Elliptic dilogarithms and parallel lines

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Abstract

We prove Boyd's conjectures relating Mahler's measures and values of L-functions of elliptic curves in the cases when the corresponding elliptic curve has conductor 14.

1 Boyd's conjectures

Rogers provided a table of relations between Mahler's measures and values of L-functions of elliptic curves of low conductors 11, 14, 15, 20, 24, 27, 32, 36 in [Rog08]. Among these relations some had been proved and some had not. According to Rogers, those relations which involve curves with complex multiplication (conductors 27, 32, 36) were all proved. Except those, only a relation with curve of conductor 11 was proved. Let us list the relations with curves of conductor 14.

Let $P \in \mathbb{C}[y, z]$. Then Mahler's measure of P is defined as

$$m(P) := (2\pi \mathrm{i})^{-2} \int_{|y|=|z|=1} \log |P(y,z)| \frac{dy}{y} \frac{dz}{z}.$$

Denote

$$\begin{aligned} n(k) &:= m(y^3 + z^3 + 1 - kyz), \\ g(k) &:= m((1+y)(1+z)(y+z) - kyz). \end{aligned}$$

Let E be the elliptic curve of conductor 14 with Weierstrass form $y^2 + yx + y = x^3 + 4x - 6$. It is isomorphic to the modular curve $X^0(14)$ with the pullback of the Néron differential $\frac{dx}{2y+x+1}$ given by the eta-product [MO97]

$$f := \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau).$$

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Then $L(E,s)=L(f,s)$ and the relations listed by Rogers are

$$n(-1)=\frac{7}{\pi^2}L(f,2), \tag{1}$$

$$n(5)=\frac{49}{2\pi^2}L(f,2), \tag{2}$$

$$g(1)=\frac{7}{2\pi^2}L(f,2), \tag{3}$$

$$g(7)=\frac{21}{\pi^2}L(f,2), \tag{4}$$

$$g(-8)=\frac{35}{\pi^2}L(f,2). \tag{5}$$

2 The regulator

Fix a smooth projective curve $C/\mathbb{C}$. An element $\sum_i\{f_i,g_i\}\in\Lambda^2\mathbb{C}(C)^\times$ will be denoted simply by $\{f,g\}$ and we will omit the corresponding "$\sum_i$" sign in expressions below to soften the notation. The regulator of $\{f,g\}\in K_2(C)$ is defined as $r_C(\{f,g\})\in H^1(C,\mathbb{R})$ whose value on $[\gamma]\in H_1(C,\mathbb{Z})$ is

$$r_C(\{f,g\})([\gamma])=\int_\gamma \log|f|d\arg g-\log|g|d\arg f.$$

Let ω be a holomorphic 1-form on C. The value of the regulator on ω is defined as follows:

$$\langle r_C(\{f,g\}),\omega\rangle=\langle r_C(\{f,g\})\cap\omega,[C]\rangle=2\int_C\log|f|d\arg g\wedge\omega. \tag{6}$$

Denote by $\mathscr{K}_n$ (resp. $\mathscr{K}_g$) the set of values of the function $\frac{y^3+z^3+1}{yz}$ (resp. $\frac{(1+y)(1+z)(y+z)}{yz}$) on the torus $|y|=|z|=1$. Then by a theorem of Deninger [Den97] for $k\notin\mathscr{K}_n$ (resp. $k\notin\mathscr{K}_g$) one can express $n(k)$ (resp. $g(k)$) as $\frac{1}{2\pi}r_C(\{y,z\})([\gamma])$ for a certain $[\gamma]\in H_1(C,\mathbb{Z})$, where C is the projective closure of the equation $y^3+z^3+1-kyz$ (resp. $(1+y)(1+z)(y+z)-kyz$). When k is on the boundary of $\mathscr{K}_n$ (resp. $\mathscr{K}_g$) Deninger's result still applies by continuity.

3 Elliptic dilogarithm

Let $E/\mathbb{C}$ be an elliptic curve. Define a map from $\Lambda^2\mathbb{C}(E)^\times$ to $\mathbb{Z}[E(\mathbb{C})]^-$ by

$$\{f,g\}\to(f)*(g)^-$$

where "$*$" and "$-$" mean the convolution and the antipode operations on divisors of an elliptic curve. Fix an isomorphism $E\cong\mathbb{C}/\langle 1,\tau\rangle$ for $\tau\in\mathfrak{H}$. Let u be the coordinate on $\mathbb{C}$. Let $x\in E(\mathbb{C})$, $x=a\tau+b$ for $a,b\in\mathbb{R}$. As in [Zag90] (it seems that the sign there is wrong) put

$$R(\tau,x)=-\frac{\mathrm{i}}{\pi}(\operatorname{Im}\tau)^2\sum_{(m,n)\neq(0,0)}\frac{\sin(2\pi(na-mb))}{(m\tau+n)^2(m\overline{\tau}+n)}.$$

We have

$$\langle r_E(\{f,g\}), du\rangle = R(\tau, (f) * (g)^-).$$

For a holomorphic 1-form ω on E put

$$R_{E,\omega}(x) = \frac{\omega}{du} R(\tau, x).$$

Then $R_{E,\omega}$ does not depend on the choice of the isomorphism $E \cong \mathbb{C}/\langle 1,\tau\rangle$ and we call $R_{E,\omega}$ the elliptic dilogarithm, while usually people call elliptic dilogarithm the real part of $R(\tau, x)$.

When E is defined over $\mathbb{R}$ and an orientation on $E(\mathbb{R})$ is chosen there is a canonical choice of the isomorphism above and we will write $R_E(x)$ for the "old dilogarithm" $R(\tau, x) = R_{E,du}(x)$.

By linearity we extend $R_{E,\omega}$ to the odd part of the group of divisors $\mathbb{Z}[E(\mathbb{C})]^-$.

The function $R_{E,\omega}(x)$ satisfies the following properties:

(i) For any $\lambda \in \mathbb{C}$ $R_{E,\lambda\omega}(x) = \lambda R_{E,\omega}(x)$.

(ii) For an isogeny $\varphi : E' \to E$ and $x \in E(\mathbb{C})$

$$R_{E,\omega}(x) = \sum_{x' \in \varphi^{-1}(x)} R_{E',\varphi^*\omega}(x'). \tag{7}$$

(iii) For a function $f \in \mathbb{C}(E)^\times$, $f \neq 1$, one has $R_{E,\omega}((f) * (1-f)^-) = 0$.

The second property is called the distribution relation, the third one is the Steinberg relation.

We expect that any algebraic relation between $R_{E,\omega}(x)$ where E, ω, x are defined over $\overline{\mathbb{Q}}$ follows from the relations listed above.

4 Beilinson's theorem for $\Gamma_0(N)$

Let N be a squarefree integer with prime decomposition $N = p_1, \ldots, p_n$. Let $f = \sum a(n)q^n$ be a newform for $\Gamma_0(N)$ of weight 2. Let W be the group of Atkin-Lehner involutions. This is a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. For $m > 0$, $m|N$ denote by w_m the Atkin-Lehner involution corresponding to m. Any cusp of $\Gamma_0(N)$ is given by $w(\infty)$ for a unique $w \in W$. The width of $w_m(\infty)$ is m. It is known that for a prime $p|N$ we have $f|_2 w_p = -a(p)f$.

Let $\mathbb{Q}[W]_0$ be the augmentation ideal of $\mathbb{Q}[W]$. For any $\alpha \in \mathbb{Q}[W]_0$, $\alpha = \sum_{w\in W} \alpha_w [w]$ consider $F_\alpha \in \mathbb{C}(X_0(N))^\times \otimes \mathbb{Q}$ such that $(F_\alpha) = \sum_{w \in W} \alpha_w [w(\infty)]$.

Let $\gamma : W \to \{\pm 1\}$ be such that $f|_2 w = \gamma(w) f$ for all $w \in W$. Let $d = \sum_{m|N} m w_m$,

$$d^{-1} = \prod_{k=1}^{n} \frac{1 - p_k w_{p_k}}{1 - p_k^2}.$$

Let γ^* be the involution of $\mathbb{Q}[W]$ which sends w to $\gamma(w)w$ for $w \in W$. Put $\alpha' = d^{-1}\alpha$, $\beta' = d^{-1}\beta$ for $\alpha, \beta \in \mathbb{Q}[W]_0$. Let $\varepsilon : \mathbb{Q}[W] \to \mathbb{Q}$ be a linear map such that $\varepsilon(w_m)$ is 0 for $m \neq 1$ and 1 for $m = 1$. Then

$$\langle r_{X_0(N)}(\{F_\alpha, F_\beta\}), 2\pi \mathrm{i} f(\tau)d\tau\rangle = -\frac{144N}{\pi}\varepsilon(w_N\alpha'\gamma^*(\beta'))L(f,1)L(f,2). \quad (8)$$

5 Parallel lines

By results stated above both sides of the conjectured identities are reduced to relations between values of the elliptic dilogarithm. To prove relations between elliptic dilogarithms one usually tries to construct rational functions f such that divisors of both f and $1-f$ are supported on a given set of points.

Let $E/\mathbb{C}$ be an elliptic curve and $Z \subset E(\mathbb{C})$ be a finite subgroup. Let us realize E as a plane cubic with equation $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{C}$. For each triple $p, q, r \in Z \setminus \{0\}$ such that $p + q + r = 0$ consider the line $l_{p,q,r}$ passing through p, q, r with equation $y + s_{p,q,r}x + t_{p,q,r} = 0$. Suppose $s_{p,q,r} = s_{p',q',r'}$ for another triple of points, which is equivalent to the lines $l_{p,q,r}$ and $l_{p',q',r'}$ being parallel. Then from equations of these lines one can obtain two functions f, g on E such that $f + g = 1$ and divisors of f and g are supported on Z. Thus we obtain (hopefully a non-trivial) relation between values of the elliptic dilogarithm at points of Z.

I propose to search for parallel lines as above in two ways. The first way, dubbed “breadth-first search”, is to fix $Z = \mathbb{Z}/m \times \mathbb{Z}/m'$ and consider the moduli space of elliptic curves E with embedding $Z \to E$. Then for any two triples p, q, r and p', q', r' the difference $s_{p,q,r} - s_{p',q',r'}$ is a function on the moduli space, which can be found explicitly, and at the points where the function is zero we obtain a relation.

Another approach, which I call “depth-first search”, is to fix a curve E and consider some large subgroup Z hoping that when Z is large enough some parallel lines will appear. However, this seems to work only for some “nice” curves.

In the proof of (1) - (5) we use identities found by the two approaches on the curve $Y^2 + YX + Y = X^3 - X$, and obtain results for isogenous curves by the distribution relation.

Finally let us mention an interesting propery of the slopes $s_{p,q,r}$.

Proposition. *There exists a unique map from $Z \setminus \{0\}$ to $\mathbb{C}$, denoted $p \to z_p \in \mathbb{C}$, such that*

(i) $z_p + z_{-p} = 0$ *for all* p,

(ii) $z_p + z_q + z_r = s_{p,q,r}$ *for all* (p, q, r) *with* $p + q + r = 0$,

moreover, we have (x_p is the x-coordinate of p)

(iii) $x_p + x_q + x_r = s^2_{p,q,r}$ *for all* (p, q, r) *with* $p + q + r = 0$.

In fact these z_p are related to Eisenstein series of weight 1 and they satisfy a certain distribution relation.

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References

[Bei85] A.A. Beilinson. Higher regulators and values of L-functions. *J. Sov. Math.*, 30:2036–2070, 1985.

[Blo00] Spencer J. Bloch. *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves.* CRM Monograph Series. 11. Providence, RI: American Mathematical Society (AMS). ix, 97 p., 2000.

[Den97] Christopher Deninger. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions. *J. Am. Math. Soc.*, 10(2):259–281, 1997.

[MO97] Yves Martin and Ken Ono. Eta-quotients and elliptic curves. *Proc. Am. Math. Soc.*, 125(11):3169–3176, 1997.

[Rog08] Mathew D. Rogers. Hypergeometric formulas for lattice sums and mahler measures. 2008.

[RV99] F. Rodriguez Villegas. Modular Mahler measures. I. Ahlgren, Scott D. (ed.) et al., Topics in number theory. In honor of B. Gordon and S. Chowla. Proceedings of the conference, Pennsylvania State University, University Park, PA, USA, July 31-August 3, 1997. Dordrecht: Kluwer Academic Publishers. Math. Appl., Dordr. 467, 17-48 (1999)., 1999.

[SS88] Norbert Schappacher and Anthony J. Scholl. Beilinson's theorem on modular curves. Beilinson's conjectures on special values of L-functions, Meet. Oberwolfach/FRG 1986, Perspect. Math. 4, 273-304 (1988)., 1988.

[Zag90] Don Zagier. The Bloch-Wigner-Ramakrishnan polylogarithm function. *Math. Ann.*, 286(1-3):613–624, 1990.