The Swiss-Cheese Operad

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Dedicated to Michael Boardman on the occasion of his sixtieth birthday.

Вороне где-то Бог послал кусочек сыру.
I. A. Krylov

ABSTRACT. We introduce a new operad, which we call the Swiss-cheese operad. It mixes naturally the little disks and the little intervals operads. The Swiss-cheese operad is related to the configuration spaces of points on the upper half-plane and points on the real line, considered by Kontsevich for the sake of deformation quantization. This relation is similar to the relation between the little disks operad and the configuration spaces of points on the plane. The Swiss-cheese operad may also be regarded as a finite-dimensional model of the moduli space of genus-zero Riemann surfaces appearing in the open-closed string theory studied recently by Zwiebach. We describe algebras over the homology of the Swiss-cheese operad.

Introduction

This is a short note whose modest purpose is to introduce a new operad, which we call the Swiss-cheese operad. It mixes naturally the little disks and the little intervals operads, which are J. Peter May's [**May77**] modifications of the little cubes operad, the remarkable discovery of J. Michael Boardman and Rainer M. Vogt [**BV73**]. The Swiss-cheese operad is related to the configuration spaces of points on the upper half-plane and points on the real line, considered by Maxim Kontsevich [**Kon97**] for the sake of deformation quantization, in the same way that the little disks operad is related to the configuration spaces of points on the complex plane. The Swiss-cheese operad may also be regarded as a finite-dimensional model of the moduli space of genus-zero Riemann surfaces appearing in the open-closed string theory studied recently by Barton Zwiebach [**Zwi97**]. Our main theorem describes algebras over the homology of the Swiss-cheese operad. Such algebraic structure is expected to be found on the physical state space of open-closed string theory.

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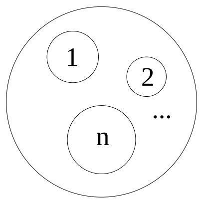
1. The Swiss-cheese operad

We will be using the standard terminology of operad theory, see [**BV73, GK94, KM95, Lod96**]. Operads will be considered in different tensor categories, such as those of manifolds, topological spaces, (graded) vector spaces, complexes of vector spaces, sets, and even modules over a given operad, depending on the context. The Swiss-cheese operad resembles the famous little disks operad. However the pictures for the Swiss-cheese operad look more like Swiss cheese than those for the little disks operad. The *little disks operad* is a collection of manifolds $D(n)$, $n \geq 1$, where each $D(n)$ is the configuration space of n nonoverlapping little disks inside the standard unit disk on the plane:

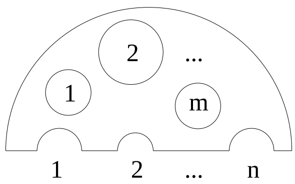


The space $D(n)$ is obviously an open set in $\mathbb{R}^{3n}$ (whence a manifold structure), each configuration being uniquely determined by the position of the centers of the disks and their radii. It is assumed that each little disk is labeled by a number from 1 through n, which defines the action of the permutation group Σ_n on $D(n)$. The operad composition

$$\gamma : D(k) \times D(n_1) \times \cdots \times D(n_k) \to D(n_1 + \cdots + n_k)$$

is given by scaling down given configurations in $D(n_1)$, ... , $D(n_k)$, gluing them into the k holes in a given configuration in $D(k)$, and erasing the seams. Thus $D = \{D(n) \mid n \geq 1\}$ becomes an operad of manifolds.

The *Swiss-cheese operad* is a collection of manifolds $S(m,n)$ $m \geq 0$, $n \geq 1$, where $S(m,n)$ is the configuration space of nonoverlapping disks labeled 1 through m and upper semidisks labeled 1 through n inside the standard unit upper semidisk, so that the little semidisks are all centered about the diameter of the big semidisk:



$S(m,n)$ is an open subset in $\mathbb{R}^{3m+2n}$. There is a natural action of the group $\Sigma_m \times \Sigma_n$ on $S(m,n)$ and two types of composition:

$$S(k,l) \times S(m_1,n_1) \times \cdots \times S(m_l,n_l) \xrightarrow{\gamma} S(k + m_1 + \cdots + m_l, n_1 + \cdots + n_l),$$
$$S(k,l) \times D(m_1) \times \cdots \times D(m_k) \xrightarrow{\Gamma} S(m_1 + \cdots + m_k, l).$$

The first one is gluing the big semidisks of given configurations in $S(m_i,n_i)$, $i = 1,\dots,l$, into the l little semidisks of a given configuration in $S(k,l)$, the second is gluing the big disks of given configurations in $D(m_i)$, $i = 1,\dots,k$, into the k little disks of a given configuration in $S(k,l)$. This looks like a new operad-type structure on first sight, but it is nothing but an *operad over an operad* or a *relative operad*, all meaning an *operad of modules over a given operad*. The point is that the collection of all $S(n) = \bigcup_{m\geq 0} S(m,n)$, $n \geq 1$, is an operad of modules over the little disks operad D, each $S(n)$ being a module over D. Recall ([**Mar96, Smi82**]) that a (*right*) *module over an operad* $\mathcal{O}$ is a collection of spaces $\mathcal{M}(m)$, $m \geq 0$, on which the action of Σ_m is given, as well as maps

$$\Gamma : \mathcal{M}(k) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_k) \to \mathcal{M}(m_1 + \cdots + m_k)$$

satisfying virtually the same axioms as those of an operad. Modules over an operad form a tensor category with respect to the following tensor product:

$$(\mathcal{M}_1 \times \mathcal{M}_2)(m) = \bigcup_{k+l=m} \mathrm{Map}_{\Sigma_k, \Sigma_l}(\Sigma_{k+l}, \mathcal{M}_1(k) \times \mathcal{M}_2(l)),$$

where $\mathrm{Map}_{\Sigma_k, \Sigma_l}$ is the set of all $\Sigma_k \times \Sigma_l$-equivariant maps $\Sigma_k \times \Sigma_l \to \mathcal{M}_1(k) \times \mathcal{M}_2(l)$, $\Sigma_k \times \Sigma_l$ acting on Σ_{k+l} by left translation. The tensor product structure is necessary to speak of operads of $\mathcal{O}$-modules.

2. Relative operads and algebras over them

Another example of a relative operad comes from linear algebra. Let V_1, V_2 be two vector spaces. The *endomorphism operad of the vector space* V_1 is the operad $\mathcal{E}nd_{V_1} = \{\mathrm{Hom}(V_1^{\otimes m}, V_1) \mid m \geq 1\}$, the permutation groups acting by permuting the inputs of multilinear operators and the operad compositions given by substitution of the outputs into the inputs. The relative version of the endomorphism operad is the *relative endomorphism operad* $\mathcal{E}nd_{V_1,V_2} = \{\mathrm{Hom}(V_1^{\otimes m} \otimes V_2^{\otimes n}, V_2) \mid m \geq 0, n \geq 1\}$. It is obviously an operad of modules over the operad $\mathcal{E}nd_{V_1}$.

That said, it is easy to define an algebra over a relative operad: it is analogous to the notion of a group representation. Let $\mathcal{M}$ be an operad over an operad $\mathcal{O}$. An *algebra over the relative operad* $\mathcal{M}$ is a pair of vector spaces V_1 and V_2 and two mappings

$$\mathcal{O} \to \mathcal{E}nd_{V_1},$$
$$\mathcal{M} \to \mathcal{E}nd_{V_1,V_2},$$

the former being a morphism of operads, which also makes $\mathcal{E}nd_{V_1,V_2}$ an $\mathcal{O}$-module, the latter being a morphism of relative operads over $\mathcal{O}$.

Another class of examples of relative operads may be obtained in the following way. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operads such that $\mathcal{O}_1$ is an *operad with commutative multiplication*, cf. [**GV95**], which is nothing but a morphism $\mathcal{C}omm \to \mathcal{O}_1$ from the commutative operad $\mathcal{C}omm$ to $\mathcal{O}_1$. A commutative multiplication can equivalently be given by an element $M \in \mathcal{O}_1(2)$ stable under the action of the symmetric group Σ_2 and such that $M \circ M := \gamma(M; M, \mathrm{id}) - \gamma(M; \mathrm{id}, M) = 0$. (Here if $\mathcal{O}_1$ is not an operad of abelian groups, the vanishing of the difference must be understood as the equality of the two terms in it). We will also assume that the commutative operad has a *multiplication unit* 1, *i.e.*, $\mathcal{C}omm(0)$ is a point. Therefore, $\mathcal{O}_1(0)$ will contain a distinguished element as well. Then the collection $\mathcal{M}(n) = \mathcal{O}_1 \times \mathcal{O}_2(n) = \bigcup_{m\geq 0} \mathcal{O}_1(m) \times \mathcal{O}_2(n)$ for $n \geq 1$ is an operad over $\mathcal{O}_1$, the multiplication in $\mathcal{O}_1$ being used to define a product

$$\mathcal{O}_1(k) \times \mathcal{O}_1(m_1) \times \cdots \times \mathcal{O}_1(m_l) \to \mathcal{O}_1(k + m_1 + \cdots + m_l)$$

needed for defining the operad composition γ for $\mathcal{M}$. The associativity of the multiplication guarantees the associativity of the operad composition γ, and the commutativity implies the equivariance of γ with respect to the action of the symmetric group.

PROPOSITION 2.1. *An algebra over the $\mathcal{O}_1$-operad $\mathcal{M} = \mathcal{O}_1 \times \mathcal{O}_2$ defined above may be equivalently described as a pair (V_1, V_2), such that V_1 is an algebra over $\mathcal{O}_1$,* *and V_2 is an algebra over $\mathcal{O}_2$, with an action*

$$\begin{aligned} V_1 \otimes V_2 &\to V_2, \\ u \otimes v &\mapsto uv, \end{aligned} \tag{2.1}$$

defining on V_2 an $\mathcal{O}_2$-algebra structure over the commutative associative algebra V_1, where V_1 acquires the structure a commutative associative algebra with a unit because of the morphism $\mathcal{C}omm \to \mathcal{O}_1$. An $\mathcal{O}_2$-algebra structure on V_2 over a commutative algebra V_1 means that

$$u_1 \dots u_m v \tag{2.2}$$

(where $u_1, \dots, u_m \in V_1$ and $v \in V_2$, uu' denoting the commutative product in V_1, and uv the action of V_1 on V_2) is independent of a possible arrangement of parentheses, and

$$f(u_1 v_1, \dots, u_n v_n) = u_1 \dots u_n f(v_1, \dots, v_n) \tag{2.3}$$

for any $f \in \mathcal{O}_2(n)$.

PROOF. 1. Given an algebra (V_1, V_2) over the $\mathcal{O}_1$-operad $\mathcal{M} = \mathcal{O}_1 \times \mathcal{O}_2$, we immediately have an $\mathcal{O}_1$-algebra structure, which induces a commutative algebra structure via $\mathcal{C}omm \to \mathcal{O}_1$, on V_1 and an $\mathcal{O}_2$-algebra structure on V_2. Take the operad units $\mathrm{id} \in \mathcal{O}_1(1)$ and $\mathrm{id} \in \mathcal{O}_2(1)$ and form their cross-product $(\mathrm{id}, \mathrm{id}) \in \mathcal{M}$. The algebra structure over the relative operad $\mathcal{M}$ will then create a linear map (2.1) corresponding to $(\mathrm{id}, \mathrm{id})$. The operad unit axiom for $\mathcal{O}_1$ and $\mathcal{O}_2$ implies that the composite map $(\mathcal{O}_1(1) \times \mathcal{O}_2(1) \times \mathcal{O}_1(m)) \times \mathcal{O}_2(n) \xrightarrow{\Gamma \times \mathrm{id}} \mathcal{O}_1(m) \times \mathcal{O}_2(1) \times \mathcal{O}_2(n) \xrightarrow{\gamma} \mathcal{O}_1(m) \times \mathcal{O}_2(n)$ evaluated at $(\mathrm{id}, \mathrm{id}) \in \mathcal{O}_1(1) \times \mathcal{O}_2(1)$ is the identity map. If we track the same composition for the relative endomorphism operad, we will see that the operation $V_1^m \times V_2^n \to V_2$ corresponding to $(f_1, f_2) \in \mathcal{O}_1(m) \times \mathcal{O}_2(n)$ is nothing but

$$f_1(u_1, \dots, u_m) f_2(v_1, \dots, v_n). \tag{2.4}$$

To show that an expression (2.2) is independent of an arrangement of parentheses, it suffices to show that

$$u_1(u_2 v) = (u_1 u_2) v.$$

This follows from the fact that the following operad compositions are equal: $\gamma((\mathrm{id}, \mathrm{id}); (\mathrm{id}, \mathrm{id})) = \Gamma((\mathrm{id}, \mathrm{id}); M)$, by definition of γ for the product relative operad $\mathcal{O}_1 \times \mathcal{O}_2$. The relation (2.3) is satisfied for the similar reason: the operad composition $\gamma : \mathcal{O}_1(0) \times \mathcal{O}_2(n) \times (\mathcal{O}_1(1) \times \mathcal{O}_2(1)) \times \dots \times (\mathcal{O}_1(1) \times \mathcal{O}_2(1)) \to \mathcal{O}_1(n) \times \mathcal{O}_2(n)$ is defined by using the commutative multiplication n times: $M^n : \mathcal{O}_1(0) \times \mathcal{O}_1(1) \times \dots \times \mathcal{O}_1(1) \to \mathcal{O}_1(n)$.

2. Suppose now we have an $\mathcal{O}_1$-algebra V_1, which in particular means V_1 is a commutative associative algebra with a unit, and an $\mathcal{O}_2$-algebra V_2 over the commutative algebra V_1, as defined in the statement of this proposition. We have to show that these data define a module (V_1, V_2) over the relative operad $\mathcal{O}_1 \times \mathcal{O}_2$. Define the corresponding morphism $\mathcal{O}_1(m) \times \mathcal{O}_2(n) \to \mathrm{Hom}(V_1^m \otimes V_2^n, V_2)$ by (2.4). What remains to see is that this is indeed a morphism of relative operads. The symmetric group equivariance follows from that for the structure operad morphisms $\mathcal{O}_1 \to \mathcal{E}nd_{V_1}$ and $\mathcal{O}_2 \to \mathcal{E}nd_{V_2}$. The compatibility with the actions of $\mathcal{O}_1$ on $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{E}nd_{V_1, V_2}$ follows from the fact that $\mathcal{O}_1 \to \mathcal{E}nd_{V_1}$ is a morphism of operads. The compatibility with the operad compositions γ for $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{E}nd_{V_1, V_2}$ follows from the fact that $\mathcal{O}_2 \to \mathcal{E}nd_{V_2}$ is a morphism of operads and the linearity (2.3). $\square$

3. Swiss-cheese algebras

Different operads are in general responsible for different algebraic structures. The little disks operad defines the important class of *Gerstenhaber or G-algebras*, which are defined by two operations, a (dot) product ab and a bracket $[a, b]$, on a graded vector space V, so that the product defines a graded commutative algebra structure on V and the bracket a graded Lie algebra structure on $V[1]$, the desuspension of the graded vector space $V = \bigoplus_n V^n$: $V[1]^n = V^{n+1}$. The bracket must be a graded derivation of the product in the following exact sense:

$$[a, bc] = [a, b]c + (-1)^{(\deg a-1)\deg b} b[a, c],$$

where $\deg a$ denotes the degree of an element $a \in V$. In other words, a G-algebra is a specific graded version of a Poisson algebra. Here and henceforth, we will consider *G-algebras with a unit* 1, which is an element in V behaving as a unit with respect to the dot product and such that $[a, 1] = 0$ for all $a \in V$.

The way the little disks operad D has relevance to G-algebras is through its homology operad $H_\bullet(D)$ and the following theorem.

Theorem 3.1 (F. Cohen [**Coh76**]). *The structure of a G-algebra on a $\mathbb{Z}$-graded vector space is equivalent to the structure of an algebra over the homology little disks operad $H_\bullet(D)$.*

Remark 3.2. Here we must take into account the $D(0)$ component of the little disks operad and a unit in a G-algebra. This is an obvious extension of Cohen's Theorem.

What is the analogue of this theorem for the Swiss-cheese operad?

Theorem 3.3. *An algebra over the homology Swiss-cheese operad $H_\bullet(S)$ is a pair of graded vector spaces V_1 and V_2, V_1 endowed with the structure of a G-algebra with a unit and V_2 with the structure of a graded associative algebra over V_1, regarded as a graded commutative algebra with respect to its dot product.*

Definition 3.4. A *Swiss-cheese algebra* is a pair (V_1, V_2) of vector spaces with the structure described in the theorem.

Proof. An algebra over a relative operad consists of a pair of graded vector spaces V_1 and V_2. By Cohen's Theorem 3.1, V_1 is a G-algebra. Each D-module $S(n)$, $n \geq 1$, is homotopy equivalent to $D \times \Sigma_n$ as a D-module. Moreover the whole collection $S = \{S(n) \mid n \geq 1\}$ is homotopy equivalent to the collection $\{D \times \Sigma_n \mid n \geq 1\}$ as a D-operad in the homotopy category of topological spaces — one just has to choose an arbitrary point in $D(2)$ as a multiplication, which will be associative and commutative up to homotopy, and provide the collection of the symmetric groups Σ_n's with the natural operad structure. This operad structure makes $\{\Sigma_n \mid n \geq 1\}$ into the associative operad $\mathcal{A}s$, whose algebras are nothing but (graded) associative ones. Passing to homology, we get exactly the situation of Proposition 2.1: (V_1, V_2) is an algebra over the product operad $H_\bullet(D) \times \Sigma_n$, except that we are now in the category of graded vector spaces, so that all operads and algebras will be graded. The proposition then describes this algebra structure as required by this theorem. □

4. Homotopy Swiss-cheese algebras

In this section we will outline a geometric construction of what we will call the homotopy Swiss-cheese operad, similar to Kontsevich's construction [**Kon94**] of the A_∞-operad. We will not provide complete details whenever the material is a straightforward generalization of the proofs and constructions of [**KSV95,**

KSV96, KVZ97]. We will first construct a relative operad homotopy equivalent to the Swiss-cheese operad. This new operad will have a filtration compatible with the relative operad structure and therefore give rise to an operad of spectral sequences converging to its homology operad. The spectral sequences will collapse at E^2, making E^1 an operad of complexes whose homology is the homology Swiss-cheese operad. It will be this operad E^1 that we will call the homotopy Swiss-cheese operad.

Consider the configuration space $\mathrm{Conf}_{m,n}$ of $m \geq 0$ labeled distinct points on the upper half-plane and $n \geq 0$ labeled distinct points on the real line on the complex plane, assuming $2m + n \geq 2$. Also consider the quotient configuration space $C_{m,n} = \mathrm{Conf}_{m,n}/G$, where $G = \{z \mapsto az + b \mid a, b \in \mathbb{R}, a > 0\}$ is the group of orientation-preserving affine transformations of the real line, acting freely because of the condition $2m+n \geq 2$. The spaces $\mathrm{Conf}_{m,n}$ and $C_{m,n}$ are homotopy equivalent to the component $S(m,n)$ of the Swiss-cheese operad. Unlike the spaces $S(m,n)$, the configuration spaces $\mathrm{Conf}_{m,n}$ or $C_{m,n}$ do not form a relative operad. We will compactify the spaces $C_{m,n}$ *à la* Kontsevich [**Kon97**], and more generally Fulton-MacPherson [**FM94**], to become smooth manifolds with corners and form a relative operad, homotopy equivalent to the Swiss-cheese operad, to which we add the component $S(1,0)$ and from which throw away the component $S(0,1)$ to satisfy the stability condition $2m + n \geq 2$.

The compactification $\overline{C}_{m,n}$ will be formed pointwise by stable configurations. A *stable configuration* is a semistable complex algebraic curve X of genus zero with $m+n$ smooth labeled punctures, $m, n \geq 0$, $2m+n \geq 2$, and a special $m+n+1$st smooth puncture labeled ∞, along with the following data:

1. The choice of a line through ∞ along with the choice of orientation on the line, which allows us to identify the corresponding irreducible component X_∞ of X with the complex plane up to a transformation in the group G. No punctures or double points are allowed strictly below this line on the component X_∞.
2. The choice of oriented lines passing through selected double points on selected irreducible components (not more than one line on each component). No punctures or double points are allowed strictly below each of these lines on the corresponding irreducible components. If a double point lies on such a line on one irreducible component, a real line must pass through this double point on the adjacent component.
3. The choice of a real tangent direction at the point ∞ and to each irreducible component at each double point not lying on a real line.

On each irreducible component X_α, one can count the number m_α of punctures and double points not lying on the real line (if any) and the number n_α of punctures and double points on the real line (if any). The stability condition means by definition that, for each component with no real line, $m_\alpha - 1 \geq 2$ and, for each component with a real line, $2m_\alpha + n_\alpha - 1 \geq 2$. Each component is considered up to the group of its conformal transformations: G for components with the choice of a real line and the group $\{z \mapsto az + b \mid a \in \mathbb{R}, a > 0, b \in \mathbb{R}^2\}$ for components with no lines.

Naively, one can think of a stable configuration as a semistable complex algebraic curve which is cut into two pieces by a cross section through a straight line passing through ∞ and a number of double points on different components, with one of the two pieces completely forgotten (discarded) and tangent directions chosen at each double point not lying on the cross section.

As a set, the *compactification* $\overline{C}_{m,n}$ may be defined as the set of stable configurations. The structure of a smooth manifold with corners may be obtained by translating the point-set description into a sequence of real blowups of the space of configurations where not all points must be distinct. The compactification we have described here is exactly the same as the one constructed by Kontsevich in [**Kon97**], which he used to prove his Formality Conjecture; we are just giving a new interpretation of points of that compactification.

The simpler similar compactifications $\overline{C}_{m,0}$ in which no real lines are involved form an operad homotopy equivalent to the little disks operad. The components $\overline{C}_{0,n}$ form an operad isomorphic (as an operad of manifolds with corners) to the Stasheff polyhedra operad, which is homotopy equivalent to the associative operad $\mathcal{A}s$. The whole collection $\overline{C}_{m,n}$ makes up a relative operad over the operad $\overline{C}_{m,0}$. The two relative operad compositions γ and Γ are given by joining stable configurations at punctures to form new double points. The obtained relative operad is homotopy equivalent to the Swiss-cheese operad.

Let us filter the spaces $\overline{C}_{m,n}$ by the number of double points:

$$\emptyset \subset F^0 \subset \cdots \subset F^{2m+n-3} = \overline{C}_{m,n},$$

where F^p consists of stable configurations that have at least $2m+n-3-p$ double points, $\dim_{\mathbb{R}} F^p = p$. The operad compositions γ and Γ respect the filtration, because they create double points. For each space $\overline{C}_{m,n}$, we get a spectral sequence. These spectral sequences form an operad of spectral sequences, see [**KSV96**], here a relative one, with the $E^1_{p,q} = H_{p+q}(F^p, F^{p-1}; \mathbb{C}) = H^{-q}(F^p \setminus F^{p-1}; \mathbb{C})$ terms forming a relative operad of complexes. The operad of complexes E^1, by analogy with the corresponding constructions for $\overline{C}_{0,n}$ and $\overline{C}_{m,0}$, may be called the *homotopy Swiss-cheese operad.* However, it is not known whether the spectral sequence collapses at E^2 and the homotopy Swiss-cheese operad is indeed a free resolution of the Swiss-cheese operad. A *homotopy Swiss-cheese algebra* may be defined as an algebra over the homotopy Swiss-cheese operad. A homotopy Swiss-cheese algebra comprises a certain algebraic structure on a pair of complexes V_1 and V_2 of vector spaces, which includes operations similar to those of a Swiss-cheese algebra and a hierarchy of higher homotopies for relations satisfied by those operations in a strict Swiss-cheese algebra, relations between the relations, etc. Since V_2 is an algebra over the operad $\overline{C}_{0,n}$, it carries the structure of an A_∞-algebra. Since V_1 is an algebra over the operad $\overline{C}_{m,0}$, it carries the structure of a homotopy G-algebra, in a sense which is different from the several versions introduced in [**VG95, GJ94**] and used in [**GV95**] and [**KVZ97**][1]. The zeroth row of the term E^1 for the operad $\overline{C}_{m,0}$ is nothing but the L_∞-operad. The last row is nothing but the C_∞-operad (see [**KSV96**]). The only ($q = 0$) row in E^1 for $\overline{C}_{0,n}$ is the A_∞-operad. Thus the homotopy Swiss-cheese algebra structure establishes an interplay between these three homotopy algebraic structures.

Remark 4.1. A similar compactification was used by Zwiebach in his work [**Zwi97**] on open-closed string theory. If one generalizes our previous work [**KSV95, KVZ97**], one may expect to obtain the algebraic structure of Ward identities encoded as the structure of a homotopy Swiss-cheese algebra, see also discussion in Stasheff's contribution [**Sta98**] to this volume.

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