"On Representations of Finite Transformation Groups of Algebraic Curves and Surfaces"

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Lemma 5. *Given k and $0 < D < \pi/\sqrt{k}$, there exists a $\chi > 0$ so that if $\bar{\gamma}_{AB}$, $\bar{\gamma}_{AC}$ are unit minimal in S_k with $0 < \alpha(\bar{\gamma}_{AB}, \bar{\gamma}_{AC}) < \pi$, $L(\bar{\gamma}_{AB}) \leq D$, and $d(B, C) \leq 3\chi$, then for any $0 < t \leq \min\, (L(\bar{\gamma}_{AB}), L(\bar{\gamma}_{AC}))$ and minimal curve $\bar{\alpha}$ from $\bar{\gamma}_{AB}(t)$ to $\bar{\gamma}_{AC}(t)$, $\max\, \{d(A, \bar{\alpha}(s))\} < t + \chi$.*

Proof. Since metric balls are convex for $k \leq 0$, we need only consider $k > 0$; by scaling the metric we reduce to $k = 1$, and clearly now we can assume $t > \pi/2$. Let $\chi > 0$ be small enough that $\cos D - (\cos (1.5\chi))(\cos (D+\chi)) > 0$. We fix curves $\bar{\gamma}_{AB}$, $\bar{\gamma}_{AC}$ as above, assume $\bar{\alpha}$ is parameterized on $[0, 1]$ and let $\sigma = d(A, \bar{\alpha}(1/2)) = \max\, \{d(A, \gamma(s)\}$. Letting $\lambda = L(\bar{\alpha})$ and applying the Cosine Law to $\alpha(\bar{\gamma}_{AB}, \bar{\alpha})$ we obtain

$$\frac{\cos \tau - (\cos t)(\cos \lambda/2)}{\sin \lambda/2} = \frac{\cos t - (\cos t)(\cos \lambda)}{\sin \lambda}$$

which reduces to $\cos \tau = \cos t / \cos \lambda/2$.

Applying the sum formula to $\cos (\tau - t)$ we see that $\tau - t$ is an increasing function of both t and λ; i.e., for fixed χ, $\tau - t$ is maximized when $L(\bar{\alpha}) = L(\beta) = t = D$. Thus we only need to prove $\cos^{-1}(\cos D / \cos (1.5\chi)) \leq \cos (D+\chi)$, and this follows from the way χ was chosen. □

For $0 < D < \pi/\sqrt{k}$, fix a closed ball $B = \bar{B}(p, D) \subset X$ and a cover U of $\bar{B}(p, 2D)$ by regions of curvature $\geq k$, and let $\chi(U) < D$ be as in Lemma 5 and also less than one eighth of a Lebesque number of U. Let $\tau(U)$ small enough that if $c\bar{\alpha}$, $\bar{\gamma}$ are unit

Introduction. Let X be a topological space with a finite group G acting on it. For suitable coefficient systems and cohomology theories, $H^*(X; \mathcal{F})$ becomes a G–representation. Study of such representations and their relationship to the symmetries of X has been the subject of extensive study. In our previous paper [A1]–[A5], we have studied such representations from the view points of group cohomology and local–global considerations. In particular, [A3] considers the integral representations on $H^2(X;\mathbb{Z})$ when X is a compact simply–connected 4–manifold. In the following paper, we continue [A3] by specializing to the case of algebraic curves and surfaces.

Historically speaking, such investigations for complex projective curves (compact Riemann surfaces) goes back to the 1893 paper of Hurwitz, in which complex representations of cyclic automorphism groups of Riemann surfaces were studied. His work was completed by Chevalley and Weil, also using analytic techniques. See Weil's collected works Vol. I, pages 529 and 532–533 for historical details and a discussion of these results. Chevalley–Weil's results were further generalized by Tamagawa to the case of curves over fields of positive characteristic with free regular automorphism groups. Tamagawa's result is formulated in terms of unramified Galois extensions of the corresponding function fields. This point of view has been further developed by number theorists, in particular, Madan and Valentini among others. (See Valentini–Madan, Journal Number Theory, Vol. 13, 1981, for a historical survey and further developments.)

Some of the results of the present paper may be considered as modest generalizations of the above–mentioned results. Such generalizations are in two directions. First, we have determined the integral representations $H^1(X,\mathbb{Z})$ for a compact Riemann surface with an arbitrary finite automorphism group (Section 4). Since the structure of $\mathbb{Z}G$–modules is a complete mystery for almost all finite groups, our formulations are in terms of group cohomology in the general case. Secondly, we have studied certain representations for suitable non–singular projective surfaces in analogy with Chevalley–Weil and Tamagawa's results. Namely, for free G–actions on projective surfaces X where $p_a(X) = p_g(X)$

(p_a = arithmetic genus and p_g = geometric genus). For curves, $p_a(X) = p_g(X)$ always. But for surfaces, this is a real restriction, and it should be compared with simple–connectivity hypothesis for complex projective surfaces. In Section 6, we have determined the kG–module $H^0(X; \mathcal{K}_X)$ (= vector space of regular 2–forms) in analogy with the case of regular 1–forms for curves. Section 5 makes a preliminary study of the $\mathbb{Z}$G–representation $H^2(X;\mathbb{Z})$ when X is compact Kähler. The general theme of sections 3–5 is to relate the topology and geometry of the underlying symmetry to the homological properties of suitable representations. In Section 2 we have gathered some definitions and a brief discussion of some of the homological notions for the convenience of the reader. Further preliminary material may be found in [A3] or in the references.

Note added in proof. Since the appearance of the first version of this paper, several related works are brought to my attention. I would like to thank Chad Schoen for discussions on his interesting results in this direction and for sending me his manuscript [Schoen]. I am also grateful to G. Ellencwejg and T. Kohono who brought to my attention the related works of S. Nakajima [Nakajima 1 & 2] which go deeper in the number theoretic direction and seem to have a slight overlap with some of our results.

Section Two. Preliminary Notions.

In the following sections, we will use the same notation and conventions as in [A3]. However, we review some of the notation for the convenience of the reader. Let G be a finite group, and R be a commutative ring with unit, e.g. $R = \mathbb{Z}$, $\hat{\mathbb{Z}}_p$ = p–adic integers, $\mathbb{F}_p$, or $\mathbb{C}$. The RG–modules are finitely generated and R–free. Finite generation may not hold for some of the RG–modules in the chain complexes used in Section 6. However, the cohomology and homology groups are all finitely generated, and this will be sufficient. Two RG–modules M_1 and M_2 are called projectively stably RG–isomorphic, denoted by $M_1 \sim M_2$, if there is a commutative diagram:

$$\begin{array}{ccc} M_1 \oplus P_1 & \xrightarrow[g]{\cong} & M_2 \oplus P_2 \\ {\scriptstyle j}\uparrow & & \downarrow{\scriptstyle \pi} \\ M_1 & \longrightarrow & M_2 \end{array}$$

where P_1 and P_2 are RG–projective, j and π are the obvious inclusion and projection, and g is an isomorphism. If P_1 and P_2 are RG–free, then we call M_1 and M_2 stably isomorphic. Stable isomorphism is an equivalence relation. Heller [Hr] has defined loop and suspension operations for RG–modules when the notions "projective cover" and "injective hull" make sense. However, projective covers do not exist, in general, for $\mathbb{Z}G$–modules although they exist for $\mathbb{F}_pG$–modules or $\hat{\mathbb{Z}}_p[G]$–modules. Here, we can define a stable version of the "Heller loop–operator", which we denote by ω, on the set of stable equivalence classes of RG–lattices (i.e. R–torsion free RG–modules). Namely, $\omega(M)$ is stably well–defined (by Schanuel's Lemma [Sw]) from the exact sequence $0 \longrightarrow \omega(M) \longrightarrow (RG)^{\alpha} \longrightarrow M \longrightarrow 0$. If we use projective RG–modules instead of $(RG)^{\alpha}$, then $\omega(M)$ is well–defined up to projective stable equivalence. Then we set

$\omega^1(M) = \omega(M)$ and $\omega^{i+1}(M) = \omega(\omega^i(M))$ inductively. For $i \in \mathbb{Z}$, this definition has a natural extension, so that $\omega^i(M)$ are stably well–defined for all $i \in \mathbb{Z}$.

We will also make use of a construction for RG–modules from cohomology classes which is explained in [A3]. Our description is a generalization and a stable version of the construction used by J. Carlson [C] in modular representation theory. Recall the Tate cohomology $\hat{H}^i(G;M)$, $i \in \mathbb{Z}$ as in e.g. Cartan–Eilenberg [CE]. Then $\widehat{\mathrm{Hom}}_G(M,R) \overset{\mathrm{def}}{\equiv} \hat{H}^0(G,\mathrm{Hom}_R(M,R))$ is isomorphic to the group of RG–homomorphisms $f : M \longrightarrow R$ modulo the subgroup of those which factor through an RG–projective. (See Mac Lane pp. 74–75 [Mc] for related discussion. It turns out that $\hat{H}^0(G,\mathrm{Hom}_R(\omega^n(M),R)) \overset{\mathrm{def}}{\equiv} \widehat{\mathrm{Hom}}(\omega^n(M),R) \cong \hat{H}^n(G;M^*)$, where $M^* = \mathrm{Hom}_R(M,R)$ with the diagonal RG–module structure. Now, given a cohomology class, $x \in \hat{H}^n(G;M^*)$, we may represent x by an RG–homomorphism $\varphi : \omega^n(M) \longrightarrow R$ which is well–defined up to factorization through RG–projectives. φ may be assumed also surjective. Define $L_\varphi \equiv \mathrm{Ker}(\varphi)$. Then L_φ is well–defined up to projective stable equivalence. (See [A3] for further discussion). The notation class (φ) will be used for the cohomology class represented by φ. The functor $\widehat{\mathrm{Ext}}{}^i_G(-,-)$ is also constructed in analogy with Tate cohomology $\hat{H}^i(G,-)$ using complete resolutions (see e.g. Cartan–Eilenberg [CE] or Carlson [C]).

An algebraic generalization of a Poincaré duality space is the notion of a chain complex with duality. Let C_* be a bounded connected chain complex of dimension n over a ring R, so that $H_0(C_*) \cong R$ and $C_i = 0$ for $i < 0$ or $i > n$ (for some $n > 0$). We call C_* a chain complex with duality of formal dimension m, if there exists a chain homotopy equivalence $h : C_{m-i} \longrightarrow C^i$ between C_* and C^*. The cellular chain complex of a Poincaré duality space or a closed oriented smooth manifold are basic examples of such complexes with duality.

In Section 5 and 6 we will need some basic facts from algebraic geometry. The standard reference for the definitions and concepts used in the following are Hartshorne [H] and Mumford [M1] [M2].

Section 3. Free Actions.

In this section we study homology representations of free actions.

3.1 Theorem. Let X_* be a (k–1)–connected bounded RG–free chain complex with duality of formal dimension $2k$. Then:

(a) The RG–module $H_k(X_*)$ is completely determined up to stable equivalence by a homology class $x \in H_{2k}(G;R)$.

(b) Let $\zeta : \omega^{-2k-1}(R) \twoheadrightarrow R$ be a representative for x. Then $H_k(X_*)$ is stably RG–isomorphic to $\omega^k L_\zeta$.

(c) Let $\varphi : \omega^{-k-1}(R) \longrightarrow \omega^k(R)$ be such that $\text{class}(\varphi) = \text{class}(\zeta) = x$ under the isomorphisms

$$\widehat{\text{Hom}}_G(\omega^{-k-1}(R), \omega^k(R)) \cong \widehat{\text{Hom}}_G(\omega^{-2k-1}(R), R) \cong H_{2k}(G;R) .$$

Then $H_k(X_*)$ is completely determined (stably) from the short exact seqeunce below:

$$0 \longrightarrow H_k(X) \longrightarrow \omega^{-k-1}(R) \xrightarrow{\varphi} \omega^k(R) \longrightarrow 0 .$$

The following corollary has been proved for $k = 2$ by Hambleton–Kreck [HK].

3.2 Corollary. A symmetric expression for $H_k(X_*)$ is obtained as follows. Let $z \in \text{Ext}^1_G(\omega^{-k-1}(R), \omega^{k+1}(R))$. Then the extension class z is represented by the short exact sequence:

$$0 \longrightarrow \omega^{k+1}(R) \longrightarrow H_k(X_*) \oplus (\text{RG–Free}) \longrightarrow \omega^{-k-1}(R) \longrightarrow 0 .$$

Proof. Since X_* is R–chain homotopic to its R–dual $X^* \equiv \mathrm{Hom}_R(X_*,R)$, we have $H_i(X_*) = 0$ for $k+1 \leq i \leq 2k-1$. Moreover, without loss of generality, we may assume that $X_i = 0$ for $i \geq 2k+1$ (see e.g. Assadi [A3] Lemma 4.2.). The connectivity of X_* in the above–mentioned dimensions gives rise to long exact sequences below:

$$0 \longrightarrow B_{k-1} \longrightarrow X_{k-1} \longrightarrow \ldots\ldots \longrightarrow X_0 \longrightarrow R \longrightarrow 0$$

$$0 \longrightarrow R \longrightarrow X_{2k} \longrightarrow \ldots\ldots \longrightarrow X_{k+1} \xrightarrow{\partial_{k+1}} X_k \longrightarrow \mathrm{coker}(\partial_{k+1}) \longrightarrow 0 .$$

We conclude that $B_{k-1} = \omega^k(R)$ and $\mathrm{coker}(\partial_{k+1}) = \omega^{-k-1}(R)$. To identify $H_k(X_*)$, we consider the commutative diagram below:

$$\text{(D)}\quad \begin{array}{ccccccccccccc}
 & & & & & & & & 0 & & 0 & & \\
 & & & & & & & & \uparrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & X_{2k} & \longrightarrow \ldots\ldots & X_{k+1} & \longrightarrow & Z_k & \xrightarrow{\alpha} & H_k(X_*) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{i} & & \downarrow{j} & & \\
0 & \longrightarrow & R & \longrightarrow & X_{2k} & \longrightarrow \ldots\ldots & X_{k+1} & \longrightarrow & X_k & \xrightarrow{\pi} & \mathrm{coker}(\partial_{k+1}) & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\partial_k} & & \downarrow{\varphi} & & \\
 & & & & & & & & B_{k-1} & \xrightarrow{=} & B_{k-1} & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & & 0 & &
\end{array}$$

The homomorphism j in the above diagram is induced from the inclusion $i : Z_k \hookrightarrow X_k$. Thus we have the short exact sequence:

$$0 \longrightarrow H_k(X_*) \longrightarrow \omega^{-k-1}(R) \xrightarrow{\varphi} \omega^k(R) \longrightarrow 0$$

of RG–modules, and $H_k(X_*)$ is stably determined by the class $(\varphi) \in \widehat{\mathrm{Hom}}_G(\omega^{-k-1}(R), \omega^k(R))$. Using the isomorphisms $\widehat{\mathrm{Hom}}_G(\omega^{-k-1}(R), \omega^k(R)) \cong \widehat{\mathrm{Hom}}_G(\omega^{-2k-1}(R),R) \cong \widehat{\mathrm{Ext}}_G^{-2k-1}(R,R) \cong H_{2k}(G;R)$, we obtain the class $x \in H_{2k}(G;R)$ corresponding to class (φ). Let $\zeta : \omega^{-2k-1}(R) \longrightarrow R$ be a representative for x. Then $L_\zeta \equiv \mathrm{Ker}(\zeta) = \omega^{-k}(\mathrm{Ker}\ \varphi)$, so that $H_k(X_*) = \omega^k L_\zeta$. This proves the Theorem. ∎

Proof of Corollary 3.2. The homomorphisms j and π of the diagram (D) in the proof of Theorem 3.1 above give rise to the following short exact sequence:

(E) $$0 \longrightarrow Z_k \xrightarrow{\alpha \oplus i} H_k(X) \oplus X_k \xrightarrow{j-\pi} \mathrm{coker}(\partial_{k+1}) \longrightarrow 0 .$$

Since $\mathrm{coker}(\partial_{k+1}) = \omega^{-k-1}(R)$ and $Z_k = \omega^{k+1}(R)$ from the exact sequence $0 \longrightarrow Z_k \longrightarrow X_k \longrightarrow \ldots\ldots \longrightarrow X_0 \longrightarrow R \longrightarrow 0$ we obtain the desired short exact sequence of the corollary. It remains to determine the extension class $z \in \mathrm{Ext}_G^1(\mathrm{coker}(\partial_{k+1}),Z_k) \equiv \widehat{\mathrm{Ext}}_G^1(\omega^{-k-1}(R), \omega^{k+1}(R)) \cong \widehat{\mathrm{Ext}}_G^{-2k-1}(R,R) \cong \hat{H}^{-2k-1}(G;R) \equiv H_{2k}(G;R)$. We apply $\widehat{\mathrm{Hom}}_G(\mathrm{coker}(\partial_{k+1}),-)$ to the exact sequences $0 \longrightarrow H_k(X_*) \longrightarrow \mathrm{coker}(\partial_{k+1}) \xrightarrow{\varphi} B_{k-1} \longrightarrow 0$ and $0 \longrightarrow Z_k \longrightarrow X_k \xrightarrow{\partial_k} B_{k-1} \longrightarrow 0$ as well as (E). We get the commutative diagram below in which δ' and δ_E are the connecting homomorphisms of the last two sequences:

$$\widehat{\operatorname{Hom}}_G(\operatorname{coker}(\partial_{k+1}), \operatorname{coker}(\partial_{k+1}))$$

$$\swarrow \varphi_* \qquad\qquad \searrow \delta_E$$

$$\widehat{\operatorname{Hom}}_G(\operatorname{coker}(\partial_{k+1}), B_{k-1}) \xrightarrow[\cong]{\delta'} \widehat{\operatorname{Ext}}^1_G(\operatorname{coker}(\partial_{k+1}), Z_k)$$

$$\downarrow = \qquad\qquad\qquad\qquad \downarrow =$$

$$\widehat{\operatorname{Hom}}_G(\omega^{-k-1}(R), \omega^k(R)) \xrightarrow{\cong} \widehat{\operatorname{Ext}}^1_G(\omega^{-k-1}(R), \omega^{k+1}(R))$$

$$\cong \searrow \qquad\qquad \swarrow \cong$$

$$H_{2k}(G,R)$$

Since $z = \delta$ (identity) and φ_*(identity) = class(φ) = class(ζ) = x , and all other isomorphisms are obtained by dimension shifting, it follows that z and x correspond under these natural isomorphisms. ■

<u>3.3 Theorem</u>. Let X be a (k–1) connected finite dimensional Poincaré complex of formal dimension 2k . Let G act freely on X and let $f : X/G \longrightarrow BG$ be the classifying map for the G–covering $X \xrightarrow{\pi} X/G$. Then:

(a) The homology class $x \equiv f_*[X/G] \in H_{2k}(BG;\mathbb{Z}) = H_{2k}(G;\mathbb{Z})$ completely determines the $\mathbb{Z}G$–module $H_k(X;\mathbb{Z})$ up to $\mathbb{Z}G$–stable isomorphism and vice versa. In fact, $H_k(X)$ is stably isomorphic to $\omega^k L_\zeta$ where class(ζ) = x as in Theorem 3.1 above.

(b) Each $x \in H_{2k}(G;\mathbb{Z})$ is realized by a free analytic G–action on a compact connected Riemann surface when k = 1 , and by a free smooth G–action on a compact simply–connected 4–manifold when k = 2 .

<u>Proof of Theorem 3.3</u>. Applying the result of Theorem 3.1 to the free $\mathbb{Z}G$–chain complex $C_*(X)$, we conclude that the stable $\mathbb{Z}G$–isomorphism class of $H_k(X)$ is determined by $x = \text{class}(\varphi) \in \widehat{\operatorname{Hom}}_G(\operatorname{coker}(\partial_{k+1}), B_{k-1}) \cong H_{2k}(G;\mathbb{Z})$. We compute x in terms of the

induced homomorphism $f_* : H_{2k}(X/G;\mathbb{Z}) \longrightarrow H_{2k}(BG;\mathbb{Z}) = H_{2k}(G;\mathbb{Z})$ as follows. Let $E_* = C_*(E_G)$, where $E_G \longrightarrow BG$ is the universal G–covering as usual, and $C_* = C_*(X)$. The RG–chain map $\gamma_\# : C_* \longrightarrow E_*$ is induced by $\gamma : X \longrightarrow E_G$. We identify (E_*, ∂'_*) as a free $\mathbb{Z}G$–resolution of $\mathbb{Z}$, Ker $\partial'_{2k} = \omega^{2k+1}(\mathbb{Z})$, and coker$(\partial'_{k+1}) = \omega^k(\mathbb{Z})$. Consider the commutative diagram below induced by γ and the above identifications:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & \omega^{2k+1}(\mathbb{Z}) & \longrightarrow & E_{2k} & \xrightarrow{\partial'} \cdots\cdots & E_{k+1} & \xrightarrow{\partial'_{k+1}} & E_k & \xrightarrow{\pi'} & \omega^k(\mathbb{Z}) & \longrightarrow 0 \\
& \uparrow \gamma_* & & \uparrow \gamma_\# & & & & \uparrow \gamma_\# & & \uparrow \lambda & \\
0 \longrightarrow & H_{2k}(X) & \longrightarrow & C_{2k} & \longrightarrow \cdots\cdots & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\pi} & Q & \longrightarrow 0 .
\end{array}$$

The class $f_*[X/G] \in H_{2k}(G;\mathbb{Z})$ is determined by $f \in \mathrm{Hom}(H_{2k}(BG)) = \mathrm{Hom}(\mathbb{Z}, H_{2k}(G;\mathbb{Z}))$. The shifting isomorphism, denoted by

$$\sigma(\omega^{k+1}) : \widehat{\mathrm{Hom}}_G(Q, \omega^k\mathbb{Z}) \xrightarrow{\cong} \widehat{\mathrm{Hom}}_G(\omega^{k+1}(Q), \omega^{k+1}(\mathbb{Z}))$$

sends class(λ) to class$(\gamma_*) = f_*$ in the diagram below:

$$\begin{array}{ccc}
\widehat{\mathrm{Hom}}_G(Q, \omega^k(\mathbb{Z})) & \xrightarrow[\cong]{\sigma(\omega^{k+1})} & \widehat{\mathrm{Hom}}_G(\omega^{k+1}(Q), \omega^{k+1}(\mathbb{Z})) \\
\downarrow \cong & & \downarrow \cong \\
H_{2k}(G;\mathbb{Z}) & \xrightarrow{\cong} & \widehat{\mathrm{Hom}}_G(H_{2k}(X), \omega^{2k+1}(\mathbb{Z})) .
\end{array}$$

Therefore, it suffices to prove that class(λ) = class(φ). Consider the commutative diagrams below in which (I) determines class(φ) :

(I)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & B_{k-1} & \xrightarrow{=} & B_{k-1} & & \\
 & & & & \uparrow & & \uparrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & B_k & \longrightarrow & C_k & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & B_k & \longrightarrow & Z_k & \xrightarrow{\alpha} & H_k(X) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(II)

$$\begin{array}{ccccc}
 & & B_{k-1} & & \\
 & \swarrow{\scriptstyle\varphi} & & \searrow{\scriptstyle\tau_{k-1}} & \\
Q & & \xrightarrow{\quad\lambda\quad} & & \omega^k(\mathbb{Z}) \\
\downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\pi'} \\
C_k & & \xrightarrow{\quad\gamma_{\#}\quad} & & E_k
\end{array}$$

(III)

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \omega^k(\mathbb{Z}) & \longrightarrow & E_{k-1} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\tau_{k-1}} & & \uparrow{\scriptstyle\gamma_{\#}} & & & & \uparrow{\scriptstyle\gamma_{\#}} & & \uparrow{\scriptstyle =} & & \\
0 & \longrightarrow & B_{k-1} & \longrightarrow & C_{k-1} & \longrightarrow & \cdots & \longrightarrow & C_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Under the shifting isomorphism $\sigma(\omega^k) : \widehat{\mathrm{Hom}}_G(\mathbb{Z},\mathbb{Z}) \xrightarrow{\pi} \widehat{\mathrm{Hom}}_G(\omega^k(\mathbb{Z}), \omega^k(\mathbb{Z})) \cong$ $\cong \widehat{\mathrm{Hom}}_G(B_{k-1}, \omega^k(\mathbb{Z}))$ in diagram (III), class($\mathrm{id}_{\mathbb{Z}}$) corresponds to class(τ_{k-1}). Thus the isomorphism $(\tau_{k-1})_*$ below:

$$\begin{array}{ccc} \widehat{\mathrm{Hom}}_G(Q,\, B_{k-1}) & \xrightarrow[(\tau_{k-1})_*]{\cong} & \widehat{\mathrm{Hom}}_G(Q,\, \omega^k(\mathbb{Z})) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & H_{2k}(G;\mathbb{Z}) & \end{array}$$

sends class(φ) to class(λ), and this is what we wanted. Thus part (a) of the theorem is proved. The proof of part (b) is included in Assadi [A3] Proposition 4.4 (c) for the case $k = 2$. For $k = 1$, the Hurwicz homomorphism $\Omega_2^{SO}(BG) \longrightarrow H_2(BG) = H_2(G;\mathbb{Z})$ is surjective, hence part (a) implies the desired conclusion. ∎

<u>3.4 Corollary</u>. For every $x \in H_2(G;\mathbb{Z})$, there exists a free projective G–action on a non–singular projective curve/$\mathbb{C}$ such that $H^1(X_{an};\mathbb{Z})$ is $\mathbb{Z}G$–stably isomorphic to $(\omega^1 L_\zeta)^*$ where $\zeta \in \mathrm{Hom}_G(\omega^{-3}(\mathbb{Z}),\mathbb{Z})$ represents x under the isomorphism $\widehat{\mathrm{Hom}}_G(\omega^{-3}(\mathbb{Z}),\mathbb{Z}) \cong \hat{H}^{-3}(G;\mathbb{Z}) = H_2(G;\mathbb{Z})$, and X_{an} is the underlying space with the usual topology.

<u>Proof</u>: According to 3.3 (b) above, there exists a compact Riemann surface Σ and a map $f : \Sigma \longrightarrow BG$ such that $f_*[\Sigma] = x \in H_2(BG;\mathbb{Z}) = H_2(G;\mathbb{Z})$. Let X be the G–covering induced by f together with the free G–action on X via covering translations. Then $H_1(X;\mathbb{Z})$ is stably $\mathbb{Z}G$–isomorphic to $\omega^1(L_\zeta)$ and class(ζ) = x by Theorem 3.1 above. Now G acts on the compact Riemann surface X by complex analytic isomorphisms, and $H^1(X_{an};\mathbb{Z})$ is $\mathbb{Z}G$–ismorphic to $\mathrm{Hom}(H_1(X),\mathbb{Z}) = (\omega^1 L_\zeta)^*$ and class(ζ) = x by Theorem 3.1 above. We may assume that the genus $(\Sigma) \geq 2$, hence genus $(X) \geq 2$, so that the canonical sheaves $\mathcal{K}_\Sigma$ and $\mathcal{K}_X$ are ample. By Serre's GAGA principle [S1], Σ and $\mathcal{K}_\Sigma$ are algebraic. Thus, X is a complete non–singular curve on which G acts by

algebraic isomorphisms, $\mathscr{K}_X$ is an ample G–line bundle on X, and $\pi : X \longrightarrow \Sigma = X/G$ is an algebraic morphism for which $\mathscr{K}_\Sigma = (\pi_* \mathscr{K}_X)^G$. Since the pluricanonical embedding $X \longrightarrow \mathbb{P}\Gamma(X, \mathscr{K}_X^{\otimes m})$ is equivariant, the G–action on X is projective.■

<u>3.5 Examples</u>. (1) If $G = \mathbb{Z}/p\mathbb{Z}$, then $H_2(G;\mathbb{Z}) = 0 = H_4(G;\mathbb{Z}) = 0$. Thus, if $\dim_{\mathbb{R}} X = 2$, then for $r = \frac{1}{p}(g-1)$ $H^1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}G)^{2r}$, and if $\dim_{\mathbb{R}} X = 4$ and $\pi_1(X) = 0$, then $H^2(X) \cong I \oplus I \oplus (\mathbb{Z}G)^s$ where I is the augmentation ideal. Since $I \cong \mathbb{Z}[\zeta]$, where ζ is a primitive p–th root of unity with the usual $\mathbb{Z}G$–module structure, then $H^2(X;\mathbb{Z}) \cong \mathbb{Z}[\zeta] \oplus \mathbb{Z}[\zeta] \oplus (\mathbb{Z}G)^s$.

(2) Suppose G has periodic cohomology, so that the p–Sylow subgroups of G are cyclic for p = odd and either cyclic or generalized quaternionic for p = 2. Then $H_2(G_p;\mathbb{Z}) = 0 = H_4(G_p;\mathbb{Z})$ for all p–Sylow subgroups $G_p \subseteq G$. Therefore, $H_2(G;\mathbb{Z}) = 0 = H_4(G;\mathbb{Z})$, and we have the following conclusions. For $\dim_{\mathbb{R}} X = 2$, $H^1(X;\mathbb{Z})$ is $\mathbb{Z}G$–isomorphic to $\omega^2(\mathbb{Z}) \oplus \omega^{-2}(\mathbb{Z}) \oplus (\mathbb{Z}G)^{2r}$. For $\dim_{\mathbb{R}} X = 4$, $\pi_1(X) = 0$, $H^2(X;\mathbb{Z}) = \omega^3(\mathbb{Z}) \oplus \omega^{-3}(\mathbb{Z}) \oplus (\mathbb{Z}G)^s$.

(3) Suppose $G = (\mathbb{Z}/p\mathbb{Z})^2$ then $H_2(G;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and $H_4(G;\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Therefore, in this case we get non–trivial examples corresponding to the non–zero elements of $H_{2k}(G;\mathbb{Z})$.

At this point, one may raise the point that the procedure in Theorem 3.3 (b) to produce free G–actions on simply–connected smooth 4–manifolds involved non–algebraic arguments. That is, surjection of $\Omega_4^{SO}(BG)$ onto $H_4(BG)$ produces $f_0 : W_0^4 \longrightarrow BG$ such that $f_{0*}[W_0^4] = x \in H_4(BG)$ and smooth surgery on the map f_0 corrects the fundamental group to give $f : W^4 \longrightarrow BG$ with $f_*[W] = x$. Then the universal cover of W, say X, is the desired smooth simply–connected 4–manifold whose homology $\mathbb{Z}G$–module $H_2(X)$ realizes the class $x \in H_4(BG)$. It is not clear if either one of these steps can be achieved using complex manifolds. Thus, we pose the following

3.6 Problem. Which homology classes $x \in H_4(G;\mathbb{Z})$ arise in Theorem 3.3 for analytic G–actions on compact complex surfaces X with $\pi_1(X) = 0$?

Section 4. Group Actions on Riemann Surfaces.

In this section, we assume that G is a finite group acting effectively on the compact Riemann surface Σ via complex analytic isomorphisms. Thus, G preserves the orientation and the isotropy subgroups $H_i \subseteq G$ are all cyclic. Moreover, for all $H_i \neq 1$, Σ^{H_i} consists of at most finitely many points of Σ. We delete the trivial subgroup (i.e. the principal isotropy subgroup for all effective finite group actions) from the list of isotropy subgroups of the action. The orbit space $\Sigma' = \Sigma/G$ is still a compact Riemann surface and $\Sigma \xrightarrow{\pi} \Sigma'$ is a ramified finite covering. We may choose a triangulation for Σ' such that the ramification points are all included in the set of vertices of Σ', and we lift this triangulation to Σ, to give Σ an equivariant triangulation. Under these circumstances, Σ becomes a G–CW complex, and the cells of Σ provide permutation bases for the cellular chain complex of Σ. This makes $C_*(\Sigma)$ into a permutation complex. In Section 3, we proved that if G acts freely on Σ, then the $\mathbb{Z}G$–module $H_1(X;\mathbb{Z})$ is stably $\mathbb{Z}G$–isomorphic to $\omega^1 L_\zeta$, where class$(\zeta) = x \in H_2(G;\mathbb{Z})$ is the image $f_*[\Sigma/G] \in H_2(BG;\mathbb{Z}) = H_2(G;\mathbb{Z})$ under the homomorphism induced by the classifying map $f : \Sigma/G \longrightarrow BG$. Moreover, every element of $H_2(G;\mathbb{Z})$ arises by such a free G–action. For instance, if $H_2(G;\mathbb{Z}) = 0$, then $H_1(\Sigma) \cong \omega^2(\mathbb{Z}) \oplus \omega^{-2}(\mathbb{Z}) \oplus (\mathbb{Z}G)^{2r}$, where r is determined by counting $\mathbb{Z}$–ranks of both sides of this equation. We proceed to determine the $\mathbb{Z}G$–module structure of $H_1(\Sigma;\mathbb{Z})$ for non–free actions in the same spirit.

First of all, the following analogue of Assadi ([A3] Theorem 5.4) is easily established.

4.1 Proposition. With the above notation, the following are equivalent:

(a) $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}G$–projective.

(b) For each prime order subgroup $C \subseteq G$, $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}C$–projective.

(c) For each prime order subgroup $C \subseteq G$, Σ^C consists of 2 points.

Furthermore, if $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}G$–projective, then p–Sylow subgroups of G are cyclic.

Proof. (a) ⇒ (b) and (b) ⇒ (c) by considering the spectral sequence $E_C \times_C \Sigma \longrightarrow BC$ and applying the localization theorem (Hsiang [Hs] or Quillen [Q]). From (c) it follows that p–Sylow subgroups of G must have one–dimensional faithful complex linear representations, hence they must be cyclic. Thus, maximal p–elementary abelian subgroups of G are isomorohic to $\mathbb{Z}/p\mathbb{Z}$. Therefore (b) ⇒ (a) by Chouinard's theorem (Chouinard [Ch] or Jackowski [J]). (c) ⇒ (b) is also possible by reversing the spectal sequence argument for (b) ⇒ (c). For a more elementary argument, consider $\Sigma_0 = \Sigma - \{x\}$ where $x \in \Sigma^C$. Then $H_1(\Sigma_0) = H_1(\Sigma)$ and $H_2(\Sigma_0) = 0$. Therefore, $H_1(\Sigma)$ is the only non-vanishing homology group in the $\mathbb{Z}G$–free chain complex $C_*(\Sigma_0,\Sigma_0^C)$. Hence, it is stably $\mathbb{Z}C$–free, and since C is cyclic, $H_1(\Sigma)$ is $\mathbb{Z}C$–free. ∎

The following lemma and the above discussion take care of $|G| = \text{prime}$.

4.2 Lemma. Let $G = \mathbb{Z}/p\mathbb{Z} = \langle t\rangle$ where p is a prime. Then $\Sigma^G \neq \phi$ if and only if $H_1(\Sigma) \cong \mathbb{Z}[\zeta]^{\alpha} \oplus (\mathbb{Z}G)^r$, where ζ is a primitive p–th root of unity and $\mathbb{Z}[\zeta]$ has the usual $\mathbb{Z}G$–module structure $\mathbb{Z}[\zeta] \equiv \mathbb{Z}[G]/(1+t+\dots+t^{p-1})$. Here $r = 2g-(p-1)\alpha$ and $\alpha = \#(\Sigma^G)-2$.

Proof. If $\Sigma^G = \phi$, then $H_1(\Sigma) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}G)^s$. Therefore, assume that $\Sigma^G \neq \phi$. Let $x_0 \in \Sigma^G$, and choose a small G–invariant disk D about x_0, and let $\Sigma_0 = \Sigma$–interior (D). First, observe that $\Sigma_0^G \neq \phi$. Otherwise, we would consider the classifying map of the regular p–fold cover $\Sigma_0 \xrightarrow{\pi} \Sigma_0/G$, say $f : \Sigma_0/G \longrightarrow BG$, and conclude that $f|\,\partial\Sigma_0/G = f' : S^1 \longrightarrow BG$ is null–homologous in $H_1(BG) \cong \mathbb{Z}/p\mathbb{Z} \cong \pi_1(BG)$, hence null–homotopic. But $\pi^{-1}(\partial\Sigma_0/G) = \partial D$ is connected, so that f' cannot be null–homotopic by covering space theory. Consequently, there exists $x_1 \in \Sigma_0^G$. Let $\Sigma^G = \{x_0,x_1,y_1,\ldots,y_\alpha\}$, and consider the permutation chain complex $C_*(\Sigma_0)$, in which $C_0(\Sigma_0) \cong C_0(x_0) \oplus C_0(\Sigma_0,x_0) = \mathbb{Z} \oplus \tilde{C}_0(\Sigma_0)$ and $\tilde{C}_0(\Sigma_0) \cong \mathbb{Z}^\alpha \oplus (\mathbb{Z}G)^r$. Since $H_2(\Sigma_0) = 0$, it follows that $\operatorname{Ker} \partial_1 = Z_1 \cong H_1(\Sigma_0) \oplus C_2(\Sigma_0)$ and $0 \longrightarrow Z_1 \longrightarrow C_1(\Sigma_0) \xrightarrow{\partial_1} \tilde{C}_0(\Sigma_0) \longrightarrow 0$ is exact. Therefore, Z_1 is stably $\mathbb{Z}G$–isomorphic to I^α, where I is the augmentation ideal of $\mathbb{Z}[G]$, which is isomorphic to $\mathbb{Z}[\zeta]$ because $G = \mathbb{Z}/p\mathbb{Z}$. Hence $H_1(\Sigma) \cong H_1(\Sigma_0) \cong \mathbb{Z}[\zeta]^\alpha \oplus (\mathbb{Z}G)^r$ as claimed.

■

Next, we assume that $\Sigma^G \neq \phi$, so that G is necessarily cyclic, but possibly having composite order. Unlike the case of $G = \mathbb{Z}/p^k\mathbb{Z}$ when p = prime, in this case Σ^G = one point is possible, as shown by Conner–Floyd [CF] (see also Ewing–Stong [ES]). Thus, we consider two cases below. Note that the case $G = \mathbb{Z}/p^k\mathbb{Z}$ is covered by the first case below since according to Atiyah–Bott and others $\Sigma^G \neq$ one point.)

<u>4.3 Proposition</u>. Suppose Σ^G has at least two points, and let $\{H_i : i = 1,\ldots,n\}$ be the collection of non–trivial isotropy subgroups considered with repetition according to the number of orbits in Σ^{H_i}, and excluding two copies of G corresponding to the first two points in Σ^G. Let ζ_i be an $|H_i|$–th root of unity and $\mathbb{Z}[\zeta_i]$ with the usual $\mathbb{Z}[H_i]$–module structure. Then $H_1(\Sigma) \cong \bigoplus_{i=1}^{n} (\mathbb{Z}G \otimes_{H_i} \mathbb{Z}[\zeta_i] \oplus (\mathbb{Z}G)^r$, where

$$r = \frac{1}{|G|}\,[\operatorname{rank} H_1(\Sigma) - \sum_{i=1}^{n} \frac{|G|}{|H_i|}(|H_i|-1)] .$$

<u>Proof</u>. Let $x_1 \in \Sigma^G$, and consider a small G–invariant disk neighborhood of x_1 (avoiding

other fixed points), called $D(x_0)$. Let $\Sigma_0 = \Sigma$–interior $(D(x_1))$. As before, Σ_0 admits a G–CW structure in which $C_1(\Sigma_0)$ and $C_2(\Sigma_0)$ are $\mathbb{Z}G$–free, and $C_0(\Sigma_0)$ is a permutation module. Let $x_0 \in \Sigma_0^G \neq \phi$ and consider the augmentation $C_0(\Sigma_0) \rightleftarrows C_0(x_0)$ which is $\mathbb{Z}G$–split via the inclusion $\{x_0\} \subset \Sigma_0$. Thus $C_0(\Sigma_0) \cong \tilde{C}_0(\Sigma_0) \oplus \mathbb{Z}$. Consider the following short exact sequences:

$$0 \longrightarrow C_2(\Sigma_0) \xrightarrow{\partial} B_1(\Sigma_0) \longrightarrow 0$$
$$0 \longrightarrow B_1(\Sigma_0) \longrightarrow Z_1(\Sigma_0) \longrightarrow H_1(\Sigma_0) \longrightarrow 0$$
$$0 \longrightarrow Z_1(\Sigma_0) \longrightarrow C_1(\Sigma_0) \longrightarrow \tilde{C}_0(\Sigma_0) \longrightarrow 0 .$$

From these, it follows that $B_1(\Sigma_0)$ is $\mathbb{Z}G$–free, and $H_1(\Sigma) \cong H_1(\Sigma_0)$ is stably isomorphic to $Z_1(\Sigma_0)$. Since all modules are $\mathbb{Z}$–free and $B_1(\Sigma_0)$ is $\mathbb{Z}G$–free, reflexivity of $\mathbb{Z}G$ implies that the second exact sequence is $\mathbb{Z}G$–split. Leaving out $\{x_0,x_1\} \subset \Sigma^G$ from the singular set of the action, it is clear that $\tilde{C}_0(\Sigma_0)$ is stably isomorphic to $\bigoplus_i \mathbb{Z}(G/H_i)$ where H_i are isotropy subgroups of the fixed points in $\Sigma - \{x_0,x_1\}$. Therefore, $Z_1(\Sigma_0)$ is stably $\mathbb{Z}G$–isomorphic to $\oplus\, \omega^1\mathbb{Z}(G/H_i)$. We also have
$\omega^1\mathbb{Z}(G/H_i) \sim \mathbb{Z}G \otimes_{H_i} \omega^1_{H_i}(\mathbb{Z}) \sim \mathbb{Z}G \otimes_{H_i} \mathbb{Z}[\zeta_i]$, where the last stable isomorphism is due to $\omega^1 H_i(\mathbb{Z}) \cong \mathbb{Z}[\zeta_i]$ as $\mathbb{Z}[H_i]$–modules, because H_i is cyclic. ∎

<u>4.4 Proposition</u>. Suppose $\Sigma^G = \{x_0\}$. Then G is a cyclic group whose order is divisible by at least two distinct primes. Let $\{H_i : i = 0,1,\dots,n\}$ be the collection of non–trivial isotropy subgroups such that $H_0 = G$ and $H_i \neq G$. Then $H_1(\Sigma)$ is completely determined by the permutation module $B = \bigoplus_{i=1}^{n} \mathbb{Z}[G/H_i]$ and an element $\theta(\Sigma) \in \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ from the exact sequence:

$$0 \longrightarrow H_1(\Sigma) \longrightarrow \mathbb{Z}[G]^{\ell} \oplus \mathbb{Z} \xrightarrow{\varphi} B \oplus \mathbb{Z}[G]^{k} \longrightarrow 0 .$$

Here, φ represents a homology class in $\widehat{\mathrm{Hom}}_G(\mathbb{Z}[G]^{\ell} \oplus \mathbb{Z}, B \oplus \mathbb{Z}[G]^{k}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ and $\mathrm{class}(\varphi) = \theta(\Sigma)$.

Proof. Consider the commutative diagram below, in which $Q = \mathrm{coker}(\partial_2)$, $C_* = C_*(\Sigma)$, and

$$C_0 \cong \mathbb{Z}[G/G] \oplus \mathrm{Im}(\partial_1) \cong \mathbb{Z} \oplus B_0 = \mathbb{Z} \oplus B \oplus (\mathbb{Z}G)^s :$$

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & B_0 & \xrightarrow{=} & B_0 & & \\
 & & & & & & \uparrow{\scriptstyle \partial_1} & & \uparrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C_2 & \longrightarrow & C_1 & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C_2 & \longrightarrow & Z_1 & \xrightarrow{\alpha} & H_1(\Sigma) & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

In the sequence $0 \longrightarrow H_1(\Sigma) \longrightarrow Q \xrightarrow{\varphi} B_0 \longrightarrow 0$, $Q \cong \mathbb{Z} \oplus (\mathbb{Z}G)^{\ell}$ and $B_0 \cong B \oplus (\mathbb{Z}G)^{k}$, with $B = \bigoplus_{i=1}^{n} \mathbb{Z}[G/H_i]$. Here, we use the fact that G must be cyclic, hence it has a periodic resolution $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$. Comparing this with the top horizontal row and applying the Schanuel Lemma (Swan [Sw]) we find out Q is stably $\mathbb{Z}G$–isomorphic to $\mathbb{Z}$. Moreover,

$\mathrm{class}(\varphi) = \theta(\Sigma) \in \widehat{\mathrm{Hom}}_G(\theta, B_0) \cong \widehat{\mathrm{Hom}}_G(\mathbb{Z}, B) = \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ determines $H_1(\Sigma)$ up to stable $\mathbb{Z}G$–isomorphism.

4.5 Corollary. In the above Proposition, if $\theta(\Sigma) = 0$, then $H_1(\Sigma) \cong \mathbb{Z} \oplus (\bigoplus_{i>0} \mathbb{Z}[\zeta_i]) \oplus \mathbb{Z}[G]^r$, where

$$r = \frac{1}{|G|}[\mathrm{rank}\, H_1(\Sigma) - 1 - \sum_{i>0} (|H_i| - 1)] .$$

Proof. If $\theta(\Sigma) = 0$, then $\mathrm{class}(\varphi) = 0$ and in the sequence $0 \longrightarrow H_1(\Sigma) \longrightarrow \mathbb{Z}[G]^{\ell} \oplus \mathbb{Z} \xrightarrow{\varphi} B \oplus [\mathbb{Z}G]^k \longrightarrow 0$, φ factors through a projective $\mathbb{Z}G$–module P, which without loss of generality we may assume to be a free $\mathbb{Z}G$–module. We form the following pull–back diagram (the left square) and complete the commutative diagram as indicated below:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \omega^1 B & \xrightarrow{=} & \omega^1 B & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\,\varphi' & \longrightarrow & T & \xrightarrow{\varphi'} & P & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\psi & \nearrow & \downarrow & & \\
0 & \longrightarrow & H_1(\Sigma) & \longrightarrow & (\mathbb{Z}G)^{\ell} \oplus \mathbb{Z} & \longrightarrow & B \oplus (\mathbb{Z}G)^k & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since P is free, φ' splits, and this gives a splitting of ψ. Therefore,

$$P \oplus \mathrm{Ker}(\varphi') \cong T \cong \mathbb{Z} \oplus (\mathbb{Z}G)^{\ell} \oplus \omega^1 B \ .$$

Identifying the terms $P = (\mathbb{Z}G)^s$, $\mathrm{Ker}\ \varphi' \cong H_1(\Sigma)$, and $\omega^1 B \sim \bigoplus_{i>0} \mathbb{Z}[\zeta_i]$, we conclude that $H_1(\Sigma)$ is stably $\mathbb{Z}G$–isomorphic to $\mathbb{Z} \oplus (\bigoplus_{i>0} \mathbb{Z}[\zeta_i]$. Since cancelation holds for $\mathbb{Z}G$–modules when G is cyclic (Swan [Sw]), the desired formula is obtained.

■

4.6 Corollary. Suppose R is a commutative ring such that RG is semisimple (e.g. a field of characteristic zero, or prime to order of G). Then, in the representation ring of RG, we have the following equation: $[H_1(\Sigma;R)] = [R] + m[RG] - \sum_{i=1}^{n} [R \otimes_H RG]$, and m is determined by counting the ranks of corresponding free R–modules.

Proof. The sequence in Proposition 4.4 splits in the representation ring of RG due to semisimplicity. ■

The above corollary for $R = \mathbb{C}$ is proved by A. Broughton [Br] using Eichler's trace formula.

The final possibility is when $\Sigma^G = \phi$ while G does not act freely on Σ. In this case, G need not be cyclic, and the formulas are somewhat more complicated:

4.7 Proposition. Suppose that G acts without fixed–points, but not freely. Let $\{H_i : i \in I\}$ be the collection of isotropy subgroups considered with multiplicities as before, and let ϵ be the augmentation homomorphism $(\epsilon(gH_i) = 1)$ and $B_0 = \mathrm{Ker}(\epsilon)$ in

$\epsilon : \bigoplus_{i \in I} \mathbb{Z}(G/H_i) \longrightarrow \mathbb{Z}$. Then, up to stable $\mathbb{Z}G$–isomorphism, the $\mathbb{Z}G$–module is determined by B_0 and a homology class $\theta(\Sigma) \in H_1(G;B_0)$. Indeed, if $\varphi : \omega^{-2}(B_0^*) \longrightarrow \mathbb{Z}$ represents $\theta(\Sigma)$ via the isomorphisms $\widehat{\mathrm{Hom}}_G(\omega^{-2}(B_0^*),\mathbb{Z}) \cong H_1(G;B_0)$, then $H^1(\Sigma) \cong \omega^{-1}(\mathrm{Ker}\ \varphi)$ and $H_1(\Sigma) \cong \omega^{-1}((\mathrm{Ker}\ \varphi)^*)$.

Proof. We have an exact sequence $0 \longrightarrow \mathrm{Im}\,\partial_1 \longrightarrow C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$ in which $C_0 \cong (\mathbb{Z}G)^{\ell} \oplus (\bigoplus_{i=1}^{m} \mathbb{Z}(G/H_i))$, $H_i \neq 1$. Let $\{e_i : 1 \leq i \leq m\}$ and $\{u_j : 1 \leq j \leq \ell\}$ be the obvious generators and basis elements for the two factors in C_0 . We choose a new basis for $(\mathbb{Z}G)^{\ell}$ factor, by fixing $H_0 \neq 1$, $e_0 \in \mathbb{Z}(G/H_0)$ its $\mathbb{Z}G$–generator, and setting $v_j = u_j - e_0$. Such an e_0 exists because the action is not free by assumption. With the new basis $\{v_j : 1 \leq j \leq \ell\}$, we observe that $\mathrm{Im}\ \partial_1$ is $\mathbb{Z}G$–stably isomorphic to B_0 in the statement of the proposition. Again, from the exact sequence

$$0 \longrightarrow H^1(\Sigma) \longrightarrow Z_1(\Sigma)^* \longrightarrow B_1(\Sigma)^* \longrightarrow 0$$

as in the preceding cases, we get the following exact sequence, up to $\mathbb{Z}G$–stable isomorphism:

$$0 \longrightarrow H^1(\Sigma) \longrightarrow \omega^{-1}(B_0) \longrightarrow \omega^1(\mathbb{Z}) \longrightarrow 0 \ .$$

From the latter, we have:

$$0 \longrightarrow \omega^{-1}H^1(\Sigma) \longrightarrow \omega^{-2}(B_0) \xrightarrow{\ \varphi\ } \mathbb{Z} \longrightarrow 0$$

and the classes of φ in $\widehat{\mathrm{Hom}}_G(\omega^{-2}(B_0),\mathbb{Z}) \cong \hat{H}^{-2}(G,B_0) \cong H_1(G,B_0)$ is the class $\theta(\Sigma)$ mentioned above. One checks that B_0 and $\theta(\Sigma)$ together determine the stable isomorphism class of $H_1(\Sigma)$. ∎

Section 5. Group Actions on Kähler Surfaces.

In this section, X denotes a simply–connected compact Kähler surface, and we assume that G is a finite group acting by complex automorphisms. Unlike general smooth manifolds, the Kähler condition imposes strong conditions on the action, and consequently on the $\mathbb{Z}G$–representation afforded by $H^2(X;\mathbb{Z})$.

5.1 Proposition. Let G be an arbitrary non–trivial finite group acting on X as above. Then $H^2(X)$ cannot be $\mathbb{Z}G$–projective if G preserves the Kähler cohomology class in $H^2(X)$.

Proof. It suffices to show this for $G = \mathbb{Z}_p$. Consider the Serre spectral sequence of the Borel fibration $E_G \times_G X \longrightarrow BG$, $H^*(G,H^*(X)) \Rightarrow H_G^*(X)$. If G acts freely on X, then $H^2(X) \cong I \oplus I \oplus (\mathbb{Z}G)^s$, hence $H^2(X)$ is not projective. Suppose $X^G \neq \phi$. Let $t \in H^2(G) \cong \mathbb{Z}_p$ and let $\alpha \in H^2(X)$ be given by the Kähler form, so that $[\alpha\Lambda\alpha] = [X]$ = cohomological orientation class of x. Consider the cup product in the spectral sequence, as well as the $H^*(G)$–algebra structure of the E_2–term in the follwoing commutative diagram:

$$\begin{array}{ccc}
H^2(G) \otimes H^0(G,H^2(X)) \otimes H^0(G,H^2(X)) & \longrightarrow & H^2(G) \otimes H^0(G,H^2(X) \otimes H^2(X)) \\
\downarrow & & \downarrow \\
H^2(G,H^2(X)) \otimes H^0(G,H^2(X)) & & H^2(G) \otimes H^0(G,H^4(X)) \\
\downarrow & & \downarrow \\
H^2(G,H^2(X) \otimes H^2(X)) & \longrightarrow & H^2(G,H^4(X))
\end{array}$$

Since $t \cdot [X] \neq 0$ in $H^2(G,H^4(X))$, we have $0 \neq t \cdot (\alpha\Lambda\alpha) = (t\alpha)\Lambda\alpha$, hence $t \cdot \alpha \neq 0$. Therefore $\alpha \in H^0(G,H^2(X)) = H^2(X)^G$ cannot be $H^*(G)$–torsion.

Consequently, $\hat{H}^*(G,H^2(X)) \neq 0$, so that $H^2(X)$ cannot be G–projective.∎

5.2 Corollary. Let $G = \mathbb{Z}_p$ act on the simply–connected Kähler surface X preserving the Kähler cohomology class. Then $\sum_{i \geq 0} \beta_i(X^G) \geq 3$, where β_i = i–th Betti number, and $X^G \neq \phi$ by hypothesis.

Proof. Since $X^G \neq \phi$, and for degree reasons, the Serre spectral sequence of $E_G \times_G X \longrightarrow BG$ collpases. (See e.g. [A3]). Now the above proof shows that $\hat{H}^*(G,H^i(X)) \neq 0$ for $i = 0,2,4$. Therefore, $H^*(G)$–rank of $H_G^*(X)$ is at least 3. The localization theorem ([HS] or [Q]) implies the desired conclusion.

∎

In the following theorem, conditions are given which guarantee that modulo $\mathbb{Z}G$–projective modules, G must act trivially on $H^2(X)$. Recall Theorem 4.14 of [A3] III.

5.3 Theorem. Suppose X is a Kähler surface, $\pi_1(X) = 0$ and G acts smooth but not freely, and $G = (\mathbb{Z}_p)^s$, $s \geq 1$. Assume that for each cyclic subgroup $C \subsetneq G$, $p > \beta_0(X^C)$ and $\beta_1(X^C) = 0$. Then the following hold:

(a) there exists an $m > 0$ such that the $\mathbb{Z}G$–module $H_2(X) \cong \mathbb{Z}^m \oplus M$ where G acts trivially on $\mathbb{Z}^m$ and M is $\mathbb{Z}G$–projective.

(b) $\chi(X^G) = \chi(X^C) = m$ for each $C \subset G$, $|C| = p$, and $\text{rank}(G) \leq 2$.

(c) If $\text{rank}(G) > 1$, then G acts freely on the set of symplectic 2–forms of X; hence G does not preserve any symplectic structure on X.

Proof. Since $\beta_1(X^C) = 0$, X^C consists of 2–spheres and isolated points. Moreover,

$H_2(X) \cong \mathbb{Z}^{r(C)} \oplus \mathbb{Z}[C]^u$ as $\mathbb{Z}C$–modules, where C acts trivially on $\mathbb{Z}^{r(C)}$, and $r(C) = \chi(X^C)-2$. Also, $G/_C$ must act trivially on $\pi_0(X^C)$ since $p > \beta_0(X^C)$ by assumption. Hence $G/_C$ must act effectively on each component S^2, and each isolated fixed point in X^C must be an isolated G–fixed point. If $G = C = \mathbb{Z}_p$, then we are done. If $\operatorname{rank}(G) > 1$, then G cannot have a free action in the punctured neighborhood of an isolated fixed point. Since $X^C \neq \phi$ for some $C \neq 1$, $G/_C$ must act effectively on each copy $S^2 \subseteq X^C$. Therefore $X^G = (X^C)^{G/C} \neq \phi$. From this (c) follows, since if C preserves some symplectic 2–form of X, then X^C must consist of exclusively isolated fixed points. (Consider the complex C–representation on the tangent space T_QX for some $Q \in X^C$, and observe that if a symplect form is preserved, then the two eigen–values of any generator of C must be distinct and not equal to 1). In view of the above observation that X^C contains copies of S^2, and that $X^C \supset X^G \neq \phi$ for each $C \neq 1$, we see that each C, and hence G, acts freely on the set of symplectic 2–forms of X. Since each $S^2 \subset X^C$ contributes one copy of $S^0 \subset (X^C)^{G/C}$, then $\chi(X^C) = \chi(X^G)$. It suffices, therefore, to prove (a). But (a) is proved in Theorem 5.6 of Assadi [A3] III.

∎

Finally, the actions considered in this section are "regular" in the terminology of [A3]. Hence, the general theorems of [A3] apply to this situation, and the same principles and argument may be used to study the $\mathbb{Z}G$–representations afforded by $H^2(X)$ for a compact Kähler surface. In particular, the fixed point set of the G–action and a suitable group cohomology element completely determine the $\mathbb{Z}G$–module $H_2(X)$ as in Proposition 4.7 of [A3] III.

Section 6. Projective Surface with Irregularity Zero.

In this section, we consider non–singular projective surfaces X defined over an algebraically closed field of arbitrary characteristic k . The analogue of simply–connectivity for complex surfaces is the condition $q(x) \equiv p_g(X) - p_a(X) = 0$ i.e. the irregularity is zero. Let $\mathcal{K}_X$ be the canonical sheaf of X , and let $\Omega^2(X) \equiv H^0(X; \mathcal{K}_X)$ be the k–vector space of "holomorphic 2–forms" of X . We compute the kG–representation $\Omega^2(X)$ for the free G–actions on X . A suitable cohomology theory is Cech cohomology using an open covering $\mathcal{U}$ of X consisting of G–invariant affine subsets of X . Such a Cech cohomology group coincides with Grothendieck's coherent cohomology, i.e.
$H^0(\mathcal{U}; \mathcal{K}_X) \cong \Omega^2(X)$. On the other hand, by Serre duality,
$\Omega^2(X) \cong \mathrm{Hom}_k(H^2(X; \mathcal{O}_X),k) \cong \mathrm{Hom}_k(H^2(\mathcal{U}; \mathcal{O}_X),k)$. Consider a free G–action on X , and observe that the variety X/G exists (Mumford [M]) and it is non–singular and projective. Moreover, the morphism $f : X \longrightarrow X/G$ is an étale principal covering. Let $\mathcal{U}_0$ be a suitable finite covering of X/G by affine open sets, and let
$\mathcal{U} = \{f^{-1}(V_0) : V_0 \in \mathcal{U}_0\}$. Then each $V = f^{-1}(V_0)$ is also affine, and we have $f^{-1} | V : V \longrightarrow V_0$ is given by a k–algebra homomorphism $\varphi : R \longrightarrow S$, i.e., $V_0 = \mathrm{Spec}(R)$, $V = \mathrm{Spec}(S)$ and ${}^a\varphi = f^{-1} | V$.

6.1 Theorem. Let X be (an irreducible) non–singular projective k–surface with $q(X) = 0$. Suppose that G acts freely on X by automorphisms. Then the kG–module $\Omega^2(X)$ is stably kG–isomorphic to $\omega_G^3(k)$.

6.2 Remark. Compare this with Corollary 3.2 which describes $H^2(X;\mathbb{Z})$ stably $\mathbb{Z}G$–isomorphic to an extension of $\omega_G^3(\mathbb{Z})$ and $\omega_G^{-3}(\mathbb{Z})$. For $k = \mathbb{C}$, the Hodge decomposition yields $H^2(X,\mathbb{C}) \cong H^{2,0}(X;\mathbb{C}) \oplus H^{0,2}(X;\mathbb{C}) \oplus H^{1,1}(X,\mathbb{C})$. Since $H^{2,0}$ and $H^{0,2}$ are dual to each other, and $\Omega^2(X) \cong H^{2,0}(X;\mathbb{C})$ the above result implies that from

$H^2(X;\mathbb{C}) \cong \Omega^2(X) \oplus \mathrm{Hom}(\Omega^2(X),\mathbb{C}) \oplus \mathbb{C}[G]^t \cong \omega_G^3(\mathbb{C}) \oplus \mathrm{Hom}(\omega_G^3(\mathbb{C}),\mathbb{C}) \oplus \mathbb{C}[G]^t$ we conclude $H^{1,1}(X;\mathbb{C}) \cong \mathbb{C}[G]^t$.

6.3 Problem. Compute the $\mathbb{Z}G$–lattices $H^{i,j}(X,\mathbb{C}) \cap H^{i+j}(X;\mathbb{Z})$.

Proof. Consider the Cech complex $C^* = C^*(\mathcal{U})$ of kG–modules for the coherent sheaf $\mathcal{O}_X$ in which $H^0(C^*) = k$, $H^1(C^*) = H^1(\mathcal{U};\mathcal{O}_X) \cong H^1(X;\mathcal{O}_X)$ since $q(X) = p_g(X) - p_a(X) = \dim_k H^1(X;\mathcal{O}_X)$ and $q(X) = 0$ by assumption. Moreover, $H^i(C^*) = H^i(\mathcal{U};\mathcal{O}_X) \cong H^i(X;\mathcal{O}_X) = 0$ for $i > 2$, and $C^i = 0$ for i sufficiently large, since $\mathcal{U}$ is a finite cover. In the case of a complex analytic manifold, we could use the analytic topology, and choose $\mathcal{U}_0$ sufficiently refined until $f^{-1}(V_0) \cong G \times V_0$ is a free orbit of V_0 up to G–isomorphism. This would imply that the Cech complex C^* is a free G–complex. In the general case at hand, we have used Zariski open sets, and we need to resort to a somewhat different argument. Consider $\varphi : R \longrightarrow S$ such that ${}^a\varphi : \mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$ is the given étale covering $f_0 = f|V : V \longrightarrow V_0$, $V = f^{-1}(V_0)$. Then $V \times_{V_0} V$ admits a section, so that $V \times_{V_0} \cong G \times V$ as V_0–schemes with free G–actions. Therefore, $S \otimes_R S$ is a free $k[G]$–module. Consider the kG–isomorphisms: $S \otimes_k S \cong S \otimes_R (R \otimes_k S) \cong S \otimes_R (S \otimes_k R) \cong (S \otimes_R S) \otimes_k R$ which shows that $S \otimes_k S$ is also kG–free. This implies, in particular, that S is kG–projective. Hence $C^0(\mathcal{U})$ is a projective kG–module. A similar argument applies to show that $C^i(\mathcal{U})$ is kG–projective. Consider the dual chain complex $C_* = \mathrm{Hom}_k(C^*,k)$ of kG–projective modules, in which $H_0(C_*) \cong k$ and $H_2(C_*) = \mathrm{Hom}_k(\Omega^2(X),k)$ are the only non–vanishing homology groups. It follows that $B_2 = \mathrm{Im}\ \partial_2 \subset C_2$ is projective over kG, since $0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots\ldots \longrightarrow C_3 \longrightarrow B_2 \longrightarrow 0$ is exact for some sufficiently large n. Moreover, $\mathbb{Z}_2 = \mathrm{Ker}\ \partial_2 \cong \omega_G^3(k)$ in view of the exact sequence:

$$0 \longrightarrow Z_2 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 \longrightarrow k \longrightarrow 0 \ .$$

The exact sequence $0 \longrightarrow B_2 \longrightarrow Z_2 \longrightarrow H_2(C_*) \longrightarrow 0$, splits, since kG is injective. Therefore, $H_2(C_*) \sim Z_2 = \omega_G^3(k)$ is an stable kG–isomorphism. Hence $\Omega^2(X) = H^2(C^*) = \mathrm{Hom}_k(\omega_G^3(k),k) = \omega_G^{-3}(k)$ as claimed. ∎

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Introduction. Let X be a topological space with a finite group G acting on it. For suitable coefficient systems and cohomology theories, $H^*(X; \mathcal{F})$ becomes a G–representation. Study of such representations and their relationship to the symmetries of X has been the subject of extensive study. In our previous paper [A1]–[A5], we have studied such representations from the view points of group cohomology and local–global considerations. In particular, [A3] considers the integral representations on $H^2(X;\mathbb{Z})$ when X is a compact simply–connected 4–manifold. In the following paper, we continue [A3] by specializing to the case of algebraic curves and surfaces.

Historically speaking, such investigations for complex projective curves (compact Riemann surfaces) goes back to the 1893 paper of Hurwitz, in which complex representations of cyclic automorphism groups of Riemann surfaces were studied. His work was completed by Chevalley and Weil, also using analytic techniques. See Weil's collected works Vol. I, pages 529 and 532–533 for historical details and a discussion of these results. Chevalley–Weil's results were further generalized by Tamagawa to the case of curves over fields of positive characteristic with free regular automorphism groups. Tamagawa's result is formulated in terms of unramified Galois extensions of the corresponding function fields. This point of view has been further developed by number theorists, in particular, Madan and Valentini among others. (See Valentini–Madan, Journal Number Theory, Vol. 13, 1981, for a historical survey and further developments.)

Some of the results of the present paper may be considered as modest generalizations of the above–mentioned results. Such generalizations are in two directions. First, we have determined the integral representations $H^1(X,\mathbb{Z})$ for a compact Riemann surface with an arbitrary finite automorphism group (Section 4). Since the structure of $\mathbb{Z}G$–modules is a complete mystery for almost all finite groups, our formulations are in terms of group cohomology in the general case. Secondly, we have studied certain representations for suitable non–singular projective surfaces in analogy with Chevalley–Weil and Tamagawa's results. Namely, for free G–actions on projective surfaces X where $p_a(X) = p_g(X)$

(p_a = arithmetic genus and p_g = geometric genus). For curves, $p_a(X) = p_g(X)$ always. But for surfaces, this is a real restriction, and it should be compared with simple–connectivity hypothesis for complex projective surfaces. In Section 6, we have determined the kG–module $H^0(X; \mathscr{K}_X)$ (= vector space of regular 2–forms) in analogy with the case of regular 1–forms for curves. Section 5 makes a preliminary study of the $\mathbb{Z}$G–representation $H^2(X;\mathbb{Z})$ when X is compact Kähler. The general theme of sections 3–5 is to relate the topology and geometry of the underlying symmetry to the homological properties of suitable representations. In Section 2 we have gathered some definitions and a brief discussion of some of the homological notions for the convenience of the reader. Further preliminary material may be found in [A3] or in the references.

Note added in proof. Since the appearance of the first version of this paper, several related works are brought to my attention. I would like to thank Chad Schoen for discussions on his interesting results in this direction and for sending me his manuscript [Schoen]. I am also grateful to G. Ellencwejg and T. Kohono who brought to my attention the related works of S. Nakajima [Nakajima 1 & 2] which go deeper in the number theoretic direction and seem to have a slight overlap with some of our results.

Section Two. Preliminary Notions.

In the following sections, we will use the same notation and conventions as in [A3]. However, we review some of the notation for the convenience of the reader. Let G be a finite group, and R be a commutative ring with unit, e.g. $R = \mathbb{Z}$, $\hat{\mathbb{Z}}_p$ = p–adic integers, $\mathbb{F}_p$, or $\mathbb{C}$. The RG–modules are finitely generated and R–free. Finite generation may not hold for some of the RG–modules in the chain complexes used in Section 6. However, the cohomology and homology groups are all finitely generated, and this will be sufficient. Two RG–modules M_1 and M_2 are called projectively stably RG–isomorphic, denoted by $M_1 \sim M_2$, if there is a commutative diagram:

$$\begin{array}{ccc} M_1 \oplus P_1 & \xrightarrow[g]{\cong} & M_2 \oplus P_2 \\ {\scriptstyle j}\uparrow & & \downarrow{\scriptstyle \pi} \\ M_1 & \longrightarrow & M_2 \end{array}$$

where P_1 and P_2 are RG–projective, j and π are the obvious inclusion and projection, and g is an isomorphism. If P_1 and P_2 are RG–free, then we call M_1 and M_2 stably isomorphic. Stable isomorphism is an equivalence relation. Heller [Hr] has defined loop and suspension operations for RG–modules when the notions "projective cover" and "injective hull" make sense. However, projective covers do not exist, in general, for $\mathbb{Z}G$–modules although they exist for $\mathbb{F}_pG$–modules or $\hat{\mathbb{Z}}_p[G]$–modules. Here, we can define a stable version of the "Heller loop–operator", which we denote by ω, on the set of stable equivalence classes of RG–lattices (i.e. R–torsion free RG–modules). Namely, $\omega(M)$ is stably well–defined (by Schanuel's Lemma [Sw]) from the exact sequence $0 \longrightarrow \omega(M) \longrightarrow (RG)^{\alpha} \longrightarrow M \longrightarrow 0$. If we use projective RG–modules instead of $(RG)^{\alpha}$, then $\omega(M)$ is well–defined up to projective stable equivalence. Then we set

$\omega^1(M) = \omega(M)$ and $\omega^{i+1}(M) = \omega(\omega^i(M))$ inductively. For $i \in \mathbb{Z}$, this definition has a natural extension, so that $\omega^i(M)$ are stably well–defined for all $i \in \mathbb{Z}$.

We will also make use of a construction for RG–modules from cohomology classes which is explained in [A3]. Our description is a generalization and a stable version of the construction used by J. Carlson [C] in modular representation theory. Recall the Tate cohomology $\hat{H}^i(G;M)$, $i \in \mathbb{Z}$ as in e.g. Cartan–Eilenberg [CE]. Then $\widehat{\mathrm{Hom}}_G(M,R) \stackrel{\mathrm{def}}{\equiv} \hat{H}^0(G,\mathrm{Hom}_R(M,R))$ is isomorphic to the group of RG–homomorphisms $f : M \longrightarrow R$ modulo the subgroup of those which factor through an RG–projective. (See Mac Lane pp. 74–75 [Mc] for related discussion. It turns out that $\hat{H}^0(G,\mathrm{Hom}_R(\omega^n(M),R)) \stackrel{\mathrm{def}}{\equiv} \widehat{\mathrm{Hom}}(\omega^n(M),R) \cong \hat{H}^n(G;M^*)$, where $M^* = \mathrm{Hom}_R(M,R)$ with the diagonal RG–module structure. Now, given a cohomology class, $x \in \hat{H}^n(G;M^*)$, we may represent x by an RG–homomorphism $\varphi : \omega^n(M) \longrightarrow R$ which is well–defined up to factorization through RG–projectives. φ may be assumed also surjective. Define $L_\varphi \equiv \mathrm{Ker}(\varphi)$. Then L_φ is well–defined up to projective stable equivalence. (See [A3] for further discussion). The notation class (φ) will be used for the cohomology class represented by φ. The functor $\widehat{\mathrm{Ext}}^i_G(-,-)$ is also constructed in analogy with Tate cohomology $\hat{H}^i(G,-)$ using complete resolutions (see e.g. Cartan–Eilenberg [CE] or Carlson [C]).

An algebraic generalization of a Poincaré duality space is the notion of a chain complex with duality. Let C_* be a bounded connected chain complex of dimension n over a ring R, so that $H_0(C_*) \cong R$ and $C_i = 0$ for $i < 0$ or $i > n$ (for some $n > 0$). We call C_* a chain complex with duality of formal dimension m, if there exists a chain homotopy equivalence $h : C_{m-i} \longrightarrow C^i$ between C_* and C^*. The cellular chain complex of a Poincaré duality space or a closed oriented smooth manifold are basic examples of such complexes with duality.

In Section 5 and 6 we will need some basic facts from algebraic geometry. The standard reference for the definitions and concepts used in the following are Hartshorne [H] and Mumford [M1] [M2].

Section 3. Free Actions.

In this section we study homology representations of free actions.

3.1 Theorem. Let X_* be a (k–1)–connected bounded RG–free chain complex with duality of formal dimension $2k$. Then:

(a) The RG–module $H_k(X_*)$ is completely determined up to stable equivalence by a homology class $x \in H_{2k}(G;R)$.

(b) Let $\zeta : \omega^{-2k-1}(R) \twoheadrightarrow R$ be a representative for x. Then $H_k(X_*)$ is stably RG–isomorphic to $\omega^k L_\zeta$.

(c) Let $\varphi : \omega^{-k-1}(R) \longrightarrow \omega^k(R)$ be such that $\text{class}(\varphi) = \text{class}(\zeta) = x$ under the isomorphisms

$$\widehat{\text{Hom}}_G(\omega^{-k-1}(R), \omega^k(R)) \cong \widehat{\text{Hom}}_G(\omega^{-2k-1}(R), R) \cong H_{2k}(G;R) .$$

Then $H_k(X_*)$ is completely determined (stably) from the short exact seqeunce below:

$$0 \longrightarrow H_k(X) \longrightarrow \omega^{-k-1}(R) \xrightarrow{\varphi} \omega^k(R) \longrightarrow 0 .$$

The following corollary has been proved for $k = 2$ by Hambleton–Kreck [HK].

3.2 Corollary. A symmetric expression for $H_k(X_*)$ is obtained as follows. Let $z \in \text{Ext}^1_G(\omega^{-k-1}(R), \omega^{k+1}(R))$. Then the extension class z is represented by the short exact sequence:

$$0 \longrightarrow \omega^{k+1}(R) \longrightarrow H_k(X_*) \oplus (\text{RG–Free}) \longrightarrow \omega^{-k-1}(R) \longrightarrow 0 .$$

<u>Proof</u>. Since X_* is R–chain homotopic to its R–dual $X^* \equiv \mathrm{Hom}_R(X_*,R)$, we have $H_i(X_*) = 0$ for $k + 1 \le i \le 2k-1$. Moreover, without loss of generality, we may assume that $X_i = 0$ for $i \ge 2k+1$ (see e.g. Assadi [A3] Lemma 4.2.). The connectivity of X_* in the above–mentioned dimensions gives rise to long exact sequences below:

$$0 \longrightarrow B_{k-1} \longrightarrow X_{k-1} \longrightarrow \ldots\ldots \longrightarrow X_0 \longrightarrow R \longrightarrow 0$$

$$0 \longrightarrow R \longrightarrow X_{2k} \longrightarrow \ldots\ldots \longrightarrow X_{k+1} \xrightarrow{\partial_{k+1}} X_k \longrightarrow \mathrm{coker}(\partial_{k+1}) \longrightarrow 0 \ .$$

We conclude that $B_{k-1} = \omega^k(R)$ and $\mathrm{coker}(\partial_{k+1}) = \omega^{-k-1}(R)$. To identify $H_k(X_*)$, we consider the commutative diagram below:

(D)
$$\begin{array}{ccccccccccccc}
 & & & & & & & & 0 & & 0 & & \\
 & & & & & & & & \uparrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & X_{2k} & \longrightarrow & \ldots\ldots X_{k+1} & \longrightarrow & Z_k & \xrightarrow{\alpha} & H_k(X_*) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow = & & \downarrow = & & \downarrow i & & \downarrow j & & \\
0 & \longrightarrow & R & \longrightarrow & X_{2k} & \longrightarrow & \ldots\ldots X_{k+1} & \longrightarrow & X_k & \xrightarrow{\pi} & \mathrm{coker}(\partial_{k+1}) & \longrightarrow & 0 \\
 & & & & & & & & \downarrow \partial_k & & \downarrow \varphi & & \\
 & & & & & & & & B_{k-1} & \xrightarrow{=} & B_{k-1} & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & & 0 & &
\end{array}$$

The homomorphism j in the above diagram is induced from the inclusion $i : Z_k \hookrightarrow X_k$. Thus we have the short exact sequence:

$$0 \longrightarrow H_k(X_*) \longrightarrow \omega^{-k-1}(R) \xrightarrow{\varphi} \omega^k(R) \longrightarrow 0$$

of RG–modules, and $H_k(X_*)$ is stably determined by the class $(\varphi) \in \widehat{\mathrm{Hom}}_G(\omega^{-k-1}(R), \omega^k(R))$. Using the isomorphisms $\widehat{\mathrm{Hom}}_G(\omega^{-k-1}(R), \omega^k(R)) \cong \widehat{\mathrm{Hom}}_G(\omega^{-2k-1}(R),R) \cong \widehat{\mathrm{Ext}}_G^{-2k-1}(R,R) \cong H_{2k}(G;R)$, we obtain the class $x \in H_{2k}(G;R)$ corresponding to class (φ). Let $\zeta : \omega^{-2k-1}(R) \twoheadrightarrow R$ be a representative for x. Then $L_\zeta \equiv \mathrm{Ker}(\zeta) = \omega^{-k}(\mathrm{Ker}\ \varphi)$, so that $H_k(X_*) = \omega^k L_\zeta$. This proves the Theorem. ■

Proof of Corollary 3.2. The homomorphisms j and π of the diagram (D) in the proof of Theorem 3.1 above give rise to the following short exact sequence:

$$\text{(E)} \qquad 0 \longrightarrow Z_k \xrightarrow{\alpha \oplus i} H_k(X) \oplus X_k \xrightarrow{j-\pi} \mathrm{coker}(\partial_{k+1}) \longrightarrow 0 \ .$$

Since $\mathrm{coker}(\partial_{k+1}) = \omega^{-k-1}(R)$ and $Z_k = \omega^{k+1}(R)$ from the exact sequence $0 \longrightarrow Z_k \longrightarrow X_k \longrightarrow \ldots\ldots \longrightarrow X_0 \longrightarrow R \longrightarrow 0$ we obtain the desired short exact sequence of the corollary. It remains to determine the extension class $z \in \mathrm{Ext}_G^1(\mathrm{coker}(\partial_{k+1}),Z_k) \equiv \widehat{\mathrm{Ext}}_G^1(\omega^{-k-1}(R), \omega^{k+1}(R)) \cong \widehat{\mathrm{Ext}}_G^{-2k-1}(R,R)$ $\cong \hat{H}^{-2k-1}(G;R) \equiv H_{2k}(G;R)$. We apply $\widehat{\mathrm{Hom}}_G(\mathrm{coker}(\partial_{k+1}),-)$ to the exact sequences $0 \longrightarrow H_k(X_*) \longrightarrow \mathrm{coker}(\partial_{k+1}) \xrightarrow{\varphi} B_{k-1} \longrightarrow 0$ and $0 \longrightarrow Z_k \longrightarrow X_k \xrightarrow{\partial_k} B_{k-1} \longrightarrow 0$ as well as (E). We get the commutative diagram below in which δ' and δ_E are the connecting homomorphisms of the last two sequences:

$$\begin{array}{ccc}
 & \widehat{\mathrm{Hom}}_G(\mathrm{coker}(\partial_{k+1}),\ \mathrm{coker}(\partial_{k+1})) & \\
 \swarrow \varphi_* & & \searrow \delta_E \\
\widehat{\mathrm{Hom}}_G(\mathrm{coker}(\partial_{k+1}), B_{k-1}) & \xrightarrow[\cong]{\delta'} & \widehat{\mathrm{Ext}}^1_G(\mathrm{coker}(\partial_{k+1}), Z_k) \\
\Big\downarrow = & & \Big\downarrow = \\
\widehat{\mathrm{Hom}}_G(\omega^{-k-1}(R),\ \omega^k(R)) & \xrightarrow{\cong} & \widehat{\mathrm{Ext}}^1_G(\omega^{-k-1}(R),\ \omega^{k+1}(R)) \\
 \cong \searrow & & \swarrow \cong \\
 & H_{2k}(G,R) &
\end{array}$$

Since $z = \delta$ (identity) and φ_*(identity) = class(φ) = class(ζ) = x , and all other isomorphisms are obtained by dimension shifting, it follows that z and x correspond under these natural isomorphisms. ■

<u>3.3 Theorem</u>. Let X be a (k–1) connected finite dimensional Poincaré complex of formal dimension 2k . Let G act freely on X and let $f : X/G \longrightarrow BG$ be the classifying map for the G–covering $X \xrightarrow{\pi} X/G$. Then:

(a) The homology class $x \equiv f_*[X/G] \in H_{2k}(BG;\mathbb{Z}) = H_{2k}(G;\mathbb{Z})$ completely determines the $\mathbb{Z}G$–module $H_k(X;\mathbb{Z})$ up to $\mathbb{Z}G$–stable isomorphism and vice versa. In fact, $H_k(X)$ is stably isomorphic to $\omega^k L_\zeta$ where class(ζ) = x as in Theorem 3.1 above.

(b) Each $x \in H_{2k}(G;\mathbb{Z})$ is realized by a free analytic G–action on a compact connected Riemann surface when k = 1 , and by a free smooth G–action on a compact simply–connected 4–manifold when k = 2 .

<u>Proof of Theorem 3.3</u>. Applying the result of Theorem 3.1 to the free $\mathbb{Z}G$–chain complex $C_*(X)$, we conclude that the stable $\mathbb{Z}G$–isomorphism class of $H_k(X)$ is determined by $x = \mathrm{class}(\varphi) \in \widehat{\mathrm{Hom}}_G(\mathrm{coker}(\partial_{k+1}), B_{k-1}) \cong H_{2k}(G;\mathbb{Z})$. We compute x in terms of the

induced homomorphism $f_*: H_{2k}(X/G;\mathbb{Z}) \longrightarrow H_{2k}(BG;\mathbb{Z}) = H_{2k}(G;\mathbb{Z})$ as follows. Let $E_* = C_*(E_G)$, where $E_G \longrightarrow BG$ is the universal G–covering as usual, and $C_* = C_*(X)$. The $\mathbb{Z}$G–chain map $\Upsilon_\#: C_* \longrightarrow E_*$ is induced by $\Upsilon: X \longrightarrow E_G$. We identify (E_*, ∂'_*) as a free $\mathbb{Z}$G–resolution of $\mathbb{Z}$, Ker $\partial'_{2k} = \omega^{2k+1}(\mathbb{Z})$, and coker$(\partial'_{k+1}) = \omega^k(\mathbb{Z})$. Consider the commutative diagram below induced by Υ and the above identifications:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & \omega^{2k+1}(\mathbb{Z}) & \longrightarrow & E_{2k} & \xrightarrow{\partial'} & \cdots\cdots & E_{k+1} & \xrightarrow{\partial'_{k+1}} & E_k & \xrightarrow{\pi'} & \omega^k(\mathbb{Z}) & \longrightarrow 0 \\
& \uparrow \Upsilon_* & & \uparrow \Upsilon_\# & & & & & \uparrow \Upsilon_\# & & \uparrow \lambda & \\
0 \longrightarrow & H_{2k}(X) & \longrightarrow & C_{2k} & \longrightarrow & \cdots\cdots & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\pi} & Q & \longrightarrow 0 .
\end{array}$$

The class $f_*[X/G] \in H_{2k}(G;\mathbb{Z})$ is determined by $f \in \mathrm{Hom}(H_{2k}(BG)) = \mathrm{Hom}(\mathbb{Z}, H_{2k}(G;\mathbb{Z}))$. The shifting isomorphism, denoted by

$$\sigma(\omega^{k+1}) : \widehat{\mathrm{Hom}}_G(Q, \omega^k\mathbb{Z}) \xrightarrow{\cong} \widehat{\mathrm{Hom}}_G(\omega^{k+1}(Q), \omega^{k+1}(\mathbb{Z}))$$

sends class(λ) to class$(\Upsilon_*) = f_*$ in the diagram below:

$$\begin{array}{ccc}
\widehat{\mathrm{Hom}}_G(Q, \omega^k(\mathbb{Z})) & \xrightarrow[\cong]{\sigma(\omega^{k+1})} & \widehat{\mathrm{Hom}}_G(\omega^{k+1}(Q), \omega^{k+1}(\mathbb{Z})) \\
\downarrow \cong & & \downarrow \cong \\
H_{2k}(G;\mathbb{Z}) & \xrightarrow{\cong} & \widehat{\mathrm{Hom}}_G(H_{2k}(X), \omega^{2k+1}(\mathbb{Z})) .
\end{array}$$

Therefore, it suffices to prove that class(λ) = class(φ). Consider the commutative diagrams below in which (I) determines class(φ):

(I)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & B_{k-1} & \xrightarrow{=} & B_{k-1} & & \\
 & & & & \uparrow & & \uparrow \varphi & & \\
0 & \longrightarrow & B_k & \longrightarrow & C_k & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & \uparrow = & & \uparrow i & & \uparrow j & & \\
0 & \longrightarrow & B_k & \longrightarrow & Z_k & \xrightarrow{\alpha} & H_k(X) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(II)

$$\begin{array}{ccccc}
 & & B_{k-1} & & \\
 & \varphi \swarrow & & \searrow \tau_{k-1} & \\
Q & & \xrightarrow{\quad\lambda\quad} & & \omega^k(\mathbb{Z}) \\
\pi \downarrow & & & & \downarrow \pi' \\
C_k & & \xrightarrow{\quad\gamma_{\#}\quad} & & E_k
\end{array}$$

(III)

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \omega^k(\mathbb{Z}) & \longrightarrow & E_{k-1} & \longrightarrow & \ldots\ldots & \longrightarrow & E_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow \tau_{k-1} & & \uparrow \gamma_{\#} & & & & \uparrow \gamma_{\#} & & \uparrow = & & \\
0 & \longrightarrow & B_{k-1} & \longrightarrow & C_{k-1} & \longrightarrow & \ldots\ldots & \longrightarrow & C_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Under the shifting isomorphism $\sigma(\omega^k) : \widehat{\mathrm{Hom}}_G(\mathbb{Z},\mathbb{Z}) \xrightarrow{\pi} \widehat{\mathrm{Hom}}_G(\omega^k(\mathbb{Z}), \omega^k(\mathbb{Z})) \cong$ $\cong \widehat{\mathrm{Hom}}_G(B_{k-1}, \omega^k(\mathbb{Z}))$ in diagram (III), class($\mathrm{id}_{\mathbb{Z}}$) corresponds to class(τ_{k-1}) . Thus the isomorphism $(\tau_{k-1})_*$ below:

$$\begin{array}{ccc} \widehat{\mathrm{Hom}}_G(Q,\, B_{k-1}) & \xrightarrow[(\tau_{k-1})_*]{\cong} & \widehat{\mathrm{Hom}}_G(Q,\, \omega^k(\mathbb{Z})) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & H_{2k}(G;\mathbb{Z}) & \end{array}$$

sends class(φ) to class(λ) , and this is what we wanted. Thus part (a) of the theorem is proved. The proof of part (b) is included in Assadi [A3] Proposition 4.4 (c) for the case $k = 2$. For $k = 1$, the Hurwicz homomorphism $\Omega_2^{SO}(BG) \longrightarrow H_2(BG) = H_2(G;\mathbb{Z})$ is surjective, hence part (a) implies the desired conclusion. ∎

3.4 Corollary. For every $x \in H_2(G;\mathbb{Z})$, there exists a free projective G–action on a non–singular projective curve/$\mathbb{C}$ such that $H^1(X_{an};\mathbb{Z})$ is $\mathbb{Z}G$–stably isomorphic to $(\omega^1 L_\zeta)^*$ where $\zeta \in \mathrm{Hom}_G(\omega^{-3}(\mathbb{Z}),\mathbb{Z})$ represents x under the isomorphism $\widehat{\mathrm{Hom}}_G(\omega^{-3}(\mathbb{Z}),\mathbb{Z}) \cong \hat{H}^{-3}(G;\mathbb{Z}) = H_2(G;\mathbb{Z})$, and X_{an} is the underlying space with the usual topology.

Proof: According to 3.3 (b) above, there exists a compact Riemann surface Σ and a map $f : \Sigma \longrightarrow BG$ such that $f_*[\Sigma] = x \in H_2(BG;\mathbb{Z}) = H_2(G;\mathbb{Z})$. Let X be the G–covering induced by f together with the free G–action on X via covering translations. Then $H_1(X;\mathbb{Z})$ is stably $\mathbb{Z}G$–isomorphic to $\omega^1(L_\zeta)$ and class(ζ) = x by Theorem 3.1 above. Now G acts on the compact Riemann surface X by complex analytic isomorphisms, and $H^1(X_{an};\mathbb{Z})$ is $\mathbb{Z}G$–ismorphic to $\mathrm{Hom}(H_1(X),\mathbb{Z}) = (\omega^1 L_\zeta)^*$ and class(ζ) = x by Theorem 3.1 above. We may assume that the genus $(\Sigma) \geq 2$, hence genus $(X) \geq 2$, so that the canonical sheaves $\mathscr{K}_\Sigma$ and $\mathscr{K}_X$ are ample. By Serre's GAGA principle [S1], Σ and $\mathscr{K}_\Sigma$ are algebraic. Thus, X is a complete non–singular curve on which G acts by algebraic isomorphisms, $\mathscr{K}_X$ is an ample G–line bundle on X , and $\pi : X \longrightarrow \Sigma = X/G$ is an algebraic morphism for which $\mathscr{K}_\Sigma = (\pi_* \ \mathscr{K}_X)^G$. Since the pluricanonical embedding $X \longrightarrow \mathbb{P}\Gamma(X, \mathscr{K}_X^{\otimes m})$ is equivariant, the G–action on X is projective.∎

<u>3.5 Examples</u>. (1) If $G = \mathbb{Z}/p\mathbb{Z}$, then $H_2(G;\mathbb{Z}) = 0 = H_4(G;\mathbb{Z}) = 0$. Thus, if $\dim_{\mathbb{R}} X = 2$, then for $r = \frac{1}{p}(g-1)$ $H^1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}G)^{2r}$, and if $\dim_{\mathbb{R}} X = 4$ and $\pi_1(X) = 0$, then $H^2(X) \cong I \oplus I \oplus (\mathbb{Z}G)^s$ where I is the augmentation ideal. Since $I \cong \mathbb{Z}[\zeta]$, where ζ is a primitive p–th root of unity with the usual $\mathbb{Z}G$–module structure, then $H^2(X;\mathbb{Z}) \cong \mathbb{Z}[\zeta] \oplus \mathbb{Z}[\zeta] \oplus (\mathbb{Z}G)^s$.

(2) Suppose G has periodic cohomology, so that the p–Sylow subgroups of G are cyclic for p = odd and either cyclic or generalized quaternionic for p = 2 . Then $H_2(G_p;\mathbb{Z}) = 0 = H_4(G_p;\mathbb{Z})$ for all p–Sylow subgroups $G_p \subseteq G$. Therefore, $H_2(G;\mathbb{Z}) = 0 = H_4(G;\mathbb{Z})$, and we have the following conclusions. For $\dim_{\mathbb{R}} X = 2$, $H^1(X;\mathbb{Z})$ is $\mathbb{Z}G$–isomorphic to $\omega^2(\mathbb{Z}) \oplus \omega^{-2}(\mathbb{Z}) \oplus (\mathbb{Z}G)^{2r}$. For $\dim_{\mathbb{R}} X = 4$, $\pi_1(X) = 0$, $H^2(X;\mathbb{Z}) = \omega^3(\mathbb{Z}) \oplus \omega^{-3}(\mathbb{Z}) \oplus (\mathbb{Z}G)^s$.

(3) Suppose $G = (\mathbb{Z}/p\mathbb{Z})^2$ then $H_2(G;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and $H_4(G;\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Therefore, in this case we get non–trivial examples corresponding to the non–zero elements of $H_{2k}(G;\mathbb{Z})$.

At this point, one may raise the point that the procedure in Theorem 3.3 (b) to produce free G–actions on simply–connected smooth 4–manifolds involved non–algebraic arguments. That is, surjection of $\Omega_4^{SO}(BG)$ onto $H_4(BG)$ produces $f_0 : W_0^4 \longrightarrow BG$ such that $f_{0*}[W_0^4] = x \in H_4(BG)$ and smooth surgery on the map f_0 corrects the fundamental group to give $f : W^4 \longrightarrow BG$ with $f_*[W] = x$. Then the universal cover of W , say X , is the desired smooth simply–connected 4–manifold whose homology $\mathbb{Z}G$–module $H_2(X)$ realizes the class $x \in H_4(BG)$. It is not clear if either one of these steps can be achieved using complex manifolds. Thus, we pose the following

<u>3.6 Problem</u>. Which homology classes $x \in H_4(G;\mathbb{Z})$ arise in Theorem 3.3 for analytic G–actions on compact complex surfaces X with $\pi_1(X) = 0$?

Section 4. Group Actions on Riemann Surfaces.

In this section, we assume that G is a finite group acting effectively on the compact Riemann surface Σ via complex analytic isomorphisms. Thus, G preserves the orientation and the isotropy subgroups $H_i \subseteq G$ are all cyclic. Moreover, for all $H_i \neq 1$, Σ^{H_i} consists of at most finitely many points of Σ. We delete the trivial subgroup (i.e. the principal isotropy subgroup for all effective finite group actions) from the list of isotropy subgroups of the action. The orbit space $\Sigma' = \Sigma/G$ is still a compact Riemann surface and $\Sigma \xrightarrow{\pi} \Sigma'$ is a ramified finite covering. We may choose a triangulation for Σ' such that the ramification points are all included in the set of vertices of Σ', and we lift this triangulation to Σ, to give Σ an equivariant triangulation. Under these circumstances, Σ becomes a G–CW complex, and the cells of Σ provide permutation bases for the cellular chain complex of Σ. This makes $C_*(\Sigma)$ into a permutation complex. In Section 3, we proved that if G acts freely on Σ, then the $\mathbb{Z}G$–module $H_1(X;\mathbb{Z})$ is stably $\mathbb{Z}G$–isomorphic to $\omega^1 L_\zeta$, where $\text{class}(\zeta) = x \in H_2(G;\mathbb{Z})$ is the image $f_*[\Sigma/G] \in H_2(BG;\mathbb{Z}) = H_2(G;\mathbb{Z})$ under the homomorphism induced by the classifying map $f : \Sigma/G \longrightarrow BG$. Moreover, every element of $H_2(G;\mathbb{Z})$ arises by such a free G–action. For instance, if $H_2(G;\mathbb{Z}) = 0$, then $H_1(\Sigma) \cong \omega^2(\mathbb{Z}) \oplus \omega^{-2}(\mathbb{Z}) \oplus (\mathbb{Z}G)^{2r}$, where r is determined by counting $\mathbb{Z}$–ranks of both sides of this equation. We proceed to determine the $\mathbb{Z}G$–module structure of $H_1(\Sigma;\mathbb{Z})$ for non–free actions in the same spirit.

First of all, the following analogue of Assadi ([A3] Theorem 5.4) is easily established.

4.1 Proposition. With the above notation, the following are equivalent:

(a) $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}G$–projective.

(b) For each prime order subgroup $C \subseteq G$, $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}C$–projective.

(c) For each prime order subgroup $C \subsetneq G$, Σ^C consists of 2 points.

Furthermore, if $H^1(\Sigma;\mathbb{Z})$ is $\mathbb{Z}G$–projective, then p–Sylow subgroups of G are cyclic.

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) by considering the spectral sequence $E_C \times_C \Sigma \longrightarrow BC$ and applying the localization theorem (Hsiang [Hs] or Quillen [Q]). From (c) it follows that p–Sylow subgroups of G must have one–dimensional faithful complex linear representations, hence they must be cyclic. Thus, maximal p–elementary abelian subgroups of G are isomorohic to $\mathbb{Z}/p\mathbb{Z}$. Therefore (b) $\Rightarrow$ (a) by Chouinard's theorem (Chouinard [Ch] or Jackowski [J]). (c) $\Rightarrow$ (b) is also possible by reversing the spectal sequence argument for (b) $\Rightarrow$ (c) . For a more elementary argument, consider $\Sigma_0 = \Sigma - \{x\}$ where $x \in \Sigma^C$. Then $H_1(\Sigma_0) = H_1(\Sigma)$ and $H_2(\Sigma_0) = 0$. Therefore, $H_1(\Sigma)$ is the only non-vanishing homology group in the $\mathbb{Z}G$–free chain complex $C_*(\Sigma_0,\Sigma_0^C)$. Hence, it is stably $\mathbb{Z}C$–free, and since C is cyclic, $H_1(\Sigma)$ is $\mathbb{Z}C$–free. ∎

The following lemma and the above discussion take care of $|G| = \text{prime}$.

4.2 Lemma. Let $G = \mathbb{Z}/p\mathbb{Z} = \langle t \rangle$ where p is a prime. Then $\Sigma^G \neq \phi$ if and only if $H_1(\Sigma) \cong \mathbb{Z}[\zeta]^{\alpha} \oplus (\mathbb{Z}G)^r$, where ζ is a primitive p–th root of unity and $\mathbb{Z}[\zeta]$ has the usual $\mathbb{Z}G$–module structure $\mathbb{Z}[\zeta] \equiv \mathbb{Z}[G]/(1+t+.....+t^{p-1})$. Here $r = 2g - (p-1)\alpha$ and $\alpha = \#(\Sigma^G)-2$.

Proof. If $\Sigma^G = \phi$, then $H_1(\Sigma) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}G)^s$. Therefore, assume that $\Sigma^G \neq \phi$. Let $x_0 \in \Sigma^G$, and choose a small G–invariant disk D about x_0 , and let $\Sigma_0 = \Sigma$–interior (D). First, observe that $\Sigma_0^G \neq \phi$. Otherwise, we would consider the classifying map of the regular p–fold cover $\Sigma_0 \xrightarrow{\pi} \Sigma_0/G$, say $f : \Sigma_0/G \longrightarrow BG$, and conclude that $f|\,\partial\Sigma_0/G = f' : S^1 \longrightarrow BG$ is null–homologous in $H_1(BG) \cong \mathbb{Z}/p\mathbb{Z} \cong \pi_1(BG)$, hence null–homotopic. But $\pi^{-1}(\partial\Sigma_0/G) = \partial D$ is connected, so that f' cannot be null–homotopic by covering space theory. Consequently, there exists $x_1 \in \Sigma_0^G$. Let $\Sigma^G = \{x_0, x_1, y_1, \ldots, y_\alpha\}$, and consider the permutation chain complex $C_*(\Sigma_0)$, in which $C_0(\Sigma_0) \cong C_0(x_0) \oplus C_0(\Sigma_0, x_0) = \mathbb{Z} \oplus \tilde{C}_0(\Sigma_0)$ and $\tilde{C}_0(\Sigma_0) \cong \mathbb{Z}^\alpha \oplus (\mathbb{Z}G)^r$. Since $H_2(\Sigma_0) = 0$, it follows that $\operatorname{Ker} \partial_1 = Z_1 \cong H_1(\Sigma_0) \oplus C_2(\Sigma_0)$ and $0 \longrightarrow Z_1 \longrightarrow C_1(\Sigma_0) \xrightarrow{\partial_1} \tilde{C}_0(\Sigma_0) \longrightarrow 0$ is exact. Therefore, Z_1 is stably $\mathbb{Z}G$–isomorphic to I^α, where I is the augmentation ideal of $\mathbb{Z}[G]$, which is isomorphic to $\mathbb{Z}[\zeta]$ because $G = \mathbb{Z}/p\mathbb{Z}$. Hence $H_1(\Sigma) \cong H_1(\Sigma_0) \cong \mathbb{Z}[\zeta]^\alpha \oplus (\mathbb{Z}G)^r$ as claimed.

∎

Next, we assume that $\Sigma^G \neq \phi$, so that G is necessarily cyclic, but possibly having composite order. Unlike the case of $G = \mathbb{Z}/p^k\mathbb{Z}$ when p = prime, in this case Σ^G = one point is possible, as shown by Conner–Floyd [CF] (see also Ewing–Stong [ES]). Thus, we consider two cases below. Note that the case $G = \mathbb{Z}/p^k\mathbb{Z}$ is covered by the first case below since according to Atiyah–Bott and others $\Sigma^G \neq$ one point.)

<u>4.3 Proposition</u>. Suppose Σ^G has at least two points, and let $\{H_i : i = 1,\ldots,n\}$ be the collection of non–trivial isotropy subgroups considered with repetition according to the number of orbits in Σ^{H_i}, and excluding two copies of G corresponding to the first two points in Σ^G. Let ζ_i be an $|H_i|$–th root of unity and $\mathbb{Z}[\zeta_i]$ with the usual $\mathbb{Z}[H_i]$–module structure. Then $H_1(\Sigma) \cong \bigoplus_{i=1}^{n} (\mathbb{Z}G \otimes_{H_i} \mathbb{Z}[\zeta_i] \oplus (\mathbb{Z}G)^r$, where

$$r = \frac{1}{|G|}\left[\operatorname{rank} H_1(\Sigma) - \sum_{i=1}^{n} \frac{|G|}{|H_i|}(|H_i|-1)\right].$$

<u>Proof</u>. Let $x_1 \in \Sigma^G$, and consider a small G–invariant disk neighborhood of x_1 (avoiding

other fixed points), called $D(x_0)$. Let $\Sigma_0 = \Sigma$–interior $(D(x_1))$. As before, Σ_0 admits a G–CW structure in which $C_1(\Sigma_0)$ and $C_2(\Sigma_0)$ are $\mathbb{Z}G$–free, and $C_0(\Sigma_0)$ is a permutation module. Let $x_0 \in \Sigma_0^G \neq \phi$ and consider the augmentation $C_0(\Sigma_0) \rightleftarrows C_0(x_0)$ which is $\mathbb{Z}G$–split via the inclusion $\{x_0\} \subset \Sigma_0$. Thus $C_0(\Sigma_0) \cong \tilde{C}_0(\Sigma_0) \oplus \mathbb{Z}$. Consider the following short exact sequences:

$$0 \longrightarrow C_2(\Sigma_0) \xrightarrow{\partial} B_1(\Sigma_0) \longrightarrow 0$$
$$0 \longrightarrow B_1(\Sigma_0) \longrightarrow Z_1(\Sigma_0) \longrightarrow H_1(\Sigma_0) \longrightarrow 0$$
$$0 \longrightarrow Z_1(\Sigma_0) \longrightarrow C_1(\Sigma_0) \longrightarrow \tilde{C}_0(\Sigma_0) \longrightarrow 0 .$$

From these, it follows that $B_1(\Sigma_0)$ is $\mathbb{Z}G$–free, and $H_1(\Sigma) \cong H_1(\Sigma_0)$ is stably isomorphic to $Z_1(\Sigma_0)$. Since all modules are $\mathbb{Z}$–free and $B_1(\Sigma_0)$ is $\mathbb{Z}G$–free, reflexivity of $\mathbb{Z}G$ implies that the second exact sequence is $\mathbb{Z}G$–split. Leaving out $\{x_0,x_1\} \subset \Sigma^G$ from the singular set of the action, it is clear that $\tilde{C}_0(\Sigma_0)$ is stably isomorphic to $\bigoplus_i \mathbb{Z}(G/H_i)$ where H_i are isotropy subgroups of the fixed points in $\Sigma - \{x_0,x_1\}$. Therefore, $Z_1(\Sigma_0)$ is stably $\mathbb{Z}G$–isomorphic to $\oplus\, \omega^1\mathbb{Z}(G/H_i)$. We also have $\omega^1\mathbb{Z}(G/H_i) \sim \mathbb{Z}G \otimes_{H_i} \omega^1_{H_i}(\mathbb{Z}) \sim \mathbb{Z}G \otimes_{H_i} \mathbb{Z}[\zeta_i]$, where the last stable isomorphism is due to $\omega^1 H_i(\mathbb{Z}) \cong \mathbb{Z}[\zeta_i]$ as $\mathbb{Z}[H_i]$–modules, because H_i is cyclic. ∎

<u>4.4 Proposition</u>. Suppose $\Sigma^G = \{x_0\}$. Then G is a cyclic group whose order is divisible by at least two distinct primes. Let $\{H_i : i = 0,1,\dots,n\}$ be the collection of non–trivial isotropy subgroups such that $H_0 = G$ and $H_i \neq G$. Then $H_1(\Sigma)$ is completely determined by the permutation module $B = \bigoplus_{i=1}^{n} \mathbb{Z}[G/H_i]$ and an element

$\theta(\Sigma) \in \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ from the exact sequence:

$$0 \longrightarrow H_1(\Sigma) \longrightarrow \mathbb{Z}[G]^{\ell} \oplus \mathbb{Z} \xrightarrow{\varphi} B \oplus \mathbb{Z}[G]^{k} \longrightarrow 0 \ .$$

Here, φ represents a homology class in $\widehat{\mathrm{Hom}}_G(\mathbb{Z}[G]^{\ell} \oplus \mathbb{Z} , B \oplus \mathbb{Z}[G]^{k}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ and $\mathrm{class}(\varphi) = \theta(\Sigma)$.

<u>Proof</u>. Consider the commutative diagram below, in which $Q = \mathrm{coker}(\partial_2)$, $C_* = C_*(\Sigma)$, and

$$C_0 \cong \mathbb{Z}[G/G] \oplus \mathrm{Im}(\partial_1) \cong \mathbb{Z} \oplus B_0 = \mathbb{Z} \oplus B \oplus (\mathbb{Z}G)^s \ :$$

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & B_0 & \xrightarrow{=} & B_0 & & \\
 & & & & & & \uparrow \partial_1 & & \uparrow \varphi & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C_2 & \longrightarrow & C_1 & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & \uparrow = & & \uparrow = & & \uparrow i & & \uparrow j & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C_2 & \longrightarrow & Z_1 & \xrightarrow{\alpha} & H_1(\Sigma) & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

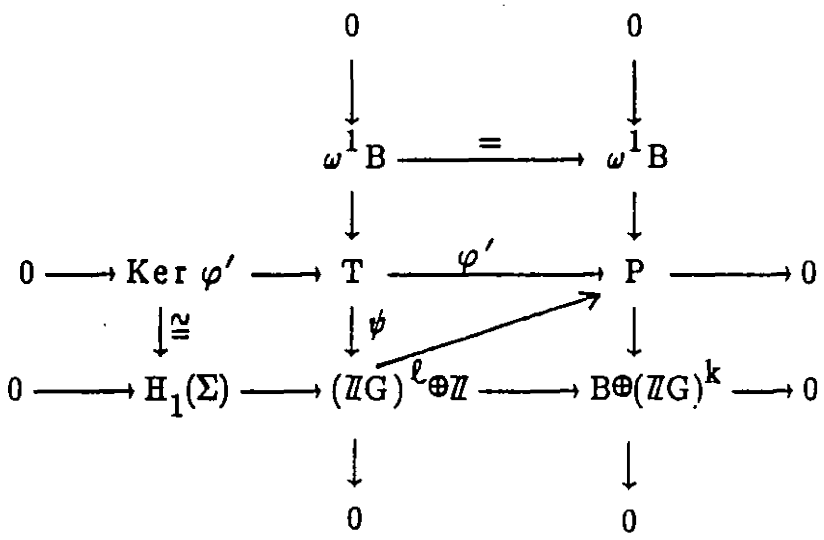
In the sequence $0 \longrightarrow H_1(\Sigma) \longrightarrow Q \xrightarrow{\varphi} B_0 \longrightarrow 0$, $Q \cong \mathbb{Z} \oplus (\mathbb{Z}G)^{\ell}$ and $B_0 \cong B \oplus (\mathbb{Z}G)^k$, with $B = \bigoplus_{i=1}^{n} \mathbb{Z}[G/H_i]$. Here, we use the fact that G must be cyclic, hence it has a periodic resolution $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$. Comparing this with the top horizontal row and applying the Schanuel Lemma (Swan [Sw]) we find out Q is stably $\mathbb{Z}G$–isomorphic to $\mathbb{Z}$. Moreover,

$\mathrm{class}(\varphi) = \theta(\Sigma) \in \widehat{\mathrm{Hom}}_G(\theta, B_0) \cong \widehat{\mathrm{Hom}}_G(\mathbb{Z}, B) = \bigoplus_{i=1}^{n} \mathbb{Z}/|H_i|\mathbb{Z}$ determines $H_1(\Sigma)$ up to stable $\mathbb{Z}G$–isomorphism.

<u>4.5 Corollary</u>. In the above Proposition, if $\theta(\Sigma) = 0$, then $H_1(\Sigma) \cong \mathbb{Z} \oplus (\bigoplus_{i>0} \mathbb{Z}[\zeta_i]) \oplus \mathbb{Z}[G]^r$, where

$$r = \frac{1}{|G|}[\mathrm{rank}\ H_1(\Sigma) - 1 - \sum_{i>0} (|H_i|-1)] \ .$$

<u>Proof</u>. If $\theta(\Sigma) = 0$, then $\mathrm{class}(\varphi) = 0$ and in the sequence $0 \longrightarrow H_1(\Sigma) \longrightarrow \mathbb{Z}[G]^{\ell} \oplus \mathbb{Z} \xrightarrow{\varphi} B \oplus [\mathbb{Z}G]^k \longrightarrow 0$, φ factors through a projective $\mathbb{Z}G$–module P , which without loss of generality we may assume to be a free $\mathbb{Z}G$–module. We form the following pull–back diagram (the left square) and complete the commutative diagram as indicated below:



Since P is free, φ' splits, and this gives a splitting of ψ . Therefore,

$$P \oplus \mathrm{Ker}(\varphi') \cong T \cong \mathbb{Z} \oplus (\mathbb{Z}G)^{\ell} \oplus \omega^1 B \ .$$

Identifying the terms $P = (\mathbb{Z}G)^s$, $\mathrm{Ker}\ \varphi' \cong H_1(\Sigma)$, and $\omega^1 B \sim \bigoplus_{i>0} \mathbb{Z}[\zeta_i]$, we conclude that $H_1(\Sigma)$ is stably $\mathbb{Z}G$–isomorphic to $\mathbb{Z} \oplus (\bigoplus_{i>0} \mathbb{Z}[\zeta_i]$. Since cancelation holds for $\mathbb{Z}G$–modules when G is cyclic (Swan [Sw]), the desired formula is obtained.

∎

4.6 Corollary. Suppose R is a commutative ring such that RG is semisimple (e.g. a field of characteristic zero, or prime to order of G). Then, in the representation ring of RG , we have the following equation: $[H_1(\Sigma;R)] = [R] + m[RG] - \sum_{i=1}^{n} [R \otimes_H RG]$, and m is determined by counting the ranks of corresponding free R–modules.

Proof. The sequence in Proposition 4.4 splits in the representation ring of RG due to semisimplicity. ∎

The above corollary for $R = \mathbb{C}$ is proved by A. Broughton [Br] using Eichler's trace formula.

The final possibility is when $\Sigma^G = \phi$ while G does not act freely on Σ. In this case, G need not be cyclic, and the formulas are somewhat more complicated:

4.7 Proposition. Suppose that G acts without fixed–points, but not freely. Let $\{H_i : i \in I\}$ be the collection of isotropy subgroups considered with multiplicities as before, and let ϵ be the augmentation homomorphism $(\epsilon(gH_i) = 1)$ and $B_0 = \mathrm{Ker}(\epsilon)$ in

$\epsilon : \bigoplus_{i \in I} \mathbb{Z}(G/H_i) \longrightarrow \mathbb{Z}$. Then, up to stable $\mathbb{Z}G$–isomorphism, the $\mathbb{Z}G$–module is determined by B_0 and a homology class $\theta(\Sigma) \in H_1(G;B_0)$. Indeed, if $\varphi : \omega^{-2}(B_0^*) \longrightarrow \mathbb{Z}$ represents $\theta(\Sigma)$ via the isomorphisms $\widehat{\mathrm{Hom}}_G(\omega^{-2}(B_0^*),\mathbb{Z}) \cong H_1(G;B_0)$, then $H^1(\Sigma) \cong \omega^{-1}(\mathrm{Ker}\ \varphi)$ and $H_1(\Sigma) \cong \omega^{-1}((\mathrm{Ker}\ \varphi)^*)$.

Proof. We have an exact sequence $0 \longrightarrow \mathrm{Im}\,\partial_1 \longrightarrow C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$ in which $C_0 \cong (\mathbb{Z}G)^{\ell} \oplus (\bigoplus_{i=1}^{m} \mathbb{Z}(G/H_i))$, $H_i \neq 1$. Let $\{e_i : 1 \leq i \leq m\}$ and $\{u_j : 1 \leq j \leq \ell\}$ be the obvious generators and basis elements for the two factors in C_0 . We choose a new basis for $(\mathbb{Z}G)^{\ell}$ factor, by fixing $H_0 \neq 1$, $e_0 \in \mathbb{Z}(G/H_0)$ its $\mathbb{Z}G$–generator, and setting $v_j = u_j - e_0$. Such an e_0 exists because the action is not free by assumption. With the new basis $\{v_j : 1 \leq j \leq \ell\}$, we observe that $\mathrm{Im}\ \partial_1$ is $\mathbb{Z}G$–stably isomorphic to B_0 in the statement of the proposition. Again, from the exact sequence

$$0 \longrightarrow H^1(\Sigma) \longrightarrow Z_1(\Sigma)^* \longrightarrow B_1(\Sigma)^* \longrightarrow 0$$

as in the preceding cases, we get the following exact sequence, up to $\mathbb{Z}G$–stable isomorphism:

$$0 \longrightarrow H^1(\Sigma) \longrightarrow \omega^{-1}(B_0) \longrightarrow \omega^1(\mathbb{Z}) \longrightarrow 0 \ .$$

From the latter, we have:

$$0 \longrightarrow \omega^{-1}H^1(\Sigma) \longrightarrow \omega^{-2}(B_0) \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0$$

and the classes of φ in $\widehat{\mathrm{Hom}}_G(\omega^{-2}(B_0),\mathbb{Z}) \cong \hat{H}^{-2}(G,B_0) \cong H_1(G,B_0)$ is the class $\theta(\Sigma)$ mentioned above. One checks that B_0 and $\theta(\Sigma)$ together determine the stable isomorphism class of $H_1(\Sigma)$. ∎

Section 5. Group Actions on Kähler Surfaces.

In this section, X denotes a simply–connected compact Kähler surface, and we assume that G is a finite group acting by complex automorphisms. Unlike general smooth manifolds, the Kähler condition imposes strong conditions on the action, and consequently on the $\mathbb{Z}G$–representation afforded by $H^2(X;\mathbb{Z})$.

5.1 Proposition. Let G be an arbitrary non–trivial finite group acting on X as above. Then $H^2(X)$ cannot be $\mathbb{Z}G$–projective if G preserves the Kähler cohomology class in $H^2(X)$.

Proof. It suffices to show this for $G = \mathbb{Z}_p$. Consider the Serre spectral sequence of the Borel fibration $E_G \times_G X \longrightarrow BG$, $H^*(G,H^*(X)) \Rightarrow H^*_G(X)$. If G acts freely on X, then $H^2(X) \cong I \oplus I \oplus (\mathbb{Z}G)^s$, hence $H^2(X)$ is not projective. Suppose $X^G \neq \phi$. Let $t \in H^2(G) \cong \mathbb{Z}_p$ and let $\alpha \in H^2(X)$ be given by the Kähler form, so that $[\alpha\Lambda\alpha] = [X]$ = cohomological orientation class of x. Consider the cup product in the spectral sequence, as well as the $H^*(G)$–algebra structure of the E_2–term in the follwoing commutative diagram:

$$\begin{array}{ccc}
H^2(G) \otimes H^0(G,H^2(X)) \otimes H^0(G,H^2(X)) & \longrightarrow & H^2(G) \otimes H^0(G,H^2(X) \otimes H^2(X)) \\
\downarrow & & \downarrow \\
H^2(G,H^2(X)) \otimes H^0(G,H^2(X)) & & H^2(G) \otimes H^0(G,H^4(X)) \\
\downarrow & & \downarrow \\
H^2(G,H^2(X) \otimes H^2(X)) & \longrightarrow & H^2(G,H^4(X))
\end{array}$$

Since $t \cdot [X] \neq 0$ in $H^2(G,H^4(X))$, we have $0 \neq t \cdot (\alpha\Lambda\alpha) = (t\alpha)\Lambda\alpha$, hence $t \cdot \alpha \neq 0$. Therefore $\alpha \in H^0(G,H^2(X)) = H^2(X)^G$ cannot be $H^*(G)$–torsion.

Consequently, $\hat{H}^*(G,H^2(X)) \neq 0$, so that $H^2(X)$ cannot be G–projective.■

5.2 Corollary. Let $G = \mathbb{Z}_p$ act on the simply–connected Kähler surface X preserving the Kähler cohomology class. Then $\sum_{i\geq 0} \beta_i(X^G) \geq 3$, where β_i = i–th Betti number, and $X^G \neq \phi$ by hypothesis.

Proof. Since $X^G \neq \phi$, and for degree reasons, the Serre spectral sequence of $E_G \times_G X \longrightarrow BG$ collpases. (See e.g. [A3]). Now the above proof shows that $\hat{H}^*(G,H^i(X)) \neq 0$ for $i = 0,2,4$. Therefore, $H^*(G)$–rank of $H^*_G(X)$ is at least 3. The localization theorem ([HS] or [Q]) implies the desired conclusion.

■

In the following theorem, conditions are given which guarantee that modulo $\mathbb{Z}G$–projective modules, G must act trivially on $H^2(X)$. Recall Theorem 4.14 of [A3] III.

5.3 Theorem. Suppose X is a Kähler surface, $\pi_1(X) = 0$ and G acts smooth but not freely, and $G = (\mathbb{Z}_p)^s$, $s \geq 1$. Assume that for each cyclic subgroup $C \subseteq G$, $p > \beta_0(X^C)$ and $\beta_1(X^C) = 0$. Then the following hold:

(a) there exists an $m > 0$ such that the $\mathbb{Z}G$–module $H_2(X) \cong \mathbb{Z}^m \oplus M$ where G acts trivially on $\mathbb{Z}^m$ and M is $\mathbb{Z}G$–projective.

(b) $\chi(X^G) = \chi(X^C) = m$ for each $C \subset G$, $|C| = p$, and $\text{rank}(G) \leq 2$.

(c) If $\text{rank}(G) > 1$, then G acts freely on the set of symplectic 2–forms of X ; hence G does not preserve any symplectic structure on X .

Proof. Since $\beta_1(X^C) = 0$, X^C consists of 2–spheres and isolated points. Moreover,

$H_2(X) \cong \mathbb{Z}^{r(C)} \oplus \mathbb{Z}[C]^u$ as $\mathbb{Z}C$–modules, where C acts trivially on $\mathbb{Z}^{r(C)}$, and $r(C) = \chi(X^C)-2$. Also, $G/_C$ must act trivially on $\pi_0(X^C)$ since $p > \beta_0(X^C)$ by assumption. Hence $G/_C$ must act effectively on each component S^2, and each isolated fixed point in X^C must be an isolated G–fixed point. If $G = C = \mathbb{Z}_p$, then we are done. If $\mathrm{rank}(G) > 1$, then G cannot have a free action in the punctured neighborhood of an isolated fixed point. Since $X^C \neq \phi$ for some $C \neq 1$, $G/_C$ must act effectively on each copy $S^2 \subsetneq X^C$. Therefore $X^G = (X^C)^{G/C} \neq \phi$. From this (c) follows, since if C preserves some symplectic 2–form of X, then X^C must consist of exclusively isolated fixed points. (Consider the complex C–representation on the tangent space $T_Q X$ for some $Q \in X^C$, and observe that if a symplect form is preserved, then the two eigen–values of any generator of C must be distinct and not equal to 1). In view of the above observation that X^C contains copies of S^2, and that $X^C \supset X^G \neq \phi$ for each $C \neq 1$, we see that each C, and hence G, acts freely on the set of symplectic 2–forms of X. Since each $S^2 \subset X^C$ contributes one copy of $S^0 \subset (X^C)^{G/C}$, then $\chi(X^C) = \chi(X^G)$. It suffices, therefore, to prove (a). But (a) is proved in Theorem 5.6 of Assadi [A3] III.

□

Finally, the actions considered in this section are "regular" in the terminology of [A3]. Hence, the general theorems of [A3] apply to this situation, and the same principles and argument may be used to study the $\mathbb{Z}G$–representations afforded by $H^2(X)$ for a compact Kähler surface. In particular, the fixed point set of the G–action and a suitable group cohomology element completely determine the $\mathbb{Z}G$–module $H_2(X)$ as in Proposition 4.7 of [A3] III.

Section 6. Projective Surface with Irregularity Zero.

In this section, we consider non–singular projective surfaces X defined over an algebraically closed field of arbitrary characteristic k . The analogue of simply–connectivity for complex surfaces is the condition $q(x) \equiv p_g(X) - p_a(X) = 0$ i.e. the irregularity is zero. Let $\mathcal{K}_x$ be the canonical sheaf of X , and let $\Omega^2(X) \equiv H^0(X; \mathcal{K}_x)$ be the k–vector space of "holomorphic 2–forms" of X . We compute the kG–representation $\Omega^2(X)$ for the free G–actions on X . A suitable cohomology theory is Cech cohomology using an open covering $\mathcal{U}$ of X consisting of G–invariant affine subsets of X . Such a Cech cohomology group coincides with Grothendieck's coherent cohomology, i.e.
$H^0(\mathcal{U}; \mathcal{K}_x) \cong \Omega^2(X)$. On the other hand, by Serre duality,
$\Omega^2(X) \cong \mathrm{Hom}_k(H^2(X; \mathcal{O}_X),k) \cong \mathrm{Hom}_k(H^2(\mathcal{U}; \mathcal{O}_X),k)$. Consider a free G–action on X , and observe that the variety X/G exists (Mumford [M]) and it is non–singular and projective. Moreover, the morphism $f : X \longrightarrow X/G$ is an étale principal covering. Let $\mathcal{U}_0$ be a suitable finite covering of X/G by affine open sets, and let
$\mathcal{U} = \{f^{-1}(V_0) : V_0 \in \mathcal{U}_0\}$. Then each $V = f^{-1}(V_0)$ is also affine, and we have $f^{-1} | V : V \longrightarrow V_0$ is given by a k–algebra homomorphism $\varphi : R \longrightarrow S$, i.e.,
$V_0 = \mathrm{Spec}(R)$, $V = \mathrm{Spec}(S)$ and ${}^a\varphi = f^{-1} | V$.

6.1 Theorem. Let X be (an irreducible) non–singular projective k–surface with $q(X) = 0$. Suppose that G acts freely on X by automorphisms. Then the kG–module $\Omega^2(X)$ is stably kG–isomorphic to $\omega_G^3(k)$.

6.2 Remark. Compare this with Corollary 3.2 which describes $H^2(X;\mathbb{Z})$ stably $\mathbb{Z}G$–isomorphic to an extension of $\omega_G^3(\mathbb{Z})$ and $\omega_G^{-3}(\mathbb{Z})$. For $k = \mathbb{C}$, the Hodge decomposition yields $H^2(X,\mathbb{C}) \cong H^{2,0}(X;\mathbb{C}) \oplus H^{0,2}(X;\mathbb{C}) \oplus H^{1,1}(X,\mathbb{C})$. Since $H^{2,0}$ and $H^{0,2}$ are dual to each other, and $\Omega^2(X) \cong H^{2,0}(X;\mathbb{C})$ the above result implies that from

$H^2(X;\mathbb{C}) \cong \Omega^2(X) \oplus \mathrm{Hom}(\Omega^2(X),\mathbb{C}) \oplus \mathbb{C}[G]^t \cong \omega_G^3(\mathbb{C}) \oplus \mathrm{Hom}(\omega_G^3(\mathbb{C}),\mathbb{C}) \oplus \mathbb{C}[G]^t$ we conclude $H^{1,1}(X;\mathbb{C}) \cong \mathbb{C}[G]^t$.

6.3 Problem. Compute the $\mathbb{Z}G$–lattices $H^{i,j}(X,\mathbb{C}) \cap H^{i+j}(X;\mathbb{Z})$.

Proof. Consider the Cech complex $C^* = C^*(\mathcal{U})$ of kG–modules for the coherent sheaf $\mathcal{O}_X$ in which $H^0(C^*) = k$, $H^1(C^*) = H^1(\mathcal{U};\mathcal{O}_X) \cong H^1(X;\mathcal{O}_X)$ since $q(X) = p_g(X) - p_a(X) = \dim_k H^1(X;\mathcal{O}_X)$ and $q(X) = 0$ by assumption. Moreover, $H^i(C^*) = H^i(\mathcal{U};\mathcal{O}_X) \cong H^i(X;\mathcal{O}_X) = 0$ for $i > 2$, and $C^i = 0$ for i sufficiently large, since $\mathcal{U}$ is a finite cover. In the case of a complex analytic manifold, we could use the analytic topology, and choose $\mathcal{U}_0$ sufficiently refined until $f^{-1}(V_0) \cong G \times V_0$ is a free orbit of V_0 up to G–isomorphism. This would imply that the Cech complex C^* is a free G–complex. In the general case at hand, we have used Zariski open sets, and we need to resort to a somewhat different argument. Consider $\varphi : R \longrightarrow S$ such that ${}^a\varphi : \mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$ is the given étale covering $f_0 = f|V : V \longrightarrow V_0$, $V = f^{-1}(V_0)$. Then $V \times_{V_0} V$ admits a section, so that $V \times_{V_0} \cong G \times V$ as V_0–schemes with free G–actions. Therefore, $S \otimes_R S$ is a free $k[G]$–module. Consider the kG–isomorphisms: $S \otimes_k S \cong S \otimes_R (R \otimes_k S) \cong S \otimes_R (S \otimes_k R) \cong (S \otimes_R S) \otimes_k R$ which shows that $S \otimes_k S$ is also kG–free. This implies, in particular, that S is kG–projective. Hence $C^0(\mathcal{U})$ is a projective kG–module. A similar argument applies to show that $C^i(\mathcal{U})$ is kG–projective. Consider the dual chain complex $C_* = \mathrm{Hom}_k(C^*,k)$ of kG–projective modules, in which $H_0(C_*) \cong k$ and $H_2(C_*) = \mathrm{Hom}_k(\Omega^2(X),k)$ are the only non–vanishing homology groups. It follows that $B_2 = \mathrm{Im}\, \partial_2 \subset C_2$ is projective over kG, since $0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots\ldots \longrightarrow C_3 \longrightarrow B_2 \longrightarrow 0$ is exact for some sufficiently large n. Moreover, $\mathbb{Z}_2 = \mathrm{Ker}\, \partial_2 \cong \omega_G^3(k)$ in view of the exact sequence:

$$0 \longrightarrow Z_2 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 \longrightarrow k \longrightarrow 0 \ .$$

The exact sequence $0 \longrightarrow B_2 \longrightarrow Z_2 \longrightarrow H_2(C_*) \longrightarrow 0$, splits, since kG is injective. Therefore, $H_2(C_*) \sim Z_2 = \omega_G^3(k)$ is an stable kG–isomorphism. Hence $\Omega^2(X) = H^2(C^*) = \mathrm{Hom}_k(\omega_G^3(k),k) = \omega_G^{-3}(k)$ as claimed. ◻

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