Varietäten, Bonner Math. Schriften Nr. **186** (1987)

[Wi] Wilson, P.M.H.: Calabi-Yau manifolds with large Picard number, Invent. Math. **98**, 139-155 (1989)

Yoshinori Namikawa : Department of Mathematics, Sophia University, Kioi-cho, Tokyo, 102, Japan,

Current adress: Max-Planck-Institute fur Mathematik, Gottfried-Claren-Strasse, Bonn, D-53225, Germany,

J.H.M. Steenbrink : Mathematical Institute, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands