Einstein metrics of cohomogeneity one

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1. Introduction.

In a recent paper [**3**] we classified Riemannian Kähler-Einstein metrics in real dimension four which were of Bianchi IX type, that is, which admitted an isometric action of SU(2) with generically three-dimensional orbits. We found that there were two families of such metrics which were complete and had negative Einstein constant. One of these consisted of metrics admitting an isometric action of $U(2)$; these metrics had previously been studied by Gibbons and Pope and by Pedersen [**5**],[**6**]. The other family, which consisted of triaxial metrics, had as its underlying topological manifold the tangent bundle of S^2.

In this note we explain how the latter family can be generalised to produce examples of complete Einstein metrics with nonpositive Einstein constant on the tangent bundle of S^n for any n greater than one. These metrics are of cohomogeneity one, that is, they admit an action of a group of isometries whose generic orbit is of codimension one. Some examples of cohomogeneity one Einstein metrics were produced by Bérard-Bergery [**1**], by defining the metric on the orbits using a Riemannian submersion with circle fibre over a Kähler-Einstein base. In the case we shall study the metrics on the orbits are not obtained in this way.

2. Cohomogeneity one metrics.

Consider a metric g in real dimension $2n+2$ admitting an isometric action of $SO(n+2)$ such that the isotropy group is generically $SO(n)$. The metric is therefore of cohomogeneity one.

We can put the metric in the form

$$g = dt^2 + g_t$$

where t is the arclength parameter along a geodesic orthogonal to the group orbits, and g_t is a homogeneous metric on the orbits.

Now, for any homogeneous space G/H we may identify the tangent space at a point with an Ad H -invariant complement $\mathfrak{p}$ for $\mathfrak{h}$ in $\mathfrak{g}$. With this identification, G-invariant metrics on G/H correspond precisely to Ad H-invariant inner products on $\mathfrak{p}$.

In our case we embed $\mathfrak{so}(n)$ in $\mathfrak{so}(n+2)$ so that if $X_{i,j}$ denotes the matrix (A_{kl}) with $A_{ij} = 1, A_{ji} = -1$ and all other entries zero, then $\{X_{i,j}; j = n+1, n+2, 1 \leq i < j\}$ spans a complement $\mathfrak{p}$ for $\mathfrak{so}(n)$ in $\mathfrak{so}(n+2)$.

Now $\mathfrak{p}$ has the following decomposition into irreducible components under the adjoint action of $SO(n)$:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$$

where $\mathfrak{p}_1, \mathfrak{p}_2$ are standard n-dimensional representations spanned by $\{X_{i,n+1} : 1 \leq i \leq n\}$ and $\{X_{i,n+2} : 1 \leq i \leq n\}$ respectively, and $\mathfrak{p}_3$ is the trivial representation spanned by $X_{n+1,n+2}$.

We shall consider metrics with respect to which the above decomposition is orthogonal, and such that

$$g_t(X_{i,n+1}, X_{k,n+1}) = a(t)^2\delta_{ik}$$

$$g_t(X_{i,n+2}, X_{k,n+2}) = b(t)^2\delta_{ik}$$

$$g_t(X_{n+1,n+2}, X_{n+1,n+2}) = c(t)^2$$

We can express this more concisely by defining g_t to be

$$\frac{1}{4}\left(a(t)^2(-B)\mid_{\mathfrak{p}_1} \oplus b(t)^2(-B)\mid_{\mathfrak{p}_2} \oplus c(t)^2(-B)\mid_{\mathfrak{p}_3}\right)$$

where B is the Killing form on $\mathfrak{so}(n+2)$ defined by $B(X,Y) = (n-2)\mathrm{Tr}XY$.

If $a^2 = b^2$ this metric is obtained using a Riemannian submersion with circle fibre over the flag manifold $SO(n+2)/SO(n) \times SO(2)$ equipped with a Kähler-Einstein metric. If a^2 and b^2 are not equal, however, g_t does not arise in this way.

Let us denote by $\tilde{Ric}$ the Ricci tensor of the homogeneous metric g_t and by Ric the Ricci tensor of the metric $g = dt^2 + g_t$.

It is convenient to introduce at this stage a new transverse coordinate ζ defined by

$$dt = a^n b^n c \, d\zeta$$

We shall denote differentiation with respect to ζ by a prime, and use $\mathcal{H}$ to denote the vector field $\frac{\partial}{\partial t} = a^{-n} b^{-n} c^{-1} \frac{\partial}{\partial \zeta}$.

The formulae of Bérard-Bergery [1] for the Ricci tensor of a metric of cohomogeneity one give

$$\begin{aligned}
Ric(X_{i,n+1}, X_{i,n+1}) &= \tilde{Ric}(X_{i,n+1}, X_{i,n+1}) - \frac{1}{a^{2n-2} b^{2n} c^2} \left(\frac{a'}{a}\right)' \\
Ric(X_{i,n+2}, X_{i,n+2}) &= \tilde{Ric}(X_{i,n+2}, X_{i,n+2}) - \frac{1}{a^{2n} b^{2n-2} c^2} \left(\frac{b'}{b}\right)' \\
Ric(X_{n+1,n+2}, X_{n+1,n+2}) &= \tilde{Ric}(X_{n+1,n+2}, X_{n+1,n+2}) - \frac{1}{a^{2n} b^{2n}} \left(\frac{c'}{c}\right)' \\
Ric(X_{i,j}, X_{k,l}) &= 0 \text{ if } i \neq k \text{ or } j \neq l \\
Ric(X_{i,j}, \mathcal{H}) &= 0
\end{aligned}$$

$$Ric(\mathcal{H}, \mathcal{H}) = -\frac{1}{a^n b^n c} \left(\frac{1}{a^n b^n c} \left(\frac{c'}{c} + n\frac{b'}{b} + n\frac{a'}{a} \right) \right)' - \frac{1}{a^{2n} b^{2n} c^2} \left(\left(\frac{c'}{c}\right)^2 + n \left(\frac{a'}{a}\right)^2 + n \left(\frac{b'}{b}\right)^2 \right)$$

Using the standard formula for the Ricci tensor of a homogeneous space [2] we can compute $\tilde{Ric}$ and we find that the Einstein equations for the cohomogeneity one metric g become

$$-\frac{1}{a^{2n-2} b^{2n} c^2} \left(\frac{a'}{a}\right)' + \frac{a^4 - (b^2 - c^2)^2}{2b^2 c^2} + n - 1 = \Lambda a^2 \quad (1)$$

$$-\frac{1}{a^{2n} b^{2n-2} c^2} \left(\frac{b'}{b}\right)' + \frac{b^4 - (c^2 - a^2)^2}{2a^2 c^2} + n - 1 = \Lambda b^2 \quad (2)$$

$$-\frac{1}{a^{2n} b^{2n}} \left(\frac{c'}{c}\right)' + \frac{n(c^4 - (a^2 - b^2)^2)}{2a^2 b^2} = \Lambda c^2 \quad (3)$$

$$-\frac{1}{a^n b^n c}\left(\frac{1}{a^n b^n c}\left(\frac{c'}{c}+n\frac{b'}{b}+n\frac{a'}{a}\right)\right)' - \frac{1}{a^{2n}b^{2n}c^2}\left(\left(\frac{c'}{c}\right)^2+n\left(\frac{a'}{a}\right)^2+n\left(\frac{b'}{b}\right)^2\right) = \Lambda \tag{4}$$

where Λ is the Einstein constant.

If $n = 1$ these equations are a presentation of the Einstein equations for a Bianchi IX metric.

It can be verified by direct calculation that equations (1)-(4) hold in particular if a, b, c satisfy the following three first-order equations

$$a' = \frac{1}{2}a^n b^{n-1}(b^2 + c^2 - a^2) \tag{5}$$

$$b' = \frac{1}{2}a^{n-1}b^n(c^2 + a^2 - b^2) \tag{6}$$

$$c' = \frac{1}{2}na^{n-1}b^{n-1}c\left(a^2 + b^2 - c^2 - \frac{2\Lambda}{n}a^2b^2\right) \tag{7}$$

It is this reduction of the Einstein equations which we shall study. The metrics arising from solutions of these equations are precisely those Einstein metrics with respect to which the almost complex structure J defined by

$$J : X_{i,n+1} \to -\frac{a}{b}X_{i,n+2}$$

$$J : \frac{\partial}{\partial t} \to \frac{1}{c}X_{n+1,n+2}$$

is Kähler.

3. Complete metrics.

We shall now demonstrate the existence of solutions to (5)-(7) which give rise to complete metrics. It will be convenient to introduce a new variable u defined by

$$du = (ab)^{n-1}d\zeta$$

so the equations (5)-(7) become

$$a_u = \frac{1}{2}a(b^2 + c^2 - a^2) \tag{8}$$

$$b_u = \frac{1}{2}b(c^2 + a^2 - b^2) \tag{9}$$

$$c_u = \frac{1}{2}nc\left(a^2 + b^2 - c^2 - \frac{2\Lambda}{n}a^2b^2\right) \tag{10}$$

The critical points of this system are precisely the points $(0, K, K), (K, 0, K), (K, K, 0)$ where $K \in \mathbb{R}$.

Assumption

From now on we shall take Λ to be less than or equal to zero.

Subject to this assumption, the linearisation of (8)-(10) about each critical point (except the origin) has one negative, one positive and one zero eigenvalue. So each of these critical points has an unstable curve.

Let us consider an unstable curve of $(K, 0, K)$ where K is nonzero. This will be a solution to (8)-(10) defined on a maximal interval $(-\infty, \eta)$ for some η (η may be ∞).

It follows from (8)-(10) that if any one of a, b or c is zero at a point in $(-\infty, \eta)$, then it is identically zero. It is clear that on the unstable curve none of a, b or c is identically zero, so none of them vanishes anywhere on $(-\infty, \eta)$. The metric and the equations (8)-(10) are invariant under changes of sign of a, b or c so from now on we can assume that a, b, c are all positive on $(-\infty, \eta)$; in particular we can take K to be positive.

The metric is therefore defined on $(-\infty, \eta)$ and to show that it is complete we need to study its behaviour as u tends to $-\infty$ and as u tends to η.

Note that if a equals b at any point on the unstable curve then a is identically equal to b, giving a contradiction. It follows that a is greater than b on the unstable curve, and so (from (9)) b is strictly increasing. It also follows from (8),(9) that $\frac{a}{b}$ is strictly decreasing, and tends to some limit $L \geq 1$ as u tends to η.

Lemma 3.1

On an unstable curve of $(K,0,K)$ we have the inequalities

$$c^2 \leq a^2 + b^2 - \frac{2\Lambda}{n} a^2 b^2 \tag{11}$$

and

$$a^2 \leq b^2 + c^2. \tag{12}$$

Proof

Suppose that $c^2 > a^2 + b^2 - \frac{2\Lambda}{n} a^2 b^2$ at u_0. The equations (8),(10) imply that a is increasing and c is decreasing at u_0. Recall also that b is increasing. We deduce that c^2 is greater than $a^2 + b^2 - \frac{2\Lambda}{n} a^2 b^2$ on $(-\infty, u_0)$, and that c is bounded away from a on $(-\infty, u_0)$. This contradicts the fact that (a,b,c) tends to $(K,0,K)$ as u tends to $-\infty$, so we have established inequality (11). The proof of inequality (12) is very similar. □

We deduce from (8),(10) that a, c are increasing on $(-\infty, \eta)$. We remarked earlier that b is strictly increasing on this interval.

Let us now study the behaviour of the metric as u tends to $-\infty$ and as u tends to η.

As u tends to $-\infty$

$$\begin{aligned} a &\simeq K \\ b &\simeq M e^{K^2 u} \\ c &\simeq K \end{aligned}$$

for some constant M.

Choosing

$$R = M e^{K^2 u}$$

as a new coordinate, and using $\sigma_{i,j}$ to denote a basis of one-forms dual to $X_{i,j}$, we find that the metric is asymptotically

$$dR^2 + R^2(\sigma_{1,n+2}^2 + \dots \sigma_{n,n+2}^2) + K^2(\sigma_{1,n+1}^2 + \dots \sigma_{n,n+1}^2 + \sigma_{n+1,n+2}^2)$$

as R tends to zero.

Now, $\sigma_{1,n+2}^2 + \dots \sigma_{n,n+2}^2$ is the standard $SO(n+1)$-invariant metric on S^n, while $\sigma_{1,n+1}^2 + \dots \sigma_{n,n+1}^2 + \sigma_{n+1,n+2}^2$ is the standard $SO(n+2)$-invariant metric on S^{n+1}. Therefore we obtain a nonsingular metric by adding in an $(n+1)$-sphere at $R = 0$.

In terms of the orbit type of the $SO(n+2)$ action, the isotropy group jumps at $R = 0$ from $SO(n)$ to $SO(n+1)$, so an n-dimensional sphere collapses to a point and the orbit at $R = 0$ is the $(n+1)$-sphere $SO(n+2)/SO(n+1)$ rather than $SO(n+2)/SO(n)$. The orbit at $R = 0$ is called a generalised bolt in the terminology of Gibbons, Page and Pope [**4**].

To examine the behaviour of the metric as u tends to η we introduce a new coordinate r, defined by

$$r = 2(ab)^{\frac{1}{2}}$$

It follows from (8),(9) that ab is increasing, so this is an allowable change of variables.

The metric is now

$$W^{-1}dr^2 + \frac{1}{4}r^2\left(\frac{a}{b}(\sigma_{1,n+1}^2 + \dots \sigma_{n,n+1}^2) + \frac{b}{a}(\sigma_{1,n+2}^2 + \dots \sigma_{n,n+2}^2) + W\sigma_{n+1,n+2}^2\right) \quad (13)$$

where $W = \frac{c^2}{ab}$.

It follows from equations (8)-(10) that

$$\frac{dW}{dr} + \frac{2(n+1)W}{r} = \frac{2n}{r}\left(\frac{a}{b} + \frac{b}{a}\right) - \Lambda r \quad (14)$$

Recall that the inequalities (11) and (12) show that a, b, c are monotonic increasing on $(-\infty, \eta)$. Suppose that the limit μ of b as u tends to η is finite. Since a/b is decreasing, and because of the estimate (11), we see that the limits λ, ν of a, c at $u = \eta$ are also finite. If $\eta = \infty$ then (λ, μ, ν) is a critical point, which leads to a contradiction as λ, μ, ν are all positive. If η is finite, then we also obtain a contradiction because $(-\infty, \eta)$ is by definition a maximal interval on which the solution exists. So we deduce that b, and hence r, tends to infinity as u tends to η.

Therefore we must study the asymptotics of the metric (13) as r tends to infinity. It follows from (14) that

$$\frac{d}{dr}\left(r^{2n+2}W\right) = 2nr^{2n+1}\left(\frac{a}{b} + \frac{b}{a}\right) - \Lambda r^{2n+3}$$

Solving for W, and recalling that $\frac{a}{b}$ decreases monotonically on $(-\infty, \eta)$ to some finite positive limit L, we see that $W = O(r^2)$ if Λ is negative and W is bounded if Λ is zero. It follows that the geodesic distance $\int^{\infty} \sqrt{W^{-1}} dr$ to $r = \infty$ is infinite.

We have shown that the metric is complete. The underlying topological manifold is the total space of a rank $n+1$ vector bundle E over S^{n+1}. In fact the sphere bundle of this vector bundle is the Stiefel manifold $SO(n+2)/SO(n)$, so E is in fact the tangent bundle of S^{n+1} **[7]**.

We summarise our results in the next theorem.

Theorem 3.2

The unstable curves of points $(K, 0, K)$, where K is nonzero, give complete Einstein metrics with nonpositive Einstein constant on TS^{n+1}.

Remarks

(i) One can also obtain complete metrics by considering the unstable curves of $(K, K, 0)$ where $n - \Lambda K^2$ is a half-integer. The condition on K is needed to ensure that the metric can be completed by adding a generalised bolt. However, for these trajectories a is identically equal to b and the metrics on the $SO(n+2)$ orbits are obtained by Riemannian submersions with circle fibres. The resulting cohomogeneity one metrics are included in the examples of Bérard-Bergery **[1]**,

(ii) The metrics obtained from unstable curves of $(0, K, K)$ are the same as those in Theorem 3.2, because the equations (8)-(10) are symmetric in a and b.

(ii) Complete Ricci-flat Kähler metrics have been shown to exist on the tangent bundles of spheres by M. Stenzel and also by Kobayashi **[8]**,

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