RAMIFIED COVERINGS OF S^2 WITH TWO DEGENERATE BRANCHING POINTS

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Abstract. In this work some Hurwitz numbers are calculated. In other words, here one compute numbers of nonequivalent ramified coverings of the Riemann sphere by another Riemann surface with two degenerate branch points for several cases.

1. Introduction

Let C_g be a connected Riemann surface of genus g, S^2 be a Riemann sphere with a fixed infinity and zero, $f : C_g \to S^2$ be a meromorphic function of degree d with k poles of orders $p_1, p_2, \ldots, p_k$ and m zeros of orders $n_1, n_2, \ldots, n_m$ $(\sum\limits_{i=1}^{k} p_i = \sum\limits_{i=1}^{m} n_i = d)$. Such a function determines a ramified covering of the Riemann sphere S^2 by the surface C_g. Two these functions f and f' (or coverings) are called equivalent if there exists a homeomorphism γ of C_g such that $f' = f \circ \gamma$.

Our aim is a calculation of numbers of nonequivalent coverings with the following condition.

A covering has only two degenerated ramification points (zero and infinity).

In other words, we assume that ramification points, except zero and infinity are simple, i.e., they have order 2. We shall say that a function (covering) is *primitive* if it satisfies this condition. The number s of simple ramification points of a primitive covering can be obtained by the Riemann–Hurwitz formula

$$s = s(g, m, k) = 2g - 2 + m + k. \tag{1-1}$$

This work extends research of several mathematicians in this branch. Our starting point it the paper [1], where the authors deal with ramified coverings of S^2 with one degenerate branching point.

Sometimes we use a notation $(1^{i_1}\, 2^{i_2}\, \ldots\, d^{i_d})$ for a partition of a number d. It means that in the partition $(1^{i_1}\, 2^{i_2}\, \ldots\, d^{i_d})$ there are i_1 numbers equal 1, i_2 numbers equal 2, and so on. For partitions α and β of a number d we denote by $CN_d(\alpha;\beta;g)$ the number of nonequivalent meromorphic functions on C_g with zeros and poles of orders α and β respectively. In the article we calculate all concrete numbers $CN_d(\alpha;\beta;g)$ for $d = 3$ (see p. 14), $d = 4$ (p. 19), and $d = 5$ (p. 31). Besides, some general formulae were written in the case when meromorphic functions have one zero of the maximal order. Namely,

$$CN_d((d);(d);g) = \begin{cases} \dfrac{d^{2g-2}\left(B_{2g+1}(d+1) - 2^{2g} B_{2g+1}(\frac{d+2}{2}) - B_{2g+1}(0)(2^{2g}-1) + \frac{d(2g+1)}{2}\right)}{2^{2g-1}(2g+1)}, & \text{for even } d, \\ \dfrac{2\cdot d^{2g-2}}{2g+1}\left(B_{2g+1}\left(\frac{d+3}{2}\right) - B_{2g+1}(0)\right) & \text{for odd } d. \end{cases}$$

If d is odd, then

$$CN_d((d);(l,d-l);g) = \frac{2d^{2g}}{l(d-l)(2g+2)}\left(B_{2g+2}\left(\frac{d+1}{2}\right) - B_{2g+2}\left(\frac{d+1-2l}{2}\right)\right).$$

If d is even and $l < \frac{d}{2}$, then

$$CN_d((d);(l,d-l);g) = \frac{d^{2g}}{(2g+2)2^{2g}l(d-l)}\times$$

$$\times\left(B_{2g+2}(d+1) - B_{2g+2}(d-2l+1) + 2^{2g+1}\left(B_{2g+2}\left(\frac{d-2l+2}{2}\right) - B_{2g+2}\left(\frac{d+2}{2}\right)\right)\right).$$

If d is even and $l = \frac{d}{2}$, then $CN_d((d);(l,d-l);g) =$

$$= \frac{d^{2g-2}}{(2g+2)2^{2g-1}}\left(B_{2g+2}(d+1) - B_{2g+2}(1) + 2^{2g+1}\left(B_{2g+2}(1) - B_{2g+2}\left(\frac{d+2}{2}\right)\right)\right) + \frac{d^{2g}}{2^{2g+1}}.$$

$$CN_d((d);(1^{d-3}\,3)) = \frac{d^(s-1)}{(d-2)!}\left(\sigma(d-1,s+2,\frac{d-1}{2}) + \frac{(d-1)(d-5)}{12}\sigma(d-1,s,\frac{d-1}{2})\right),$$

where $s = 2g+d-3$.

In formulae above $B_p(x)$ denote the Bernoulli polynomial and the numbers $\sigma(n,s,a)$ have the property: $\sum\limits_{s=0}^{\infty}\sigma(n,s,a)\frac{x^s}{s!} = e^{ax}(1-e^{-x})^n$.

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2. Method of calculation

2.1. Consider a primitive covering with fixed orders of zeros: $\alpha = (n_1,\ldots,n_m)$ and of poles: $\beta = (p_1,\ldots,\pi_n)$. Denote by $R_1, R_2, \ldots, R_s$ simple ramification points. A small circle, drowning around a ramification point gives a permutation of covering sheets. If we enumerate the sheets, we can associate to each ramification point an element σ of the symmetric group S_d. In particular, each simple ramification point R_i corresponds to a transposition π_i, zero corresponds to a permutation σ^1_α of a cycle type $\alpha = (n_1,\ldots,n_m)$, and infinity corresponds to a permutation σ_β of a cycle type $\beta = (p_1,\ldots,p_k)$. Obviously,

$$\sigma^1_\alpha\pi_1\cdots\pi_s = \sigma_\beta^{-1}. \tag{2-1}$$

Note that permutations σ and σ^{-1} have the same cycle type.

It is known that any collection of permutation, satisfying (2-1) can be realized by some primitive covering. Moreover two such collections corresponds to equivalent coverings iff there exists a permutation $\sigma \in S_d$ such that

$$\sigma'^1_\alpha = \sigma\sigma^1_\alpha\sigma^{-1},\ \pi'_i = \sigma\pi_i\sigma^{-1}, \qquad i = 1,2,\ldots,s.$$

Let $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$ be the set of ordered collections of transpositions $\{\pi_i\} = (\pi_1, \dots, \pi_s)$, where $s = s(g, m, k)$, such that π_i's are permutation of the symmetric group S_d, $\sigma_\alpha^1 \in S_d$ is a fixed permutation and the product $\sigma_\alpha^1 \pi_1 \cdots \pi_s$ has the same cycle type as σ_β is. By definition, put

$$x_{d,s}(\sigma_\alpha^1, \sigma_\beta) = \left|\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)\right| \text{ (the number of the set's elemets).}$$

Denote by $St_{\sigma_\alpha^1}$ the subgroup of S_d consisting of permutations commuting with σ_α^1. This subgroup acts on the set $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$ by the rule:

$$\sigma : \{\pi_i\} \mapsto \{\sigma \pi_i \sigma^{-1}\}.$$

Obviously, an orbits set of this action one to one corresponds to equivalent classes of coverings with 2 degenerated ramification points of orders α (a cycle type of σ_α^1) and β (a cycle type of σ_β). We'll say that collections $\{\pi_i\}, \{\pi_i'\} \in \Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$ are equivalent, if they belong to one orbit of the $St_{\sigma_\alpha^1}$ action. The number of all equivalent classes of the collections we denote by $N_d((\alpha); (\beta); g)$.

On the other hand, this number equals a quantity of equivalent classes of coverings $f : C_g \to S^2$ with two degenerated ramification points of orders $(n_1, \dots, n_m)$ and $(p_1, \dots, p_k)$, but may be disconnected. Therefore, we have the following identity:

$$N_d(\alpha; \beta; g) = N_d(\beta; \alpha; g).$$

Let $\{\pi_i\}$ be a collection of $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$. Denote by M the subgroup of S_d generated by σ_α^1 and π_i's. This subgroup is called a monodromy group of a corresponding covering. We see that the covering is connected iff the group M transitive acts on the set $\{1, 2, \dots, d\}$. Let us call a collection of $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$ connected if a corresponding covering is connected too and disconnected in the converse case.

For a given set $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta)$ we denote by D the subset of all disconnected collections. Then the number $CN_d(\alpha; \beta; g)$ of all connected nonequivalent coverings is coincides with quantity of $St_{\sigma_\alpha^1}$ orbits in $\Pi_d^s(\sigma_\alpha^1, \sigma_\beta) \setminus D$.

2.2. We see that the first calculation step of Hurwitz numbers is computation of numbers $x_{d,s}(\sigma_\alpha^1, \sigma_\beta)$. Let $A_{\beta\alpha}$ be the number $x_{d,1}(\sigma_\alpha^1, \sigma_\beta)$. In other words, $A_{\beta\alpha}$ is the number of transpositions π such that $\sigma_\alpha^1 \pi = \sigma_\beta$.

In this notation we have

$$x_{d,s}(\sigma_\alpha^1, \sigma_\beta\Big) = \sum_{\gamma \vdash d} x_{d,s-1}(\sigma_\alpha^1, \sigma_\gamma) x_{d,1}(\sigma_\gamma^1, \sigma_\beta) = \sum_{\gamma \vdash d} A_{\beta\gamma} x_{d,s-1}(\sigma_\alpha^1, \sigma_\gamma), \tag{2-2}$$

or in a matrix form,

$$X_{d,s} = A X_{d,s-1}, \quad \text{where } A = (A_{\beta\alpha}),\ X_{d,s} = [x_{d,s}(\sigma_\alpha^1, \sigma_\beta)].$$

We can consider the set of the numbers $A_{\beta\alpha}$ as a linear operator $\mathcal{A}$ on the vector space $Z\mathbb{C}[S_d]$ (the center of group algebra). This operator acts by the rule

$$\mathcal{A} : w_\alpha \mapsto \sum_\beta A_{\beta\alpha} w_\beta,$$

where w_α is an element of $Z\mathbb{C}[S_d]$ equal to sum of all permutations of cycle type α.

2.3. To solve the recursive relations (2-2) it is convenient to know eigenvectors and eigenvalues of the operator $\mathcal{A}$. Suppose that C_α's are the eigenvectors and μ_α's are the eigenvalues. Additionally, assume that

$$C_\alpha = \sum_\beta c_{\beta\alpha} w_\beta.$$

Then recursive relations (2-2) has the form $X'_s = A' X'_{s-1}$, where

$$X'_s = C^{-1} X_{ds},\ C = (c_{\beta\alpha}),\ A' = \operatorname{diag}(\mu_\alpha).$$

Therefore, $X'_s = (A')^s X'_0$, or $x'_{s,\alpha} = \mu_\alpha^s x'_{0,\alpha}$ in the element form. Note that the initial data $\{x'_{0,\alpha}\}$ are determined by the initial data

$$x_{d,0}(\sigma^1_\alpha, \sigma_\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta \end{cases} :$$

$X'_0 = C^{-1} X_{d,0}$, i.e.,

$$x'_{0,\gamma} = \sum_{\beta \vdash d} c'_{\gamma\beta} x_{d,0}(\sigma^1_\alpha, \sigma_\beta) = c'_{\gamma\alpha},$$

where $C^{-1} = (c'_{\gamma\beta})$. Moreover, returning to the old basis, we get

$$x_{d,s}(\sigma^1_\alpha, \sigma_\beta) = \sum_\gamma c_{\beta\gamma} \mu_\gamma^s c'_{\gamma\alpha}.$$

Consider another linear operator $\mathcal{B}$ on the same vector space $Z(\mathbb{C}S_d)$.

$$\mathcal{B} : w_\alpha \mapsto w_\alpha \star w_2 = \sum_{\beta \vdash d} B_{\beta\alpha} w_\beta,$$

where '$\star$' denotes the product of $\mathbb{C}S_d$ and w_2 is the sum of all transpositions.

2.4. Lemma. *For any two partitions α and β of d we have*

$$B_{\beta\alpha} = A_{\alpha\beta}.$$

Proof. Let us calculate a product $w_\alpha \star w_2$.

$$w_\alpha = \sum_i \sigma^i_\alpha, \quad w_2 = \sum_j \pi_j, \quad w_\alpha w_2 = \sum_{ij} \sigma^i_\alpha \pi_j = \sum_{\beta \vdash d} B_{\beta\alpha} w_\beta.$$

Here $B_{\beta\alpha}$ is a number of pairs (σ^i_α, π_j) such that $\sigma^i_\alpha \pi_j = \sigma^1_\beta$. But the last equality is equivalent to $\sigma^i_\alpha = \sigma^1_\beta \pi_j$. Therefore, the number $B_{\beta\alpha}$ is a quantity of transpositions π's such that $\sigma^1_\beta \pi = \sigma_\alpha$, i.e. $B_{\beta\alpha} = A_{\alpha\beta}$.

2.5. Known facts about representations of $\mathbf{S_d}$. In this subsection we calculate eigenvectors and eigenvalues of the operator $\mathcal{B}$ (see 2.4), using standard facts of the group representation theory. The set of complex irreducible representations of the symmetric group S_d one to one corresponds to the set of all partitions of the number d. Denote by χ_α the character of an irreducible representation corresponding to a partition α.

The set of all characters χ_α's farm a basis of the vector space $Z(\mathbb{C}S_d)^*$, dual to the center of group algebra $\mathbb{C}S_d$. By the known theorem characters of irreducible representations satisfy the following orthogonal property

$$\langle \chi_\alpha | \chi_\beta \rangle = \frac{1}{d!} \sum_{\sigma \in S_n} \chi_\alpha(\sigma) \overline{\chi_\beta(\sigma)} = \delta_{\alpha\beta} \quad \forall \alpha, \beta \vdash d. \tag{2-3}$$

This fact is true for irreducible representations of any finite group. In our case values of characters are real and we will omit the symbol of the complex conjugate.

By definition, put $\widetilde{\chi}_\alpha = \frac{1}{d!} \sum\limits_{\sigma \in S_d} \chi_\alpha(\sigma)\sigma \in Z(\mathbb{C}S_d)$. It follows from (2-3) that

$$\chi_\alpha(\widetilde{\chi_\beta}) = \delta_{\alpha\beta}, \tag{2-4}$$

i.e., the set of all $\widetilde{\chi_\alpha}$'s form a basis of $Z(\mathbb{C}S_d)$, dual to characters.

Let V_α be a vector space of an irreducible representation $R_\alpha : S_d \to \operatorname{Aut} V_\alpha$ corresponding to a partition α, then any element $\widetilde{\chi_\beta}$ can be considering as a linear operator on the space V_α. Moreover, $\widetilde{\chi_\beta} \in Z(\mathbb{C}S_d)$ and for any representation R_α, and $\sigma \in S_d$ we have,

$$R_\alpha(\sigma) R_\alpha(\widetilde{\chi_\beta}) = R_\alpha(\widetilde{\chi_\beta}) R_\alpha(\sigma).$$

But if A and B are commuting linear operators, then an eigensubspase of the operator a is invariant w.r.t. the operator B. Therefore, an eigensubspase of $R_\alpha(\widetilde{\chi_\beta})$ is invariant w.r.t. $R_\alpha(\sigma)$ for any σ. This contradicts to irreducibility of R_α. Hence, $R_\alpha(\widetilde{\chi_\beta}) = \lambda_{\alpha\beta} \mathrm{Id}_{V_\alpha}$.

On the other hand, $\operatorname{tr} R_\alpha(\widetilde{\chi_\beta}) = \dim V_\alpha \cdot \lambda_{\alpha\beta}$. In addition, be definition $\operatorname{tr} R_\alpha(\widetilde{\chi_\beta}) = \chi_\alpha(\widetilde{\chi_\beta})$. Therefore,

$$R_\alpha(\widetilde{\chi}_\beta) = \begin{cases} \frac{1}{\dim V_\alpha} \mathrm{Id}_{V_\alpha}, & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

With help of this fact we can compute a product $\widetilde{\chi_\alpha} \star \widetilde{\chi_\beta}$ as elements of the center. To do this we search its coordinates w.r.t. the basis $\{\widetilde{\chi_\alpha}\}$. We have $R_\gamma(\widetilde{\chi_\alpha} \star \widetilde{\chi_\beta}) = R_\gamma(\widetilde{\chi_\alpha}) \circ R_\gamma(\widetilde{\chi_\beta}) = 0$ unless $\gamma = \alpha = \beta$, i.e., $\chi_\gamma(\widetilde{\chi_\alpha} \star \widetilde{\chi_\beta}) = 0$ in these cases. Besides, $R_\alpha(\widetilde{\chi_\alpha} \star \widetilde{\chi_\alpha}) = (R_\alpha(\widetilde{\chi_\alpha}))^2 = \frac{1}{(\dim V_\alpha)^2} \mathrm{Id}_{V_\alpha}$. Whence, $\chi_\alpha(\widetilde{\chi_\alpha} \star \widetilde{\chi_\alpha}) = \frac{1}{\dim V_\alpha}$. Thus,

$$\widetilde{\chi}_\alpha \star \widetilde{\chi}_\beta = \begin{cases} \frac{\widetilde{\chi}_\alpha}{\dim V_\alpha}, & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{2-5}$$

Using dual basises, we obtain $w_2 = \sum_{\alpha \vdash d} \chi_\alpha(w_2) \widetilde{\chi}_\alpha$. Note that, if we denote the set of all cycle type α permutations by W_α (W_2 is the set of all transpositions), then

$$\chi_\alpha(w_2) = \chi_\alpha \left(\sum_{\pi \in W_2} \pi \right) = \chi_\alpha(\pi) \cdot |W_2|.$$

Finally,

$$\widetilde{\chi}_\alpha \star w_2 = \frac{\chi_\alpha(w_2)}{\dim V_\alpha} \widetilde{\chi}_\alpha.$$

In other words, $\mu_\alpha = \frac{\chi_\alpha(w_2)}{\dim V_\alpha} = |W_2|\frac{\chi_\alpha(\pi)}{\chi_\alpha(w_1)}$ is an eigenvalue and $\widetilde{\chi}_\alpha$ is an eigenvector of the linear operator $\mathcal{B}$.

2.6. Let us calculate the conversion matrix $F = (f_{\alpha\beta})$ from the basis $\{w_\alpha\}$ to the eigenbasis $\{\widetilde{\chi}_\beta\}$, i.e., $\widetilde{\chi}_\beta = \sum\limits_\alpha f_{\alpha\beta} w_\alpha$.

$$\widetilde{\chi}_\beta = \frac{1}{d!}\sum_{\sigma\in S_d}\chi_\beta(\sigma)\sigma = \frac{1}{d!}\sum_{\alpha\vdash d}\chi_\beta(\sigma_\alpha)|w_\alpha.$$

Hence,

$$f_{\alpha\beta} = \frac{\chi_\beta(\sigma_\alpha)}{d!}, \qquad \text{and } C^{-1} = (c'_{\alpha\beta}) = \left(\frac{\chi_\alpha(\sigma_\beta)}{d!}\right). \tag{2-6}$$

Taking into account 2.3, lemma 2.4, and a standard fact of linear algebra, we need calculate $F^{-1} = (g_{\alpha\gamma})$. We use the orthogonal relations of characters (2-3).

$$\delta_{\alpha\beta} = \frac{1}{d!}\sum_{\sigma\in S_d}\chi_\alpha(\sigma)\chi_\beta(\sigma) = \frac{1}{d!}\sum_{\gamma\vdash d}\chi_\alpha(\sigma_\gamma)\chi_\beta(\sigma_\gamma)|W_\gamma| = \sum_{\gamma\vdash d}\big(d!|W_\gamma|f_{\gamma\alpha}\big)f_{\gamma\beta}.$$

Whence,

$$g_{\alpha\gamma} = d!|W_\gamma|f_{\gamma\alpha}, \text{ and } C = (F^{-1})^t = (c_{\alpha\beta}) = (d!|W_\alpha|f_{\alpha\beta}). \tag{2-7}$$

Combining together our calculation, we have $x'_{0,\gamma} = \frac{1}{d!}\chi_\gamma(\sigma_\alpha)$ and

$$x_{d,s}(\sigma^1_\alpha,\sigma_\beta) = \frac{|W_\beta|}{d!}\sum_{\gamma\vdash d}\left(\chi_\gamma(\sigma_\beta)\chi_\gamma(\sigma_\alpha)\left(\frac{\chi_\gamma(w_2)}{\chi_\gamma(w_1)}\right)^s\right). \tag{2-8}$$

It is convenient for concrete calculations to rewrite the last formula in the matrix form. Assume that we ordered partitions of the number d and denote by χ the matrix $\big(\chi_\alpha(\beta)\big)$ such that in α-th row and β-th column is a value of the character χ_α on a permutation σ_β of the cycle type β.

Let W and μ be diagonal matrixes such that $W_{\alpha\alpha} = |W_\alpha|$ and $\mu_{\alpha\alpha} = \frac{\chi_\gamma(w_2)}{\chi_\gamma(w_1)}$. In this notation we get

$$\big(x_{d,s}(\sigma^1_\alpha,\sigma_\beta)\big) = \frac{1}{d!}\chi^t\mu^s\chi W. \tag{2-9}$$

3. The case of one zero.

In this section we deal with a covering of the sphere by a Riemann surface of genus g with two degenerate ramification points, one of them has the maximal order. In notation of the introduction, we calculate numbers $CN_d((d),\beta;g)$. We fix symbol τ for a cycle of maximal length, belonging to S_d.

3.1. It is known, that if χ_λ is a character of irreducible representation of S_d corresponding to a partition $\lambda = (1^{i_1}\,2^{i_2}\,\ldots\,d^{i_d})$, then

$$\chi_\lambda(\tau) = \begin{cases} (-1)^{k+1}, & \text{if } \lambda = (1^{k-1}\,(d-k+1)),\\ 0, & \text{in other cases.}\end{cases}$$

Therefore, the general formula (2-8) can be rewritten as follows

$$x_{d,s}(\tau,\sigma_\beta) = \frac{|W_\beta|}{d!}\sum_{\gamma\vdash d}(\chi_\gamma(\sigma_\beta)\chi_\gamma(\tau)\left(\frac{\chi_\gamma(w_2)}{\chi_\gamma(w_1)}\right)^s) = \frac{|W_\beta|}{d!}\sum_{k=1}^{d}\chi_k^d(\sigma_\beta)(-1)^{k+1}\mu_{d,k}^s, \tag{3-1}$$

where $\chi_k^d = \chi_{(1^{k-1}\,(d-k+1))}$ and $\mu_{d,k} = \frac{\chi_k^d(w_2)}{\chi_k^d(w_1)}$.

Let us calculate $\chi_k^d(12)$, i.e. the value of the character χ_k^d on the transposition (12). It is known, that the representation, corresponding to the partition $\lambda_k = (1^{k-1}\,(d-k+1))$ is the $(k-1)$th exterior power of the simplicial representation, if $k \geq 2$, and the trivial one-dimensional representation, if $k=1$. Therefore, $\chi_1^d(12) = 1$.

Let us recall the construction of simplicial representation of the symmetric group S_d. Let V be a $(d-1)$-dimensional vector space with a basis $e_1,\dots e_{d-1}$. By definition put power$e_d = -\sum\limits_{i=1}^{d-1} e_i$. Then the simplicial representation of S_d on the space V can be defined by the rule:

$$\sigma(e_i) = e(\sigma(i)).$$

Therefore, to calculate $\chi_2^d(12)$ we need to fined basis vectors such that $(12)e_i = e_i$. This condition is true for $i = 3,4,\dots,(d-1)$. Hence,

$$\chi_2^d(12) = d-3.$$

The $(k-1)$th exterior of the simplicial representation can be defined as

$$\sigma : e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_{k-1}} \mapsto e_{\sigma(i_1)}\wedge e_{\sigma(i_2)}\wedge\dots\wedge e_{\sigma(i_{k-1})}.$$

We have

$$(12) : e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_{k-1}} \mapsto \begin{cases} -e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_{k-1}}, & \text{if } i_1 = 1,\ i_2 = 2, \\ e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_{k-1}}, & \text{if } i_3 \geq 3, \\ \neq \pm e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_{k-1}}, & \text{in other cases.} \end{cases}$$

This implies that for $k\in[3,d-2]$ the following formulae takes place

$$\chi_k^d(12) = \binom{d-3}{k-1} - \binom{d-3}{k-3} = \frac{(d-2)!(d-2k+1)}{(k-1)!(d-k)!}. \tag{3-2}$$

But assuming that $\binom{b}{a} = 0$ whenever $a > b$ or $a < 0$, the last formula will true for any k.

Now we recall that $\chi_k^d(w_2)$ equals $|W_2|\chi_k^d(12) = \frac{d(d-1)}{2}\chi_k^d(12)$ and $\chi_k^d(w_1)$ is dimension of the space of the corresponding representation, i.e., $\chi_k^d(w_1) = \binom{d-1}{k-1}$. Thus,

$$\mu_{d,k} = \frac{\chi_k^d(w_2)}{\chi_k^d(w_1)} = \frac{d(d-2k+1)}{2}, \tag{3-3}$$

and formula (3-1) can be simplified.

$$x_{d,s}(\tau,\sigma_\beta) = \frac{|W_\beta| d^s}{2^s d!} \sum_{k=1}^{d} \chi_k^d(\sigma_\beta)(-1)^{k+1}\nu_k^s, \tag{3-4}$$

where $\nu_k = d - 2k + 1$.

3.2. Note that each a collection $\{\pi_i\} \in \Pi_d^s(\tau,\sigma_\beta)$ is connected. Therefore, to calculate a number $CN_d((d);\beta;g)$ it is sufficient to compute the number of St_τ orbits in the set $\Pi_d^s(\tau,\sigma_\beta)$.

At first we suppose that d is odd. In this case the subgroup St_τ does not contain order 2 elements, since order of element should divide order of the subgroup. Moreover, each element of this subgroup acts on the set $\{1,2,\ldots,d\}$ without fixed points. Hence, any element of St_τ does not commute with a transposition and the subgroup acts free on the set $\Pi_d^s(\tau,\sigma_\beta)$ in this case. Therefore,

$$CN_d(\tau;\beta;g) = \frac{1}{d} x_{d,s}(\tau,\sigma_\beta)$$

for some s depending of g,β.

Next we consider the case of even d, i.e. $d = 2r$. We have, $\tau^r = \pi_1\pi_2\cdots\pi_r$, where π_i's are transpositions. Denote be M the set of these transpositions. It is clear, that an element of St_τ commutes with such a transposition iff it is τ^r. Therefore,

if a collection from $\Pi_d^s(\tau,\sigma_\beta)$ consists of $\pi_i \in M$ only, then the orbit containing this collection has r points exactly (such a collection we call a collection of the first type);

if a collection from $\Pi_d^s(\tau,\sigma_\beta)$ contains at least one a transposition which does not belong to M, then its orbit has $d = 2r$ points (such a collection we call a collection of the second type).

Let us calculate the number of the first type collections in the set $\Pi_d^s(\tau,\sigma_\beta)$ in accordance with β.

Lemma. *We can enumerate elements π_i's of the set M such that $\tau\pi_i\tau^{-1} = \pi_{i\dot{+}1}$, where $\dot{+}$ denotes the sum in the ring $\mathbb{Z}_r$.*

Proof. Note that the product $\tau\pi\tau^{-1}$ belongs to the set M whenever $\pi \in M$. In fact, this product equals a transposition and commutes with any power of τ. Suppose now, that $\tau^p\pi\tau^{-p} = \tau^q\pi\tau^{-q}$ for some integers $0 \le p < q < r$. Then $\tau^{p-q}\pi\tau^{q-p} = \pi$, i.e. the number $(p-q)$ is divided by r. This contradiction proves the lemma.

Starting at this moment we use indexing of transpositions from M as in the previous lemma.

Lemma. $\tau\pi_i\pi_{i\dot{+}1} = \pi_{i\dot{+}1}\tau\pi_{i\dot{+}1}$.

Proof. The lemma is proved by direct calculation:

$$\tau\pi_i\tau^{-1} = \pi_{i\dot{+}1} \;\Leftrightarrow\; \tau\pi_i = \pi_{i\dot{+}1}\tau \;\Leftrightarrow\; \tau\pi_i\pi_{i\dot{+}1} = \pi_{i\dot{+}1}\tau\pi_{i\dot{+}1}.$$

Let us denote by G the subgroup of S_d which is generated by transpositions of the set M. Obviously, the subgroup G is commutative.

Lemma. *For any pair of transpositions π_i and $\pi_{i\dot{+}p}$ there exists an element $g \in G$ such that*

$$\tau\pi_i\pi_{i\dot{+}p} = g\tau g.$$

Proof. The proof is by induction on p. The case $p = 1$ follows from the previous lemma. Let us do the inductive step.

$$\tau\pi_i\pi_{i\dot{+}p} = (\tau\pi_i\pi_{i\dot{+}(p-1)})\pi_{i\dot{+}(p-1)}\pi_{i\dot{+}p} = (g_1\tau g_1)\pi_{i\dot{+}(p-1)}\pi_{i\dot{+}p} =$$

$$= g_1(\tau\pi_{i\dot{+}(p-1)}\pi_{i\dot{+}p})g_1 = g_1\pi_{i\dot{+}p}\tau\pi_{i\dot{+}p}g_1 = g\tau g.$$

Lemma. *Let $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{2k-1}}, \pi_{i_{2k}}$ be transpositions of the set M. The there exists an element $g \in G$ such that*

$$\tau\pi_{i_1}\pi_{i_2}\cdots\pi_{i_{2k-1}}\pi_{i_{2k}} = g\tau g.$$

Proof. The proof is by induction on k. The case $k = 1$ follows from the previous lemma. Let us do the inductive step.

$$\tau\pi_{i_1}\pi_{i_2}\cdots\pi_{i_{2k-1}}\pi_{i_{2k}} = (\tau\pi_{i_1}\pi_{i_2}\cdots\pi_{i_{2k-3}}\pi_{i_{2k-2}})\pi_{i_{2k-1}}\pi_{i_{2k}} =$$

$$= (g_1\tau g_1)\pi_{i_{2k-1}}\pi_{i_{2k}} = g_1(\tau\pi_{i_{2k-1}}\pi_{i_{2k}})g_1 = g_1(g_2\tau g_2)g_1 = g\tau g.$$

Lemma. *If $\pi \in M$, then cycle type of the product $\tau\pi$ is (r, r).*

Proof. Sins $\tau = (1, 2, \dots, 2r) = (2, 3, \dots, 2r, 1) = \cdots = (2r, 1, \dots, 2r-1)$ and the transposition π has the form $(i, i+r)$, we can assume, that $\pi = (1, r+1)$. Now the lemma follows from calculation:

$$(1, 2, \dots, 2r)(1, r+1) = (1, r+2, r+3, \dots, 2r)(r+1, 2, 3 \dots, r).$$

Corollary. *Let $\pi_1, \pi_2, \dots, \pi_k$ be transpositions from M, then cycle type of the product $\tau\pi_1\pi_2\cdots\pi_k$ is either*

$(2r)$, if k is even, or

(r, r), if k is odd.

3.3. Proposition.

$$CN_{2r+1}(\tau, \sigma_\beta) = \frac{1}{2r+1}x_{2r+1,s}(\tau, \sigma_\beta).$$

$$CN_{2r}(\tau, \sigma_\beta) = \begin{cases} \frac{1}{2r}x_{2r,s}(\tau, \sigma_\beta) + \frac{r^{s-1}}{2}, & \text{if } \beta = (2r) \text{ or } \beta = (r, r) \\ \frac{1}{2r}x_{2r,s}(\tau, \sigma_\beta), & \text{in other cases.} \end{cases}$$

Proof. As we show in the beginning of the section, the subgroup St_τ acts on the set $\Pi_d^s(\tau, \sigma_\beta\big)$ free whenever $d = 2r + 1$. This implies the first formula.

Assume that $d = 2r$. Then the set $\Pi_d^s(\tau, \sigma_\beta\big)$ has a collection of the first type only if cycle type σ_β is either $(2r)$ or (r, r). This fact follows from the corollary. Therefore, the subgroup St_τ acts on the set $\Pi_d^s(\tau, \sigma_\beta\big)$ free unless either $\beta = (2r)$ or $\beta = (r, r)$.

Suppose now, that β is one of the singular partitions. Then the set $\Pi_d^s(\tau, \sigma_\beta\big)$ has collections of the first type. Moreover, quantity of such collections is r^s (in both cases).

These arguments prove the proposition.

3.4. Here we use formula (3-4) to calculate $CN_d((d); (d); g)$. At first, $\left|W_{(d)}\right| = (d-1)!$ and

$$x_{d,s}(\tau, \tau') = \frac{d^{s-1}}{2^s}\sum_{k=1}^{d}(d - 2k + 1)^s,$$

where s is the number of simple ramification point and it equals $2g$ in our situation. In particular, s is even.

By definition put $\sum\limits_{k=1}^{d} \nu_k^s = D_s(d)$ and consider two cases.

The first case is $d = 2n+1$. Then

$$D_s(d) = 2^s \sum_{k=1}^{2n+1} (n-k+1)^s = 2^{s+1} S_s(n+1),$$

where $S_s(m)$ is the standard notation of a sum $\sum\limits_{k=1}^{m-1} k^s$. This sum can be rewritten with help of the Bernoulli polynomials (see [2]):

$$S_s(m) = \frac{1}{s+1}(B_{s+1}(m) - B_{s+1}(0)), \tag{3-5}$$

where $B_p(x) = \sum\limits_{k=0}^{p} \binom{p}{k} b_k x^{m-k}$ is the Bernoulli polynomial and b_k's are Bernoulli numbers.

Thus,

$$x_{d,s}(\tau,\tau') = \frac{2\cdot d^{s-1}}{s+1}\Big(B_{s+1}\left(\frac{d+3}{2}\right) - B_{s+1}(0)\Big).$$

Further, taking into account the proposition on page 9, we have

$$CN_d((d);(d);g) = \frac{2\cdot d^{2g-2}}{2g+1}\Big(B_{s+1}\left(\frac{d+3}{2}\right) - B_{s+1}(0)\Big) \qquad \text{(if } d \text{ is odd).}$$

The second case is $d = 2n$. In this case $D_s(d) = \sum\limits_{k=1}^{2n}(2n-2k+1)^s = 2\sum\limits_{k=1}^{2n}(2k-1)^s$. Denote by $O_s(n)$ the sum $\sum\limits_{k=1}^{n}(2k-1)^s$ and by $E_s(n)$ the sum $\sum\limits_{k=1}^{n}(2k)^s$. Then $O_s(n)+E_s(n) = S_s(2n+1)$.

On the other hand, $E_s(n) = 2^s S_s(n+1)$. Hence,

$$O_s(n) = S_s(2n+1) - 2^s S_s(n+1) = \frac{B_{s+1}(2n+1) - 2^s B_{s+1}(n+1) + B_{s+1}(0)(2^s-1)}{s+1}. \tag{3-6}$$

Therefore,

$$x_{d,2g}(\tau,\tau') = \frac{d^{2g-1}(B_{2g+1}(d+1) - 2^{2g}B_{2g+1}(\frac{d+2}{2}) - B_{2g+1}(0)(2^{2g}-1))}{2^{2g-1}(2g+1)}.$$

Using the proposition on page 9, we get

$$CN_d((d);(d);g) = \frac{d^{2g-2}(B_{2g+1}(d+1) - 2^{2g}B_{2g+1}(\frac{d+2}{2}) - B_{2g+1}(0)(2^{2g}-1) + \frac{d(2g+1)}{2})}{2^{2g-1}(2g+1)}.$$

Combining together our calculations, we can formulate a proposition.

Proposition.

$$CN_d((d);(d);g) = \begin{cases} \frac{d^{2g-2}(B_{2g+1}(d+1) - 2^{2g}B_{2g+1}(\frac{d+2}{2}) - B_{2g+1}(0)(2^{2g}-1) + \frac{d(2g+1)}{2})}{2^{2g-1}(2g+1)} & \text{for even } d, \\ \frac{2\cdot d^{2g-2}}{2g+1}\Big(B_{2g+1}\left(\frac{d+3}{2}\right) - B_{2g+1}(0)\Big) & \text{for odd } d. \end{cases}$$

3.5. Denote by σ_l a permutation from the group S_d of the cycle type $(l, d-l)$ with $l \le d-l$. Then

$$\chi_k^d(\sigma_l) = \begin{cases} (-1)^{k+1}, & \text{if } k \le l, \\ 0, & \text{if } l < k \le d-l, \\ (-1)^k, & \text{if } d-l < k. \end{cases} \tag{3-7}$$

Lemma. In the case $\sigma_\beta = \sigma_l$ formula (3-4) has the form

$$x_{d,s}(\tau, \sigma_l) = \frac{|W_l| d^s}{d! 2^{s-1}} \sum_{k=1}^{l} (d-2k+1)^s. \tag{3-8}$$

Proof. By definition put $\nu_k = (d-2k+1)$ and consider two cases.

The first. If $d = 2n-1$, then the permutation τ is even and for any l the permutation σ_l is odd. Therefore, s is odd and we have the following relations

$$(-1)^{k+1} = \chi_k^d(\tau) = \chi_{d+1-k}^d(\tau) = (-1)^{k+1}, \quad \chi_k^d(\sigma_l) = -\chi_{d+1-k}^d(\sigma_l), \quad \nu_k^s = -\nu_{d+1-k}^s.$$

This implies that $\chi_k^d(\tau)\chi_k^d(\sigma_l)\nu_k^s = \chi_{d+1-k}^d(\tau)\chi_{d+1-k}^d(\sigma_l)\nu_{d+1-k}^s$.

The second. If $d = 2n$, then the permutation τ is odd and for any l the permutation σ_l is even. Therefore, s is odd and we have

$$(-1)^{k+1} = \chi_k^d(\tau) = -\chi_{d+1-k}^d(\tau) = -(-1)^{d+-k}, \quad \chi_k^d(\sigma_l) = \chi_{d+1-k}^d(\sigma_l), \quad \nu_k^s = -\nu_{d+1-k}^s.$$

In particular, the previous relation is true too. Now the proof follows from (3-7).

Proposition. If d is odd, then

$$CN_d((d);(l,d-l);g) = \frac{2d^{2g}}{l(d-l)(2g+2)} \left(B_{2g+2}\left(\frac{d+1}{2}\right) - B_{2g+2}\left(\frac{d+1-2l}{2}\right) \right).$$

If d is even and $l < \frac{d}{2}$, then

$$CN_d((d);(l,d-l);g) = \frac{d^{2g}}{(2g+2)2^{2g} l(d-l)} \times$$

$$\times \left(B_{2g+2}(d+1) - B_{2g+2}(d-2l+1) + 2^{2g+1}\left(B_{2g+2}\left(\frac{d-2l+2}{2}\right) - B_{2g+2}\left(\frac{d+2}{2}\right) \right) \right).$$

If d is even and $l = \frac{d}{2}$, then $CN_d((d);(l,d-l);g) =$

$$= \frac{d^{2g-2}}{(2g+2)2^{2g-1}} \left(B_{2g+2}(d+1) - B_{2g+2}(1) + 2^{2g+1}\left(B_{2g+2}(1) - B_{2g+2}\left(\frac{d+2}{2}\right) \right) \right) + \frac{d^{2g}}{2^{2g+1}}.$$

Proof. We calculate the numbers s with help of the Riemann–Hurwitz formulae: $s = 2g+1$.

Suppose that $d = 2n-1$ is odd. Then $\nu_k = d-2k+1 = 2(n-k)$ and $\sum_{k=1}^{l} \nu_k^s = 2^s \sum_{k=1}^{l} (n-k)^s = 2^s(S_s(n) - S_s(n-l))$. In addition, $\left|W_{(l,d-l)}\right| = \frac{d!}{l(d-l)}$. Finally, by the proposition on page 9 we have

$$CN_d((d);(d,d-l);g) = \frac{1}{d} x_{d,2g+1}(\tau, \sigma_l).$$

Now the first formula of the proposition follows from (3-5) and the previous lemma.

Suppose that $d = 2n$ is even. In this case $\nu_k = d - 2k + 1 = 2(n-k) + 1$ and $\sum_{k=1}^{l} \nu_k^s = O_s(n) - O_s(n-l)$, where $O_s(n) = \sum_{k=1}^{n} (2k-1)^s$. Besides,

$$\left|W_{(l,d-l)}\right| = \begin{cases} \frac{d!}{l(d-l)} & \text{if } l < d/2 \\ \frac{d!}{2l^2} & \text{if } l = d/2. \end{cases}$$

Further, taking into account the proposition on the page 9, formulae (3-5), (3-6) and the previous lemma we obtain the last formulae of the proposition.

3.6. Let us research the case when σ_β is a 3-cycle, i.e., $\sigma_\beta = (abc)$. Then

$$\chi_k^d = \binom{d-4}{k-4} + \binom{d-4}{k-1}.$$

It can be checked that

$$\binom{d-4}{k-4} + \binom{d-4}{k-1} = \frac{1}{d-2}\binom{d-1}{k-1}\left(\frac{3}{4}\nu_k^2 + \frac{(d-1)(d-5)}{4}\right),$$

where $\nu_k = d + 1 - 2k$ as usual. On the other hand, $|W_3| = \frac{d(d-1)(d-2)}{3}$. Therefore,

$$x_{d,s}(\tau,(abc)) = \frac{d^s}{3 \cdot 2^s (d-2)!} \sum_{k=1}^{d} (-1)^{k+1} \binom{d-1}{k-1} \left(\frac{3}{4}\nu_k^2 + \frac{(d-1)(d-5)}{4}\right) \nu_k^s.$$

To write the number $NC_d((d),(1^{d-3}\,3));g$, we use the notation:

$$\sigma(n,s,\alpha) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (\alpha - k)^s.$$

These numbers have the following property.

Lemma ([1]). $\sum_{s=0}^{\infty} \sigma(n,s,\alpha)\frac{x^s}{s!} = e^{\alpha x}(1 - e^{-x})^n$.

Taking into account the lemma, we get

$$\sum_{k=1}^{d} (-1)^{k+1} \binom{d-1}{k-1} \nu_k^s = 2^s \sigma(d-1, s, \frac{d-1}{2}).$$

Hence,

$$x_{d,s}(\tau,(abc)) = \frac{d^s}{(d-2)!}\left(\sigma(d-1, s+2, \frac{d-1}{2}) + \frac{(d-1)(d-5)}{12}\sigma(d-1,s,\frac{d-1}{2})\right).$$

Combining this calculation with the proposition on the page 9 and the fact that $s = 2g + d - 3$, we obtain

Proposition.

$$CN_d((d);(1^{d-3}\,3)) = \frac{d(s-1)}{(d-2)!}\left(\sigma(d-1,s+2,\frac{d-1}{2}) + \frac{(d-1)(d-5)}{12}\sigma(d-1,s,\frac{d-1}{2})\right),$$

where $s = 2g + d - 3$.

4. Coverings of degree 3.

4.1. Here we deal with primitive coverings of degree 3. There exists only one possibility for orders of degenerate ramification points: (3). In the following table we give number s(g,m,k) of simple ramification points for degree 3 primitive coverings.

	(3)	$(1,2)$
(3)	$2g$	$2g+1$
$(1,2)$	$2g+1$	$2g+2$

We start our calculations with numbers $x_{3,s}(\sigma^1_\alpha, \sigma_\beta)$ with help of formula (2-9).

Order partitions of 3: 1^3; $1\,2$; 3 and write the matrixes χ, μ and W (for character values we use the book [3]).

$$\chi = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \mu = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Hence,

$$\begin{aligned} x_{3,s}((12),(ab)) &= 3^s, & x_{3,s}((12),(abc)) &= 2\cdot 3^{s-1} \\ x_{3,s}((123),(ab)) &= 3^s, & x_{3,s}((123),(abc)) &= 2\cdot 3^{s-1} \end{aligned} \tag{4-1}$$

4.2. At first we calculate the number $N_3((1\,2);(1\,2);g)$. Fix the permutation $\sigma^1_\alpha = (12)$ and consider the set $\Pi_3^{2g+2}((12);(ab))$. By formula (4-1) and the definition of $x_{3,s}((12);(ab))$ we obtain

$$|\Pi_3^{2g+2}((12);(ab))| = 3^{2g+2}.$$

The subgroup $St_{(12)}$ acts on the set $\Pi_3^{2g+2}((12);(ab))$ with a unique fixed point $\{\pi_i\} = \{(12)\}$. In addition, order of $St_{(12)}$ is 2. Therefore,

$$N_3((1\,2);(1\,2);g) = \frac{1}{2}(x_{2,2g+2}((12);(ab)) - 1) + 1 = \frac{1}{2}(3^{2g+2} - 1) + 1.$$

Further, the subgroup M generated by the transpositions (12) and π_i's acts on the set $\{1,2,3\}$ transitive whenever at least one of the π_i's does not equal (12). Therefore,

$$CN_3((1\,2);(1\,2);g) = \frac{1}{2}(x_{3,2g+2}((12);(ab)) - 1) = \frac{1}{2}(3^{2g+2} - 1).$$

4.3. The next number we calculate is $N_3((1\,2);(3);g)$. To do this we hold fixed a transposition (12) and consider the set $\Pi_3^{2g+1}((12);(abc))$. According to (4-1),

$$\left|\Pi_3^{2g+1}((12);(abc))\right| = 2\cdot 3^{2g}.$$

The subgroup $St_{(12)}$ acts on the set $\Pi_3^{2g+1}((12);(abc))$ without fixed points. Its order is 2. Hence,

$$N_3((1\,2);(3);g) = \frac{1}{2}\left|\Pi_3^{2g+1}((12);(abc))\right| = 3^{2g}.$$

Moreover, since the monodromy group of a corresponding covering has a cycle of the maximal length, all the coverings are connected, i.e.,

$$CN_3((1\,2);(3);g) = N_3((1\,2);(3);g) = 3^{2g}.$$

4.4. The last number we need to compute is $N_3((3);(3);g)$. It follows from (4-1) that the number $\left|\Pi_3^{2g}((123);(abc))\right|$ equals $2\cdot 3^{2g-1}$. The subgroup $St_{(123)}$ acts free on the set $\Pi_3^{2g}((123);(abc))$ and $\left|St_{(123)}\right| = 3$. Moreover, any collection of this set is connected. Thus,

$$CN_3((3);(3);g) = N_3((3);(3);g) = 2\cdot 3^{2g-2}.$$

These calculations can be summarized in the following tables.

$N_3(\alpha;\beta;g)$

	(3)	$(1,2)$
(3)	$2\cdot 3^{2g-2}$	3^{2g}
$(1,2)$	3^{2g}	$\frac{3^{2g+2}-1}{2}+1$

$CN_3(\alpha;\beta;g)$

	(3)	$(1,2)$
(3)	$2\cdot 3^{2g-2}$	3^{2g}
$(1,2)$	3^{2g}	$\frac{3^{2g+2}-1}{2}$

5. Coverings of degree 4.

5.1. For degree 4 coverings there exist the following possible orders of degenerated ramification points:

$$(1,3), \quad (2,2), \quad (4).$$

Using the Riemann-Hurwitz formula (1-1), find quantities $s = s(m,k,g)$ of simple ramification points.

$\mathbf{s = s(m,k,g)}$

	$(1,1,2)$	$(1,3)$	$(2,2)$	(4)
$(1,1,2)$	$2g+4$	$2g+3$	$2g+3$	$2g+2$
$(1,3)$		$2g+2$	$2g+2$	$2g+1$
$(2,2)$			$2g+2$	$2g+1$
(4)				$2g$

As in the case of degree 3 we order partitions of 4: 1^4; $1^2\,2$; 2^2; $1\,3$; 4 and take character values from the book [3]. We have

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & 0 & -1 \\ 2 & 0 & 2 & -1 & 0 \\ 3 & -1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \mu = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix}$$

and

$$
\begin{aligned}
&x_{4,s}((12),(ab)) = 2^{s-1}(3^s+1), && x_{4,s}((12),(ab)(cd)) = 2^{s-2}(3^s-1),\\
&x_{4,s}((12),(abc)) = 2^{s+1}3^{s-1}, && x_{4,s}((12),(abcd)) = 2^{s-1}(3^s-1),\\
&x_{4,s}((12)(34),(ab)) = 2^{s-1}(3^s-1), && x_{4,s}((12)(34),(ab)(cd)) = 2^{s-2}(3^s+1),\\
&x_{4,s}((12)(34),(abc)) = 2^{s+1}3^{s-1}, && x_{4,s}((12)(34),(abcd)) = 2^{s-1}(3^s+1),\\
&x_{4,s}((123),(ab)) = 2^{s-1}3^{s}, && x_{4,s}((123),(ab)(cd)) = 2^{s-2}3^{s},\\
&x_{4,s}((123),(abc)) = 2^{s+1}3^{s-1}, && x_{4,s}((123),(abcd)) = 2^{s-1}3^{s},\\
&x_{4,s}((1234),(ab)) = 2^{s-1}(3^s-1), && x_{4,s}((1234),(ab)(cd)) = 2^{s-2}(3^s+1),\\
&x_{4,s}((1234),(abc)) = 2^{s+1}3^{s-1}, && x_{4,s}((1234),(abcd)) = 2^{s-1}(3^s+1),
\end{aligned} \tag{5-1}
$$

5.2. Now we are ready to calculate the Hurwitz numbers for coverings of degree 4. Let us start with $N_4((1^2\,2);(1^2\,2);g)$. Recall that this number is the quantity of all the topological nonequivalent coverings $f: C_g \to S^2$ with $2g+6$ simple ramification points. Taking into account (5-1), we have

$$\left|\Pi_4^{2g+4}((12),(ab))\right| = 2^{2g+3}(3^{2g+4}+1).$$

In the set $\Pi_4^{2g+4}((12),(ab))$ there exist fixed collections w.r.t. the $St_{(12)}$ action. The fixed collections consist of the transpositions (12) and (34). Therefore, the number of the fixed collections is 2^{2g+4}. All other orbits have 4 different points. Hence,

$$N_4((1^2\,2);(1^2\,2);g) = 2^{2g+4} + \frac{1}{4}\left(\left|\Pi_4^{2g+4}((12),(ab))\right| - 2^{2g+4}\right) = 2^{2g+1}(3^{2g+4}+7).$$

To calculate the number $CN_4((1^2\,2);(1^2\,2);g)$ consider subsets of $\Pi_4^{2g+4}((12),(ab))$ consisting of disconnected collections.

$$D_1 = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_4^{2g+4}((12),(ab)) \,|\, \pi_i = (12)\ \forall i.\right\},$$

$$D_2 = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_4^{2g+4}((12),(ab)) \,|\, \pi_i \in \{(12),(34)\}\ \forall i\right\},$$

$$D_3 = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_4^{2g+4}((12),(ab)) \,|\, \pi_i(4) = 4\ \forall i\right\},$$

$$D_4 = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_4^{2g+4}((12),(ab)) \,|\, \pi_i(3) = 3\ \forall i,\right\}.$$

We see that a collection $\{\pi_i\} \in \Pi_4^{2g+4}((12),(ab))$ corresponds to a connected covering whenever $\{\pi_i\} \notin D = \bigcup\limits_{\alpha=1}^{4} D_\alpha$. Besides,

$$|D_1| = 1,\ |D_2| = 2^{2g+4},\ |D_3| = |D_4| = \left|\Pi_3^{2g+4}((12),(ab))\right| = 3^{2g+4} \text{ (see } (4-1));$$

$$D_1 \subset D_\alpha\ \forall\alpha, \quad D_\alpha \cap D_\beta = D_1 \text{ for } \alpha \neq \beta.$$

Thus,

$$CN_4((1^2\,2);(1^2\,2);g) \;= \frac{1}{4}\left(\left|\Pi_4^{2g+4}((12),(ab))\right| - 2|D_3| - |D_2| + 2\right) =$$

$$= \frac{1}{2}(3^{2g+4}-1)(2^{2g+2}-1).$$

5.3. The next number is $N_4((1^2\,2);(1\,3);g)$. It follows from (5-1) that

$$\left|\Pi_4^{2g+3}((12),(abc))\right| = 2^{2g+4}\cdot 3^{2g+2}.$$

On the set $\Pi_4^{2g+3}((12),(abc))$ the group $St_{(12)}$ acts free. Since all orbits of this action have 4 collections exactly, we obtain

$$N_4((1^2\,2);(1\,3);g) = 2^{2g+2}\cdot 3^{2g+2}.$$

Now we compute the number $CN_4((1^2\,2);(1\,3);g)$ of connected coverings. Consider the subsets of $\Pi_4^{2g+3}((12),(abc))$ consisting of disconnected collections.

$$D_3 = \left\{(\pi_1,\ldots,\pi_{2g+3} \in \Pi_4^{2g+3}((12),(abc))|\ \pi_i(3)=3\ \forall i\right\},$$

$$D_4 = \left\{(\pi_1,\ldots,\pi_{2g+3} \in \Pi_4^{2g+3}((12),(abc))|\ \pi_i(4)=4\ \forall i\right\}.$$

It is clear that $|D_3| = |D_4|$, $D_3\cap D_4 = \varnothing$ and $|D_4| = \left|\Pi_3^{2g+3}((12),(abc))\right| = 2\cdot 3^{2g+2}$ (see 4-1). Therefore,

$$CN_4((1^2\,2);(1\,3);g) = \frac{1}{4}\left(\left|\Pi_3^{2g+3}((12),(abc))\right| - 2|D_4|\right) = 3^{2g+2}(2^{2g+2}-1).$$

5.4. Let us calculate the number $N_4((1^2\,2);(2^2);g)$. By (5-1) the quantity of elements of the set $\Pi_4^{2g+3}((12);(ab)(cd))$ is $2^{2g+1}(3^{2g+3}-1)$. As before, the group $St_{(12)}$ acts on the set $\Pi_4^{2g+3}((12),(ab)(cd))$ by conjugations and we need to determine number of the action orbits. A collection $\{\pi_i\}$ is fixed w.r.t. this action iff $\pi_i \in \{(12),(34)\}$. On the other hand such a collection belongs to $\Pi_4^{2g+3}((12),(ab)(cd))$ only if quantity of the transpositions (12) into it is even and, consequently, quantity of (34) is odd. Thus the subset of fixed collections (i.e., one point orbits) has 2^{2g+2} elements. Another orbit has 4 elements. Hence,

$$N_4((1^2\,2);(2^2);g) = 2^{2g+2} + 2^{2g-1}(3^{2g+3}-3) = 2^{2g-1}(3^{2g+3}+5).$$

Further, to compute $CN_4((1^2\,2);(2^2);g)$ we remark that only fixed collections w.r.t. $St_{(12)}$ action are disconnected. Therefore,

$$CN_4((1^2\,2);(2^2);g) = 2^{2g-1}(3^{2g+3}-3).$$

5.5. The next number is $N_4((1^2\,2);(4);g)$. Recall that $\Pi_4^{2g+2}((12),(abcd))$ is the set of all collections of transpositions $(\pi_1,\ldots,\pi_{2g+2})$ satisfying (2-1) with $\sigma_\alpha^1 = (12)$ and $\sigma_\beta = (abcd)$, a 4-cycle. By formula (5-1) we have

$$\left|\Pi_4^{2g+2}((12),(abcd))\right| = 2^{2g+1}(3^{2g+2}-1).$$

The subgroup $St_{(12)}$ acts on the set $\Pi_4^{2g+2}((12),(abcd))$ free and each collection of this set is connected. Thus,

$$CN_4((1^2\,2);(4);g) = N_4((1^2\,2);(4);g) = 2^{2g-1}(3^{2g+2}-1).$$

5.6. Let us compute the number $N_4((1\,3);(1\,3);g)$. By relations (5-1),

$$\left|\Pi_4^{2g+2}((123),(abc))\right| = x_{4,2g+2}((123),(abc)) = 2^{2g+3}\cdot 3^{2g+1}.$$

The subgroup $St_{(123)}$ acts free on the set $\Pi_4^{2g+2}((123),(abc))$. Its order equals 3. Whence,

$$N_4((1\,3);(1\,3);g) = 2^{2g+3}\cdot 3^{2g}.$$

To calculate the number of connected coverings, we need reject from $\Pi_4^{2g+2}((123),(abc))$ collections $\{\pi_i\}$ such that $\pi_i(4)=4\ \forall i$. Denote the set of all such collections by D_4. It is clear, that $D_4 = \Pi_3^{2g+2}((123),(abc))$. Hence, $|D_4| = 2\cdot 3^{2g+1}$ (see (4-1)). Thus,

$$CN_4((1\,3);(1\,3);g) = 3^{2g}(2^{2g+3}-2).$$

5.7. Now we wont to search the number $N_4((1\,3);(2^2);g)$. The set $\Pi_4^{2g+2}((123),(ab)(cd))$ has $x_{4,2g+2}((123),(ab)(cd)) = 3^{2g+2}\cdot 2^{2g}$ elements (see (5-1)). The subgroup $St_{(123)}$ acts on this set free. Hence,

$$N_4((1\,3);(2^2);g) = 3^{2g+1}\cdot 2^{2g}.$$

Moreover, any covering with degenerate ramification points of such a type is connected. Therefore,

$$CN_4((1\,3);(2^2);g) = N_4((1\,3);(2^2);g) = 3^{2g+1}\cdot 2^{2g}.$$

5.8. $N_4((1\,3);(4);g)$. It follows from (5-1) that the set $\Pi_4^{2g+1}((123),(abcd))$ has $3^{2g+1}\cdot 2^{2g}$ elements. The group $St_{(123)}$ acts on this set free and any collection of $\Pi_4^{2g+1}((123),(abcd))$ is connected. We have

$$CN_4((1\,3);(4);g) = N_4((1\,3);(4);g) = 3^{2g}\cdot 2^{2g}.$$

5.9. Each a collection $\{\pi_i\}\in\Pi_4^{2g+2}((12)(34),(ab)(cd))$ corresponds to a meromorphic function $f: C_g\to S^2$ with 2 zeros of multiplicities 2 and 2, and 2 poles with the same orders. The subgroup $S_{(12)(34)}$ consists of the following permutations:

$$St_{(12)(34)} = \{id,\ (12),\ (34),\ (12)(34),\ (1324),\ (4231),\ (13)(24),\ (14)(23)\}.$$

In particular, $\left|St_{(12)(34)}\right| = 8$. This group acts on the set $\Pi_4^{2g+2}((12)(34),(ab)(cd))$ by conjugations. Suppose that a collection $\{\pi_i\}\in\Pi_4^{2g+2}((12)(34),(ab)(cd))$ is fixed w.r.t this action. Then there exists a transposition π such that $\sigma\pi\sigma^{-1}=\pi$ for any σ of the subgroup $St_{(12)(34)}$. But $|St_\pi| = 4$. This contradiction proves that the action of the subgroup is free and

$$N_4((2^2);(2^2);g) = \frac{1}{8}x_{4,2g+2}((12)(34),(ab)(cd)) = 2^{2g-3}(3^{2g+2}+1).$$

As usually, D is the subset of $\Pi_4^{2g+2}((12)(34),(ab)(cd))$ consisting of disconnected collections. It is clear that

$$D=\left\{\{\pi_i\}\in\Pi_4^{2g+2}((12)(34),(ab)(cd))|\ \pi_i\in\{(12),\ (34)\}\ \forall i\right\}.$$

A collection $\{\pi\}$ consisting of (12) and (34) belongs to D iff the number of transpositions (12)'s into it is even. Hence, $|D|=2^{2g+1}$ and

$$CN_4((2^2);(2^2);g)=\frac{1}{8}(2^{2g}(3^{2g+2}+1)-2^{2g+1})=2^{2g-3}(3^{2g+2}-1).$$

5.10. Finally, the group $St_{(12)(34)}$ acts free on the set $\Pi_4^{2g+1}((12)(34),(abcd))$ and each a collection of this set corresponds to a connected covering. We get

$$CN_4((2^2);(4);g)=N_4((2^2);(4);g)=2^{2g-3}(3^{2g+1}+1).$$

5.11. To finish our considerations of degree 4 coverings it remains only to calculate

$$CN_4((4);(4);g)=N_4((4);(4);g).$$

It follows from (5-1) that $\left|\Pi_4^{2g}((1234),(abcd))\right|=2^{2g-1}(3^{2g}+1)$. Consider the subgroup $St_{(1234)}$ of S_4 consisting of permutations commuting with (1234). If we denote (1234) by σ, then $St_{(1234)}=\{id,\sigma,\sigma^2,\sigma^{-1}\}$. It is clear, that for any transpositions π the inequality $\sigma\pi\sigma^{-1}\neq\pi$ is true. But in the case $\pi\in\{(13),(24)\}$ we have $\sigma^2\pi\sigma^{-2}=\pi$.

This observation and the definition of the set D imply

$$D=\left\{\{\pi_i\}\in\Pi_4^{2g}((1234),(abcd))|\ \pi_i\in\{(13),(24)\}\ \forall i\right\}.$$

Obviously, $|D|=2^{2g}$ and each an orbit of the $St_{(1234)}$ actions belonging to D has two points exactly. Besides, the subgroup $St_{(1234)}$ acts on the set $\Pi_4^{2g}((1234),(abcd))\setminus D$ free. Therefore,

$$CN_4((4);(4);g)=N_4((4);(4);g)=2^{2g-1}+\frac{1}{4}(2^{2g-1}(3^{2g}+1)-2^{2g})=2^{2g-3}(3^{2g}+3).$$

Rewrite the Hurwitz numbers for degree 4 primitive coverings of S^2 in the following tables.

$N_4(\alpha;\beta;g)$

	$(1^2\,2)$	$(1\,3)$	(2^2)	(4)
$(1^2\,2)$	$2^{2g+1}(3^{2g+4}+7)$	$2^{2g+2}3^{2g+2}$	$2^{2g-1}(3^{2g+3}+5)$	$2^{2g-1}(3^{2g+2}-1)$
$(1\,3)$		$2^{2g+3}3^{2g}$	$2^{2g}3^{2g+1}$	$2^{2g}3^{2g}$
(2^2)			$2^{2g-3}(3^{2g+2}+1)$	$2^{2g-3}(3^{2g+1}+1)$
(4)				$2^{2g-3}(3^{2g}+3)$

$CN_4(\alpha;\beta;g)$

	$(1^2\,2)$	$(1\,3)$	(2^2)	(4)
$(1^2\,2)$	$\frac{(3^{2g+4}-1)(2^{2g+2}-1)}{2}$	$3^{2g+2}(2^{2g+2}-1)$	$2^{2g-1}(3^{2g+3}-3)$	$2^{2g-1}(3^{2g+2}-1)$
$(1\,3)$		$3^{2g}(2^{2g+3}-2)$	$2^{2g}3^{2g+1}$	$2^{2g}3^{2g}$
(2^2)			$2^{2g-3}(3^{2g+2}-1)$	$2^{2g-3}(3^{2g+1}+1)$
(4)				$2^{2g-3}(3^{2g}+3)$

6. The case of degree 5

6.1. Let us put in order partitions of integer 5 in the following way:

$$\alpha_1 = 1^5,\ \alpha_2 = 1^3\,2,\ \alpha_3 = 1\,2^2,\ \alpha_4 = 1^2\,3,\ \alpha_5 = 2\,3,\ \alpha_6 = 1\,4,\ \alpha_7 = 5;$$

We use characters values of the symmetric group S_5 from the book [3] to write the matrixes:

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & 1 & -1 & 0 & -1 \\ 5 & 1 & 1 & -1 & 1 & -1 & 0 \\ 6 & 0 & -2 & 0 & 0 & 0 & 1 \\ 5 & -1 & 1 & -1 & -1 & 1 & 0 \\ 4 & -2 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 \end{pmatrix},$$

$$\mu = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 \end{pmatrix}.$$

Then, taking into account (2-9), we have

$$\begin{aligned} &x_{5,s}((12345),(ab)) = \tfrac{1}{3}5^s(2^{s-1}-1), && x_{5,s}((12345),(ab)(cd)) = 5^s2^{s-2}, \\ &x_{5,s}((12345),(abc)) = \tfrac{1}{3}5^s(2^s-1), && x_{5,s}((12345),(ab)(cde)) = \tfrac{1}{3}5^s(2^s+1), \\ &x_{5,s}((12345),(abcd)) = 5^s2^{s-1}, && x_{5,s}((12345),(abcde)) = 5^{s-1}(2^{s+1}+2). \end{aligned} \tag{6-1}$$

$$\begin{aligned} &x_{5,s}((1234),(abcd)) = 2^{s-1}(5^s+1), && x_{5,s}((1234),(ab)(cde)) = \tfrac{1}{3}2^s(5^s-1), \\ &x_{5,s}((1234),(abc)) = \tfrac{1}{3}2^s(5^s+1), && x_{5,s}((1234),(ab)(cd)) = 2^{s-2}(5^s-1), \\ &x_{5,s}((1234),(ab)) = \tfrac{1}{3}2^{s-1}(5^s-1), \end{aligned} \tag{6-2}$$

$$x_{5,s}((12)(345),(ab)(cde)) = \tfrac{1}{3}(10^s+5^s+2^s), \quad x_{5,s}((12)(345),(abc)) = \tfrac{1}{3}(10^s-5^s-2^s),$$
$$x_{5,s}((12)(345),(ab)(cd)) = 2^{s-2}(5^s+1), \qquad x_{5,s}((12)(345),(ab)) = \tfrac{1}{3}(2^{s-1}(5^s+1)-5^s); \tag{6-3}$$

$$x_{5,s}((123),(abc)) = \tfrac{1}{3}(10^s+5^s+2^s), \qquad x_{5,s}((123),(ab)(cd)) = 2^{s-2}(5^s-1),$$
$$x_{5,s}((123),(ab)) = \tfrac{1}{3}(2^{s-1}(5^s-1)+5^s); \tag{6-4}$$

$$x_{5,s}((12)(34),(ab)(cd)) = 2^{s-2}(5^s+1), \quad x_{5,s}((12)(34),(ab)) = \tfrac{1}{3}2^{s-1}(5^s+1); \tag{6-5}$$

$$x_{5,s}((12),(ab)) = \frac{1}{3}(2^{s-1}(5^s+1)+2\cdot 5^s). \tag{6-6}$$

6.2. Number of simple ramification points. We use Riemann–Hurwitz formulae (1-1) to fill the table of numbers of simple ramification points for corresponding coverings.

	$1^3\,2$	$1\,2^2$	$1^2\,3$	$2\,3$	$1\,4$	5
$1^3\,2$	$2g+6$					
$1\,2^2$	$2g+5$	$2g+4$				
$1^2\,3$	$2g+5$	$2g+4$	$2g+4$			
$2\,3$	$2g+4$	$2g+3$	$2g+3$	$2g+2$		
$1\,4$	$2g+4$	$2g+3$	$2g+3$	$2g+2$	$2g+2$	
5	$2g+3$	$2g+2$	$2g+2$	$2g+1$	$2g+1$	$2g$

6.3. Now we calculate numbers $N_5((5);\beta;g)$. It is clear, that

$$N_5((5);\beta;g) = CN_5((5);\beta;g).$$

The subgroup $St_{(12345)}$ acts on sets $\Pi_5^s((12345),\sigma_\beta)$ free and each an orbit of this action has 5 points. Therefore,

$$CN_5((5);(5);g) = N_5((5);(5);g) = \tfrac{1}{5}x_{5,2g}((12345),(abcde)) = 5^{2g-2}(2^{2g+1}+2);$$
$$CN_5((5);(1\,4);g) = N_5((5);(1\,4);g) = \tfrac{1}{5}x_{5,2g+1}((12345),(abcd)) = 5^{2g}2^{2g};$$
$$CN_5((5);(2\,3);g) = N_5((5);(2\,3);g) = \tfrac{1}{5}x_{5,2g+1}((12345),(ab)(cde)) = 5^{2g}\tfrac{2^{2g+1}+1}{3};$$
$$CN_5((5);(1^2\,3);g) = N_5((5);(1^2\,3);g) = \tfrac{1}{5}x_{5,2g+2}((12345),(abc)) = 5^{2g+1}\tfrac{2^{2g+2}-1}{3};$$
$$CN_5((5);(1\,2^2);g) = N_5((5);(1\,2^2);g) = \tfrac{1}{5}x_{5,2g+2}((12345),(ab)(cd)) = 5^{2g+1}2^{2g};$$
$$CN_5((5);(1^3\,2);g) = N_5((5);(1^3\,2);g) = \tfrac{1}{5}x_{5,2g+3}((12345),(ab)) = 5^{2g+2}\tfrac{2^{2g+2}-1}{3}.$$

6.4. Applying the proposition on page 9, we compute the numbers $N_5((1\,4);\beta;g)$.

$$N_5((1\,4);(1\,4);g)=\frac{1}{4}(x_{5,2g+2}((1234),(abcd))-2^{2g+2})+2^{2g+1}=2^{2g-1}(5^{2g+2}+3).$$

$$N_5((1\,4);(1\,2^2);g)=\frac{1}{4}(x_{5,2g+3}((1234),(ab)(cd))-2^{2g+3})+2^{2g+2}=2^{2g-1}(5^{2g+3}+3).$$

$$CN_5((1\,4);(2\,3);g)=N_5((1\,4);(2\,3);g)=\frac{1}{4}x_{5,2g+2}((1234),(ab)(cde))=2^{2g}\frac{5^{2g+2}-1}{3}.$$

$$N_5((1\,4);(1^2\,3);g)=\frac{1}{4}x_{5,2g+3}((1234),(abc))=2^{2g+1}\frac{5^{2g+3}+1}{3}.$$

$$N_5((1\,4);(1^3\,2);g)=\frac{1}{4}x_{5,2g+4}((1234),(ab))=2^{2g+1}\frac{5^{2g+4}-1}{3}.$$

Note that if a transposition $\tau\in S_5$ satisfy the condition: $\tau(5)=5$, then

$$\Pi_5^{2g+3}((1234),\tau)\supset\Pi_4^{2g+3}((1234),\tau).$$

Moreover, the action of the subgroup $St_{(1234)}$ on this sets commutes with the inclusion, and disconnected collections belong to $\Pi_4^{2g+3}((1234),\tau)$. Therefore,

$$CN_5((1\,4);(1,p_2,\dots,p_m);g)=N_5((1\,4);(1,p_2,\dots,p_m);g)-N_4((4);(p_2,\dots,p_m);g+1).$$

We have

$$CN_5((1\,4);(1\,4);g)=2^{2g-1}(5^{2g+2}-3^{2g+2});$$

$$CN_5((1\,4);(1\,2^2);g)=2^{2g-1}(5^{2g+3}-3^{2g+3}+2);$$

$$CN_5((1\,4);(1^2\,3);g)=2^{2g+1}\frac{5^{2g+3}-2\cdot3^{2g+3}+1}{3}.$$

$$CN_5((1\,4);(1^3\,2);g)=2^{2g+1}\frac{5^{2g+4}-3^{2g+5}+2}{3}.$$

6.5. Lemma. Let M^r be the set of ordered collections $(a_1,\dots,a_r)$ such that $a_i\in\{-1,1,2,3\}$. By definition put,

$$M^r_+=\{(a_1,\dots,a_r)|a_1\cdot a_2\cdots a_r>0\},\quad M^r_-=\{(a_1,\dots,a_r)|a_1\cdot a_2\cdots a_r<0\}.$$

Then

$$|M^r|=4^r,\qquad |M^r_+|=\frac{4^r+2^r}{2},\qquad |M^r_-|=\frac{4^r-2^r}{2}.$$

Proof. The first equality is trivial. We'll prove the other equalities for even r only. The case of odd r is proved similarly. For a collection $(a_1, \dots, a_r)$ we denote by $n(a_1, \dots, a_r)$ a quality of its elements a_i's such that $a_i = -1$. Besides, we denote by $M_+^r(n)$ the subset of M_+^r consisting of collections $(a_1, \dots, a_r)$ such that $n(a_1, \dots, a_r) = 2n$. Then $M_+^{2r} = \bigcup\limits_{n=0}^{r} M_+^{2r}(n)$.

It is clear that $|M_+^{2r}(n)| = 3^{2r-2n}\binom{2r}{2r-2n}$, i.e.,

$$|M_+^{2r}| = \sum_{n=0}^{r} 3^{2r-2n}\binom{2r}{2r-2n}.$$

By the binomial formula we have

$$4^{2r} = (1+3)^{2r} = \sum_{n=0}^{r} 3^{2r-2n}\binom{2r}{2r-2n} + \sum_{n=0}^{r-1} 3^{2r-2n-1}\binom{2r}{2r-2n-1}.$$

$$2^{2r} = (1-3)^{2r} = \sum_{n=0}^{r} 3^{2r-2n}\binom{2r}{2r-2n} - \sum_{n=0}^{r-1} 3^{2r-2n-1}\binom{2r}{2r-2n-1}.$$

Therefore, $|M_+^{2r}| = \frac{4^{2r}+2^{2r}}{2}$. This completes the proof.

6.6. The number $N_5((2\,3);(2\,3);g)$. On the set $\Pi_5^{2g+2}((12)(345);(ab)(cde);g)$ the subgroup $St_{(12)(345)}$ acts by conjugations. To compute number of orbits of this action we consider the set Π consisting of ordered collections $(\pi_1, \dots, \pi_{2g+2})$, where π_i's are transpositions. Let us divide the set Π into 3 subsets:

Π_6 is a join of all length 6 $St_{(12)(345)}$ orbits;
Π_3 is a join of all length 3 $St_{(12)(345)}$ orbits;
Π_1 is a join of all length 1 $St_{(12)(345)}$ orbits.

It is clear, that Π_1 includes one point only. Namely, $\Pi_1 = \{(12), (12), \dots, (12)\}$. Collections of transpositions, belonging to the subset Π_3 consist of transpositions from $\{(12), (34), (35), (45)\}$.

We see that $\Pi_1 \subset \Pi_5^{2g+2}((12)(345);(ab)(cde);g)$, since $(12)(345)\underbrace{(12)\cdots(12)}_{2g+2} = (12)(345)$.

In addition, if a collection $(\pi_1, \dots, \pi_{2g+2})$ belongs to $\Pi_3 \cap \Pi_5^{2g+2}((12)(345);(ab)(cde);g)$ then it has even quality of the transposition (12). It follows from lemma 6.5, that $|\Pi_3 \cup \Pi_1| = \frac{4^{2g+2}+2^{2g+2}}{2}$, i.e. $|\Pi_3| = \frac{4^{2g+2}+2^{2g+2}}{2} - 1$.

By definition put,

$$\Pi_3(id) = \{(\pi_1, \dots, \pi_{2g+2}) \in \Pi_3 |\ (12)(345)\pi_1 \cdots \pi_{2g+2} = id\},$$

$$\Pi_3(345) = \{(\pi_1, \dots, \pi_{2g+2}) \in \Pi_3 |\ (12)(345)\pi_1 \cdots \pi_{2g+2} = (345)\},$$

$$\Pi_3(543) = \{(\pi_1, \dots, \pi_{2g+2}) \in \Pi_3 |\ (12)(345)\pi_1 \cdots \pi_{2g+2} = (543)\},$$

We have

$$\Pi_3 \cap \Pi_5^{2g+2}((12)(345);(ab)(cde);g) = \Pi_3(345) \cup \Pi_3(543).$$

Let φ be the map from Π_3 to Π_3 taking a collection $(\pi_1, \dots, \pi_{2g+1})$ to the collection, that can be obtained by the following rule. We change the last transposition different from (12) (π_j for example) to the product $\pi_j(34)(35)$. We see, that φ is bijective and

$$\varphi : \Pi_3(id) \to \Pi_3(543), \quad \varphi : \Pi_3(543) \to \Pi_3(345), \quad \varphi : \Pi_3(345) \to \Pi_3(id).$$

Therefore, $|\Pi_3(id)| = |\Pi_3(345)| = |\Pi_3(543)| = \frac{1}{3}|\Pi_3|$. Thus,

$$|\Pi_3 \cap \Pi_5^{2g+2}((12)(345); (ab)(cde); g)| = \frac{4^{2g+2} + 2^{2g+2} - 2}{3}.$$

Now we are ready to calculate the number $N_5((2,3);(2,3);g)$.

$$N_5((2\,3);(2\,3);g) = \frac{1}{6}\Big(x_{5,2g+2}((12)(345)) - |\Pi_3 \cap \Pi_5^{2g+2}((12)(345);(ab)(cde);g)| - |\Pi_1|\Big) +$$

$$+\frac{1}{3}|\Pi_3 \cap \Pi_5^{2g+2}((12)(345);(ab)(cde);g)| + 1 =$$

$$= \frac{1}{18}(10^{2g+2} + 5^{2g+2} + 4^{2g+2} + 2^{2g+3} + 13).$$

Remark now that a connected collection from the set $\Pi_5^{2g+2}((12)(345);(ab)(cde))$ includes at least one transposition different from (12), (34), (35) and (45). Therefore, the subset of all connected collections is

$$\Pi_5^{2g+2}((12)(345);(ab)(cde)) \setminus (\Pi_3 \cup \Pi_1) \text{ and}$$

$$CN_5((2\,3);(2\,3);g) = \frac{1}{18}(10^{2g+2} + 5^{2g+2} - 4^{2g+2} - 1).$$

6.7. The number $N_5((2\,3);(1^2\,3);g)$. Here we denote by the symbol Π the set of ordered collections $(\pi_1,\ldots,\pi_{2g+3})$ with π_i's are transpositions of the group S_5. As before the subgroup $St_{(12)(345)}$ acts on this set and we divide the set into 3 subsets $\Pi = \Pi_6 \cup \Pi_3 \cup \Pi_1$, where Π_k is a join of all length k $St_{(12)(345)}$ orbits. Obviously,

$$\Pi_1 = \{(12),(12),\ldots,(12)\} \in \Pi_5^{2g+3}((12)(345);(abc))$$

and a collection from $\Pi_3 \cap \Pi_5^{2g+3}((12)(345);(abc))$ consists of transpositions (12), (34), (35), (45). Moreover, a quantity of the transpositions (12) in it is odd. We denote by Π_3^- the subset of collections from Π_3, satisfying the last condition. By lemma 6.5 we get $|\Pi_3^- \cup \Pi_1| = \frac{1}{2}(4^{2g+3} - 2^{2g+3})$, i.e., $|\Pi_3^-| = \frac{4^{2g+3}-2^{2g+3}-2}{2}$.

As in the previous section we divide the set Π_3^- into 3 classes:

$\Pi_3^-(id) = \{(\pi_1,\ldots,\pi_{2g+3}) \in \Pi_3^- \mid (12)(345)\cdot\pi_1\cdots\pi_{2g+3} = id\}$;

$\Pi_3^-(345) = \{(\pi_1,\ldots,\pi_{2g+3}) \in \Pi_3^- \mid (12)(345)\cdot\pi_1\cdots\pi_{2g+3} = (345)\}$;

$\Pi_3^-(543) = \{(\pi_1,\ldots,\pi_{2g+3}) \in \Pi_3^- \mid (12)(345)\cdot\pi_1\cdots\pi_{2g+3} = (543)\}$.

As before, we have $|\Pi_3^-(id)| = |\Pi_3^-(345)| = |\Pi_3^-(543)| = \frac{1}{3}|\Pi_3^-|$ and

$$|\Pi_3 \cap \Pi_5^{2g+3}((12)(345),(abc))| = \frac{2}{3}|\Pi_3^-| = \frac{4^{2g+3} - 2^{2g+3} - 2}{3}.$$

Whence,

$$N_5((2\,3);(1^2\,3);g) =$$

$$= \frac{1}{6}\Big|\Pi_5^{2g+3}((12)(345),(abc)) \setminus (\Pi_3 \cup \Pi_1)\Big| + \frac{1}{3}\Big|\Pi_5^{2g+3}((12)(345),(abc)) \cap \Pi_3\Big| + |\Pi_1| =$$

$$= \frac{1}{18}(10^{2g+3} - 5^{2g+3} + 4^{2g+3} - 2^{2g+4} + 13).$$

It is remains to note that all connected collections belong to $\Pi_5^{2g+3}((12)(345),(abc)) \setminus (\Pi_3 \cup \Pi_1)$, i.e.

$$CN_5((2\,3);(1^2\,3);g) = \frac{1}{18}(10^{2g+3} - 5^{2g+3} - 4^{2g+3} - 1).$$

6.8. The number $N_5((2\,3);(1\,2^2);g)$. We use notations $\Pi,\ \Pi_1,\ \Pi_3$, from the previous section. As above, the set Π_3 includes collections consisting of the transpositions $(12),\ (34),\ (35),\ (45)$. But in this case $\Pi_1 \cap \Pi_5^{2g+3}((12)(345),(ab)(cd)) = \emptyset$ and a collection belonging to $\Pi_3 \cap \Pi_5^{2g+3}((12)(345),(ab)(cd))$ has even quantity of transposition equal to (12), i.e.,

$$\Pi_3 \cap \Pi_5^{2g+3}((12)(345),(ab)(cd)) \subset \Pi_3^+.$$

On the other hand, if $(\pi_1,\dots,\pi_{2g+3}) \in \Pi_3^+$, then the product $(12)(345)\cdot\pi_1\cdots\pi_{2g+3}$ is a permutation of cycle type $(2,2)$. Therefore, $\Pi_3 \cap \Pi_5^{2g+3}((12)(345),(ab)(cd)) = \Pi_3^+$. Taking into account lemma 6.5, we get $|\Pi_3^+| = \frac{4^{2g+3}+2^{2g+3}}{2}$. Thus,

$$N_5((2\,3);(1\,2^2);g) = \frac{1}{6}\left|\Pi_5^{2g+3}((12)(345),(ab)(cd)) \setminus \Pi_3^+\right| + \frac{1}{3}\left|\Pi_3^+\right| =$$

$$= 2^{2g}\frac{5^{2g+3} + 2^{2g+4} + 3}{3}$$

and

$$CN_5((2\,3);(1\,2^2);g) = \frac{1}{6}\left|\Pi_5^{2g+3}((12)(345),(ab)(cd)) \setminus \Pi_3^+\right| = 2^{2g}\frac{5^{2g+3} - 2^{2g+4} - 1}{3}.$$

6.9. The number $N_5((2\,3);(1^3\,2);g)$. Let Π be the set of ordered collections $(\pi_1,\dots,\pi_{2g+4})$. The subgroup $St_{(12)(345)}$ acts on this set and we denote by Π_k the subset of Π as usual. At first we note, that $\Pi_1 \cap \Pi_5^{2g+4}((12)(345);(ab)) = \emptyset$.

Let Π_3^+ be the subset of Π_3 consisting of collections such that a quantity of the transpositions equal to (12) in it is even. In addition, we put $\Pi_3^- = \Pi_3 \setminus \Pi_3^+$. It is easy to prove, that

$$\Pi_3 \cap \Pi_5^{2g+4}((12)(345);(ab)) = \Pi_3^- \cup \Pi_3^+(id),$$

where

$$\Pi_3^+(id) = \{(\pi_1,\dots,\pi_{2g+4}) \in \Pi_3^+ |\ (345)\pi_1\cdots\pi_{2g+4} = id\}.$$

Besides,

$$\left|\Pi_3^+(id)\right| = \frac{1}{3}\left(\left|\Pi_3^+\right| - 1\right) = \frac{4^{2g+4} + 2^{2g+4} - 2}{6} \quad \text{and} \quad \left|\Pi_3^-\right| = \frac{4^{2g+4} - 2^{2g+4}}{2}.$$

Therefore,

$$\left|\Pi_3 \cap \Pi_5^{2g+4}((12)(345);(ab))\right| = \frac{4^{2g+5} - 2^{2g+5} - 2}{6}.$$

Thus,

$$N_5((2\,3);(1^3\,2)) = \frac{1}{6}\left|\Pi_5^{2g+4}((12)(345);(ab)) \setminus \Pi_3\right| + \frac{1}{3}\left|\Pi_3 \cap \Pi_5^{2g+4}((12)(345);(ab))\right| =$$

$$= \frac{1}{36}(10^{2g+4} - 2\cdot 5^{2g+4} + 4^{2g+5} - 2^{2g+4} - 2).$$

$$CN_5((2\,3);(1^3\,2)) = \frac{1}{6}\left|\Pi_5^{2g+4}((12)(345);(ab)) \setminus \Pi_3\right| =$$

$$= \frac{1}{36}(10^{2g+4} - 2\cdot 5^{2g+4} - 4^{2g+5} + 3\cdot 2^{2g+4} + 2).$$

6.10. The number $N_5((1^2\,3);(1^2\,3);g)$. Let Π be the set of ordered length $2g+4$ collections of transpositions from the group S_5. The order 6 subgroup $St_{(123)}$ acts on the set $\Pi_5^{2g+4}((123);(abc)) \subset \Pi$ by conjugations and the number $N_5((1,1,3);(1,1,3);g)$ is the quantity of $St_{(123)}$ orbits in this set. We divide the set Π into 3 subsets: $\Pi = \Pi_1 \cup \Pi_3 \cup \Pi_6$, where Π_k is a join of length k $St_{(123)}$ orbits in the set Π.

We see that $\Pi_1 = \{(45),(45),\ldots,(45)\} \subset \Pi_5^{2g+4}((123);(abc))$, a collection from Π_3 consists of the transpositions (12), (13), (23) and (45). Moreover, $\Pi_3 \cap \Pi_5^{2g+4}((123);(abc)) \subset \Pi_3^+$, where Π_3^+ is subset of Π_3, consisting of collections $(\pi_1,\ldots,\pi_{2g+4})$ such that the number of (45) in them is even. By lemma 6.5,

$$\left|\Pi_3^+ \cup \Pi_1\right| = \frac{4^{2g+4}+2^{2g+4}}{2}, \quad \text{i.e.,} \quad \left|\Pi_3^+\right| = \frac{4^{2g+4}+2^{2g+4}-2}{2}.$$

By definition put

$$\Pi_3^+(id) = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_3^+ \mid (123)\pi_1\cdots\pi_{2g+4} = id\right\}$$
$$\Pi_3^+(123) = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_3^+ \mid (123)\pi_1\cdots\pi_{2g+4} = (123)\right\}$$
$$\Pi_3^+(321) = \left\{(\pi_1,\ldots,\pi_{2g+4}) \in \Pi_3^+ \mid (123)\pi_1\cdots\pi_{2g+4} = (321)\right\}.$$

By the same argument, as before, we obtain

$$\left|\Pi_3^+(321)\right| = \left|\Pi_3^+(123)\right| = \frac{1}{3}\left|\Pi_3^+\right| = \frac{4^{2g+4}+2^{2g+4}-2}{6}$$

and

$$\Pi_3 \cap \Pi_5^{2g+4}((123),(abc)) = \Pi_3^+(123) \cup \Pi_3^+(321),$$

i.e.,

$$\left|\Pi_3 \cap \Pi_5^{2g+4}((123),(abc))\right| = \frac{4^{2g+4}+2^{2g+4}-2}{3}.$$

Therefore,

$$N_5((1^2\,3);(1^2\,3);g) =$$

$$= \frac{1}{6}\left|\Pi_5^{2g+4}((123),(abc)) \setminus (\Pi_3 \cup \Pi_1)\right| + \frac{1}{3}\left|\Pi_5^{2g+4}((123),(abc)) \cap \Pi_3)\right| + 1 =$$

$$= \frac{1}{18}(10^{2g+4} + 5^{2g+4} + 4^{2g+4} + 2^{2g+5} + 13).$$

The subset D of disconnected collections in the set $\Pi_5^{2g+4}((123),(abc))$ can be represented as a join $D = D_5 \cup D_4 \cup D_{45}$, where

$$D_5 = \left\{(\pi_1, \dots, \pi_{2g+4}) \in \Pi_5^{2g+4}((123),(abc)) |\ \forall i\ \pi_i(5) = 5\right\};$$
$$D_4 = \left\{(\pi_1, \dots, \pi_{2g+4}) \in \Pi_5^{2g+4}((123),(abc)) |\ \forall i\ \pi_i(4) = 4\right\};$$
$$D_{45} = \left\{(\pi_1, \dots, \pi_{2g+4}) \in \Pi_5^{2g+4}((123),(abc)) |\ \forall i\ \pi_i(\{4.5\}) = \{4,5\}\right\}.$$

Since the subgroup $St_{(123)}$ acts on the set $\Pi_5^{2g+4}((123),(abc)) \setminus D$ free, we have

$$CN_5((1^2\,3);(1^2\,3);g) = \frac{1}{6}\left(x_{5,2g+4}((123),(abc)) - |D|\right) \text{ and}$$

$$|D| = 2|D_5| - |D_5 \cap D_4| + |D_{45} \setminus (D_5 \cup D_4)|.$$

Furthermore,

$$|D_4| = |D_5| = x_{4,2g+4}((123),(abc)) = 2^{2g+5} \cdot 3^{2g+3}; \quad \text{(see (5-1))}$$

$$|D_5 \cap D_4| = x_{3,2g+4}((123),(abc)) = 2 \cdot 3^{2g+3} \quad \text{(see (4-1))};$$

$$D_{45} \setminus (D_5 \cup D_4) = \left(\Pi_3 \cap \Pi_5^{2g+4}((123),(abc))\right) \setminus (D_4 \cap D_5), \text{ and}$$

$$|D_{45} \setminus (D_5 \cup D_4)| = \frac{1}{3}(4^{2g+4} + 2^{2g+4} - 2) - 2 \cdot 3^{2g+3}.$$

Thus,

$$CN_5((1^2\,3);(1^2\,3);g) = \frac{1}{18}(10^{2g+4} - 4 \cdot 6^{2g+4} + 5^{2g+4} - 4^{2g+4} + 4 \cdot 3^{2g+4} + 2).$$

6.11. The number $N_5((1^2\,3);(1\,2^2);g)$. We use the notations Π, Π_1, Π_3 of the previous section. It is clear, that $\Pi_1 \cap \Pi_5^{2g+4}((123),(ab)(cd)) = \emptyset$ and

$\Pi_3 \cap \Pi_5^{2g+4}((123),(ab)(cd)) = \Pi_3^-$ (collections with odd quantity of the transpositions (45)). Using lemma 6.5 to calculate the number $\left|\Pi_3^-\right|$, we have

$$N_5((1^2\,3);(1\,2^2);g) = \frac{1}{6}\left|\Pi_5^{2g+4}((123),(ab)(cd)) \setminus \Pi_3\right| + \frac{1}{3}\left|\Pi_3^-\right| =$$

$$= \frac{1}{6}(25 \cdot 10^{2g+2} - 3 \cdot 2^{2g+2} + 2 \cdot 4^{2g+3}).$$

Moreover, the set disconnected collections coincides with $\Pi_3 \cap \Pi_5^{2g+4}((123),(ab)(cd))$, i.e.,

$$CN_5((1^2\,3);(1\,2^2);g) = \frac{1}{6}(25 \cdot 10^{2g+2} + 2^{2g+2} - 2 \cdot 4^{2g+3}).$$

6.12. The number $N_5((1^2\,3);(1^3\,2);g)$. Let Π be the set of ordered collections $(\pi_1, \dots, \pi_{2g+5})$, where π_i's are transpositions, Π_k be a subset of Π equal to a join of length k $St_{(123)}$ orbits. We see that $\Pi_1 \cap \Pi_5^{2g+5}((123),(ab)) = \emptyset$. If Π_3^+ denotes the subset of Π_3, consisting of collections with even quantity of the transpositions (54), then $\Pi_3^+ \subset \Pi_3 \cap \Pi_5^{2g+5}((123),(ab))$ and

$$\left|\Pi_3^+\right| = \frac{4^{2g+5} + 2^{2g+5}}{2}.$$

By definition put $\Pi_3^- = \Pi_3 \setminus \Pi_3^+$,

$$\Pi_3^-(id) = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_3^- \mid (123)\pi_1\cdots\pi_{2g+5} = (45)\Big\};$$
$$\Pi_3^-(123) = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_3^- \mid (123)\pi_1\cdots\pi_{2g+5} = (45)(123)\Big\};$$
$$\Pi_3^-(321) = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_3^- \mid (123)\pi_1\cdots\pi_{2g+5} = (45)(321)\Big\}.$$

Then,

$$\left|\Pi_3^- \cup \Pi_1\right| = \frac{4^{2g+5} - 2^{2g+5}}{2}, \quad \Pi_3^- \cap \Pi_5^{2g+5}((123),(ab)) = \Pi_3^-(id) \quad \text{and}$$
$$\Pi_3^-(id) = \frac{1}{3}\left|\Pi_3^-\right| = \frac{4^{2g+5} - 2^{2g+5} - 2}{6}.$$

We obtain

$$\left|\Pi_3 \cap \Pi_5^{2g+5}((123),(ab))\right| = \left|\Pi_3^+\right| + \left|\Pi_3^-(id)\right| = \frac{1}{6}(4^{2g+6} + 2^{2g+6} - 2).$$

Therefore,

$$N_5((1^2\,3);(1^3\,2);g) = \frac{1}{6}\left|\Pi_5^{2g+5}((123),(ab)) \setminus \Pi_3\right| + \frac{1}{3}\left|\Pi_5^{2g+5}((123),(ab)) \cap \Pi_3\right| =$$
$$= \frac{1}{36}(10^{2g+5} + 2\cdot 5^{2g+5} + 4^{2g+6} + 2^{2g+5} - 2).$$

To calculate the number $CN_5((1^2\,3);(1^3\,2);g)$ we apply the method of section 6.10. Namely, we consider the subset of $\Pi_5^{2g+5}((123),(ab))$ consisting of disconnected collections:

$$D_5 = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_5^{2g+5}((123),(ab)) \mid \forall i\ \pi_i(5) = 5\Big\};$$
$$D_4 = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_5^{2g+5}((123),(ab)) \mid \forall i\ \pi_i(4) = 4\Big\};$$
$$D_{45} = \Big\{(\pi_1,\ldots,\pi_{2g+5}) \in \Pi_5^{2g+5}((123),(ab)) \mid \forall i\ \pi_i(\{4.5\}) = \{4,5\}\Big\}.$$

The subset D of all disconnected collections in $\Pi_5^{2g+5}((123),(ab))$ is $D_4 \cup D_5 \cup D_{45}$. The subgroup $St_{(123)}$ acts on the complement $\Pi_5^{2g+5}((123),(ab)) \setminus (D_4 \cup D_5 \cup D_{45})$ free and the number of orbits of this action is $CN_5((1,1,3);(1,1,1,2);g)$. Furthermore,

$$|D_4| = |D_5| = x_{4,2g+5}((123),(ab) = 2^{2g+4}3^{2g+5} \text{ (see (5-1);}$$

$$|D_5 \cap D_4| = x_{3,2g+5}((123),(ab) = 3^{2g+5} \text{ (see (4-1),}$$
$$D_{54} \setminus (D_5 \cup D_4) = \Big(\Pi_3 \cap \Pi_5^{2g+5}((123),(ab))\Big) \setminus (D_5 \cap D_4),$$

$$|D_{54} \setminus (D_5 \cup D_4)| = \frac{4^{2g+6} + 2^{2g+6} - 2 \cdot 3^{2g+6} - 2}{6}.$$

Therefore,

$$CN_5((1^2\,3); (1^3\,2); g) =$$

$$= \frac{1}{6}(\left|\Pi_5^{2g+5}((123), (ab)\right| - 2|D_4| + |D_4 \cap D_5| - |D_{54} \setminus (D_4 \cup D_5)|) =$$

$$= \frac{1}{36}(10^{2g+5} - 6^{2g+6} + 2 \cdot 5^{2g+5} - 4^{2g+6} + 4 \cdot 3^{2g+6} - 3 \cdot 2^{2g+5} + 2).$$

6.13. The number $N_5((1\,2^2); (1\,2^2); g)$. The subgroup $St_{(12)(34)}$ has order 8. It can be proved that

$$St_{(12)(34)} \cap St_{(ab)} = \begin{cases} \langle (12), (34) \rangle, & \text{if } (ab) = (12) \text{ or } (ab) = (34), \\ id & \text{in other cases.} \end{cases}$$

Hence, the set Π of collections $(\pi_1 \ldots, \pi_{2g+4})$ can be divided into 2 subsets $\Pi = \Pi_2 \cup \Pi_8$, where Π_k is a join of length k $St_{(12)(34)}$ orbits.

The subset Π_2 consists of collections of the transpositions (12) and (34). Moreover, if a collection $(\pi_1, \ldots, \pi_{2g+4}) \in \Pi_2 \cap \Pi_5^{2g+4}((12)(34), (ab)(cd))$, then the quantity of transpositions π_i's equal to (12) is even. Hence,

$$\left|\Pi_2 \cap \Pi_5^{2g+4}((12)(34), (ab)(cd)\right| = 2^{2g+3}.$$

Thus,

$$N_5((1\,2^2); (1\,2^2); g) = \frac{1}{8}\left|\Pi_5^{2g+4}((12)(34), (ab)(cd) \setminus \Pi_2\right| + \frac{1}{2}\left|\Pi_5^{2g+4}((12)(34), (ab)(cd) \cap \Pi_2\right| =$$

$$= 2^{2g-1}5^{2g+4}.$$

To calculate $CN_5((1\,2^2); (1\,2^2); g)$ by definition put

$$D_5 = \left\{(\pi_1, \ldots, \pi_{2g+4}) \in \Pi_5^{2g+4}((12)(34), (ab)(cd))|\ \forall i\ \pi_i(5) = 5\right\};$$

$$D_{12} = \left\{(\pi_1, \ldots, \pi_{2g+4}) \in \Pi_5^{2g+4}((12)(34), (ab)(cd))|\ \forall i\ \pi_i(\{1, 2\}) = \{1, 2\}\right\};$$

$$D_{34} = \left\{(\pi_1, \ldots, \pi_{2g+4}) \in \Pi_5^{2g+4}((12)(34), (ab)(cd))|\ \forall i\ \pi_i(\{3, 4\}) = \{3, 4\}\right\}.$$

The subset D of all disconnected collections belonging to $\Pi_5^{2g+4}((12)(34), (ab)(cd))$ is $D_5 \cup D_{12} \cup D_{34}$.

We see that

$$|D_5| = x_{4,2g+4}((12)(34), (ab)(cd)) = 2^{2g+2}(3^{2g+4} + 1);$$

$$|D_{12}| = |D_{34}|, \quad D_{12} \cap D_{34} = D_5 \cap D_{12} = D_5 \cap D_{34} = \Pi_2 \cap \Pi_5^{2g+4}((12)(34), (ab)(cd)).$$

Therefore, $|D_{12} \cap D_{34}| = 2^{2g+3}$.

Let us calculate $|D_{12}|$. If $(\pi_1, \ldots, \pi_{2g+4}) \in D_{12}$, then $\pi \in \{(12), (34), (35), (45))$ and the quantity of (12) in it is even. Taking into account lemma 6.5, we get $|D_{12}| = \frac{4^{2g+4}+2^{2g+4}}{2}$.

Thus

$$CN_5((1\,2^2);(1\,2^2);g) = \frac{1}{8}\left(|\Pi_5^{2g+4}((12)(34);(ab)(cd))| - |D_5| - 2|D_{12}| + 2|D_{12}\cap D_{34}|\right) =$$

$$= 2^{2g-1}(5^{2g+4} - 3^{2g+4}) - 2\cdot 4^{2g+2}.$$

6.14. The number $N_5((1\,2^2);(1^3\,2);g)$. A $St_{(12)(34)}$ orbit in the set $\Pi_5^{2g+5}((12)(34),(ab))$ has either 2 or 8 points. The subset $\Pi_2 \subset \Pi_5^{2g+5}((12)(34),(ab))$ of all length 2 $St_{(12)(34)}$ orbits contains collections $(\pi_1,\ldots,\pi_{2g+5})$ such that $\forall i\ \pi_i \in \{(12),(34)\}$. Therefore, $|\Pi_2| = 2^{2g+5}$ and

$$N_5((1\,2^2);(1^3\,2);g) = \frac{1}{8}\left|\Pi_5^{2g+5}((12)(34),(ab))\setminus\Pi_2\right| + \frac{1}{2}|\Pi_2| = 2^{2g+1}\frac{5^{2g+5}+19}{3}.$$

As before by definition put

$$D_5 = \left\{(\pi_1,\ldots,\pi_{2g+5})\in\Pi_5^{2g+5}((12)(34),(ab))|\ \forall i\ \pi_i(5)=5\right\};$$
$$D_{12} = \left\{(\pi_1,\ldots,\pi_{2g+5})\in\Pi_5^{2g+5}((12)(34),(ab))|\ \forall i\ \pi_i(\{1,2\})=\{1,2\}\right\};$$
$$D_{34} = \left\{(\pi_1,\ldots,\pi_{2g+5})\in\Pi_5^{2g+5}((12)(34),(ab))|\ \forall i\ \pi_i(\{3,4\})=\{3,4\}\right\}.$$

The subset D of all disconnected collections belonging to $\Pi_5^{2g+5}((12)(34),(ab))$ is $D = D_5 \cup D_{12}\cup D_{34}$. Besides,

$$|D_5| = x_{4,2g+5}((12)(34),(ab)) = 2^{2g+4}(3^{2g+5}-1);\quad |D_{12}| = |D_{34}|;$$

$$D_5\cap D_{12} = D_5\cap D_{34} = D_{12}\cap D_{34} = \Pi_2.$$

Denote by M the set of collections $(\pi_1,\ldots,\pi_{2g+5})$ such that $\forall i\ \pi_i\in\{(12),(34),(35),(45)\}$. Then $D_{12}\subset M$. Let M^- be the subset of M, consisting of collections with odd quantity of the transpositions (12) and $M^+ = M\setminus M^-$. Then $M^-\subset D_{12}$ and $|M^-| = \frac{1}{2}(4^{2g+5}-2^{2g+5})$. By definition put

$$M^+(id) = \{(\pi_1,\ldots,\pi_{2g+5})\in M^+|\ (34)\pi_1\cdots\pi_{2g+5} = id\},$$
$$M^+(345) = \{(\pi_1,\ldots,\pi_{2g+5})\in M^+|\ (34)\pi_1\cdots\pi_{2g+5} = (345)\},$$
$$M^+(543) = \{(\pi_1,\ldots,\pi_{2g+5})\in M^+|\ (34)\pi_1\cdots\pi_{2g+5} = (543)\}.$$

Then $M^+\cap D_{12} = M^+(id)$, $|M^+(id)| = \frac{1}{3}|M^+| = \frac{4^{2g+5}+2^{2g+5}}{6}$. Therefore, $|D_{12}| = |M^+(id)| + |M^-| = \frac{4^{2g+6}-2^{2g+6}}{6}$.

Thus,

$$CN_5((1\,2^2);(1^3\,2);g) = \frac{1}{8}(|\Pi_5^{2g+5}((12)(34),(ab))| - |D_5| - 2|D_{12}| + 2|D_{12}\cap D_{34}|) =$$

$$= \frac{2^{2g+1}}{3}(5^{2g+5} - 3^{2g+6} - 2^{2g+8} + 20).$$

6.15. The last number $N_5((1^3\,2);(1^3\,2);g)$. Order of the subgroup $St_{(54)}$ is 12. Obviously,

$$St_{(54)}\cap St_{(ab)}=\begin{cases}\langle(54),(ab)\rangle, & \text{if } (ab)\in St_{(45)};\\ id, & \text{in the convers case.}\end{cases}$$

Let Π be the set of ordered collections $(\pi_1,\dots,\pi_{2g+6})$ where π_i's are transpositions. We divide the set Π into 3 subsets: $\Pi=\Pi_1\cup\Pi_3\cup\Pi_{12}$, where as usual, Π_k is a join of length k $St_{(45)}$ orbits. It is clear, that

$$\Pi_1=\{((45),\dots,(45))\}\subset\Pi_5^{2g+6}((45),(ab)).$$

In particular, $|\Pi_1|=1$.

By definition put $\Pi_3(ab)=\{(\pi_1,\dots,\pi_{2g+6})\in\Pi_3|\ \pi_i\in\{(45),(ab)\}\}$. Then

$$\Pi_3=\Pi_3(12)\cup\Pi_3(13)\cup\Pi_3(23)=\Pi_3\cap\Pi_5^{2g+6}((45),(ab)).$$

We see that

$$|\Pi_3(12)|=|\Pi_3(13)|=|\Pi_3(23)|=2^{2g+6}-1.$$

Therefore,

$$N_5((1^3\,2);(1^3\,2);g)=\frac{1}{12}(x_{5,2g+6}((45),(ab))-|\Pi_3|-|\Pi_1|)+\frac{1}{3}|\Pi_3|+1=$$

$$=\frac{1}{12}(25\cdot10^{2g+4}+5^{2g+6}+31\cdot2^{2g+4}+3).$$

The subset D of disconnected collections in $\Pi_5^{2g+6}((45),(ab))$ can be represented as a join $D=D_{45}\cup D_{12}\cup D_{23}\cup D_{13}$, where

$$D_{\alpha\beta}=\{(\pi_1,\dots,\pi_{2g+6})\in\Pi_5^{2g+6}((45),(ab))|\ \forall i\ \pi_i(\{\alpha,\beta\})=\{\alpha,\beta\}\}.$$

The subset D_{45} contains collections $(\pi_1,\dots,\pi_{2g+6})$, with $\pi_i\in\{(45),(12),(23),(13)\}$ only. Moreover, such a collection belongs to D_{45} in the two cases:

(i) the number of (45) in it is odd (we denote the corresponding subset by D_{45}^-),

(ii) the number of (45) is even and the product $\pi_1\cdots\pi_{2g+6}=id$ (we denote the corresponding subset by $D_{45}^+(id)$).

By lemma 6.5

$$|D_{45}^-|=\frac{4^{2g+6}-2^{2g+6}}{2},\qquad |D_{45}^+(id)|=1+\frac{4^{2g+6}+2^{2g+6}-2}{6}.$$

Therefore,

$$|D_{45}|=\frac{4^{2g+7}-2^{2g+7}+4}{6}.$$

The subset D_{12} consists of all ordered length $2g+6$ collections of the transpositions with $(12),(34),(35),(45)$ such that either a quantity of the transpositions (12) in them is even and the product of all elements of collection equals id, or

a quantity of the transpositions (12) in them is odd and the product of all elements of collection equals (45).

The first subset we denote by $D_{12}^{+}(id)$ and the second one by $D_{12}^{-}(45)$, i.e., $D_{12} = D_{12}^{+}(id) \cup D_{12}^{-}(45)$.

Taking into account lemma 6.5, and the standard trick, we get

$$|D_{12}^{+}(id)| = \frac{4^{2g+6} + 2^{2g+6} + 4}{6}, \quad |D_{12}^{-}(45)| = \frac{4^{2g+6} - 2^{2g+6}}{6}.$$

Therefore, $|D_{12}| = \frac{4^{2g+6}+2}{3}$. Similarly,

$$|D_{23}| = |D_{13}| = \frac{4^{2g+6} + 2}{3}.$$

The intersection $D_{45} \cap D_{12}$ consists of all length $2g+6$ collections of the transpositions (45) and (12). Whence, $|D_{45} \cap D_{12}| = 2^{2g+6}$. Likewise,

$$|D_{45} \cap D_{13}| = |D_{45} \cap D_{23}| = 2^{2g+6}.$$

It is clear, that

$$D_{12} \cap D_{13} = D_{12} \cap D_{23} = D_{23} \cap D_{13} = \{((45), \ldots, (45))\}, \text{ i.e.,}$$

$$|D_{12} \cap D_{13}| = |D_{12} \cap D_{23}| = |D_{23} \cap D_{13}| = 1.$$

In addition, an intersection of any 3 different subsets D_{ij} has one point and the intersection of all D_{ij}'s is one point set too. Finally, the subgroup $St_{(45)}$ acts free on the set $\Pi_5^{2g+6}((45); (ab)) \setminus D$. Thus,

$$CN_5((1^3\,2); (1^3\,2)) = \frac{1}{12}(|\Pi_5^{2g+6}((45), (ab))| - |D_{45}| - 3|D_{13}| + 3|D_{45} \cap D_{23}| + 3 - 4 + 1) =$$

$$= \frac{1}{72}(10^{2g+6} + 4 \cdot 5^{2g+6} - 10 \cdot 4^{2g+6} + 21 \cdot 2^{2g+6} - 16).$$

6.16. Let us collect all the numbers $CN_5(\alpha; \beta; g)$ together.

$CN_5((5); (5); g) = 5^{2g-2}(2^{2g+1} + 2)$;

$CN_5((5); (1\,4); g) = 5^{2g}2^{2g}$;

$CN_5((5); (2\,3); g) = 5^{2g}\frac{2^{2g+1}+1}{3}$;

$CN_5((5); (1^2\,3); g) = 5^{2g+1}\frac{2^{2g+2}-1}{3}$;

$CN_5((5); (1\,2^2); g) = 5^{2g+1}2^{2g}$;

$CN_5((5); (1^3\,2); g) = 5^{2g+2}\frac{2^{2g+2}-1}{3}$;

$CN_5((1\,4); (1\,4); g) = 2^{2g-1}(5^{2g+2} - 3^{2g+2})$;

$CN_5((1\,4); (2\,3); g) = 2^{2g}\frac{5^{2g+2}-1}{3}$;

$CN_5((1\,4); (1\,2^2); g) = 2^{2g-1}(5^{2g+3} - 3^{2g+3} + 2)$;

$CN_5((1\,4); (1^2\,3); g) = 2^{2g+1}\frac{5^{2g+3}-2\cdot 3^{2g+3}+1}{3}$;

$CN_5((1\,4); (1^3\,2); g) = 2^{2g+1}\frac{5^{2g+4}-3^{2g+5}+2}{3}$;

$CN_5((2\,3);(2\,3);g) = \frac{1}{18}(10^{2g+2} + 5^{2g+2} - 4^{2g+2} - 1);$

$CN_5((2\,3);(1^2\,3);g) = \frac{1}{18}(10^{2g+3} - 5^{2g+3} - 4^{2g+3} - 1);$

$CN_5((2\,3);(1\,2^2);g) = 2^{2g}\frac{5^{2g+3}-2^{2g+4}-1}{3}$

$CN_5((2\,3);(1^3\,2)) = \frac{1}{36}(10^{2g+4} - 2\cdot 5^{2g+4} - 4^{2g+5} + 3\cdot 2^{2g+4} + 2)$

$CN_5((1^2\,3);(1^2\,3);g) = \frac{1}{18}(10^{2g+4} - 4\cdot 6^{2g+4} + 5^{2g+4} - 4^{2g+4} + 4\cdot 3^{2g+4} + 2);$

$CN_5((1^2\,3);(1\,2^2);g) = \frac{1}{6}(25\cdot 10^{2g+2} + 2^{2g+2} - 2\cdot 4^{2g+3});$

$CN_5((1^2\,3);(1^3\,2);g) = \frac{1}{36}(10^{2g+5} - 6^{2g+6} + 2\cdot 5^{2g+5} - 4^{2g+6} + 4\cdot 3^{2g+6} - 3\cdot 2^{2g+5} + 2)$

$CN_5((1\,2^2);(1\,2^2);g) = 2^{2g-1}(5^{2g+4} - 3^{2g+4}) - 2\cdot 4^{2g+2};$

$CN_5((1\,2^2);(1^3\,2);g) = \frac{2^{2g+1}}{3}(5^{2g+5} - 3^{2g+6} - 2^{2g+8} + 20);$

$CN_5((1^3\,2);(1^3\,2)) = \frac{1}{72}(10^{2g+6} + 4\cdot 5^{2g+6} - 10\cdot 4^{2g+6} + 21\cdot 2^{2g+6} - 16).$

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