# A FEW EXAMPLES OF ELLIPTIC THREEFOLDS WITH TRIVIAL CANONICAL BUNDLE

by

D.G.Markushevich

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Federal Republic of Germany

Department of Mathematics
Israel Institute of
Technology
Technion, Haifa 32000
Israel

MPI/91- 41

Resume. A few examples of simply connected complex projective threefolds with trivial canonical bundle (that is, Calabi-Yau manifolds) are constructed, which are elliptic fibrations over certain rational surfaces. They are obtained as embedded bundles of cubic curves in $\mathbb{P}^2$ or their degenerations admitting a resolution of singularities with trivial canonical bundle. Two examples with the Euler characteristic zero and $h^{11}=11$ and 18 are obtained.

A few thousand examples of Calabi-Yau manifolds are known at present. They are mostly obtained by simple projective constructions, such as taking complete intersections in products of weighted projective spaces, or ramified coverings of Fano threefolds, and so on. There is also a series of examples which can be represented as families of cubic curves given by the Weierstrass normal equation $Y^2Z = X^3 + aXZ^2 + bZ^3$ for $a \in H^0(S, \mathcal{O}(-4K_S))$, $b \in H^0(S, \mathcal{O}(-6K_S))$, where S is a rational surface, $K_S$ the canonical divisor on S, and it is supposed that $\mathcal{O}_S(-4K_S)$ and $\mathcal{O}_S(-6K_S)$ have sufficiently many sections to provide a non-singular family (see, e.g., [1], or [2] for K3 surfaces with elliptic pencil). By Miranda's criterion of smoothness of the elliptic fibration [3] and the Bertini theorem [4], it is sufficient that the linear systems $|-4K_S|$ and $|-6K_S|$ have no common base points. So, one can easily see that over any Del Pezzo surface, and over the Hirzebruch surface $\mathbb{F}_2$, there always exists a smooth elliptic fibration in the Weierstrass normal form which is a Calabi-Yau manifold. The case of $S = \mathbb{P}^2$ was studied in detail by A.Albano [5]; for the corresponding elliptic Calabi-Yau manifold we have the following invariants: $\chi(X) = -540$, $h^{11} = 4$, $h^{21} = 274$. The computation of Hodge numbers is not as easy as for other constructions mentioned above, because the elliptic threefold X is not an ample divisor in the corresponding projective bundle (that is why $h^{11} \neq 2$).

We are going to provide a few more examples of elliptic Calabi-Yau manifolds over certain rarional surfaces using more general families of elliptic curves as well as their degenerations and crepant (= retaining the triviality of the canonical bundle) resolutions of singularities.

## 1. Definitions and basic constructions

**1.1. Definition.** A Calabi—Yau manifold (CYM) is a simply connected complex projective threefold X with $K_X=0$.

**1.2. Definition.** An elliptic Calabi-Yau manifold (ECYM) is a flat projective morphism $f:V \longrightarrow S$, such that: (i) S is a non-singular projective surface; (ii) V is a CYM.

By adjunction formula, a generic fiber of an ECYM is an elliptic curve.

**1.3. Weierstrass normal form (WNF).** Let S be a projective manifold, $\mathcal{M}$ any line bundle on S, $a \in H^0(S, \mathcal{M}^{-4})$, $b \in H^0(S, \mathcal{M}^{-6})$. Let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ be the projectivization of the split vector bundle $\mathcal{E} = \mathcal{O}\oplus\mathcal{M}^2\oplus\mathcal{M}^3$, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}/S}(1)$ the tautological Grothendieck line bundle. For any line bundle $\mathcal{N}$ on S and the bundle map $\mathcal{N} \hookrightarrow \mathcal{E}$ there is a natural homomorphism of sheaves $\pi^*\mathcal{N} \longrightarrow \mathcal{L}$, where $\pi : \mathbb{P} \longrightarrow \mathbb{P}^2$ is the natural projection. The latter homomorphism defines a section of $\pi^*\mathcal{N}^{-1}\otimes\mathcal{L}$. Thus, for the natural inclusions of direct summands $\mathcal{O}$, $\mathcal{M}^2$, $\mathcal{M}^3$ into $\mathcal{E}$ we obtain three sections of $\mathcal{L}$, $\mathcal{L}\otimes\pi^*\mathcal{M}^{-2}$, $\mathcal{L}\otimes\pi^*\mathcal{M}^{-3}$, which will be denoted by Z, X, Y respectively. In this notation, the expression $s = Y^2Z - X^3 - aXZ^2 - bZ^3$ defines a section of the line bundle $\mathcal{L}^3\otimes\pi^*\mathcal{M}^{-6}$.

**1.4. Definition.** When we say that V is defined by the equation

$$Y^2Z = X^3 + aXZ^2 + bZ^3 \tag{1.1}$$

in the WNF, this means that V is the scheme of zeros in $\mathbb{P}$ of the section $s \in H^0(\mathbb{P}, \mathcal{L}^3 \otimes \pi^* \mathcal{M}^{-6})$ ([6], 9.7.9.1):

$$V = Z_{\mathbb{P}}(s) .$$

**1.5.Proposition.** Adopt the notation Of **1.3**, **1.4**. Then we have:

(i) The Grothendieck dualizing sheaf $\omega_V$ is trivial if and only if $\mathcal{M} \cong \omega_S = \mathcal{O}_S(K_S)$, the canonical sheaf of S.

(ii) V is a smooth manifold for generic a and b, if the base loci of linear systems $|\mathcal{M}^{-4}|$ and $|\mathcal{M}^{-6}|$ have no common points. In this case $f = \pi|V$ is flat, and all the fibers of f are reduced irreducible cubic curves in $\mathbb{P}^2$. We have:

$$f^{-1}(P) \simeq \begin{cases} \text{a cuspidal cubic, if } a(P) = b(P) = 0 \\ \text{a nodal cubic, if } \Delta(P) = 0, \ a(P) \neq 0 \\ \text{a non-singular cubic, if } \Delta(P) \neq 0 \end{cases}$$

where $\Delta = 4a^3 + 27b^2$ is the discriminant.

**Proof.** (i) is proved by standard adjunction technique and in using the Euler exact sequence for projectivized vector bundles. See, e.g. [5]. (ii) easily follows from Proposition (2.1) of [3] and the Bertini theorem [4].

**1.6. Corollary.** The equation (1.1) with generic a,b and $\mathcal{M} \cong \omega_S$ defines an ECYM over every surface S from the following list: $\mathbb{P}^2$; $\mathbb{P}^1 \times \mathbb{P}^1$; Hirzebruch surfaces $\mathbb{F}_d$ for d = 1, 2; Del Pezzo surfaces $S_d$(d = 1,...,7) obtained from $\mathbb{P}^2$ by blowing up k = 9 - d points satisfying certain conditions of generic position. The Euler number of these ECYM's is given in Table 1.

Table 1

| $S$ | $\mathbb{P}^2$ | $\mathbb{P}^1\times\mathbb{P}^1$ | $\mathbb{F}_1$ | $\mathbb{F}_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi(V)$ | -540 | -480 | -480 | -480 | -60 | -120 | -180 | -240 | -300 | -360 | -420 |

**Proof.** For generic a,b the discriminant curve $C = \{\Delta=0\}$ in $S$ has ordinary cusps at points of intersection of the curves $C_1 = \{a=0\}$ and $C_2=\{b=0\}$, and no other singularities. So, we have the picture plotted on Fig. 1. Hence, by additivity of $\chi$,

$$\chi(V) = \chi(S\setminus C)\chi(E) + \chi(C\setminus\{C_1\cap C_2\})\chi(\text{nodal cubic}) + \#\{C_1\cap C_2\}\cdot\chi(\text{cuspidal cubic}) = -60K_S^2 .$$

**1.7. Generalization of the WNF construction.** Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}\oplus \mathcal{O}_{\mathbb{P}^2}(-m)\oplus\mathcal{O}_{\mathbb{P}^2}(-n)$ $(0\le m\le n)$, $\mathbb{P}=\mathbb{P}(\mathcal{E})$, and $\mathcal{L}$, $\pi$ be as above. Choose fiber homogeneous coordinates $X$, $Y$, $Z$ on $\mathbb{P}$ as canonical sections of $\mathcal{L}$, $\mathcal{L}\otimes\pi^*\mathcal{O}_{\mathbb{P}^2}(m)$, $\mathcal{L}\otimes\pi^*\mathcal{O}_{\mathbb{P}^2}(n)$ corresponding to the embeddings of line summands of $\mathcal{E}$ into itself. Then we define:

$$s = \sum_{\substack{i+j+k=3\\ i,j,k\ge0}} a_{ijk}X^iY^jZ^k \in H^0(\mathbb{P}, \mathcal{L}^3\otimes\mathcal{O}_{\mathbb{P}^2}(m+n+3)), \qquad V = Z_{\mathbb{P}}(s),$$

where $a_{ijk} \in H^0(\mathbb{P}, \pi^*\mathcal{O}_{\mathbb{P}^2}((i-1)n+(j-1)(n-m)+3))$.

**1.8. Proposition.** The scheme $V$ of zeros of the section $s$ is a smooth ECYM for generic coefficients $a_{ijk}$ if and only if the pair $(m,n)$ occurs in the following list:

(0,0), (0,1), (0,3), (1,1), (1,2), (1,3), (1,4), (2,2),

(2,3), (2,5), (3,3), (3,6), (4,7), (5,8), (6,9).

In this case $\chi(V) = -162-6(m^2-mn+n^2)$.

**Proof.** By the Bertini theorem, singularities may occur only in the base locus of the linear system $M = |\mathcal{L}^3 \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(m+n+3)|$. Since the divisors of the non-zero coefficients $a_{ijk}$ move in their linear systems without base points, the base locus of M projects to the whole of $\mathbb{P}^2$ via $\pi$. So, it must be the union of some "coordinate axes" $X=0$, $Y=0$, $Z=0$, or their intersections (which are cross-sections of $\pi$ over $\mathbb{P}^2$).

Denote $d_{ijk} = (i-1)n+(j-1)(n-m)+3$, where $(ijk)$ is the exponent of a monomial $X^iY^jZ^k$. We have $d_{ijk} \geq 0$ for the monomials $X^3$, $X^2Y$, $X^2Z$, $XY^2$, $XYZ$ independently of the values of m,n. So, these monomials always occur in a generic s with a non-zero coefficient. They all are divisible by X, and thence, for the generic fiber of $\pi$ to be irreducible, we should impose the condition that $d_{ijk} \geq 0$ for at least one monomial non-divisible by X. As $d_{030} \geq d_{021} \geq d_{012} \geq d_{003}$, the weakest condition of such kind is $d_{030} \geq 0$.

If this condition is satisfied, then the monomials $X^3$ and $Y^3$ have non-zero coefficients, and for the generic fiber to be non-singular, at least one of the monomials $Z^3$, $XZ^2$, $YZ^2$ must have a non-zero coefficient.

If $Z^3$ does, then M is base point free, and hence the generic V is non-singular. This happens when $d_{003}=-2n+m+3 \geq 0$ (as $d_{030} \geq d_{003}$, the inequality $d_{030} \geq 0$ is automatically satisfied), that is when the pair (m,n) is an element of the set

$$\Pi_1 = \{ (m,n) \in \mathbb{Z}^2 \mid 0 \leq m \leq n, \ -2n+m+3 \geq 0 \} .$$

If $XZ^2$ has a non-zero coefficient, then we have $d_{102}=m-n+3 \geq 0$, and $d_{030}=n-2m+3 \geq 0$ (the latter inequality guarantees the existence of a monomial non-divisible by X with a non-zero coefficient). Under these conditions, the generic fiber of $\pi$ is non-singular, but the non-singularity of V at the points of the base locus $\{X=Y=0\}$ is to be verified.

The equations defining the singular locus of V are

$$\frac{\partial s}{\partial u_i} = 0 \ (i = 0,\ 1,\ 2), \qquad \frac{\partial s}{\partial X} = \frac{\partial s}{\partial Y} = \frac{\partial s}{\partial Z} = 0, \qquad (1.2)$$

where $(u_0, u_1, u_2)$ are homogeneous coordinates on the $\mathbb{P}^2$ which is the image of $\pi$. Taking into account that singular points may occur only in the cross-section X=Y=0, we see, that (1.2) is equivalent to

$$X = Y = 0, \quad a_{102}(u) = a_{012}(u) = 0 .$$

So, for V to be non-singular, one of the two coefficients $a_{102}$ or $a_{012}$ must be nowhere vanishing, that is a constant section of the trivial line bundle over $\mathbb{P}^2$. Denote

$$\Pi_2 = \{(m,n) \in \mathbb{Z}^2 \mid 0\le m\le n,\ n-2m+3\ge 0,\ n-m+3=0\}$$

and

$$\Pi_3 = \{(m,n) \in \mathbb{Z}^2 \mid 0\le m\le n,\ d_{012}=-n+3=0\} .$$

We have shown, that the non-singularity of the generic V holds only when $(m,n) \in \Pi_1\cup\Pi_2\cup\Pi_3$. This is exactly the set listed in the statement of the Proposition.

The formula for $\chi$ is proved via Gauss - Bonnet Theorem $\chi(V) = \deg c_3(T_V)$.

**1.9. Remark.** For ECYM's of Proposition 1.8, we have $-540\le\chi\le-162$, where $-540$ is attained by the WNF, and $-162$ by the hypersurface of bidegree (3,3) in $\mathbb{P}^2\times\mathbb{P}^2$ (m=n=0).

## 2. Crepant resolution of singularities

**2.1. Definition.** Let Y be a complex projective variety with only Gorenstein singularities, and $f : V \longrightarrow Y$ a birational morphism such that V is non-singular. The variety V (or the morphism f) will be called a crepant resolution of singularities of V, if the following isomorphism holds:

$$f^{*}\omega_{Y} \cong \omega_{V} ,$$

where $\omega_V$ , $\omega_Y$ are canonical sheaves of V, Y respectively. The resolution f is called small if the codimension of its exceptional locus in V is greater than one. It is small over a poont $P \in \mathrm{Sing}\ Y$, if this holds locally near $f^{-1}(P)$.

It is obvious, that a small resolution of a Gorenstein singularity is crepant.

**2.2 Definition.** A Du Val (or an ADE-singularity) is a surface singularity which is analytically isomorphic to the germ at the origin of one of the following hypersurfaces in $\mathbb{C}^3$:

$$X^2+Y^2+Z^{n+1}=0 \qquad (A_n ,\ n\geq 1)$$
$$X^2+Y^2Z+Z^{n-1}=0 \qquad (D_n ,\ n\geq 4)$$
$$X^2+Y^3+Z^4=0 \qquad (E_6)$$
$$X^2+Y^3+YZ^3=0 \qquad (E_7)$$
$$X^2+Y^3+Z^5=0 \qquad (E_8)$$

**2.3. Proposition** ([7]). A singular surface Y admits a crepant resolution if and only if it has only Du Val singularities.

**2.4. Definition.** A compound Du Val (cDV) singularity is a germ of a singular three-dimensional complex space analytically isomorphic to the germ at the origin of the hypersurface in $\mathbb{C}^4$ defined by an equation of the form

$$f(X, Y, Z) + Tg(X, Y, Z, T) = 0,$$

where f is one of the above ADE-polynomials, and g is an arbitrary polynomial.

In other words, a cDV singularity is analytically equivalent to a hypersurface singularity, such that its generic hyperplane section through the origin is a Du Val singularity. If the generic hyperplane section has the singularity of type $A_n$ , $D_n$ , resp. $E_n$ , then the three-dimensional singularity is said to be of type $cA_n$ , $cD_n$ , resp. $cE_n$ .

**2.5. Proposition.** Let Y be a three-dimensional projective variety admitting a crepant resolution $f : V \longrightarrow Y$ . Then the following assertions hold:

(i) Y is normal, and at a generic point of each one-dimensional component of Sing Y the singularity is analytically isomorphic to $\mathbb{A}^1 \times$\{Du Val singularity\}.

(ii) If $P \in \text{Sing } Y$ is an isolated cDV singular point, then f is small over P, and $f^{-1}(P)_{red}$ is a chain of smooth rational curves with possible triple points of embedding codimension 2. Furthermore, in this case algebraic and analytic divisor class groups of the germ (Y,P) are non-trivial and isomorphic.

For proofs, see [8, 9, 10].

**2.6. Basic example.** Any isolated $cA_1$ singularity is analytically equivalent to the singularity of the form $X^2+ Y^2+ Z^2+$

$T^k = 0$ $(k \geq 2)$. The latter admits a small resolution if and only if k is even, and in this case the divisor class group is $\mathbb{Z}$ and generated by the class of a smooth surface $H = \{X + \sqrt{-1}Y = Z + \sqrt{-1}T^{k/2} = 0\}$. The surface H is a Weil divisor in the hypersurface $Q = \{X^2 + Y^2 + Z^2 + T^k = 0\}$. The small resolution is obtained by blowing up H. Such blow up is an isomorphism over $Q \setminus \{0\}$, that is in all the points in which H is locally Cartier, and $f^{-1}(0) \cong \mathbb{P}^1$. Thus, f is indeed small.

A projective threefold with the $cA_1$ singularity admits a small resolution if and only if the property of the existence of a smooth surface H passing through the singular point $P \in Y$ holds not only local-analytically, but also in the algebraic category.

## 3. Constructions of ECYM's involving a crepant resolution of singularities

**3.1. Resolving singularities of generic V's in the base locus X=Y=0.** We adopt here the notation of 1.7 and 1.8. It may turn out, that for some pairs (m,n) the generic V is not smooth, but acquires certain cDV singularities.

The first possible case is when $d_{102} > 0$, $d_{012} > 0$, and $d_{003} < 0$. Then there are $k = d_{102} d_{012}$ ordinary double points defined by $a_{102}(u) = a_{012}(u) = 0$, $X = Y = 0$. The surface $H = \{X = Y = 0\}$ is a smooth Weil divisor passing through all the singular points. So, according to 2.6, the blow up of H is a small, and hence crepant resolution of singularities. There is the only one such case: m=0, n=2. Then $d_{102} = 1$, $d_{012} = 1$, so there is one singular point.

The second possible case is when $d_{102} > 0$, $d_{012} < 0$, but $d_{030} \geq 0$.

Then we have a singular $cA_1$-curve $X=Y=0$, $a_{102}(u)=0$.

Putting together the above cases, we come to the following statement:

**3.2. Proposition.** In the notation of Proposition 1.8, the following pairs of integers $(m,n)$ give rise to singular generic varieties $V = Z_{\mathbb{P}}(s)$ admitting a crepant resolution, which is an ECYM:

$$(0,2),\ (2,4),\ (3,4),\ (3,5),\ (4,5),\ (4,6),\ (5,7).$$

**Proof.** For $(0,2)$, see the above discussion. All the remaining pairs form exactly the set singled out by the enequalities $d_{030}\geq 0$, $d_{012}<0$, $d_{102}>0$. Under these conditions, the equation of V has the form

$$\begin{aligned}&a_{300}X^3+ a_{210}X^2Y + a_{201}X^2Z + a_{120}XY^2+ a_{111}XYZ + a_{102}XZ^2+\\&a_{030}Y^3+ a_{021}Y^2Z = 0\end{aligned} \tag{3.1}$$

As we are interested in the singularities occuring in the neighbourhood of the cross-section $X=Y=0$, we pass to the inhomogeneous coordinates by setting $Z=1$, and rewrite the equation (3.1) in the following form:

$$\begin{aligned}&x(a_{102} + a_{111}y + a_{120}y^2+ a_{201}x + a_{210}xy + a_{300}x^2) +\\&y^2(a_{021} + a_{030}y) = 0\end{aligned} \tag{3.2}$$

So, we have a cDV singularity along the curve $x=y=a_{102}=0$. There are the following two possibilities:

**3.2.1.** $a_{021}\not\equiv 0$ ($d_{021}\geq 0$). Then either $d_{021}=0$, and then $a_{021}$ is a constant section of the trivial line bundle, so that (3.2) defines a locally trivial family of $A_1$ singularities, or $d_{021}>0$, and then the local triviality fails at the $d_{021}d_{102}$ dissident points

(M.Reid's term) in which $a_{021}$ vanishes.

In the first case, it follows from Proposition 2.3 that the blow up of the smooth curve $C = \{x=y=a_{102}=0\}$ gives a crepant resolution of singularities, say $f : \tilde{V} \longrightarrow V$. In the second case this holds generically. We have to check, what happens near dissident points.

Let P be one of them. As $a_{102}$ and $a_{021}$ are generic polynomials on $\mathbb{P}^2$ of corresponding degrees, we can use them as a part of local coordinate system near P (upon passing to inhomogeneous coordinates in $\mathbb{P}^2$). So, the following functions form a local coordinate system near P:

$$u = \bar{a}_{102} + \bar{a}_{111}y + \bar{a}_{120}y^2 + \bar{a}_{201}x + \bar{a}_{210}xy + \bar{a}_{300}x^2 ,$$

$$v = \bar{a}_{021}(\bar{a}_{030})^{-2/3}, \ \bar{x} = x, \ \bar{y} = y(\bar{a}_{030})^{1/3} ,$$

the bar over a indicating that the corresponding polynomials are taken in inhomogeneous coordinates obtained by setting one of the $u_i$ equal to 1. In these coordinates, the local equation of V has the form:

$$\bar{x}u + v\bar{y}^2 + \bar{y}^3 = 0.$$

It is easily checked that the blow up of C resolves this $cA_2$ singularity, so in this case $\tilde{V}$ is also non-singular. Since $f : \tilde{V} \longrightarrow V$ is crepant over the generic point of C, it is crepant everywhere, and we are done.

**3.2.2.** $a_{021} \equiv 0$ $(d_{021} < 0)$. Then either $d_{030}=0$, and (3.2) defines a locally trivial family of $A_2$ singularities, or $d_{030} > 0$, and there are $d_{030}d_{102}$ dissident points in which $a_{030}=0$ and the local triviality fails. Similar to 3.2.1, we can choose local coordinates near a dissident point P in such a way, that the local equation of V will have the form

$$xu + vy^3 = 0 .$$

Let us blow up the curve $C = \{x=y=0\}$. In the coordinate patch $U = \{ u_1 = \frac{x}{y} , u_2 = y, u_3 = \frac{u}{y} \}$ we get the ordinary double singularity

$$u_1 u_3 + v u_2 = 0.$$

This singularity is resolved by blowing up the smooth Weil divisor $u_1 = u_2 = 0$, which is globally defined as the intersection of the proper transform of the "coordinate axis" $x=0$ and the exceptional divisor. So, the composition of the crepant blow up $f_1 : V_1 \longrightarrow V$ of C and the small resolution $f_2 : \tilde{V} \longrightarrow V_1$ of the $d_{030}d_{102}$ singular points of $V_1$ via blow ups of smooth surfaces passing through these points gives a crepant resolution $f = f_2 \circ f_1 : \tilde{V} \longrightarrow V$ of singularities of V.

This finishes the proof of Proposition 3.2.

### 3.3 Crepant resolution of non-generic V's

**3.3.1. An example of an ECYM over $\mathbb{P}^2$ with 9 points blown up and $\chi=0$ , $h^{11}= 11$.** The first case (m,n)=(0,0) of Proposition 1.8 is the case of a generic hypersurface of bidegree (3,3) in $\mathbb{P}^2 \times \mathbb{P}^2$. If we take a special bidegree (3,3) hypersurface V defined by the equation

$$g_1(u_0,u_1,u_2)f_1(x,y,z) + g_2(u_0,u_1,u_2)f_2(x,y,z) = 0,$$

where $g_i$, $f_i$ are generic cubic polynomials, then the projection $\pi : V \longrightarrow \mathbb{P}^2$ onto the first factor is not a flat morphism, and V is singular. It has 81 ordinary double singular points $f_1=f_2=g_1=g_2=0$, which all lie in 9 projective planes $\pi^{-1}(P_i)$ (i=1,...,9), where $\{P_1, \ldots, P_9\} = \{g_1=g_2=0\}$ is the set of points in which the two cubic curves $g_1=0$ and $g_2=0$ meet. All the singularities of V are resolved by a small resolution, consisting in blowing up the ideal $(g_1, g_2)$ in V. The map $\pi$ pulls back to the blow up $\tilde{\mathbb{P}}^2_9$ of the same ideal in $\mathbb{P}^2$ giving rise to a flat elliptic fibration $\tilde{\pi} : \tilde{V} \longrightarrow \tilde{\mathbb{P}}^2_9$.

The base of $\tilde{\pi}$ is itself a fibration of cubic curves over $\mathbb{P}^1$. Over any cubic of this fibration, the restriction of $\tilde{\pi}$ is a trivial fibration. Indeed, if $C'_{\lambda,\mu} = \{g_1:g_2=\lambda:\mu\}$ is the fiber of $\tilde{\mathbb{P}}^2_9$ over the point $(\lambda,\mu) \in \mathbb{P}^1$, then $\pi^{-1}(C'_{\lambda,\mu}) \cong C'_{\lambda,\mu}\times C''_{-\mu,\lambda}$, where $C''_{\kappa,\nu} = \{f_1:f_2=\kappa:\nu\}$.

As the cubic polynomials $g_i$, $f_i$ have been chosen to be generic, one of the two curves, either $C'_{\lambda,\mu}$, or $C''_{-\mu,\lambda}$ is always non-singular, and hence $\chi(\pi^{-1}(C'_{\lambda,\mu})) = 0$. This proves that $\chi(V)=0$.

The Picard group of $\tilde{V}$ is generated by those 2 Cartier divisors which come from V (they are pull backs of the hyperplane sections via projections to both factors of $\mathbb{P}^2\times\mathbb{P}^2$), plus 9 Weil divisors on V which become Cartier on $\tilde{V}$, hence, in total 11 generators. This proves the assertion about $h^{11}$.

There is another way to see that $\chi=0$ in looking at $\tilde{V}$ as a result of surgery applied to the smoothening $V_\varepsilon$ of V. $V_\varepsilon$ is a non-singular (3,3) hypersurface in $\mathbb{P}^2\times\mathbb{P}^2$, and hence $\chi(V_\varepsilon)=-162$, as follows from the formula of Proposition 1.8. The degeneration of $V_\varepsilon$ to V with subsequent small resolution can be represented topologically as the replacement of 81 Milnor fibers $S^3\times D^3$ in $V_\varepsilon$ near singular points of V by the neighbourhoods of the exceptional $\mathbb{P}^1$ of a small blow up of a non-degenerate quadratic singular point, which is topologically $S^2\times D^4$. As $S^3\times D^3$ and $S^2\times D^4$ have the same boundary, namely $S^3\times S^2$, we can cut out the Milnor fibers and paste in $S^2 \times D^4$ instead of them along the common boundary. This will increase the Euler number by the quantity $81(\chi(S^2\times D^4)-\chi(S^3\times D^3))=162$, which proves that $\chi(\tilde{V})=0$.

**3.3.2. The extension of 3.3.1 to WNF ($\chi$=0, $h^{11}$=18).** Consider the hypersurface V in $\mathbb{P}^2\times\mathbb{P}^2$ defined by a generic equation of the form

$$f_1(Y^2Z-X^3-\gamma XZ^2-\delta Z^3) + f_2(\alpha XZ^2+\beta Z^3) = 0,$$

where $f_i=f_i(u_0,u_1,u_2)$ are cubic forms, $\alpha,\beta,\gamma,\delta$ are complex numbers, and denote by $\pi$ the projection to the first factor $\mathbb{P}^2_{(u_0,u_1,u_2)}$. Fibers of $\pi$ are singular over the discriminant curve $C=\{\Delta=0\}$, where $\Delta = 4(f_1\gamma-f_2\alpha)^3 + 27f_1(f_1\delta-f_2\beta)^2$, and over the curve $C_1=\{f_1=0\}$. Over $C_1$ , V is singular along the curve $f_1$=Z=X=0. Local calculations show that it is a locally trivial $cA_6$ singularity. It can be resolved by three subsequent blow ups with centers in non-singular curves, so that the fiber over each $cA_6$ singular point will be the chain of six smooth rational curves. Blowing up further 9 planes $f_1=f_2=0$ and the corresponding 9 points in $\mathbb{P}^2_{(u_0,u_1,u_2)}$ , we will get the flat elliptic fibration $\tilde{\pi} : \tilde{V} \longrightarrow \tilde{\mathbb{P}}^2_9$ with $\tilde{V}$ a smooth ECYM.

Now, let us investigate the structure of the fibration $\tilde{\pi}$ in the whole. The equation of C can be written in the factorized form

$$(f_1-\kappa_1 f_2)(f_1-\kappa_2 f_2)(f_1-\kappa_3 f_2) = 0 ,$$

where $\{\kappa_i\}$ is a generic set of complex constants depending algebraically on $\alpha,\beta,\gamma,\delta$. The proper transform of this curve in $\tilde{\mathbb{P}}^2_9$ is the union of three disjoint elliptic curves $E_1$, $E_2$, $E_3$ in $\tilde{\mathbb{P}}^2_9$ , and over every point of these curves the fiber of $\tilde{\pi}$ is a nodal cubic curve. The proper transform of the elliptic curve $C_1$ (which will be denoted by the same letter) is disjoint from $E_i$, and over each point of $C_1$ the fiber of $\tilde{\pi}$ is Kodaira's fiber $III_*$. Thus the structure of the fibration is described by Fig. 2.

As $\chi(E_i)=\chi(C_1)=\chi(E)=0$, we obtain the Euler number $\chi(\tilde{V})=0$. The generators of the Picard group are the same as in 3.3.1, plus any 7 of 8 ruled surfaces over $C_1$; so, $h^{11}=11+7=18$.

**3.3.3. Degenerate WNF in non-trivial projective bundles.**

Again, adopt the notation of 1.7, and take m=4, n=6. Consider a generic equation of the form

$$u_1(Y^2Z-X^3-\gamma XZ^2-\delta Z^3) + u_2(\alpha XZ^2+\beta Z^3) = 0, \qquad (3.3)$$

where $u_1$, $u_2$ are coordinate linear forms in $\mathbb{P}^2_{(u_0,u_1,u_2)}$, and $\alpha,\gamma \in H^0(\mathbb{P}^2, \mathcal{O}(8))$, $\beta,\delta \in H^0(\mathbb{P}^2, \mathcal{O}(12))$. The left hand side of (3.3) is a (non-generic) section of the line bundle $\mathcal{L}^3\otimes\pi^*\mathcal{O}_{\mathbb{P}^2}(13)$ over $\mathbb{P}$. The fibers of the projection $\pi : V \longrightarrow \mathbb{P}^2_{(u_0,u_1,u_2)}$ are non-singular over the complement of the two curves $C_1=\{u_1=0\}$ and $C=\{\Delta=0\}$, where $\Delta = 4(u_1\gamma-u_2\alpha)^3+ 27u_1(u_1\delta-u_2\beta)^2$ is the discriminant. Over $C_1$, V has singularities along the curve $X=Z=u_1=0$ which are generically locally trivial $cA_6$, but there are also 8 dissident $cA_7$ points in which $\alpha$ vanishes. The $cA_6$ points are resolved similar to the above, by three subsequent blow ups, but the result of these blow ups has 8 isolated ordinary double points over dissidents, lying in one of the two exceptional ruled surfaces which were pasted in by the last blow up. Blowing up this exceptional surface yields a small resolution of the eight singular points.

Next thing to do is to blow up the proper transform of the plane $u_1=u_2=0$, which is a smooth Weil divisor in V passing through the 9 singular points

$$\{ u_1 = u_2 = Y^2Z-X^3-\gamma XZ^2-\delta Z^3 = \alpha XZ^2+\beta Z^3 = 0 \} .$$

This blow up is a small resolution of the 9 singular points, and the resulting ECYM $\tilde{V}$ admits a flat projection $\tilde{\pi}$ to the projective plane $\tilde{\mathbb{P}}^2_1$ with one point $\{u_1=u_2=0\}$ blown up.

The following properties are easily verified:

(i) Define $S_1= \{P \in C \mid P \text{ is a cusp of } C,\ P \notin C_1\}$, $S_2= C\cap C_1$, $S= S_1\cup S_2$ . Then

$$\pi^{-1}(P) = \begin{cases} \text{a nodal cubic, if } P \in C\setminus S \\ \text{a cuspidal cubic, if } P \in S_1 \end{cases}$$

(ii) $S_2$ consists of 8 points in which C osculates $C_1$ up to the third order, defined by the equations $u_1=\alpha=0$, and one triple point of C in which three smooth branches meet, at $u_1=u_2=0$. The latter point gives rise to three distinct points on the proper transform $\tilde{C} \subset \tilde{\mathbb{P}}^2_1$ , forming the set $\bar{S}_2$ , and the former 8 are those, over which dissidents sit, forming the set $\bar{\bar{S}}_2$ .

(iii) The fiber of $\tilde{\pi}$ over any non-dissident point of the proper transform $\tilde{C}_1$ of $C_1$ in $\tilde{\mathbb{P}}^2_1$ is Kodaira's $III_*$ (it is shown on Fig. 2), and over every dissident point, the stucture of the fiber is shown on Fig. 3.

This type is not present in Kodaira's list, because there is no smooth surface in $\tilde{V}$ containing this fiber. If we take a singular one, and resolve the singularity, we will introduce one more component, and the fiber will turn to Kodaira's $II_*$.

Putting all together, we can compute the Euler characteristic of the ECYM $\tilde{V}$ . We have:

$$\chi(\tilde{V}) = \underbrace{\chi(\mathbb{P}^2\setminus(C\cup C_1))\chi(E)}_{\substack{\| \\ 0}} + \chi(C\setminus S)\chi(\text{nodal cubic}) +$$

$$\#S_1\cdot\chi(\text{cuspidal cubic}) + \#\bar{S}_2\cdot\chi(\text{nodal cubic}) + \#\bar{\bar{S}}_2\cdot\chi(\mathrm{II}^{\prime}_{*}) +$$

$$\chi(\tilde{C}_1\setminus\bar{\bar{S}}_2)\chi(\mathrm{III}_{*}) = -284$$

A similar investigation in the case when m=2, n=3 for the equation

$$f_1(Y^2Z-X^3-\gamma XZ^2-\delta Z^3) + f_2(\alpha XZ^2+\beta Z^3) = 0,$$

where $f_1$ and $f_2$ are quadratic forms of variables $u_0$, $u_1$, $u_2$ , yields an ECYM $\tilde{V}$ fibered over the blow up $\tilde{\mathbb{P}}^2_4$ of $\mathbb{P}^2$ in 4 points $f_1=f_2=0$. Its Euler number is $\chi(\tilde{V})=-104$.

## R E F E R E N C E S


1. Reid, M., "The moduli space of 3-folds with K=0 may nevertheless be irreducible", Math. Ann. **278**, 329-334(1987).

2. Iskovskih, V.A. and Shafarevich, I.R., "Algebraic surfaces", in Contemporary Problens in Mathematics VINITI AN SSSR, vol. 35, Moscow, 1989, 131-271 (in Russian).

3. Miranda, R., "Smooth models for elliptic threefolds", in The Birational Geometry of Degenerations, Birkhauser, Boston, 1983.

4. Griffiths, P.A. and Harris, J., Principles of Algebraic Geometry, Wiley, New York, 1978.

5. Albano, A., "Infinite generation of the Griffiths group: a local proof", PhD Thesis, University of Utah, 1986.

6. Grothendieck, A. and Dieudonné, J., Eléments de Géométrie Algébrique I, Springer-Verlag, 1971.

7. Durfée, A., "Fifteen characterizations of rational double points and simple critical points", Enseignement Math.(2), **25**, 131-163(1979).

8. Reid, M., "Canonical 3-folds", in Journées de Géométrie Algébrique d'Angers (A. Beauville, editor), Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, p. 273-310.

9. Pinkham, H., "Factorization of Birational Maps in Dimension Three", in Proc.Symp.Pure Math. **40**, 343-372(1981).

10. Morrison, D., "The birational geometry of surfaces with rational double points", Math.Ann. **271**, 415-438(1985).

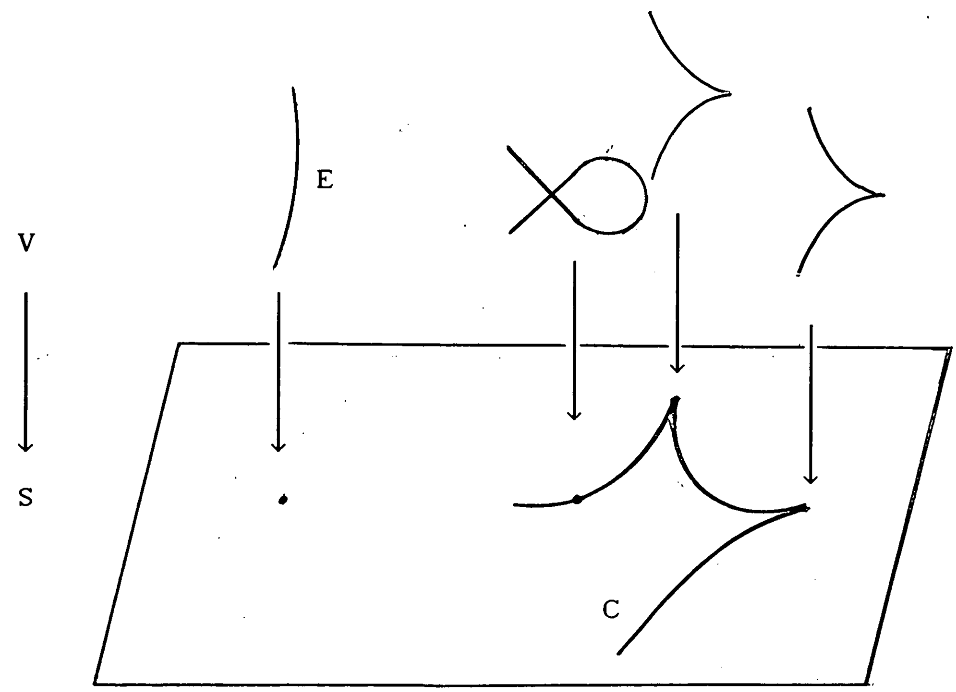


Fig. 1. The structure of fibration f. The generic fiber E is a smooth elliptic curve. Fibers over smooth points of the discriminant curve C are nodal cubics, and those over cusps of C are cuspidal ones.

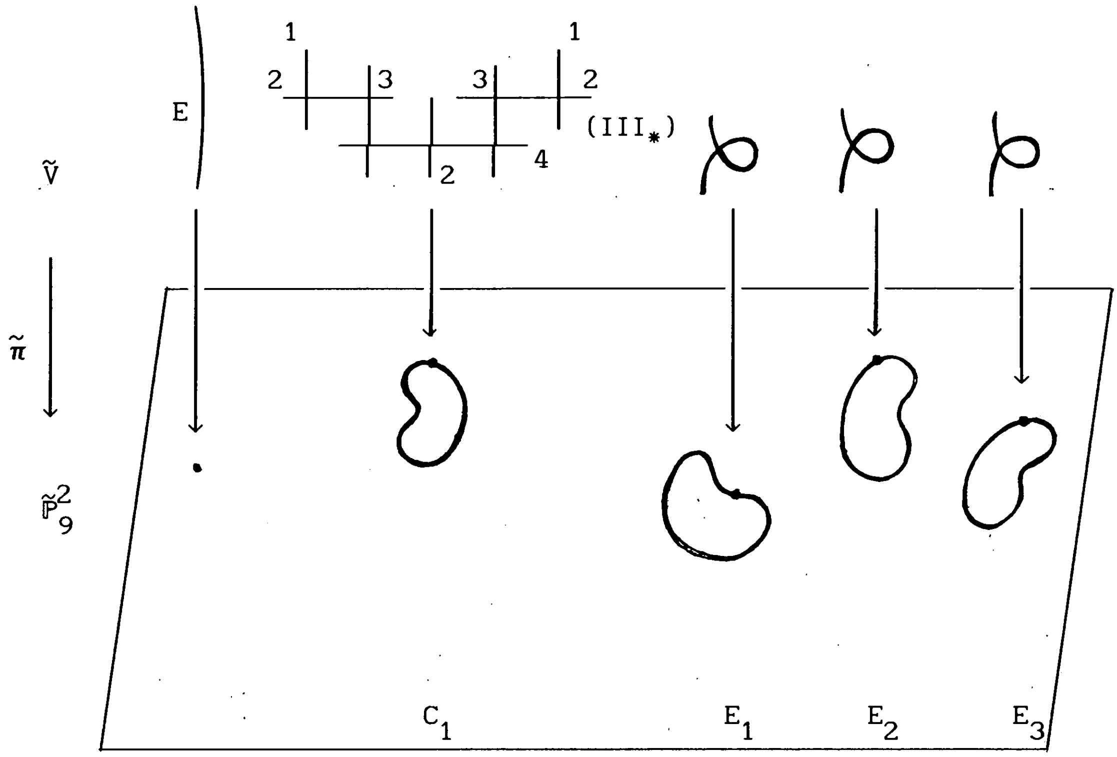


Fig. 2. The structure of fibration $\tilde{\pi}$. The generic fiber is E, a smooth elliptic curve. The numbers near components of the fiber $III_*$ indicate their multiplicity in the scheme-theoretic inverse image $\tilde{\pi}^{-1}(P)$.

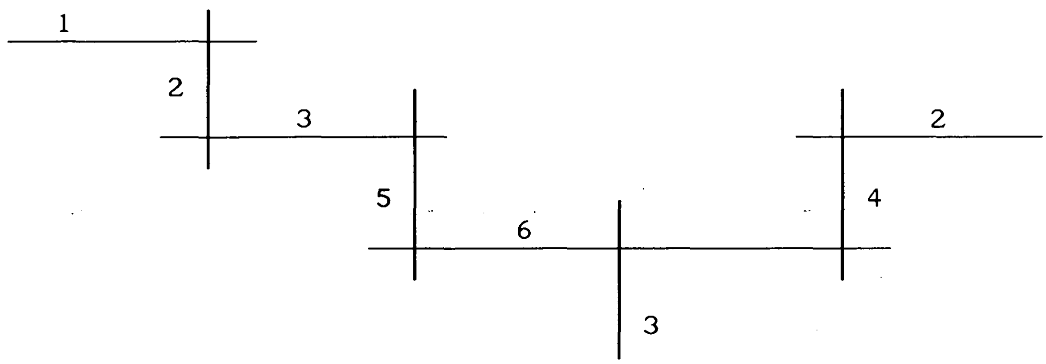


Fig.3. The dissident fiber $II'_*$ . It is an incompleted Kodaira's fiber $II_*$ .