TATE CYCLES ON A PRODUCT OF TWO HILBERT MODULAR SURFACES

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§1. Introduction

Let X be a smooth projective variety defined over a number field K. Let L be a finite extension of K and denote by $Z_j(X;L)$ the free abelian group generated by codimension j subvarieties of X which are defined over L. There is a cycle class map

$$Z_j(X;L)\otimes\mathbb{Q}_\ell \longrightarrow H_\ell^{2j}(\bar{X})(j)^{Gal(\bar{\mathbb{Q}}/L)}.$$

A conjecture of Tate asserts that this map is surjective. It has been shown to hold for many varieties. Let us denote by

$$\mathrm{T}a_\ell(X;L) \;=\; H_\ell^{2j}(\bar{X})(j)^{Gal(\bar{\mathbb{Q}}/L)}$$

the space of Tate cycles on X defined over L and by

$$\mathrm{T}a_\ell(X) = \cup_L \mathrm{T}a_\ell(X;L)$$

the space of all Tate cycles on X. The aim of this paper is to describe all the Tate cycles on the product of two Hilbert modular surfaces, however we are unable to say if these Tate cycles come from algebraic cycles.

Let F be a real quadratic field and let S denote a Hilbert modular surface corresponding to this field. Thus, $S = S_K$ is a surface defined over $\mathbb{Q}$ which is the smooth compactification of an open surface S^o which satisfies

$$S^o(\mathbb{C}) = \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/K_\infty K$$

The first author would like to thank IAS for their support through NSF grant DMS 9304580, and would also like to thank NSERC for their support during the course of this course. The second author would like to thank Max-Planck-Institute, Bonn where this work was completed.

[1]Steacie Fellow

where $G = R_{F/\mathbb{Q}} GL_{2/F}$, K is a compact open subgroup of $G(\mathbb{A}_f)$ and $K_\infty = K_\infty^c Z$ where K_∞^c is the connected component of the identity of a maximal compact subgroup of $GL_2(F \otimes \mathbb{R})$,

$$K_\infty^c \cong SO_2(\mathbb{R}) \times SO_2(\mathbb{R}),$$

and Z is the center of $GL_2(F \otimes \mathbb{R})$.

Let F_1 and F_2 be two real quadratic fields and let S_1 and S_2 denote corresponding Hilbert modular surfaces with respect to K_1 and K_2 (respectively). In this paper we show that $\mathrm{T}a_\ell(S_1 \times S_2)$ is spanned by $\mathrm{T}a_\ell(S_1) \otimes \mathrm{T}a_\ell(S_2)$ and certain codimension 2 cycles which we shall construct. Since Tate cycles on a Hilbert modular surface are known to be algebraic ([HLR], [MR], [K]), the Tate conjecture for $S_1 \times S_2$ is therefore equivalent to proving the algebraicity of these cycles.

§2. Preliminaries on the tensor product of two dimensional representations

In this section we prove several results about the tensor product of two 2 dimensional representations. All the results proved are of a rather elementary nature and are surely well-known but for lack of suitable reference, we have included all the proofs. The results of this section are for abstract representations of a general group.

Theorem 2.1. *The tensor product of two 2 dimensional irreducible representations is reducible only if either both the representations are dihedral, or they are a twist of each other by a character.*

The proof of this theorem will be completed in several steps which we break in the following lemmas and propositions. Some of these will be of independent interest to us in later sections.

Lemma 2.2. *Let π_1 and π_2 be two representations of a group into $O(V, \mathbb{C})$ (or, $GO(V, \mathbb{C})$) which become equivalent in $GL(V, \mathbb{C})$. Then if π_1 (and therefore π_2) leaves invariant up to scaling a quadratic form unique up to scaling, π_1 is equivalent to π_2 in $O(V, \mathbb{C})$ (or, $GO(V, \mathbb{C})$).*

Proof: We omit the totally straightforward proof.

Lemma 2.3. *Let V be a 4-dimensional representation of a group G such that for a choice of a quadratic form unique up to scalars, the representation of G lands inside*

$GO(V)$. Then if $V \cong V_1 \otimes V_2$, and also $V \cong W_1 \otimes W_2$, then there exists a character χ of G and $i \in \{1,2\}$ such that $V_1 \cong W_i \otimes \chi$ and $V_2 \cong W_j \otimes \chi^{-1}$ for $j \neq i$.

Proof: We have the exact sequence

$$0 \to \mathbb{C}^* \to GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \to GSO(4,\mathbb{C}) \to 0$$

which is obtained by taking the external tensor product of the standard 2 dimensional representation of $GL(2,\mathbb{C})$ with itself. It follows that if the representation of G inside $GSO(V,\mathbb{C})$ is written as $V_1 \otimes V_2$ and also as $W_1 \otimes W_2$, then these correspond to two ways of lifting the representation of G into $GSO(V,\mathbb{C})$ to $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$. Since the kernel of the mapping of $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ to $GSO(V,\mathbb{C})$ is central, the ambiguity in such a lifting is by a character into $\mathbb{C}^*$. By the previous lemma, the representation V is well defined up to conjugacy inside $GO(V,\mathbb{C})$, and that concludes the proof of this lemma.

Remark. That the previous lemma is not true without some hypothesis is shown by the following example. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. It has a unique irreducible representation ρ of dimension 2. Then it is easy to see that $\rho \otimes \rho \cong \{1 \oplus \alpha\} \otimes \{1 \oplus \beta\}$, where α and β are any two distinct non-trivial characters of G.

Lemma 2.4. *Let σ be an irreducible 2-dimensional representation of a group G. If for a non-trivial character χ of G, $\sigma \cong \sigma \otimes \chi$, then χ is of order 2 and σ is induced from a character on the kernel of χ.*

Proof: Since $\det(\sigma \otimes \chi) = \det(\sigma)\chi^2 = \det(\sigma)$, $\chi^2 = 1$. Let H be the kernel of χ. Let $A \in \mathrm{Hom}_G(\sigma, \sigma \otimes \chi)$ be a non-zero element. Since $\chi^2 = 1$, A^2 is an intertwining operator from σ to itself. After scaling A, we can therefore assume that $A^2 = 1$. Since $\chi \neq 1$, A is not the identity map. (We think of A as an endomorphism of the vector space underlying σ.) This implies that the eigenspaces of A with eigenvalues ± 1 are non-zero, and invariant under H. The action of H on these define two characters of H either of which induce to give the representation σ of G.

Remark. The same proof as above can be used when $\dim(\sigma) = 3$ to prove that if $\sigma \cong \sigma \otimes \chi$ for a non-trivial character χ, then χ is of order 3, and σ is induced from a character of a subgroup of index 3 defined by χ.

Corollary 2.5. If the tensor product $\sigma_1 \otimes \sigma_2$ of two 2-dimensional irreducible representations of a group G contains two characters then both σ_1 and σ_2 are induced from a subgroup H of G of index 2.

Proof: Write $\sigma_1 \otimes \sigma_2 = \chi_1 \oplus \chi_2 \oplus \tau$. By Schur's lemma, $\chi_1 \neq \chi_2$, and we have $\sigma_1 \cong \sigma_2^* \otimes \chi_1$, and $\sigma_1 \cong \sigma_2^* \otimes \chi_2$ for distinct characters χ_1 and χ_2. This implies that σ_1 and σ_2 are dihedral.

Corollary 2.6. $Sym^2(\sigma)$ is reducible if and only if σ is dihedral.

Proof : Since $\sigma \otimes \sigma = Sym^2(\sigma) \oplus \Lambda^2(\sigma)$, if $Sym^2(\sigma)$ is reducible, $\sigma \otimes \sigma$ must contain two characters of G, and we are done by the previous corollary.

Lemma 2.7. *Let G be a group, N a subgroup of index 2, and H a subgroup of index 2 of N. Let σ be a 2-dimensional irreducible representation of G which is a sum of 2 characters when restricted to H. Then the representation σ of G is dihedral.*

Proof : Before we begin the proof we note the following elementary fact which will be used many times in the proof without explicit mention. The fact is that if we have a character of a normal subgroup with cyclic quotient which is invariant under the whole group, then the character extends to the group. Now, let $\sigma|_H = \chi_1 \oplus \chi_2$.

Case 1: H is normal in G. In this case all the conjugates of χ_1 under the inner conjugation action of G on H occurs in the restriction of σ to H. Therefore there is an index 2 subgroup of G leaving χ_1 stable. χ_1 will extend to a character of this index 2 subgroup, and σ will be induced from such an extended character.

Case 2: H is not normal in G. In this case there is a unique subgroup $H' \neq H$ in N of index 2 which is a conjugate of H under the action of G. Therefore $H \cap H'$ is a normal subgroup of G. Since H and H' are conjugate in G, σ restricted to H' is also the sum of two characters. If $\chi_1 \neq \chi_2$ on $H \cap H'$, then the eigenspaces of $H \cap H'$ corresponding to χ_1 and χ_2 are unique lines on which the actions of H and H' must extend, therefore N will also leave stable these two lines, and σ will be induced from the character of N on one of these two lines. If $\chi_1 = \chi_2$ on $H \cap H'$, then G operates trivially on χ_1 restricted to $H \cap H'$. Let H'' be the index 2 subgroup of N containing $H \cap H'$ but distinct from H and H'. Then H'' is a normal subgroup of G of index 4 to which χ_1 will extend to, and we are done as in case 1.

Proposition 2.8. *Let σ_1 and σ_2 be two 2-dimensional irreducible representations of a group G. Then if $\sigma_1 \otimes \sigma_2$ is a sum of two 2-dimensional irreducible representations then both σ_1 and σ_2 are dihedral representations.*

Proof: Suppose that $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ with both τ_1 and τ_2 irreducible. Since both σ_1 and σ_2 are two dimensional representations, σ_1 and σ_2 preserve alternating forms up to scaling on a two dimensional vector space. Taking the tensor product of the alternating forms, we get a quadratic form on the vector space underlying $\sigma_1 \otimes \sigma_2$ which $\sigma_1 \otimes \sigma_2$ preserves up to scaling.

Define the nullity of a quadratic space to be the dimension of the maximal subspace which is perpendicular to the whole space under the bilinear form. The maximal null space is invariant under the similitude group. Since τ_1 and τ_2 are irreducible representations, the nullity of the quadratic spaces underlying τ_1 and τ_2 must be either 0 or 2. If the nullity of the vector space underlying τ_1 is 2, i.e., the quadratic form on τ_1 is identically zero, the bilinear form on $\tau_1 \times \tau_2 \to k$ must be non-degenerate. This implies that as representations of G, $\tau_2^* \cong \tau_1 \otimes \chi$ for a character χ of G. Therefore $\tau_1 \oplus \tau_2 \cong \tau_1 \otimes [1 \oplus (\det \tau_1)^{-1}\chi^{-1}]$. If τ_1 is not a dihedral representation, then there is up to scaling a unique quadratic form on $\tau_1 \oplus \tau_2$ which is left invariant up to scaling by G. This implies by Lemma 2.3 that one of σ_1 or σ_2 is reducible, contrary to our assumption. Therefore τ_1 and τ_2 must be dihedral. This is also the case when both τ_1 and τ_2 are non-degenerate subspaces. Therefore in all cases if $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ with both τ_1 and τ_2 irreducible, the representations corresponding to τ_1 and τ_2 land inside $GO(2)$. Since $GSO(2)$ is of index 2 inside $GO(2)$, the representations τ_1 and τ_2 define subgroups H_1 and H_2 of G of index 2. However $H_1 = H_2$ as the representation $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ is inside $GSO(4)$. Since $GSO(2)$ is abelian, we find from Corollary 2.5 combined with Lemma 2.7 that the representations σ_1 and σ_2 are dihedral.

This completes the proof of the theorem at the beginning of the section. We next note the following lemma.

Lemma 2.9. *For 2 dimensional irreducible non-dihedral representations σ_1 and σ_2 of a group G, $Sym^2\sigma_1 \cong Sym^2\sigma_2$ if and only if $\sigma_1 \cong \sigma_2 \otimes \chi$ for a quadratic character χ of G.*

Proof : Taking the determinant of $Sym^2\sigma_1$ and $Sym^2\sigma_2$, we have that $\det \sigma_1^3 = \det \sigma_2^3$. Now the vector space underlying $Sym^2\sigma_1$ has a quadratic form on it for which the similitude factor is $(\det \sigma_1)^2$. Therefore from the isomorphism of $Sym^2\sigma_1$ with $Sym^2\sigma_2$, $(\det \sigma_1)^2 = (\det \sigma_2)^2$. Combining this with the earlier identity $(\det \sigma_1)^3 = (\det \sigma_2)^3$, we get that $\det \sigma_1 = \det \sigma_2$. Therefore $\det \sigma_1^{-1} Sym^2\sigma_1 \cong \det \sigma_2^{-1} Sym^2\sigma_2$. Or, $Ad(\sigma_1) \cong Ad(\sigma_2)$. Therefore $\sigma_1 \cong \sigma_2 \otimes \chi$ for a character χ of G of order 2.

Remark 2.10. More generally, exactly the same argument as above yields that if $Sym^2\sigma_1 \cong Sym^2\sigma_2 \otimes \mu$ for a character μ of G, then $\sigma_1 \cong \sigma_2 \otimes \chi$ for a character χ of G with $\mu = \chi^2$.

§3. Tensor Induction

From the work of many mathematicians culminating in the work of R.Taylor, one knows that to a cohomological automorphic form π on $GL(2)$ of a totally real number field k, there is a 2-dimensional ℓ-adic representation σ_π of $Gal(\overline{\mathbb{Q}}/k)$ with the same L-function as π. If the degree of k over $\mathbb{Q}$ is d, then the automorphic representation π contributes a 2^d dimensional ℓ-adic representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ to the dth cohomology of the corresponding Hilbert modular variety. The process of going from a 2-dimensional representation of $Gal(\overline{\mathbb{Q}}/k)$ to the 2^d-dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is a general one which we review now.

Given any finite dimensional representation V of dimension n of a subgroup H of any group G there is a representation of G denoted by $M(V)$ of dimension n^d which is called *tensor induction* or *multiplicative induction*. We will not recall the definition of $M(V)$ here but refer the reader to [C-R]. However we note that if H is a normal subgroup of G then the representation $M(V)$ of G when restricted to H is the tensor product of the various conjugates of V under the action of G/H. The representation $M(V)$ has the following properties.

(1) $M(V_1 \otimes V_2) \cong M(V_1) \otimes M(V_2)$.

(2) $M(V)^* \cong M(V^*)$

(3) $M(\chi)$ for a character χ of H is the transfer of χ to G, i.e., it is the composite of χ under the transfer map $G/[G,G] \to H/[H,H]$.

We also recall that the transfer map from the Weil group W_k to the subgroup W_K is given by the inclusion of the idele group of k into that of K.

The tensor induction has the property that if H is a normal subgroup of G, then

$$M(V) \cong M(V^g)$$

for the conjugation by any element g of G on any representation V of H.

Finally, for our purposes, if H is a subgroup of G of index 2 and V is a representation of G, then

$$M(V|_H) \cong Sym^2(V) \oplus \omega_{G/H}\Lambda^2 V$$

where $\omega_{G/H}$ is the non-trivial character of G trivial on H.

§4. Cohomology of a Hilbert modular surface

Let S be a Hilbert modular surface associated to a real quadratic field F. We have the decomposition

$$H^2_\ell(S) = IH^2_\ell(\bar{S}) \;\oplus\; H^2_\ell(S^\infty)$$

where $\bar{S}$ is the Baily-Borel compactification of S^o, IH denotes intersection cohomology and S^∞ denotes the divisor at the cusps such that $S = S^o \cup S^\infty$. The action of the Hecke algebra induces a decomposition of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ modules

$$IH^2_\ell(\bar{S}) = \oplus H^2_\ell(\pi).$$

The automorphic representations π of $GL(2)$ over F which appear in the above decomposition have the component at infinity π_∞ to be the lowest discrete series of $PGL(2, F \otimes \mathbb{R})$; in particular, the central character at infinity of such π is trivial.

For representations π as above, there is a representation σ_π of $Gal(\overline{\mathbb{Q}}/F)$ of dimension 2 with the property that

$$L(\sigma_\pi, s) = L(\pi, s)$$

where the L function on the right is the standard L-function associated to π. Automorphic representations π for which there exists a Galois representation σ_π with the above equality of L-functions are called automorphic representations with associated Galois representations.

We have $H^2_\ell(\pi) = M(\sigma_\pi)$ as $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ representations where $M(\sigma_\pi)$ is the 4 dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ obtained from the representation σ_π of $Gal(\overline{\mathbb{Q}}/F)$ by the process of tensor induction of last section. In particular, the representation $H^2_\ell(\pi)$ restricted to $Gal(\overline{\mathbb{Q}}/F)$ is $\sigma_\pi \otimes \sigma_\pi^\tau$ where τ is the non-trivial element of the Galois group of F over $\mathbb{Q}$ operating by conjugation on σ_π.

The work of Harder, Langlands, Rapoport describes the Tate classes in the 2nd cohomology of a Hilbert modular surface. We review some of their work here. We begin with the following curious lemma.

Lemma 4.1. *Suppose that π is a cuspidal, non-CM automorphic representation of $GL(2)$ over a number field K which has associated to it a Galois representation. Suppose K is a quadratic extension k with τ as the Galois automorphism of K over k. If $\pi^\tau \cong \pi \otimes \chi$ for a Grössencharacter χ of K, then χ is trivial when restricted to the ideles of k.*

Proof: Let σ_π be the two dimensional representation of the Galois group $Gal(\overline{\mathbb{Q}}/K)$ of K associated to the automorphic representation π of $GL(2)$ of K, and let $M(\sigma_\pi)$ be the 4-dimensional representation of the Galois group $Gal(\overline{\mathbb{Q}}/k)$ of k which is obtained from σ_π by the process of tensor induction from $Gal(\overline{\mathbb{Q}}/K)$ to $Gal(\overline{\mathbb{Q}}/k)$. The restriction of $M(\sigma_\pi)$ to $Gal(\overline{\mathbb{Q}}/K)$ is isomorphic to $\sigma_\pi \otimes \sigma_\pi^\tau$. Therefore since σ_π is irreducible and is non-CM, $\sigma_\pi \otimes \sigma_\pi^\tau$ contains atmost 1 one-dimensional representation of $Gal(\overline{\mathbb{Q}}/K)$. Therefore under the condition $\sigma_\pi^\tau \cong \sigma_\pi \otimes \chi$, $M(\sigma_\pi)$ contains exactly 1 one-dimensional character of $Gal(\overline{\mathbb{Q}}/k)$. So, $M(\sigma_\pi) = A \oplus \mu$, where A

is a 3 dimensional irreducible representation of $Gal(\overline{\mathbb{Q}}/k)$ and μ is a character of $Gal(\overline{\mathbb{Q}}/k)$. Now we have the isomorphisms

$$M(\sigma_\pi) \cong M(\sigma_\pi^\tau) \cong M(\sigma_\pi \otimes \chi) \cong M(\sigma_\pi) \otimes \chi|_{J_k}.$$

This implies that

$$A \oplus \mu \cong (A \oplus \mu) \otimes \chi|_{J_k}.$$

Comparing the 1-dimensional components, we find that χ restricted to the idele class group of k is trivial.

Lemma 4.2. *Let π_1 and π_2 be two cuspidal representations of $GL(2)$ over a number field k which have associated Galois representations. If π_1 and π_2 are both non-CM and are twists of each other by a Grössencharacter over an extension of k, then π_1 and π_2 are twists of each other over k itself.*

Proof : Let σ_{π_1} and σ_{π_2} be the two dimensional representations of $Gal(\overline{\mathbb{Q}}/k)$ associated to π_1 and π_2 respectively. Since π_i are non-CM, σ_{π_i} remain irreducible and non-dihedral over any number field by the method of Ribet, cf [HLR]. Since σ_{π_1} and σ_{π_2} are twists of each other over an extension which we can assume to be Galois over k, we find that $\sigma_{\pi_1} \otimes \sigma_{\pi_2}^*$ contains a 1-dimensional representation when restricted to a normal extension of k. However, by the non-CM hypothesis and Corollary 2.5, $\sigma_{\pi_1} \otimes \sigma_{\pi_2}^*$ cannot contain more than one 1-dimensional representation. This implies that the corresponding vector in $\sigma_{\pi_1} \otimes \sigma_{\pi_2}^*$ must be invariant under $Gal(\overline{\mathbb{Q}}/k)$, i.e., σ_{π_1} and σ_{π_2} are twists of each other over k.

Remark 4.3. Since the proofs of Lemma 4.1 and 4.2 use existence of Galois representations, we are able to prove the above lemmas only for such automorphic representations (though the statement of the lemmas makes sense more generally). Indeed a very recent preprint of E. Lapid and J. Rogawski [L-R], received by the authors after this paper was written, uses trace formula to prove Lemma 4.1 in general. Since we use only cohomological automorphic representations, Lemmas 4.1 and 4.2 suffice for our purposes. We have abused notation to use χ for both a Grössencharacter and the associated ℓ-adic representation. Such an abuse of notation is done at many other places in the paper.

Proposition 4.4. *Suppose that π is a cuspidal, non-CM automorphic representation of $GL(2)$ over a real quadratic field F which has associated to it a Galois representation. Then, for a number field k, the representation $M(\sigma_\pi)$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ contains a vector on which $Gal(\overline{\mathbb{Q}}/k)$ acts trivially if and only if there exists a representation π_0 of $GL(2)$ over $\mathbb{Q}$ such that*

$$\pi = BC(\pi_0) \otimes \chi$$

with $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{\mathbb{Q}^*} = 1$ *on the image of the ideles of* k *inside the ideles of* $\mathbb{Q}$ *under the norm mapping, and where* ω_{π_0} *is the central character of* π_0, $\omega_{F/\mathbb{Q}}$ *is the character of the ideles of* $\mathbb{Q}$ *associated by class field theory to the quadratic extension* F.

Proof : If $M(\sigma_\pi)$ has a vector on which $Gal(\overline{\mathbb{Q}}/k)$ acts trivially, then in particular $\sigma_\pi \otimes \sigma_\pi^\tau$ has a vector on which $Gal(\overline{\mathbb{Q}}/kF)$ acts trivially. This implies that the representations σ_π and σ_π^τ are twists of each other over kF, and therefore by Lemma 4.2 over F. This implies that

$$\sigma_\pi^\tau \cong \sigma_\pi \otimes \alpha$$

for some Grössencharacter α of $Gal(\overline{\mathbb{Q}}/F)$ which is trivial on the ideles of $\mathbb{Q}$ by Lemma 4.1. Therefore α can be written as $\alpha = \chi^\tau/\chi$. So,

$$(\sigma_\pi \otimes \chi^{-1})^\tau \cong \sigma_\pi \otimes \chi^{-1}.$$

Therefore $\sigma_\pi \otimes \chi^{-1}$ can be written as a base change, i.e., there exists π_0 automorphic on $GL(2)$ over $\mathbb{Q}$ without CM with $\sigma_\pi \otimes \chi^{-1} = \sigma_{\pi_0}|_F$, or $\pi = BC(\pi_0) \otimes \chi$.

Therefore,

$$M(\sigma_\pi) \cong M(\sigma_{\pi_0}|_{Gal(\overline{\mathbb{Q}}/F)}) \otimes \chi|_{J_\mathbb{Q}}.$$

So,

$$M(\sigma_\pi) \cong (Sym^2\sigma_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}}) \otimes \chi|_{J_\mathbb{Q}}.$$

It follows that $M(\sigma_\pi)$ contains a vector on which $Gal(\overline{\mathbb{Q}}/k)$ operates trivially if and only if $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{J_\mathbb{Q}} = 1$ on the image of the ideles of k inside the ideles of $\mathbb{Q}$ under the norm mapping.

Remark 4.5. The ℓ-adic character α of $Gal(\overline{\mathbb{Q}}/F)$ which appears in the above proof (and also in the proof of Lemma 4.2) has been identified to a Grössencharacter of F. This seems to need some argument as not all ℓ-adic characters come from Grössencharacters. Since both the ℓ-adic representations σ_π and σ_π^τ come from cohomology of algebraic varieties, so does the character α of $Gal(\overline{\mathbb{Q}}/K)$ with $\sigma_\pi^\tau \cong \sigma_\pi \otimes \alpha$. Now, it can be shown [cf. B1] that ℓ-adic characters which appear in cohomology of algebraic varieties come from Grössencharacters. We note that two cuspidal automorphic representations of $GL(n), n \geq 3$ with the same adjoint L-function (of degree n^2) are not necessarily twists of each other by a Grössencharacter because of examples of Blasius in [B2].

Question 4.6. Suppose that π_1 and π_2 are two cuspidal automorphic representations of $\mathrm{GL}(n)$ with associated Galois representations σ_1 and σ_2. Suppose that there is an ℓ-adic character χ of the Galois group such that $\sigma_1 = \sigma_2 \cdot \chi$. Then is χ associated to a Grössencharacter?

The next theorem due to Harder, Langlands and Rapaport, and which is a consequence of Proposition 4.4, gives a complete parametrization of Tate classes on a Hilbert Modular surface coming from non-CM automorphic forms.

Theorem 4.7. *Suppose that π is a cuspidal non-CM automorphic representation of $GL(2)$ over a real quadratic field F. Assume that π contributes to the 2nd cohomology of the corresponding Hilbert modular surface. Then this contribution of π to the 2nd cohomology of the Hilbert modular surface contains a Tate class if and only if $\pi \cong BC(\pi_0) \otimes \chi$ for an automorphic representation π_0 of $GL(2)$ over $\mathbb{Q}$, and a Grössencharacter χ of F such that $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{J_\mathbb{Q}} \cdot \nu^{-1}$ is of finite order (ν is the norm character on the ideles). This finite order character defines an abelian extension of $\mathbb{Q}$ which is the field of definition of the corresponding Tate class.*

§5. Tate classes on the product in the non-CM case

Since the first cohomology of a Hilbert modular surface vanishes, we need only consider the Tate cycles which are contained in $H^4_\ell(S_1 \times S_2)$. Moreover, the essential component of this is

$$H^2_\ell(S_1) \otimes H^2_\ell(S_2).$$

Decomposing this $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-module according to the action of the Hecke algebra, we are reduced to considering

$$H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$$

for certain cuspidal automorphic forms π_i on $GL(2, F_i)$.

Theorem 5.1. *Assume that π_1 and π_2 are cuspidal, non-CM automorphic forms on $GL(2)$ over F_1 and F_2 respectively. Then $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ has a Tate class over a number field k which does not come as a product of Tate classes from individual factors if and only if either*

(i) $F_1 = F_2$, and π_1 and π_2 are twists of each other, say $\pi_1 = \pi_2 \otimes \chi$ with the property that

$$(\omega_{\pi_2}\chi)|_{J_\mathbb{Q}} \otimes \nu^{-2}$$

is a finite order character of the idele class group of $\mathbb{Q}$ which is trivial on the image of J_k inside $J_\mathbb{Q}$ under the norm mapping;

or,

(ii) π_1 and π_2 are up to twist by characters base change from $\mathbb{Q}$ to F_1 and F_2 respectively of the same cuspidal representation on $\mathbb{Q}$, say

$$\pi_1 \cong BC(\Pi) \otimes \nu_1$$

$$\pi_2 \cong BC(\Pi^*) \otimes \nu_2,$$

for a cuspidal representation Π *on* $GL(2)$ *over* $\mathbb{Q}$, *and Grössencharacters* ν_1 *and* ν_2 *of* F_1 *and* F_2 *respectively, with the property that*

$$(\nu_1\nu_2)|_{J_{\mathbb{Q}}} \otimes \nu^{-2}$$

is a finite order character of the idele class group of $\mathbb{Q}$ *which is trivial on the image of* J_k *inside* $J_{\mathbb{Q}}$ *under the norm mapping.*

Proof : Assume that there is a Tate class in $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ defined over k; then in particular, $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ has a one-dimensional subspace invariant under $Gal(\overline{\mathbb{Q}}/kF_1F_2)$. By hypothesis π_i are non-CM and therefore σ_{π_i} remain irreducible, non-dihedral over any number field. Let θ_1 be an element of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ which restricts to the non-trivial automorphism of F_1, and if $F_1 \neq F_2$, it restricts to the trivial automorphism of F_2; define θ_2 similarly. From section 2 we know that $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$ restricted to kF_1F_2 is either irreducible, or is the sum of a one dimensional representation and an irreducible 3 dimensional representation. If $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$ contains a 1 dimensional representation when restricted to kF_1F_2, it follows from Lemma 4.2 that $\sigma^{\theta_i}_{\pi_i} \cong \sigma_{\pi_i} \otimes \chi_i$ over F_i for certain characters χ_i of F_i. By Lemma 4.1, χ_i are trivial on the ideles of $\mathbb{Q}$, and therefore $\pi_i = BC(\Pi_i) \otimes \mu_i$. Here BC is base change to F_1 for π_1, and base change to F_2 for π_2.

On the other hand if $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$ is irreducible for $i = 1, 2$, then under the hypothesis that $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ has a Tate cycle over k, we have the isomorphism of $Gal(\overline{\mathbb{Q}}/kF_1F_2)$ modules

$$\sigma_{\pi_1} \otimes \sigma^{\theta_1}_{\pi_1} \cong (\sigma_{\pi_2} \otimes \sigma^{\theta_2}_{\pi_2})^* \otimes \nu^2.$$

Here ν is the norm character of the ideles of $\mathbb{Q}$ which comes up because a Tate cycle in dimension $2n$ carries a twisting by the n-th power of the cyclotomic character.

By Lemmas 2.3 and 4.2, we assume without loss of generality that there is an isomorphism of $Gal(\overline{\mathbb{Q}}/F_1F_2)$-modules

$$\sigma_{\pi_1} \cong \sigma_{\pi_2} \otimes \chi$$

for some Grössencharacter χ of F_1F_2.

Assume that $F_1 \neq F_2$. Since θ_1 operates trivially on F_2, it operates trivially on the representation σ_{π_2} restricted to $Gal(\overline{\mathbb{Q}}/F_1F_2)$. Therefore applying θ_1 to the isomorphism $\sigma_{\pi_1} \cong \sigma_{\pi_2} \otimes \chi$, we find that σ_{π_1} and $\sigma^{\theta_1}_{\pi_1}$ are twists of each other when restricted to F_1F_2, and therefore σ_{π_1} and $\sigma^{\theta_1}_{\pi_1}$ are twists of each other over F_1 contradicting the irreducibility of $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$.

It follows that if $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ has a Tate cycle over a number field k and either $F_1 \neq F_2$, or if $F_1 = F_2$, π_1 and π_2 are not twists of each other, then there are automorphic forms Π_1 and Π_2 for $GL(2)$ over $\mathbb{Q}$, and Grössencharacters μ_1 and μ_2 of F_1 and F_2 respectively, such that

$$\pi_1 \cong BC(\Pi_1) \otimes \mu_1$$

$$\pi_2 \cong BC(\Pi_2) \otimes \mu_2.$$

Now,

$$H^2_\ell(\pi_1) = M(\sigma_{\pi_1}) \cong (Sym^2\sigma_{\Pi_1} \oplus \det \sigma_{\Pi_1} \cdot \omega_{F_1/\mathbb{Q}}) \otimes \mu_1|_{J_\mathbb{Q}},$$

$$H^2_\ell(\pi_2) = M(\sigma_{\pi_2}) \cong (Sym^2\sigma_{\Pi_2} \oplus \det \sigma_{\Pi_2} \cdot \omega_{F_2/\mathbb{Q}}) \otimes \mu_2|_{J_\mathbb{Q}}.$$

Therefore, $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ contains a Tate cycle over k which is not a product of Tate cycles on individual factors if there is an isomorphism of $Gal(\overline{\mathbb{Q}}/k)$ modules

$$\mu_1 \otimes Sym^2\sigma_{\Pi_1} \cong (\mu_2 \otimes Sym^2\sigma_{\Pi_2})^* \otimes \nu^2.$$

From remark 2.10, if the symmetric squares of two non-dihedral representations differ by a character, then the representations themselves differ by a character. So, we can assume that there is cusp form Π on $GL(2)$ of $\mathbb{Q}$, and Grössencharacters ν_1 and ν_2 of F_1 and F_2 respectively, such that

$$\pi_1 \cong BC(\Pi) \otimes \nu_1$$

$$\pi_2 \cong BC(\Pi^*) \otimes \nu_2.$$

Now the condition above for the existence of a Tate class in $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ over k translates into the condition that the character

$$(\nu_1\nu_2)|_{J_\mathbb{Q}} \otimes \nu^{-2}$$

of the idele class group of $\mathbb{Q}$ is trivial on the image of J_k in $J_\mathbb{Q}$ under the norm mapping.

Finally, if $F_1 = F_2$, and $\pi_1 = \pi_2 \otimes \chi$, then

$$M(\sigma_{\pi_1}) \otimes M(\sigma_{\pi_2}) = M(\chi \cdot \omega_{\pi_2} \oplus \chi \cdot Sym^2\sigma_{\pi_2}).$$

This contains a Tate class if and only if the character $(\omega_{\pi_2} \cdot \chi)|_{J_\mathbb{Q}} \otimes \nu^{-2}$ of the idele class group of $\mathbb{Q}$ is trivial on the image of J_k in $J_\mathbb{Q}$ under the norm mapping, proving the theorem.

Corollary 5.2. All the Tate cycles arising out of non-CM forms on a product of two Hilbert modular surfaces are defined over abelian extensions of $\mathbb{Q}$.

The Tate cycles constructed in case (ii) of the previous theorem will be called special Tate cycles, and as mentioned in the introduction, we have not been able to find algebraic cycles corresponding to them.

Definition 5.3. Let Π be a cusp form on $GL(2)$ over $\mathbb{Q}$, μ_1, μ_2 Grössencharacters on F_1, and F_2 respectively, and $\pi_1 = BC(\Pi) \otimes \mu_1$, and $\pi_2 = BC(\Pi^*) \otimes \mu_2$ be cusp forms on $GL(2)$ over F_1 and F_2 respectively. Assume that $(\mu_1\mu_2)|_{J_{\mathbb{Q}}}\nu^{-2}$ is a finite order character. Then in $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ there is a Tate cycle, called *special Tate Cycle.* It is defined over an abelian extension of $\mathbb{Q}$ corresponding to this finite order character.

Remark 5.4. Assume that the cuspidal automorphic representations π_i of $GL(2)$ over F_i are base change of cuspidal automorphic representations Π_i on $GL(2)$ over $\mathbb{Q}$. The special classes occur in

$$H^2(\pi_1) \otimes H^2(\pi_2) \simeq H^1(\Pi_1)^{\otimes 2} \otimes H^1(\Pi_2)^{\otimes 2}$$

where Π_1 and Π_2 contribute to the cohomology of Abelian varieties A_1 and A_2 (say). These Abelian varieties belong to a family for which one knows that all Tate cycles are algebraic. Indeed, A_i has multiplication by a field E_i satisfying $dim\ A_i = [E_i : \mathbb{Q}]$. In [M1] it was proved that for such Abelian varieties which do not have complex multiplication, the ring of Tate cycles is algebraic and generated by the classes of divisors. If A_i has complex multiplication, by a result of Shimura, it is isogenous over $\overline{\mathbb{Q}}$ to a power of an elliptic curve. It is easy to show that if one of the A_i has complex multiplication, then so does the other. Hence, over $\overline{\mathbb{Q}}$, $A_1 \times A_2$ is isogenous to a product of elliptic curves. It is well-known that for such an Abelian variety, all Tate cycles are algebraic. (See [M2], for a proof.) Therefore if the isomorphism $H^2(\pi_i) \simeq H^1(\Pi_i)^{\otimes 2}$ is induced from an algebraic cycle, the Tate cycles that we construct will be algebraic.

Remark 5.5. Our construction of special cycles is very general and seems to be yet another example of a modular construction of Tate cycles which has no apparent geometric realization. An example of a modular construction (unrelated to ours) which *has* been proved algebraic can be found in [EG].

§6. Tate classes in the CM-case

Suppose that π_1 is of CM-type. Then over a sufficiently large field, $H^2_\ell(\pi_1)$ decomposes as a sum of four one-dimensional representations. From this it is easy to see that if there is a Tate class in $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ which is not the product of Tate classes on individual factors, then π_2 must also be a CM form.

Hence we assume in this section that π_1 and π_2 are CM cuspidal automorphic representations on $GL(2)$ over F_1 and F_2 respectively. Let us write $\pi_i = Ind(\psi_i)$ for Grössencharacters ψ_i of an imaginary quadratic extension M_i of F_i. We denote by $V_\ell(\psi_i)$ the ℓ-adic representation of $Gal(\bar{\mathbb{Q}}/M_i)$ associated to ψ_i. The ℓ-adic representation σ_{π_i} is equal to $\mathrm{Ind}_{M_i}^{F_i} V_\ell(\psi_i)$, and therefore the restriction of σ_{π_i} to M_i is $V_\ell(\psi_i) \oplus V_\ell(\bar{\psi}_i)$ where $\bar{\psi}_i$ is the complex conjugate of ψ_i. If θ_i denotes an element of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ which is non-trivial when restricted to F_i, then $\sigma_{\pi_i}^{\theta_i}$ is induced from a Grössencharacter ψ_i' on M_i' ($M_i' = \theta_i(M_i)$) obtained from the Grössencharacter ψ_i on M_i by "transport de structure". Over the field $\tilde{M}_1 = M_1M_1'$, we have the decomposition

$$\begin{aligned} H^2_\ell(\pi_1) &\simeq (V_\ell(\psi_1) \oplus V_\ell(\bar{\psi}_1)) \otimes (V_\ell(\psi_1') \oplus V_\ell(\bar{\psi}_1')), \\ &\simeq V_\ell(\psi_1\psi_1') \oplus V_\ell(\psi_1\bar{\psi}_1') \oplus V_\ell(\bar{\psi}_1\psi_1') \oplus V_\ell(\bar{\psi}_1\bar{\psi}_1'). \end{aligned}$$

We note that the restriction of a Grössencharacter to a field extension corresponds to composition with the norm mapping. So, in the above decomposition over $\tilde{M}_1$, we have abused notation to denote $\psi_1\psi_1'$, for instance, for the Grössencharacter on $\tilde{M}_1$ which is the product of two Grössencharacters on $\tilde{M}_1$ which are obtained from ψ_1 on M_1 and ψ_1' on M_1' via composition with the norm mapping.

For a number field F with normal closure $\tilde{F}$, set $G = Gal(\tilde{F}/\mathbb{Q})$ and $H = Gal(\tilde{F}/F)$. We note that the infinity type of a Grössencharacter on F is the same as an integer valued function on G/H (as G/H can be identified to the set of embeddings of F into $\mathbb{C}$). The advantage of this notation for us is that the infinity type of a Grössencharacter ψ on F thought of as a function on G/H when thought of as a function on G gives the infinity type of the Grössencharacter on $\tilde{F}$ obtained from ψ by composing with the norm mapping from $\tilde{F}$ to F. We also recall that if a Grössencharacter χ of a CM extension K of a totally real field F contributes to the cohomology of the corresponding Hilbert modular variety, then the infinity type of χ is a set of embeddings of K in $\mathbb{C}$ whose restriction to F is precisely the set of embeddings of F into $\mathbb{C}$.

We will use the following lemma several times in the proof of the next proposition.

Lemma 6.1.

(1) *Let f_1, f_2 be two functions on a group G right invariant under subgroups H_1, H_2 of G. If the function $f_1 + f_2$ on G is invariant under the right action of the subgroup H generated by H_1 and H_2 inside G, then f_1 and f_2 are also invariant under the right action of H.*

(2) *Let f_1, f_2, f_3 be three functions on a group G right invariant under subgroups H_1, H_2, H_3 of G. Assume that $f_1 + f_2 = f_3$. If the inner conjugation action of H_1 leaves H_2 and H_3 invariant, and if H_1 is a finite group which is contained in the subgroup generated by H_2 and H_3, then f_2 is invariant under the subgroup generated by H_1 and H_2.*

Proof : We only prove part (2) as part (1) is rather trivial. We have for any $g \in G$, and $h \in H_1$

$$f_1(g) + f_2(g) = f_3(g),$$

$$f_1(gh) + f_2(gh) = f_3(gh).$$

Therefore for any $g \in G$ and $h \in H_1$ we have,

$$f_2(g) - f_2(gh) = f_3(g) - f_3(gh).$$

Since H_1 leaves H_2 and H_3 invariant under the inner conjugation action, this implies that the function $f_2(g) - f_2(gh)$ is invariant under H_2 and H_3, and therefore under H_1. Since H_1 is a finite group, this implies that this function must be identically zero, i.e., $f_2(g)$ is invariant under H_1.

Proposition 6.2. *If $H^2_\ell(\pi_1)\otimes H^2_\ell(\pi_2)$ supports a Tate class which does not come as the tensor product of Tate classes on individual factors, then the M_i are biquadratic extensions of $\mathbb{Q}$ with $M_1 \cap M_2 = M$, a quadratic imaginary extension of $\mathbb{Q}$. Moreover, the infinity type of the Grössencharacters ψ_1 and ψ_2 are invariant under the Galois automorphism of M_1 over M and of M_2 over M.*

Proof : The field M_i is either Galois over $\mathbb{Q}$, or its normal closure $\tilde{M}_i$ is of degree 8 over $\mathbb{Q}$ with Galois group the dihedral group $D_8 = \{x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1}\}$. We know that $H^2_\ell(\pi_i)$ is a sum of 4 Grössencharacters on $\tilde{M}_i$. Therefore $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ is a sum of 16 Grössencharacters on $\tilde{M}_1\tilde{M}_2$. If this is to contain a Tate cycle, then the product of a Grössencharacter χ_1 appearing in $H^2_\ell(\pi_1)$ and a Grössencharacter χ_2 appearing in $H^2_\ell(\pi_2)$ must have the constant infinity type on $\tilde{M}_1\tilde{M}_2$. By Lemma 6.1(1) applied to $G = Gal(\tilde{M}_1\tilde{M}_2/\mathbb{Q}), H_1 = Gal(\tilde{M}_1\tilde{M}_2/\tilde{M}_1), H_2 = Gal(\tilde{M}_1\tilde{M}_2/\tilde{M}_2)$, f_1 the infinity type χ_1, f_2 the infinity type of χ_2, we find that the infinity type of the Grössencharacter χ_1 is invariant under $Gal(\tilde{M}_1/\tilde{M}_1 \cap \tilde{M}_2)$. If $\tilde{M}_1 \cap \tilde{M}_2 = \mathbb{Q}$, then χ_1 and χ_2 themselve correspond to Tate classes, and we need not consider this case. If $\tilde{M}_1 \cap \tilde{M}_2 \neq \mathbb{Q}$, we will need to consider several cases depending on this intersection. The Grössencharacter χ_1 is itself the product of two characters ψ_1 and ψ_1' (or, ψ_1 and $\bar{\psi}_1'$ etc.). We will apply Lemma 6.1(2) to this situation to deduce some properties of ψ_1 and ψ_2 which will complete the proof. We will prove the proposition assuming $F_1 \neq F_2$.

If $\tilde{M}_1 \cap \tilde{M}_2 = M$, a quadratic extension of $\mathbb{Q}$, then if $M_1 = \tilde{M}_1$, and $M_2 = \tilde{M}_2$, both M_1 and M_2 are biquadratic extensions $M_1 = F_1M$, and $M_2 = F_2M$, and since M_i are CM extensions of F_i, M is a quadratic imaginary field. By Lemma 6.1(1), the infinity type of the Grössencharacters χ_1 on M_1 and χ_2 on M_2 are the pull back of the infinity type of Grössencharacters on M. From this it is easy to see that the infinity type of the Grössencharacters ψ_1 and ψ_2 are pull back of Grössencharacters on M.

If $\tilde{M}_1 \neq M_1$, and $\tilde{M}_1 \cap \tilde{M}_2 = M$ a quadratic extension of $\mathbb{Q}$, let M_1' be the image of M_1 under an element of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $M_1' \neq M_1$. Let σ (resp. τ) be the non-trivial element of $\mathrm{Gal}(\tilde{M}_1/\mathbb{Q})$ stabilising M_1 (resp. M_1'). We claim that σ does not act trivially on M, because otherwise τ which is a conjugate of σ will also act trivially on M. This would imply that M is contained in F_1 which is not possible. By Lemma 6.1(2) applied to $G = Gal(\tilde{M}_1/\mathbb{Q}), H_1 =< \sigma >, H_2 =< \tau >, H_3 = \mathrm{Gal}(\tilde{M}_1/M)$, and $f_1 =$ infinity type of ψ_1, $f_2 =$ infinity type of ψ_1', $f_3 =$ infinity type of χ_1, we find that the infinity type of ψ_1 is the pull back of an infinity type from F_1. Such infinity types do not correspond to cohomological representations.

If $\tilde{M}_1 \cap \tilde{M}_2 = M$ is of degree 4 over $\mathbb{Q}$, then neither M_1 nor M_2 is Galois over $\mathbb{Q}$, and M must be the biquadratic field F_1F_2. An application of Lemma 6.1(1) implies that the infinity type of the Grössencharacter χ_1 on $\tilde{M}_1$ is pullback from an infinity type on M. Application now of Lemma 6.1(2) to $G = Gal(\tilde{M}_1/\mathbb{Q}), H_1 =< \sigma >, H_2 =< \tau >, H_3 = \mathrm{Gal}(\tilde{M}_1/M)$, and the same functions as in the last paragraph, implies that the infinity type of the Grössencharacter ψ_1 restricted to $\tilde{M}_1$, and therefore the infinity type of ψ_1 is pull back from F_1. Again this is not allowed as we are considering cohomological representations only. This completes the proof of the proposition.

Since any two Grössencharacters with the same infinity type differ by a finite order character, and since we can construct a Grössencharacter of a number field with a given infinity type (with obvious constraints arising out of Dirichlet Unit theorem), the previous proposition implies the following theorem.

Theorem 6.3. *If π_1 and π_2 are CM forms on $GL(2)$ over F_1 and F_2 respectively such that $H_\ell^2(\pi_1) \otimes H_\ell^2(\pi_2)$ contains a Tate cycle which does not come as the product of Tate cycles from individual factors, then π_1 and π_2 come from Grössencharacters ψ_1 and ψ_2 on biquadratic fields $M_1 = MF_1$ and $M_2 = MF_2$ where M is an imaginary quadratic extension of $\mathbb{Q}$. Moreover the Grössencharacters ψ_1 and ψ_2 are obtained up to finite order characters on the idele class group of M_1 and M_2 respectively, from Grössencharacters ϕ_1 and ϕ_2 of M via the norm mapping, where ϕ_1 corresponds to an embedding of M into $\mathbb{C}$, and ϕ_2 also corresponds to an embedding of M into $\mathbb{C}$. Conversely, such a construction gives rise to a Tate cycle.*

The dimension of the Tate cycles in $H^2_\ell(\pi_1)\otimes H^2_\ell(\pi_2)$ is 6 of which a 4 dimensional subspace is spanned by the tensor product of Tate cycles on individual factors.

Remark 6.4. The automorphic form π_1 (resp. π_2) of $GL(2)$ over F_1 (resp. F_2) in the above theorem is not in general the base change of an automorphic form on $GL(2)$ over $\mathbb{Q}$ even after twisting by a Grössencharacter on F_1 (resp. F_2) unlike in the non-CM case earlier. The Tate cycles in this theorem are not in general defined over abelian extensions of $\mathbb{Q}$ again unlike the non-CM case.

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