An Asymptotic Formula for the Derivatives of a Function for which the Riemann Hypothesis holds and which is associated with the Riemann Zeta Function

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Abstract

An asymptotic formula for the derivatives at zero as the order of the derivative tends to infinity is found for the function introduced by P. R. Taylor, for which the Riemann hypothesis on the zeros holds. As a corollary an asymptotic formula for the derivatives at any point of the function which plays an important role in the theory of the Riemann zeta-function (main term in Riemann's ξ-function) as the order of the derivative tends to infinity is derived. The comparison of each of the obtained asymptotic formulae with that found earlier by L. D. Pustyl'nikov for the Riemann ξ-function is given.

Keywords and Phrases Riemann hypothesis, zeta-function, asymptotic formula, Riemann function.

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1 The Basic Result

We derive an asymptotic formula for the derivatives of the function $G(s) = \xi_1(s+1/2) - \xi_1(s-1/2)$ at the point $s = 0$ as the order of the derivative tends to the infinity. Here $\xi_1(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, $\Gamma(s)$ is the gamma function, and $\zeta(s)$ is the Riemann zeta function. It is proved in [1] that for the function $G(s)$ the Riemann hypothesis on the zeros holds, i.e. that all nontrivial zeros of G(s) are on the critical line $\Re s = 1/2$. On the other hand, in [2] and [3] for the Riemann function $\xi(s)$ which is by definition

$$\xi(s) = (1/2)s(s-1)\xi_1(s) \tag{1}$$

an asymptotic formula for its derivatives at the point $s = 1/2$ as the order of the derivative tends to the infinity has been found.

Hence it is of interest in relation of the Riemann hypothesis on the zeros to compare these asymptotic formulae. To this end we first find asymptotic formulae of the derivatives of the functions

$$G_1(s) = \xi_1(s+1/2) \text{ and } G_2(s) = \xi_1(s-1/2) \tag{2}$$

at the point $s = 0$ as the order of the derivative tends to the infinity. These formulae give then the corresponding asymptotics for $G(s)$.

Remark Because of the equality $\xi(s) = \xi(1-s)$ and (1) the odd derivatives of $G_1(s)$ at the point $s = 0$ are all equal to zero.

The main result of our paper is the following.

Theorem *For the derivative of the function $G(s)$ at the point $s = 0$ the following asymptotic expression holds as the order of the derivative r tends to the infinity*:

$$\frac{d^rG}{ds^r}(0) \sim \begin{cases} -2^{r+1}r!(1-\frac{1}{3^{r+1}}) & \text{if } r \text{ is odd,} \\ 2^{r+1}r!(1+\frac{1}{3^{r+1}}) & \text{if } r \text{ is even .} \end{cases}$$

(We use the notation $a(r) \sim b(r)$ for $a(r)/b(r) \to 1$ as $r \to \infty$.) The theorem clearly follows from the remark above, Theorem 1 and Theorem 2 given below which rely on a basic estimate of integrals given in Theorem 3 involving these functions. The methods of proofs are similar to that developed in [3] .

2 An Asymptotic Formula for the Function $G_1(s)$

The following theorem holds based on Theorem 3 proved in Section 3.

Theorem 1 *For the even-order derivative of the function $G_1(s)$ at the point $s = 0$ the following asymptotic expression holds as the order r of the derivative tends to the infinity*:

$$\frac{d^rG_1}{ds^r}(0) \sim -2^{r+2}r! \, . \tag{3}$$

Proof

Let us consider the well-known relation (see [4])

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = 1/(s(s-1)) + f(s) + g(s) \ , \qquad (4)$$

where $s \neq 0,\ 1$, and

$$f(s) = \int_1^\infty x^{s/2-1}\omega(x)\,dx, \quad g(s) = \int_1^\infty x^{-s/2-1/2}\omega(x)\,dx, \qquad (5)$$

and $\omega(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$. It is clear that the relation

$$f(s) = g(1-s) \qquad (6)$$

holds. From (5) and (6) for each even natural number r follows

$$\frac{d^r f}{ds^r}\left(\frac{1}{2}\right) + \frac{d^r g}{ds^r}\left(\frac{1}{2}\right) = 2\frac{d^r f}{ds^r}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{r-1}\int_1^\infty (\ln^r x)x^{-\frac{3}{4}}\omega(x)\,dx \ . \quad (7)$$

Further, one clearly sees that for each even natural r

$$\frac{d^r}{ds^r}\frac{1}{s(s-1)}\,\Big|_{s=1/2} = -2^{r+2}r! \qquad (8)$$

holds. Now Theorem 1 easily follows from the definition of the function $G_1(s)$, the equalities (2), (7), (8) and Corollary of Theorem 3 given below.

∎

3 An Asymptotic Formula for the Function $G_2(s)$

The following theorem holds based on Theorem 3 proved in Section 3.

Theorem 2 *For the derivative of the function $G_2(s)$ at the point $s = 0$ the following asymptotic expression holds as the order r of the derivative tends to the infinity*:

$$\frac{d^r G_2}{ds^r}(0) \sim 2^{r+1}r!(1 - \frac{1}{3^{r+1}}) \ . \qquad (9)$$

Proof In view of (4 – 6) we have

$$\frac{d^r}{ds^r}G_2(s)\,|_{s=0} \ = \frac{d^r}{ds^r}\left(1/(s(s-1)) + f(s) + f(1-s)\right)\big|_{s=-1/2} \ . \qquad (10)$$

Further we have

$$\frac{d^r}{ds^r}\frac{1}{s(s-1)}\Big|_{s=-1/2} = 2^{r+1}r!(1-\frac{1}{3^{r+1}}) \ . \tag{11}$$

Then Theorem 2 follows from (10), (5), (11) and Corollary of Theorem 3 given below. ∎

4 Formulation and Proof of Theorem 3

1. For every natural $r \geq 2$ and for each real a let

$$I_{r,a} = \int_1^\infty (\ln^r x) x^a \omega(x)\, dx \tag{12}$$

with $\omega(x)$ as in Section 2.

Theorem 3 *We have the following asymptotic expression as* $r \to \infty$:

$$I_{r,a} \sim (\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + \beta)^r \exp\left(-r(\ln\frac{r}{\pi})^{-1}e^{\beta}\right) \times \left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}\right)^{a+1} \frac{\sqrt{\pi}}{\sqrt{r\left(\frac{1}{2(\ln\frac{r}{\pi}-\ln\ln\frac{r}{\pi})^2}+\frac{1}{2\ln\frac{r}{\pi}}\right)}}\ , \tag{13}$$

where the function $\beta = \beta(r)$ *satisfies* $\lim_{r\to\infty}\beta(r) = 0$.

The proof of Theorem 3 is given in subsections 2 - 6.

2. Set $u = \ln x$. Then

$$I_{r,a} = \int_0^\infty u^r e^{u(a+1)}\omega(e^u)\, du = \int_0^\infty e^{F(u)}\, du\ , \tag{14}$$

where

$$F(u) = r\ln u + u(a+1) - \pi e^u + \ln(1+\psi(e^u)),\ \ \psi(x) = \sum_{n=2}^{\infty} e^{-(n^2-1)\pi x}\ . \tag{15}$$

Differentiation of the function $F(u)$ results in the relations

$$\frac{dF}{du}(u) = \frac{r}{u} + a + 1 - \pi e^u + \frac{d\ln(1+\psi(e^u))}{du}\ , \tag{16}$$

$$\frac{d^2F}{du^2}(u) = -\frac{r}{u^2} - \pi e^u + \frac{d^2\ln(1+\psi(e^u))}{du^2}\ . \tag{17}$$

Let us consider the equation

$$\frac{dF}{du}(u) = \frac{r}{u} + a + 1 - \pi e^u + \frac{d\ln(1+\psi(e^u))}{du} = 0 \tag{18}$$

for $u \geq 0$. Since

$$\lim_{u\to 0}\frac{dF}{du}(u) = +\infty \ \ , \lim_{u\to +\infty}\frac{dF}{du}(u) = -\infty$$

and the inequality $\frac{d^2F}{du^2}(u) < 0$ holds for $u \geq 0$ for large r, equation (18) has a unique solution for $u > 0$ and large r.

We consider the "approximating equation"

$$\frac{r}{u} - \pi e^u = 0\ . \tag{19}$$

By Lemma 1.2 in [3], its solution $u = \hat{u}$ has the form

$$\hat{u} = \ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_1, \tag{20}$$

where the function $c_1 = c_1(r)$ satisfies the condition $\lim_{r\to\infty} c_1(r) = 0$. Therefore, by (20), (18), and (17), the solution $u = \tilde{u}$ of equation (18) can be written as

$$\tilde{u} = \ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2, \tag{21}$$

where the function $c_2 = c_2(r)$ satisfies the same condition as $c_1 = c_1(r)$, that is,

$$\lim_{r\to\infty} c_2(r) = 0\ . \tag{22}$$

Let ε_r be a constant such that $0 < \varepsilon_r < \tilde{u}$. We represent the integral $I_{r,a}$ in the form

$$I_{r,a} = \int_0^{\tilde{u}-\varepsilon_r} e^{F(u)}\,du + \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\,du + \int_{\tilde{u}+\varepsilon_r}^{\infty} e^{F(u)}\,du\ . \tag{23}$$

Substitution of (21) in the first equation in (15) gives

$$\begin{aligned} e^{F(\tilde{u})} &= \left(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2\right)^r \\ &\times \exp\left(-r\left(\ln\frac{r}{\pi}\right)^{-1} e^{c_2}\right)\left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1} e^{c_2}\right)^{a+1} \\ &\times \left(1+\psi\left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1} e^{c_2}\right)\right)\ . \end{aligned} \tag{24}$$

Next, by (22), the asymptotic relation

$$\left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}e^{c_2}\right)^{a+1} \sim \left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}\right)^{a+1} \tag{25}$$

holds as $r \to \infty$, and

$$\lim_{r\to\infty} \psi\left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}e^{c_2}\right) = 0\ .$$

Therefore, by (24), we obtain the asymptotic formula

$$\begin{aligned} e^{F(\tilde{u})} &\sim \left(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2\right)^r \exp\left(-r\left(\ln\frac{r}{\pi}\right)^{-1}e^{c_2}\right) \\ &\times \left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}\right)^{a+1} \end{aligned} \tag{26}$$

as $r \to \infty$. Substitution of (21) in (17) yields

$$\begin{aligned} \frac{d^2F}{du^2}(\tilde{u}) = &- \frac{r}{\left(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2\right)^2} \\ &- \pi\exp\left(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2\right) \\ &+ \frac{d^2\ln(1+\psi(\exp(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2)))}{du^2} \\ = &- \frac{r}{(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2)^2} \\ &- r\left(\ln\frac{r}{\pi}\right)^{-1}e^{c_2} + o(1) < -c_3\, r\left(\ln\frac{r}{\pi}\right)^{-1}\ , \end{aligned} \tag{27}$$

where $c_3 > 0$ does not depend on r and $\lim_{r\to\infty} o(1) = 0$.

Let us estimate $|d^3F(u)/du^3|$ for $\tilde{u} - \varepsilon_r \le u \le \tilde{u} + \varepsilon_r$. Differentiating (17), we obtain

$$\frac{d^3F}{du^3}(u) = \frac{2r}{u^3} - \pi e^u + \frac{d^3\ln(1+\psi(e^u))}{du^3}\ .$$

Therefore, by (21),

$$\sup_{\tilde{u}-\varepsilon_r\le u\le\tilde{u}+\varepsilon_r}\left|\frac{d^3F}{du^3}(u)\right| < c_4\left(\frac{r}{(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2 - \varepsilon_r)^3} + r\left(\ln\frac{r}{\pi}\right)^{-1}e^{\varepsilon_r}\right)\ , \tag{28}$$

where $c_4 > 0$ is a constant not depending on r.

For $\tilde{u}-\varepsilon_r \le u \le \tilde{u}+\varepsilon_r$, the equality

$$F(u) = \tilde{F}(u) + \hat{F}(u) \tag{29}$$

holds, where

$$\tilde{F}(u) = F(\tilde{u}) + \frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})(u-\tilde{u})^2, \quad \hat{F}(u) = \frac{1}{6}\frac{d^3F}{du^3}(\zeta_1)(u-\tilde{u})^3\,, \tag{30}$$

and $\tilde{u}-\varepsilon_r \le \zeta_1 \le \tilde{u}+\varepsilon_r$. Applying Lemma 2.1 from [3] and formula (27), we obtain

$$\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} \exp\left(\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})(u-\tilde{u})^2\right)\, du = \frac{\sqrt{\pi}}{\sqrt{|\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})|}}(1+R_{\varepsilon_r})\,,$$

where

$$|R_{\varepsilon_r}| < \frac{\exp(\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})\varepsilon_r^2)}{1+\sqrt{1-\exp(\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})\varepsilon_r^2)}}\,.$$

Therefore, by (27),

$$\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} \exp\left(\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})(u-\tilde{u})^2\right)\, du \;= \frac{\sqrt{\pi}}{\sqrt{r(A+o(1))}}(1+R_{\varepsilon_r})\,, \tag{31}$$

where

$$A = \frac{1}{2(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2)^2} + \frac{\exp(c_2)}{2\ln\frac{r}{\pi}}$$

and

$$|R_{\varepsilon_r}| < \frac{\exp(-r\varepsilon_r^2 A + \varepsilon_r o(1))}{1+\sqrt{1-\exp(-r\varepsilon_r^2 A + \varepsilon_r o(1))}} \tag{32}$$

By (29), we have

$$\exp(F(u)) = \exp(\tilde{F}(u)+\hat{F}(u)) = \exp(\tilde{F}(u)) + \exp(\tilde{F}(u))(\exp(\hat{F}(u))-1)\,.$$

Consequently,

$$\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du = \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{\tilde{F}(u)}\, du + R'_{\varepsilon_r}\,, \tag{33}$$

where, according to (30),

$$|R'_{\varepsilon_r}| < e^{F(\tilde{u})} \sup_{\tilde{u}-\varepsilon_r\le u\le\tilde{u}+\varepsilon_r} |\exp(\hat{F}(u))-1| \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} \exp\left(\frac{1}{2}\frac{d^2F}{du^2}(\tilde{u})(u-\tilde{u})^2\right) du\,. \tag{34}$$

Applying the relation for $\hat{F}(u)$ in (30) and inequality (28), and assuming that $\varepsilon_r^3 r$ does not exceed a constant not depending on r, we derive the inequality

$$\sup_{\tilde{u}-\varepsilon_r \le u \le \tilde{u}+\varepsilon_r} |\exp(\hat{F}(u)) - 1| < c_5 \varepsilon_r^3 \frac{r}{\ln \frac{r}{\pi}}$$

where c_5 is a constant not depending on r. But if the relation

$$\lim_{r\to\infty} \varepsilon_r^3 r = 0 \qquad (35)$$

holds, then the previous inequality implies that

$$|R'_{\varepsilon_r}| = e^{\tilde{F}(u)} \left(\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} \exp\left(\frac{1}{2} \frac{d^2 F}{du^2}(\tilde{u})(u-\tilde{u})^2 \right) du \right) o(1) ,$$

and, by (33), the asymptotic relation

$$\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du \sim e^{F(\tilde{u})} \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} \exp\left(\frac{1}{2} \frac{d^2 F}{du^2}(\tilde{u})(u-\tilde{u})^2 \right) du \qquad (36)$$

holds as $r \to \infty$.

We now apply (36), (26), (31), and (32) and, assuming (35) and the relation

$$\lim_{r\to\infty} \varepsilon_r^2 r = \infty , \qquad (37)$$

obtain the asymptotic expression

$$\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du \sim \left(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2 \right)^r \exp\left(-r(\ln\frac{r}{\pi})^{-1} e^{c_2} \right)$$
$$\times \left(\frac{r}{\pi}\left(\ln\frac{r}{\pi} \right)^{-1} \right)^{a+1} \frac{\sqrt{\pi}}{\sqrt{r\left(\frac{1}{2(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi})^2} + \frac{1}{2\ln\frac{r}{\pi}} \right)}} \qquad (38)$$

as $r \to \infty$, where the function $c_2 = c_2(r)$ satisfies the inequality (22).

3. We write $u_+ = \tilde{u} + \varepsilon_r$ and $u_- = \tilde{u} - \varepsilon_r$. Since $u = \tilde{u}$ is a solution of equation (18), we have

$$F(u_+) = F(\tilde{u}) + \frac{\varepsilon_r^2}{2} \frac{d^2 F}{du^2}(\zeta_+), \quad F(u_-) = F(\tilde{u}) + \frac{\varepsilon_r^2}{2} \frac{d^2 F}{du^2}(\zeta_-), \qquad (39)$$

where the numbers ζ_+ and ζ_- satisfy the inequalities

$$\tilde{u} \le \zeta_+ \le \tilde{u} + \varepsilon_r, \quad \tilde{u} - \varepsilon_r \le \zeta_- \le \tilde{u} . \qquad (40)$$

The application of (17), (21) and (40) results in

$$\max\left(\frac{d^2F}{du^2}(\zeta_-), \frac{d^2F}{du^2}(\zeta_+)\right) \quad < \quad -\frac{r}{(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2 + \varepsilon_r)^2} - \frac{r}{e^{c_2+\varepsilon_r}\ln\frac{r}{\pi}} + c_6 \ , \tag{41}$$

where c_6 is a constant not depending on r. Now let

$$\varepsilon_r = r^{-(1/2-\delta)}, \quad 0 < \delta < \frac{1}{6} \ . \tag{42}$$

Then (35) and (37) hold and, by (39) and (41), we have the inequalities

$$F(\tilde{u}) - F(u_+) > c_7 r^\delta, \quad F(\tilde{u}) - F(u_-) > c_7 r^\delta,$$
$$e^{F(u_+)} = e^{F(\tilde{u})-(F(\tilde{u})-F(u_+))} < \frac{e^{F(\tilde{u})}}{e^{c_7 r^\delta}} \ , \tag{43}$$
$$e^{F(u_-)} = e^{F(\tilde{u})-(F(\tilde{u})-F(u_-))} < \frac{e^{F(\tilde{u})}}{e^{c_7 r^\delta}} \ ,$$

where $c_7 > 0$ is a constant not depending on r.

By (17), the inequality $\frac{d^2F}{du^2} < 0$ holds for r large enough and $0 \le u \le \tilde{u}$, and thus we have $\frac{dF}{du}(\tilde{u}) = 0$, and $F(u)$ is a monotone increasing function as u grows. Therefore formulae (43) and (21) imply the inequality

$$\int_0^{\tilde{u}-\varepsilon_r} e^{F(u)}\,du \quad < \quad c_8\tilde{u}e^{F(u_-)} \quad < \quad c_8\frac{(\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + c_2)e^{F(\tilde{u})}}{e^{c_7 r^\delta}} \ , \tag{44}$$

where $c_8 > 0$ is a constant not depending on r.

4. Suppose that a positive number $\tilde{\tilde{x}}_r$ satisfies the equation

$$\ln^r \tilde{\tilde{x}}_r = \exp(\pi\delta\tilde{\tilde{x}}_r) \ , \tag{45}$$

where δ is the parameter introduced in (42). Then the relation

$$\frac{r}{\pi\delta} = \frac{\tilde{\tilde{x}}_r}{\ln\ln\tilde{\tilde{x}}_r} \ , \tag{46}$$

holds. From this we deduce

$$\ln\frac{r}{\pi\delta} = \ln\tilde{\tilde{x}}_r - \ln\ln\ln\tilde{\tilde{x}}_r = \tilde{\tilde{y}}_r - \ln\ln\tilde{\tilde{y}}_r \ ,$$

where

$$\tilde{\tilde{y}}_r = \ln \tilde{\tilde{x}}_r \ . \tag{47}$$

Let us set

$$\mu = \ln \frac{r}{\pi\delta} \ , \quad f(\tilde{\tilde{y}}_r) = \tilde{\tilde{y}}_r - \ln\ln \tilde{\tilde{y}}_r \ .$$

Then $\mu = f(\tilde{\tilde{y}}_r)$. Let $\hat{y}_r$ satisfy the equation

$$\mu = \hat{y}_r - \ln \hat{y}_r \ . \tag{48}$$

By (46) and (48), we have

$$f(\hat{y}_r) > \mu \ , \quad f(\mu) < \mu \ ,$$

and, since $f(y)$ is a monotone increasing function as y grows, relation (46) implies that $\tilde{\tilde{y}}_r$ satisties the inequality

$$\mu < \tilde{\tilde{y}}_r < \hat{y}_r \tag{49}$$

According to (46) and Lemma 1.1 from [3], the solution $\hat{y}_r$ of equation (48) has the form

$$\hat{y}_r = \ln \frac{r}{\pi\delta} + \ln\ln \frac{r}{\pi\delta} + c_9 \ ,$$

and the function $c_9 = c_9(r)$ satisfies the condition $\lim_{r\to\infty} c_9(r) = 0$. Consequently, by (49) and (47), we have

$$\ln \frac{r}{\pi\delta} < \ln \tilde{\tilde{x}}_r < \ln \frac{r}{\pi\delta} + \ln\ln \frac{r}{\pi\delta} + c_9 \ , \quad \lim_{r\to\infty} c_9(r) = 0 \ . \tag{50}$$

The application of (43) and (50) gives

$$\begin{aligned} \left| \int_{u_+}^{\ln \tilde{\tilde{x}}_r} e^{F(u)} \, du \right| \quad &< \quad e^{F(u_+)} |\ln \tilde{\tilde{x}}_r - u_+| \\ &< \quad \frac{e^{F(\tilde{u})}}{e^{c_7 r^\delta}} \left(\ln \frac{r}{\pi\delta} + \ln\ln \frac{r}{\pi\delta} + c_9 - u_+ \right) \ . \end{aligned} \tag{51}$$

5. Let us estimate the integral

$$\tilde{\tilde{I}}_r \stackrel{def}{=} \int_{\tilde{\tilde{x}}_r}^{\infty} (\ln^r x) x^a \omega(x) \, dx \ , \tag{52}$$

where $\tilde{\tilde{x}}_r$ is defined by (45) and $\omega(x)$ is the function introduced at the beginning of this section. According to (45), (50) and the definitions of

$\omega(x)$ and $\tilde{\tilde{x}}_r$, we have

$$\begin{aligned} \tilde{\tilde{I}}_r &< \int_{\tilde{\tilde{x}}_r}^{\infty} x^a \left(\sum_{n=1}^{\infty} e^{-\pi x (n^2 - \delta)} \right) dx \\ &< \int_{r/(\pi\delta)}^{\infty} x^a \left(\sum_{n=1}^{\infty} e^{-\pi x (n^2 - \delta)} \right) dx \ . \end{aligned} \tag{53}$$

6. According to the definition of the integral $I_{r,a}$ the following inequality holds:

$$I_{r,a} = \int_0^{\tilde{u}-\varepsilon_r} e^{F(u)}\, du + \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du + \int_{\tilde{u}+\varepsilon_r}^{\ln \tilde{\tilde{x}}_r} e^{F(u)}\, du + \int_{\ln \tilde{\tilde{x}}_r}^{\infty} du \ , \tag{54}$$

where $\tilde{u}$, ε_r, and $\tilde{\tilde{x}}_r$ are the numbers introduced above. For ε_r as in (42) we obtain, by (44) and (51)–(53), the inequality

$$\begin{aligned} \left| \int_0^{\tilde{u}-\varepsilon_r} e^{F(u)}\, du + \int_{\tilde{u}+\varepsilon_r}^{\infty} e^{F(u)}\, du \right| &< c_{10} \frac{(\ln \frac{r}{\pi} - \ln\ln \frac{r}{\pi} + c_{11}) e^{F(\tilde{u})}}{e^{c_7 r^{\delta}}} \\ + \int_{r/(\pi\delta)}^{\infty} x^a \left(\sum_{n=1}^{\infty} e^{-\pi x (n^2 - \delta)} \right) dx \end{aligned} \tag{55}$$

for all r sufficiently large, where c_{10} and c_{11} are constants not depending on r.

As $r \to \infty$ the first term on the right-hand side of (55), by (36), (31), and (32), is

$$o\left(\int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du \right)$$

and the second term there is $o(1)$. Therefore the asymptotic relation

$$I_{r,a} \sim \int_{\tilde{u}-\varepsilon_r}^{\tilde{u}+\varepsilon_r} e^{F(u)}\, du$$

holds as $r \to \infty$ and, by (38), the same is true for the asymptotic relation

$$\begin{aligned} I_{r,a} &\sim (\ln \frac{r}{\pi} - \ln\ln \frac{r}{\pi} + c_2)^r \exp\left(-r (\ln \frac{r}{\pi})^{-1} e^{c_2} \right) \\ &\times \left(\frac{r}{\pi} \left(\ln \frac{r}{\pi} \right)^{-1} \right)^{a+1} \frac{\sqrt{\pi}}{\sqrt{r \left(\frac{1}{2(\ln \frac{r}{\pi} - \ln\ln \frac{r}{\pi})^2} + \frac{1}{2 \ln \frac{r}{\pi}} \right)}} \ , \end{aligned} \tag{56}$$

where the function $c_2 = c_2(r)$ satisfies (22).

Thus, Theorem 3 with $\beta(r) = c_2(r)$ is proved. ∎

Corollary *We have the following asymptotic expression as* $r \to \infty$:

$$\frac{d^r f}{ds^r}(s)\,|_{s=b} \sim \left(\frac{1}{2}\right)^r (\ln\frac{r}{\pi} - \ln\ln\frac{r}{\pi} + \beta)^r \exp\left(-r(\ln\frac{r}{\pi})^{-1}e^{\beta}\right) \times \left(\frac{r}{\pi}\left(\ln\frac{r}{\pi}\right)^{-1}\right)^{a+1} \frac{\sqrt{\pi}}{\sqrt{r\left(\frac{1}{2(\ln\frac{r}{\pi}-\ln\ln\frac{r}{\pi})^2}+\frac{1}{2\ln\frac{r}{\pi}}\right)}}, \quad (57)$$

with the function $f(s)$ *defined in (5),* $a = b/2 - 1$ *and the function* $\beta = \beta(r)$ *for which* $\lim_{r\to\infty}\beta(r) = 0$ *holds.*

5 Discussion

In [3] the following asymptotic formula for the even-order derivative of the Riemann function $\xi = \xi(s + 1/2)$ at the point $s = 0$ as the order r of the derivative tends to ∞ was found:

$$\frac{d^r}{ds^r}\xi(s+1/2)\,|_{s=0} \sim 2^{-(r-2)}\left(\ln\frac{r-2}{\pi} - \ln\ln\frac{r-2}{\pi} + \gamma\right)^{r-2} \times \exp\left(-(r-2)\left(\ln\frac{r-2}{\pi}\right)^{-1}e^{\gamma}\right)(r-2)^{1/4}r(r-1) \times \left(\ln\frac{r-2}{\pi}\right)^{-1/4}\frac{\pi^{1/4}}{\sqrt{(r/2-1)\left(\frac{1}{(\ln\frac{r-2}{\pi}-\ln\ln\frac{r-2}{\pi})^2}+\frac{1}{\ln\frac{r-2}{\pi}}\right)}}, \quad (58)$$

where the function $\gamma = \gamma(r)$ satisfies the condition $\lim_{r\to\infty}\gamma(r) = 0$.

One can easily compare this formula first with that given here in our main Theorem for P. R. Taylor's function $G(s)$ for which the Riemann hypothesis holds (see [1]) and then with (57) given in Corollary above for $b = 1/2$. In the first case we see that the absolute value of the even-order derivatives of $G(s)$ at the point $s = 0$ growfor $r \to \infty$ more quickly than that of the Riemann function $\xi = \xi(s + 1/2)$ at the same point $s = 0$. In the second case for the function $f(s)$ both formulae are very similar and the main term in (57) can be obtained from the main term in (58) by replacing in it $r - 2$ by r.

References

[1] P. R. Taylor. On the Riemann Zeta Function. In *Quarterly Journal Math.*, volume 16, pages 1–21. 1945.

[2] L. D. Pustyl'nikov. On the asymptotic behaviour of the Taylor series coefficients of $\xi(s)$. In *Russian Math. Surveys*, volume 55, pages 349–350. 2000.

[3] L. D. Pustyl'nikov. An asymptotic formula for the Taylor coefficients of the function $\xi(s)$. In *Izvestya: Mathematics*, volume 65:1, pages 85–98. 2001.

[4] A. A. Karatsuba. Principles of analytic number theory. 2nd ed., Nauka, Moscow, Russian, 1983.