The combinatorics of Harish–Chandra bimodules

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Hochgeehrter Herr Professor!

Schon seit recht langer Zeit strebten wir danach, Ihnen die Ergebnisse, die wir in der von Ihnen geschaffenen Theorie der topologischen Räume gefunden haben, mitzuteilen. Wir erlauben uns die Hoffnung auszusprechen, dass Sie die Gefälligkeit haben werden, uns zu gestatten, hier einige derselben zu nennen. Ein Teil der gewonnenen Resultate haben wir neuerdings in drei Noten („Bull. Internat. de l'Académie Polonaise" 1923) ohne Beweise formuliert; sie bilden die Anfangszüge einer Theorie, deren Darstellung die Redaktion der Zeitschrift „Fundamenta Mathematicae" von uns zu erhalten erwünscht hat.

Das Wesen der kompakten topologischen Räume ist das erste, was wir einer systematischen Untersuchung unterwerfen wollten. In dieser Hinsicht hatten wir zuerst die sogenannten bikompakten Räume herauszuheben, die durch eine jede der drei folgenden äquivalenten Eigenschaften charakterisiert werden können:

1°. Eine jede abnehmende wohlgeordnete Menge nichtleerer abgeschlossener Mengen besitzt einen nichtleeren Durchschnitt.

2°. Eine jede unendliche Menge $\mathfrak{M}$ besitzt wenigstens einen vollständigen Häufungspunkt ξ (d. i. dass $\mathfrak{D}(\mathfrak{M}, U_\xi)$ dieselbe Mächtigkeit wie $\mathfrak{M}$ hat, welche auch die Umgebung U_ξ von ξ sein möge)

3°. Der verschärfte Borelsche Satz (vgl. Satz VI, S. 272 Ihrer „Grundzüge")

Die bikompakten Räume besitzen mehrere bemerkenswerte Eigenschaften, sowohl mengentheoretischer, als topologischer Natur. Insbesondere sei auf Folgendes hingewiesen: Jede perfekte Menge besitzt daselbst die Mächtigkeit $\geq 2^{\aleph_0}$. Sie besitzt insbesondere genau die Mächtigkeit $2^{\aleph_0}$, wenn im Raume jedes F ein G_δ ist. Die letztere Bedingung (immer in bikompakten Räumen), die keineswegs dem Axiome „F" (II. Abzählbarkeitsaxiom) äquivalent ist, wohl aber aus dem letzten folgt, hat zur Folge das Axiom „E"; sie genügt um mehrere Mächtigkeitsfragen zu erledigen; z. B. lässt sich, unter der erwähnten Bedingung, jede abgeschlossene Menge in zwei Mengen zerspalten, deren eine perfekt,

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1 Introduction

1.1 The object of study

Let $\mathbf{g}$ be a complex semisimple Lie algebra and $\mathbf{U} = \mathbf{U}(\mathbf{g})$ its enveloping algebra. On any U-bimodule $M \in \mathbf{U} - mod - \mathbf{U}$ we define the adjoint $\mathbf{g}$-action $ad : \mathbf{g} \rightarrow End_{\mathbf{C}}M$ via $(adX)m = Xm - mX \forall X \in \mathbf{g}, m \in M$. A bimodule is called "locally $ad\mathbf{g}$-finite" if and only if any $m \in M$ is "$ad\mathbf{g}$-finite", i.e. contained in a finite dimensional $ad\mathbf{g}$-stable subspace.

In this article we study the category $\mathcal{HC}$ of all U-bimodules $M \in \mathbf{U} - mod - \mathbf{U}$ which are (1) locally $ad\mathbf{g}$-finite and (2) of finite lenght as bimodules. These are the Harish-Chandra bimodules of the title.

1.2 Motivation

The representation theory of complex semisimple Lie groups like $G = SL(n, \mathbf{C})$ leads one naturally to study such bimodules with $\mathbf{g} = LieG = sl(n, \mathbf{C})$. In the following discussion of how this comes about we will often want to forget the complex structure on $\mathbf{g}$ and regard it just as a real Lie algebra. In these instances we denote it by $\mathbf{g}^r$.

Let $\pi : G \to Aut_{\mathbf{C}}E$ be an admissible representation of G in a complex Banach space E. We choose a maximal compact subgroup K in G, like $K = SU(n)$, with Lie algebra $\mathbf{k} = Lie K \subset \mathbf{g}^r$. On the K-finite vectors

$$E_K = \{v \in E \mid dim \mathbf{C} K v < \infty\}$$

of E acts $\mathbf{g}^r$ in a natural way. This space E_K with the actions of K and $\mathbf{g}^r$ is called the Harish-Chandra module of E. The $\mathbf{R}$-linear action of $\mathbf{g}^r$ on E_K leads to a $\mathbf{C}$-linear action of $\mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C}$ on E_K whose restriction to $\mathbf{k} \otimes_{\mathbf{R}} \mathbf{C}$ is locally finite.

Now $\mathbf{k} \subset \mathbf{g}^r$ consists just of the fixed points in $\mathbf{g}^r$ of some Cartan involution $\theta : \mathbf{g}^r \to \mathbf{g}^r$, given in our example by $\theta(A) = -\bar{A}^t$. We may choose an isomorphism of complex Lie algebras $\mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{g} \times \mathbf{g}$ such that $\theta \otimes_{\mathbf{R}} \mathbf{C}$ corresponds to switching the two components $(X, Y) \mapsto (Y, X)$ of $(X, Y) \in \mathbf{g} \times \mathbf{g}$. Then $\mathbf{k} \otimes_{\mathbf{R}} \mathbf{C}$ corresponds to the diagonal in $\mathbf{g} \times \mathbf{g}$. Now $\mathbf{U}(\mathbf{g} \times \mathbf{g}) \cong \mathbf{U} \otimes \mathbf{U}$ canonically (we always write $\otimes_{\mathbf{C}} = \otimes$) and the principal antiautomorphism $X \mapsto -X$ of $\mathbf{g}$ leads to an isomorphism $\mathbf{U} \to \mathbf{U}^{opp}$. Thus we have canonically

$$\begin{aligned} \mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C} - mod &\cong \mathbf{g} \times \mathbf{g} - mod \\ &\cong \mathbf{U} \otimes \mathbf{U} - mod \\ &\cong \mathbf{U} \otimes \mathbf{U}^{opp} - mod \\ &\cong \mathbf{U} - mod - \mathbf{U} \end{aligned}$$

and clearly via this equivalence $\mathbf{k} \otimes_{\mathbf{R}} \mathbf{C}$-locally finite $\mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C}$-modules correspond to $ad\mathbf{g}$-locally finite $\mathbf{U}$-bimodules.

The reason we prefer to work with $\mathbf{U}$-bimodules rather than with $\mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C}$-modules is that such bimodules can be tensored with each other as well as with arbitrary $\mathbf{U}$-modules. These operations are of great importance and would look awkward when expressed in terms of $\mathbf{g}^r \otimes_{\mathbf{R}} \mathbf{C}$-modules.

So from any admissible representation E of G we obtain via a differentiation process followed by some algebraic manipulations a locally $ad\mathbf{g}$-finite $\mathbf{U}$-bimodule E_K. One shows that E is irreducible if and only if E_K is, and that for E a principal series still E_K has finite lenght, i.e. is an object of $\mathcal{HC}$.

Consider for example the action of $G = SL(n, \mathbf{C})$ on the full flag variety $F = \{\mathbf{C}^n = V^n \supset V^{n-1} \supset \ldots \supset V^0 = 0 \mid dim V^i = i\}$. It induces an action of G on the Banach space $E = L^\infty(F)$ of continuous functions $F \to \mathbf{C}$. This is a principal series representation. The corresponding bimodule E_K is the "$ad\mathbf{g}$-finite dual of $\mathbf{U}/Z^+\mathbf{U}$" which we define presently. Namely we denote by $Z \subset \mathbf{U}$ the center and let $Z^+ = Ann_Z\mathbf{C}$ be the central annihilator of the trivial representation $\mathbf{C}$ of $\mathbf{g}$. Then $\mathbf{U}/Z^+\mathbf{U}$ is a $\mathbf{U}$-bimodule, and so is its (algebraic) dual. The $ad\mathbf{g}$-finite dual is now the subspace of all $ad\mathbf{g}$-finite vectors in the algebraic dual space.

Certainly a central problem in representation theory is to compute the composition factors of principal series representations or, equivalently, of their duals, i.e. of $\mathbf{U}$-bimodules like $\mathbf{U}/Z^+\mathbf{U}$. This problem is solved by the Kazhdan-Lusztig conjectures, which by now are a theorem due to Beilinson-Bernstein and Brylinski-Kashiwara [BB, BK, Sp].

We approach this problem from another side, translating it down roughly speaking to a statement on Z-bimodules. Although this translated problem looks much easier than the original one, we have to invoke the Kazhdan-Lusztig conjectures to solve it. Nevertheless the method has the benefit of allowing deeper insight in the structure of the category $\mathcal{HC}$ and thus ultimately of principal series representations.

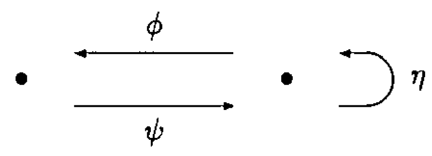
1.3 Example

Take $\mathbf{g} = sl(2, \mathbf{C})$ and consider in $\mathcal{HC}$ the subcategory

$$\mathcal{H} = \{M \in \mathcal{HC} \mid (Z^+)^n M = M(Z^+)^n = 0 \text{ for } n \gg 0\}.$$

Our dual principal series $\mathbf{U}/Z^+\mathbf{U}$ lies in this category as well as the trivial bimodule $\mathbf{C}$. There is an obvious surjection $\phi : \mathbf{U}/Z^+\mathbf{U} \to \mathbf{C}$, whose kernel $L = ker\phi$ can be shown to be irreducible. In fact, up to isomorphism L and $\mathbf{C}$ are the only irreducible objects in $\mathcal{H}$.

As was shown by Gelfand-Ponomarev [GP], the $\mathbf{C}$-category $\mathcal{H}$ is equivalent to the category of finite dimensional complex representations of the quiver



with relations $\eta\psi = 0 = \phi\eta$ and η, $\phi\psi$ nilpotent. This approach to the combinatorics of $\mathcal{H}$ is certainly most clear and beautiful. It has been generalized to Lie algebras of rank two by Irving [Ir], but it seems hard to go further.

The approach followed in this paper does not look quite as neat for $\mathfrak{g} = sl(2, \mathbf{C})$ but generalizes to arbitrary $\mathfrak{g}$. The description of $\mathcal{H}$ looks now as follows: We consider the complex plane $\mathbf{C}^2$ with coordinate functions X and Y, so that the ring of all regular functions on $\mathbf{C}^2$ is $R(\mathbf{C}^2) = \mathbf{C}[X, Y]$. Then inside $\mathbf{C}^2$ we consider the diagonal $\Delta_e = \{(x, x)\}$, the other diagonal $\Delta_s = \{(x, -x)\}$ and their union $\Delta_e \cup \Delta_s$. The regular functions $R(\Delta_e)$, $R(\Delta_s)$ and $R(\Delta_e \cup \Delta_s)$ on these sets are modules over $R(\mathbf{C}^2) = \mathbf{C}[X, Y]$ and we form

$$A = End_{\mathbf{C}[X,Y]}(R(\Delta_e) \oplus R(\Delta_e \cup \Delta_s)).$$

Obviously $\mathbf{C}[X, Y] \subset A$. We will show that $\mathcal{H}$ is equivalent to the category of all finite dimensional A-modules on which X and Y act nilpotently. To explain how this generalizes to arbitrary $\mathfrak{g}$, we need some results on Hecke algebras.

1.4 Hecke algebras and bimodules

Let $(\mathcal{W}, \mathcal{S})$ be any Coxeter system. For simplicity assume $\mathcal{S}$ to be finite. We have the Hecke algebra $\mathbf{H} = \mathbf{H}(\mathcal{W}, \mathcal{S}) = \bigoplus_{x \in \mathcal{W}} \mathbf{Z}[t, t^{-1}]\mathbf{T}_x$ as in [KL]. The multiplication is given by the formulas $\mathbf{T}_x \mathbf{T}_y = \mathbf{T}_{xy}$ for all $x, y \in \mathcal{W}$ such that $l(x) + l(y) = l(xy)$ and $\mathbf{T}_s^2 = t^2 + (t^2 - 1)\mathbf{T}_s$ for all $s \in \mathcal{S}$. Let E be the geometric representation of the Coxeter group $\mathcal{W}$ defined in [Bou], Ch.5, §4, and let $V = E \otimes_{\mathbf{R}} \mathbf{C}$ be its complexification. Let $S = S(V^*) = R(V)$ be the symmetric algebra in V^* alias the regular functions on V. This is given a grading such that $degV^* = 2$, thus $S = \bigoplus_{i \geq 0} S^i$ with $S^i = 0$ for odd i, $S^0 = \mathbf{C}$, $S^2 = V^*$.

For any additive category $\mathcal{A}$ form the split Grothendieck group $< \mathcal{A} >$. This is the free abelian group on the objects modulo the usual relations for each split short exact sequence. Any $A \in \mathcal{A}$ defines $< A > \in < \mathcal{A} >$. We consider the category $S - Molf - S$ of graded S-bimodules which are finitely generated as left S-modules and write $Hom_{S \otimes S}$ for bimodule homomorphisms. The group $< S - Molf - S >$ forms even a ring under $\otimes_S$.

For any graded object $M = \bigoplus M^i$ define the shifted objects $M(n)$ by $(M(n))^i = M^{i-n}$. For any $s \in \mathcal{S}$ consider the s-invariants $S^s \subset S$.

Theorem 1 *There is a ring homomorphism* $\mathcal{E} : \mathbf{H} \to < S - Molf - S >$ *such that* $\mathcal{E}(t) = < S(1) >$, $\mathcal{E}(\mathbf{T}_s + 1) = < S \otimes_{S^s} S > \forall s \in \mathcal{S}$.

Remember Kazhdan and Lusztig [KL] defined a new basis $\{\mathbf{C}'_x\}_{x\in\mathcal{W}}$ of $\mathbf{H}$ over $\mathbf{Z}[t,t^{-1}]$. The following theorem is one of the main results. It is proved in section 4.

Theorem 2 *Suppose $\mathcal{W}$ is cristallographic [Bou], i.e. a Weyl group.*

1. *For all $x \in \mathcal{W}$ there are objects $\mathbf{B}_x \in S - Molf - S$, well defined up to isomorphism, such that $\mathcal{E}(\mathbf{C}'_x) =< \mathbf{B}_x >$.*
2. *The $\mathbf{B}_x$ are indecomposable.*
3. *Form the graded algebra $A = A(\mathcal{W}, \mathcal{S}; V) = End_{S\otimes S}(\bigoplus_x \mathbf{B}_x)$. Then $A = \bigoplus_{i\geq 0} A^i$ lives only in positive degrees. Furthermore the projections 1_x onto the $\mathbf{B}_x$ form a basis of A^0.*
4. *The $Hom_{S\otimes S}(\mathbf{B}_x, \mathbf{B}_y)$ are graded free right S-modules of finite rank, via the right action of S on $\mathbf{B}_x$ or $\mathbf{B}_y$ equivalently. For any commutative (not necessarily graded) S-algebra S' the canonical map*

$$Hom_{S\otimes S}(\mathbf{B}_x, \mathbf{B}_y) \otimes_S S' \to Hom_{S\otimes S'}(\mathbf{B}_x \otimes_S S', \mathbf{B}_y \otimes_S S')$$

is an isomorphism. Analoguous statements hold from the left.

Remarks:

1. To see that the $\mathbf{B}_x$ are well defined the reader should prove a Krull-Remak-Schmidt theorem for finitely generated graded modules over polynomial rings.
2. Together 3.) and 4.) imply even a much stronger statement than 2.). Namely, if we consider the S-algebra $\mathbf{C} = S^0$ then $\mathbf{B}_x \otimes_S \mathbf{C}$ is an indecomposable S-module even if we forget about grading.
3. The theorem would imply the Kazhdan-Lusztig conjectures. Thus it is a pity we need these conjectures to prove it. In fact I conjecture the theorem to hold without the assumption that $\mathcal{W}$ is a Weyl group.
4. In case $\mathbf{C}'_x = t^{-l(x)} \sum_{y\leq x} \mathbf{T}_y$ the bimodule $\mathbf{B}_x$ has a very simple description. Namely consider for any $y \in \mathcal{W}$ the twisted diagonal $\Delta_y = \{(yv, v)\}$ in $V \times V$. The regular functions $R(\Delta_{\leq x})$ on $\Delta_{\leq x} = \cup_{y\leq x}\Delta_y$ form a graded module over $R(V \times V) = S \otimes S$. If we consider this as

an S-bimodule and shift the grading down by $l(x)$ we obtain $\mathbf{B}_x$. In formulas, $\mathbf{B}_x \cong R(\Delta_{\leq x})(-l(x))$. For arbitrary x still $\mathbf{B}_x$ has support in the closed reduced subscheme $\Delta_{\leq x} \subset V \times V$.

5. In general, $\mathbf{C}'_x = t^{-l(x)} \sum_{z \leq x} P_{z,x}(t^2)\mathbf{T}_z$ and $Hom_{S\otimes S}(\mathbf{B}_x, \mathbf{B}_y)$ is free as a right S-module of rank $\sum_z P_{z,x}(1)P_{z,y}(1)$.

6. The last point of the theorem can be interpreted as follows: Consider A as an algebra over $\mathbf{C} \otimes S = S$, i.e. as a family of algebras over $SpecS$. Then the family A is flat and over the generic point it is just a sum of $|\mathcal{W}|$ matrix algebras of various sizes. On the contrary over the closed point $0 \in \mathbf{h}^* \subset SpecS$ our family A specializes to "the algebra of category $\mathcal{O}$", as section 1.6 will show.

1.5 Notations for categories of modules

For any $\mathbf{C}$-algebra R let $R{-}mod \supset R{-}mof \supset R{-}mod^e$ denote the categories of all R-modules, finitely generated R-modules and finite dimensional R-modules respectively. If R is graded, we denote by $R - Mod \supset R - Mof \supset R - Mod^e$ the analoguous categories of graded modules. These notations generalize in an obvious way to bimodules. We often identify $S - mod - R$ and $S \otimes R^{opp} - mod$ etc. If we require bimodules to be finitely generated from the left, we write $S - molf - R$ etc.

1.6 Harish-Chandra bimodules

Let us again go into the general situation. Let $\mathbf{g} \supset \mathbf{b} \supset \mathbf{h}$ be a complex semisimple Lie algebra, a Borel and a Cartan and $(\mathcal{W}, \mathcal{S})$ the associated Coxeter system. Let $\mathbf{U} \supset Z \supset Z^+$ be the enveloping algebra, its center and the kernel of the trivial central character. We will restrict our attention to the direct summand $\mathcal{H}$ of $\mathcal{HC}$ given by

$$\mathcal{H} = \{M \in \mathcal{HC} \mid (Z^+)^n M = M(Z^+)^n = 0 \text{ for } n \gg 0\}.$$

At the center of our interest is an exact functor $\mathbf{V} : \mathcal{H} \to \mathbf{C} - mod$. It can be characterized (up to nonunique isomorphism) by the property that it annihilates all irreducibles except those with maximal Gelfand-Kirillov dimension, and maps those (i.e. the irreducible principal series module $L \in$

$\mathcal{H}$) to a onedimensional vector space. Certainly the Z-actions on $X \in \mathcal{H}$ give rise to a Z-bimodule structure on $\mathbf{V}X$. We can (and will) thus always regard $\mathbf{V}$ as a functor $\mathbf{V} : \mathcal{H} \to Z - mod - Z$. We prove

Proposition 1 *There is a natural equivalence* $\mathbf{V}(X \otimes_{\mathbf{U}} Y) \cong \mathbf{V}(X) \otimes_Z \mathbf{V}(Y)$ *of functors* $\mathcal{H} \times \mathcal{H} \to Z - mod - Z$.

The category $\mathcal{H}$ has not enough projectives. However let $I \subset Z$ be a Z^+-primary ideal. Then the category $\mathcal{H}^I = \{X \in \mathcal{H} \mid XI = 0\}$ has enough projectives and one of our main results is:

Theorem 3 *Let* $Q \in \mathcal{H}^I$ *be projective. Then for any* $M \in \mathcal{H}$ *the functor* $\mathbf{V}$ *induces an isomorphism* $Hom_{\mathcal{H}}(M, Q) \to Hom_{Z \otimes Z}(\mathbf{V}M, \mathbf{V}Q)$.

Remark: Certainly the same holds when we replace $\mathcal{H}^I$ by ${}^I\mathcal{H} = \{X \in \mathcal{H} \mid IX = 0\}$. It does not hold however if we restrict the Z-actions on both sides simultaneously, i.e. for categories ${}^I\mathcal{H}^{I'}$ with general I, I'.

Let us consider $S = S(\mathbf{h}) = R(\mathbf{h}^*)$ and denote by $\xi : Z \to S$ be the Harish-Chandra homomorphism, characterized by $\xi(z) - z \in \mathbf{U}\mathbf{n}$ where $\mathbf{n}$ is the nilradical of $\mathbf{b}$. If we apply the preceding section with $V = \mathbf{h}^*$ it produces for us certain S-bimodules $\mathbf{B}_x$, $x \in \mathcal{W}$. Via the isomorphisms $Z/(Z^+)^n = S/(S^+)^n$ induced by ξ our ideal $I \subset Z$ with $I \supset (Z^+)^n$ produces an ideal $I_S \subset S$ with $I_S \supset (S^+)^n$. In the following theorem we use ξ to quietly restrict S-bimodules to Z-bimodules.

Theorem 4 *Let* P_x^I *be the indecomposable projectives of* $\mathcal{H}^I$, *suitably parametrized by* $x \in \mathcal{W}$. *Then* $\mathbf{V}P_x^I \cong \mathbf{B}_x/\mathbf{B}_x I_S$ *as* Z*-bimodules.*

The reason we are so interested in homomorphisms between projectives is that for any artinian $\mathbf{C}$-category $\mathcal{A}$ with a projective generator $P \in \mathcal{A}$ actually $Hom_{\mathcal{A}}(P, \ ) : \mathcal{A} \to mod^e - End_{\mathcal{A}}P$ is an equivalence of categories. Now recall the $S \otimes S$-algebra $A = A(\mathcal{W}, S; \mathbf{h}^*)$ from theorem 2. The two preceding theorems establish an equivalence of categories $\mathcal{H}^I \cong mod^e - A/(S \otimes I_S)$. If we put $I = Z^+$ the above theorems specialize to results of [So3].

1.7 Reformulation in the setup of projective functors

I want to rephrase these theorems in terms of projective functors. Put $\mathcal{M} = \{M \in \mathbf{g} - mod \mid \forall m \in M, \exists n \gg 0 \text{ such that } (Z^+)^n m = 0\}$ and consider the category $\mathcal{P}$ of projective functors $F : \mathcal{M} \to \mathcal{M}$ in the sense of [BG]. Let F_x, $x \in \mathcal{W}$ be the indecomposable ones, suitably parametrized such that $F_e = id$ and $F_{w_\circ}$ corresponds to the antidominant projective. Let $\widehat{Z}$ be the completion of Z at Z^+. It acts on $\mathcal{M}$. Thus a right and a left action of $\widehat{Z}$ on $\mathcal{P}$.

For any graded object M (bounded below) let $\widehat{M}$ be its completion "along the graduation". For example $\widehat{S}$ is the completion of S at S^+. Certainly ξ induces an isomorphism $\widehat{Z} = \widehat{S}$.

Theorem 5 *1. $End_{\mathcal{P}} F_{w_\circ} = S \widehat{\otimes_{S^{\mathcal{W}}}} S$ canonically. Namely the multiplication $\widehat{Z} \otimes \widehat{Z} \to End_{\mathcal{P}} F_{w_\circ}$ and the obvious map $\widehat{Z} \otimes \widehat{Z} \to S \widehat{\otimes_{S^{\mathcal{W}}}} S$ are both surjections with the same kernel.*

2. Identify $S \widehat{\otimes_{S^{\mathcal{W}}}} S - mod \subset \widehat{S} - mod - \widehat{S}$ as a full subcategory. The functor $\mathbf{V} = Hom_{\mathcal{P}}(F_{w_\circ}, \;) : \mathcal{P} \to \widehat{S} - mod - \widehat{S}$ is fully faithful. We have $\mathbf{V}(F \circ G) \cong \mathbf{V}F \otimes_{\widehat{S}} \mathbf{V}G$ for all $F, G \in \mathcal{P}$.

3. For a suitable parametrization of the F_x we have $\mathbf{V}F_x \cong \widehat{\mathbf{B}_x}$.

This theorem is merely a reformulation of the theorems in the preceding section and will not be proved.

1.8 Some extensions of perverse sheaves

For any complex algebraic variety X let $\mathcal{D}(X)$ be the derived category with bounded algebraically constructible cohomology of sheaves of complex vector spaces on X^{an}. For $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$ define the graded vector space $Hom^\bullet_{\mathcal{D}}(\mathcal{F}, \mathcal{G})$ with $Hom^i_{\mathcal{D}}(\mathcal{F}, \mathcal{G}) = Hom_{\mathcal{D}}(\mathcal{F}, \mathcal{G}[i])$. Let $H(X)$ be the cohomology ring of X with complex coefficients. We have the hypercohomology $\mathbf{H} : \mathcal{D}(X) \to H(X) - Mod$. Now let $G \supset B$ be connected complex algebraic groups with Lie algebras $\mathfrak{g} \supset \mathfrak{b}$. On $X = G/B \times G/B$ consider the diagonal G-action. The following theorem will be proved at the very end of this paper.

Theorem 6 *Let $\mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$ be both the intersection cohomology complex of the closure of some G-orbit. Then the canonical map*

$$Hom^\bullet_{\mathcal{D}}(\mathcal{F}, \mathcal{G}) \to Hom_{H(X)}(\mathbf{H}\mathcal{F}, \mathbf{H}\mathcal{G})$$

is an isomorphism of graded vector spaces.

Remarks:

1. The theorem will hold as well for $X = G/P \times G/Q$ with $P, Q \subset G$ any two parabolic subgroups.

2. In [So3] we computed the $\mathbf{H}\mathcal{F} \in H(X) - Mod$. I describe the result in the notations of this paper. Let us put $R = R(\mathbf{h}) = S(\mathbf{h}^*)$ and $C = C(\mathbf{h}^*, \mathcal{W}) = R/(R^+)^{\mathcal{W}} R$. The section on Hecke algebras produces for us certain $\mathbf{B}_x \in R - Mod - R$. The Borel picture gives us a surjection $R \otimes R \twoheadrightarrow H(X)$. Then $\mathbf{B}_x \otimes_R C = C \otimes_R \mathbf{B}_x \cong \mathbf{H}\mathcal{F}$ as R-bimodules, for suitable $x \in \mathcal{W}$ depending on $\mathcal{F}$.

1.9 A duality conjecture

I want to state a duality conjecture closely related to Beilinson-Ginsburg duality and motivated by unpublished work of Ginsburg. First some generalities. For any additive category $\mathcal{B}$ let $K^b(\mathcal{B})$ be the homotopy category of bounded complexes. For any abelian category $\mathcal{A}$ let $D^b(\mathcal{A})$ be the bounded derived category. Let $p(\mathcal{A}) \subset \mathcal{A}$ be the full additive subcategory of projective objects. Under suitable hypothesis canonically $K^b(p(\mathcal{A})) = D^b(\mathcal{A})$.

Now let $K \subset Z$ be the kernel of the composition $Z \to S \to C(\mathbf{h}, \mathcal{W})$. Then $\mathcal{H}^K = {}^K\mathcal{H}^K = {}^K\mathcal{H}$. Let $\mathcal{B} \subset C - mod^e - C$ (resp. $\tilde{\mathcal{B}} \subset C - Mod^e - C$) be the full additive subcategory generated by the $B_x = \mathbf{B}_x \otimes_S C$ with $x \in \mathcal{W}$ (resp. the $\mathbf{B}_x \otimes_S C(i)$ with $x \in \mathcal{W}, i \in \mathbf{Z}$). Then $\mathbf{V}$ induces an equivalence $p({}^K\mathcal{H}^K) \cong \mathcal{B}$ and we get an equivalence $D^b({}^K\mathcal{H}^K) \cong K^b(\mathcal{B})$ where $\otimes^L_{\mathbf{U}/K\mathbf{U}}$ corresponds to $\otimes_C$.

Now as we know already ${}^K\mathcal{H}^K \cong mod^e - A$ with $A = End_{C \otimes C}(\bigoplus_x B_x)$. This is a graded algebra and we put ${}^K\tilde{\mathcal{H}}^K = Mod^e - A$. I regard it as a mixed version of ${}^K\mathcal{H}^K$. Again $D^b({}^K\tilde{\mathcal{H}}^K) \cong K^b(\tilde{\mathcal{B}})$ and the tensor product is a functor $\otimes_C : K^b(\tilde{\mathcal{B}}) \times K^b(\tilde{\mathcal{B}}) \to K^b(\tilde{\mathcal{B}})$. I want to simplify notation and set $\tilde{\mathcal{T}} = K^b(\tilde{\mathcal{B}})$ the "bounded derived category of mixed Harish-Chandra bimodules" (with suitable restriction on the action of the center). It is equipped with shifts of complex-degree $[n]$, shifts of grading of graded modules (i) and convolution $\otimes_C$ to be denoted $*$ from now on.

On the dual side there ought to exist a triangulated (but not full) subcategory $\mathcal{G}$ in the bounded derived category $D(G/B \times G/B)$ of all mixed Hodge modules on this space (for more canonicity we should take here in fact the flag manifold of the Langlands dual group) such that (1) if we restrict objects of $\mathcal{G}$ to G-orbits we get semisimple complexes consisting of sums of Tate sheaves only and (2) $\mathcal{G}$ is stable under convolution and Tate twist. For M I denote by $M(2)$ the once Tate twisted object usually denoted by $M(1)$. Now construct $\tilde{\mathcal{G}}$ by formally adding a root of the Tate twist. Namely put $\tilde{\mathcal{G}} = \mathcal{G} \times \mathcal{G}$ and set $(M, N)(1) = (N(2), M)$. This is a triangulated category with automorphisms $(i), [n]$ and convolution $\circ$.

Conjecture 1 *There should be an equivalence of triangulated categories $\kappa : \tilde{\mathcal{T}} \cong \tilde{\mathcal{G}}$ such that $\kappa(M[n]) = (\kappa M)[n]$, $\kappa(M(i)) = (\kappa M)[-i](-i)$ and $\kappa(M * N) = (\kappa M) \circ (\kappa N)$. It should transform suitably shifted projective mixed Harish-Chandra bimodules to intersection cohomology complexes of closures of G-orbits.*

The equivalent categories $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{G}}$ with convolution could be regarded as something like a ring. Then the representation theory of real reductive Lie groups should be investigated as something like a module over this ring. This would lift the actions of Hecke algebras via projective functors or dually via convolution with intersection cohomology complexes to a higher structural level, and one may hope to be able to identify these two actions or, more precisely, their mixed versions, via some "Koszul duality" similar to the duality conjectured above.

Dreaming in another direction I want to mention that may-be (cf. [BGi]) even $A = A(\mathcal{W}, \mathcal{S}; \mathbf{h}^*)$ itself is a Koszul algebra, however with a grading different from the one we defined on it. In the $sl(2)$-case our grading gives on the quiver the loop arrow degree two and the two other arrows degree one, whereas in case we give all three arrows degree one we obtain a Koszul algebra.

1.10 Generalizations

To save time and indices I have not written this paper in the maximal possible generality. Let me nevertheless formulate the (slightly conjectural) results in full generality. One may define an exact functor $\mathbf{V} : \mathcal{HC} \to \mathbf{C} - mod$ characterized (up to non-unique isomorphism) by the property that it annihilates all irreducibles except those of maximal Gelfand-Kirillov dimension, and maps those to a onedimensional vector space.

Again this can and will be regarded as a functor $\mathbf{V} : \mathcal{HC} \to Z - mod - Z$ and proposition 1 continues to hold with $\mathcal{H}$ replaced by $\mathcal{HC}$, i.e. $\mathbf{V}(X \otimes_{\mathrm{U}} Y) = \mathbf{V}(X) \otimes_Z \mathbf{V}(Y)$ naturally. Theorem 3 continues to hold as well when we take for $\mathcal{H}$ any block of $\mathcal{HC}$ and for $I \subset Z$ any ideal of finite codimension. To generalize theorem 4 to the case of regular (but possibly non-integral) central character is also rather straightforward. Basically we ought to replace $\mathcal{W}$ by the integral Weyl group. To include singular central characters into the picture as well, we ought to first generalize the section on Hecke algebras and bimodules, but I think the paper is already thick enough.

1.11 Thanks

I thank Jens Carsten Jantzen and Henning Haahr Andersen for pointing out errors in a preliminary version.

2 Hecke algebras and bimodules

2.1 Realization of the Hecke algebra via bimodules

For any Coxeter system $(\mathcal{W}, \mathcal{S})$ the Hecke algebra

$$\tilde{\mathbf{H}} = \tilde{\mathbf{H}}(\mathcal{W}, \mathcal{S}) = \bigoplus_{x \in \mathcal{W}} \mathbf{Z}[q, q^{-1}]\mathbf{T}_x$$

is defined over $\mathbf{Z}[q, q^{-1}]$ by generators $\{\mathbf{T}_s\}_{s \in \mathcal{S}}$ and relations $\mathbf{T}_s^2 = (q-1)\mathbf{T}_s + q \forall s \in \mathcal{S}$, $\mathbf{T}_s\mathbf{T}_t \ldots \mathbf{T}_t = \mathbf{T}_t\mathbf{T}_s \ldots \mathbf{T}_s$ (resp. $\mathbf{T}_s\mathbf{T}_t \ldots \mathbf{T}_s = \mathbf{T}_t\mathbf{T}_s \ldots \mathbf{T}_t$) with n factors on both sides in case $s, t \in \mathcal{S}$ are distinct, st is of order n and n is even (resp. odd). Later we set $\mathbf{H} = \tilde{\mathbf{H}} \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Z}[t, t^{-1}]$ with $q = t^2$.

We assume from now on that $\mathcal{S}$ is finite. As in the introduction let V be the complexified geometric representation of $\mathcal{W}$, but since we work with $\tilde{\mathbf{H}}$ we have to grade S as usual, $S^1 = V^*$. So we are interested in the split Grothendieck group of left S-finite graded S-bimodules $< S - Molf - S >$. This group is even a ring under $\otimes_S$. We want to prove:

Theorem 7 *There is a ring homomorphism $\mathcal{E} : \tilde{\mathbf{H}} \to < S - Molf - S >$ such that $\mathcal{E}(q) = < S(1) >$, $\mathcal{E}(\mathbf{T}_s + 1) = < S \otimes_{S^s} S > \forall s \in \mathcal{S}$.*

Proof: Let us interpret $S - mod - S$ as the category of all quasicoherent sheaves on $V \times V$. Consider in $V \times V$ the twisted diagonals $\Delta_x = \{(xv, v) \mid v \in V\}$ for all $x \in \mathcal{W}$. For any subset $A \subset \mathcal{W}$ define $\Delta_A = \cup_{x \in A} \Delta_x$ and consider the ring of regular functions $R(A) = R(\Delta_A) \in S - Molf - S$ on Δ_A. For example, $R(x) \otimes_S R(y) \cong R(xy) \forall x, y \in \mathcal{W}$. The proof of the theorem relies on the following proposition:

Proposition 2 *Suppose $\mathcal{S} = \{s, t\}$ and $\sharp \mathcal{W} < \infty$, i.e. $\mathcal{W}$ is a finite dihedral group. Choose $x \in \mathcal{W}$ and set $A = \{w \leq x\} \subset \mathcal{W}$. Then in $S - Molf - S$ there is an isomorphism*

$$S \otimes_{S^s} R(A) \cong R(A \cup sA) \oplus R(A \cap sA)(1).$$

Remark: The assumption $\sharp \mathcal{W} < \infty$ should be superfluous.
Proof: Postponed to the next subsection.
We deduce the theorem. Without restriction of generality we assume that $\mathcal{W}$ is a dihedral group. Let us abbreviate notation and set $R(\leq x) = R(\{w \leq x\})$ for any $x \in \mathcal{W}$. Certainly we can define an homomorphism of abelian groups $\mathcal{E} : \tilde{\mathbf{H}} \to < S - Molf - S >$ by the prescription

$$\mathcal{E}(q^n \sum_{w \leq x} \mathbf{T}_w) = < R(\leq x)(n) > \forall x \in \mathcal{W}, n \in \mathbf{Z}.$$

Then $\mathcal{E}(q) = < S(1) >$ and $\mathcal{E}(\mathbf{T}_s + 1) = < R(\leq s) > = < S \otimes_{S^s} S >$ for all $s \in \mathcal{S}$, the latter equality by the proposition with $x = e$ the identity of $\mathcal{W}$.

We just have to show that this $\mathcal{E}$ is an algebra homomorphism. For this it is sufficient to check for all $x \in \mathcal{W}$, $s \in \mathcal{S}$ the equality

$$\mathcal{E}((\mathbf{T}_s + 1) \sum_{w \leq x} \mathbf{T}_w) = < S \otimes_{S^s} R(\leq x) > .$$

Now set again $A = \{w \leq x\}$. A short calculation in $\tilde{\mathbf{H}}$ shows that

$$(\mathbf{T}_s + 1) \sum_{w \leq x} \mathbf{T}_w = \sum_{w \in A \cup sA} \mathbf{T}_w + q \sum_{v \in A \cap sA} \mathbf{T}_v.$$

We compare with the above proposition and are through. *q.e.d.*
Further remarks to the theorem:

1. Suppose $\sharp \mathcal{W} < \infty$. As above let $R(\mathcal{W})$ denote the ring of regular functions on the union of all twisted diagonals. Then $\mathcal{E}$ factors over $< R(\mathcal{W}) - Mof >$.

2. Consider for any $\mathcal{F} \in S - mof - S$ and $x \in \mathcal{W}$ the dimension $d_x(\mathcal{F})$ of its (geometric) stalk at the generic point of Δ_x and define the "cycle map" $\mathcal{C} :< S - mof - S >\to \mathbf{Z}[[\mathcal{W}]]$ by $\mathcal{F} \mapsto \sum d_x(\mathcal{F})x$. Then the composition $\mathcal{C} \circ \mathcal{E} : \tilde{\mathbf{H}} \to \mathbf{Z}[[\mathcal{W}]]$ is the evaluation at $q = 1$ (and factors in particular over $\mathbf{Z}[\mathcal{W}] \subset \mathbf{Z}[[\mathcal{W}]]$).

2.2 Deformation of Schubert calculus

In this subsection I suppose always $\sharp\mathcal{W} < \infty$. To prove the proposition we first have to develop some generalities. Any reflection $s : V \to V$ gives $s : S \to S$. If we choose an equation $\alpha \in V^*$ of the reflecting hyperplane V^s, we may define the "twisted derivation" $\partial_s = \partial_s^\alpha : S \to S, f \mapsto (2\alpha)^{-1}(f - sf)$. If $X \subset V$ is closed and s-stable, then $s : S \to S$ induces $s : R(X) \to R(X)$ and $R(X)$ decomposes into eigenspaces $R(X) = R(X)^+ \oplus R(X)^-$. If in addition no irreducible component of X lies inside V^s then even $\partial_s : S \to S$ induces $\partial_s : R(X) \to R(X)$ and we see that multiplication by α and ∂_s are mutually inverse isomorphisms $R(X)^- \leftrightarrow R(X)^+$ compatible with the S^s-module structures.

Now instead of V let us consider $V \times V$, with the reflection $s \in \mathcal{W}$ acting only on the first factor. The above considerations give us $s, \partial_s : S \otimes S \to S \otimes S$ and even $s, \partial_s : R(A) \to R(A)$ in case $A \subset \mathcal{W}$ is s-stable. These are homomorphisms in $S^s - mod - S$.

Lemma 1 *Let $A \subset \mathcal{W}$ be s-stable. Then $S \otimes_{S^s} R(A) \cong R(A) \oplus R(A)(1)$ in $S - Mod - S$.*

Proof: $R(A) = R(A)^+ \oplus R(A)^-$, $S \otimes_{S^s} R(A)^+ \cong R(A)$ by multiplication and $\alpha : R(A)^+(1) \to R(A)^-$ is an isomorphism. *q.e.d.*

Consider the ring $R(\mathcal{W})$ of regular functions on the union of all twisted diagonals. Obviously the left and right actions of $S^{\mathcal{W}}$ on $R(\mathcal{W})$ coincide. Therefore a surjection $S \otimes_{S^{\mathcal{W}}} S \to R(\mathcal{W})$.

Lemma 2 *The surjection $S \otimes_{S^{\mathcal{W}}} S \to R(\mathcal{W})$ is an isomorphism.*

Proof: Let K be the kernel. Since $dim(S \otimes_{S^{\mathcal{W}}} Quot S) = \sharp\mathcal{W} = dim(R(\mathcal{W}) \otimes_S Quot S)$ we have $K \otimes_S Quot S = 0$. But $S \otimes_{S^{\mathcal{W}}} S$ is torsionfree as a right S-module, and so is K. This implies $K = 0$. *q.e.d.*

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition 3 *There is $\phi \in R(\mathcal{W})$ homogeneous of degree $l(w_\circ)$ such that $\phi|\Delta_x = 0 \Leftrightarrow x \neq w_\circ$.*

Proof[Proposition]: We start with some preparatory lemmata. For $x \in \mathcal{W}$ we choose a reduced expression $x = s_1 \cdots s_r, s_i \in \mathcal{S}$, and form $\partial_x = \partial_{s_1} \ldots \partial_{s_r} : R(\mathcal{W}) \to R(\mathcal{W})$. Following [BGG] the ∂_x are well defined up to scalars. They commute with the right S-action.

Lemma 3 *For all $f \in R(\mathcal{W})$ the element $\partial_{w_\circ} f$ belongs to the image of $1 \otimes S$ in $R(\mathcal{W})$.*

Proof: For all $h \in R(\mathcal{W})$ and $s \in \mathcal{S}$ the element $\partial_s h \in R(\mathcal{W})$ is fixed by s. So $\partial_{w_\circ} f$ is fixed by all $s \in \mathcal{S}$, hence by $\mathcal{W}$. This proves the lemma. *q.e.d.*

Lemma 4 *Let $I \subset R(\mathcal{W})$ be an ideal. Then $I + \partial_s I$ is an ideal as well, for all $s \in \mathcal{S}$.*

Proof: We need to show that $I + \partial_s I$ is stable under left and right multiplication by $f \in S$. For the right multiplication this is clear since ∂_s commutes with $(\cdot f)$. For the left multiplication use the formula $\partial_s(fm) = (\partial_s f)m + (sf)(\partial_s m) \forall f \in S,\ m \in R(\mathcal{W})$. *q.e.d.*

After these preparatory lemmas let us now prove the proposition. It will be important to distinguish $hg = h(1 \otimes g)$ and $gh = (g \otimes 1)h$ for $h \in R(\mathcal{W}), g \in S$. Choose $f \in R(\mathcal{W})$ homogeneous of degree d such that $f|\Delta_x = 0 \forall x \neq w_\circ$. Certainly fS is an ideal of $R(\mathcal{W})$, and using the preceding lemma repeatedly we find that $\sum_x (\partial_x f)S$ is an ideal of $R(\mathcal{W})$ as well.

Let $i : V \to \cup \Delta_x,\ v \mapsto (v, v)$ be the diagonal, $i^\sharp : R(\mathcal{W}) \to S$ the corresponding comorphism. I claim that

$$(*) \qquad \sum_x (\partial_x f)S = R(\mathcal{W}) i^\sharp(\partial_{w_\circ} f).$$

Here the inclusion $\supset$ is evident since $\partial_{w_\circ} f = 1 \cdot i^\sharp(\partial_{w_\circ} f)$ by lemma 3. If $f = 0$ equality is evident as well. If not, we need

Lemma 5 *Suppose $f \in R(\mathcal{W})$ is such that $f|\Delta_x \neq 0 \Leftrightarrow x = w_\circ$. Then $\partial_y f|\Delta_{yw_\circ} \neq 0$ and from $\partial_y f|\Delta_{xw_\circ} \neq 0$ follows $x \leq y$.*

Proof: From the definition of ∂_s we deduce (1) $f|\Delta_x = f|\Delta_{sx} = 0 \Rightarrow \partial_s f|\Delta_x = \partial_s f|\Delta_{sx} = 0$ and (2) $f|\Delta_x = 0, f|\Delta_{sx} \neq 0 \Rightarrow \partial_s f|\Delta_x \neq 0, \partial_s f|\Delta_{sx} \neq 0$. The lemma follows by induction. *q.e.d.*

So if $f|\Delta_x \neq 0 \Leftrightarrow x = w_\circ$ the $\partial_x f$ are linearly independent for the right S-action on $R(\mathcal{W})$ and the equality $(*)$ follows by counting dimensions in each degree. Thus indeed $\sum_x (\partial_x f)S = R(\mathcal{W})i^\sharp(\partial_{w_\circ} f)$. This says in particular that $f = \phi(\partial_{w_\circ} f)$ for suitable $\phi = \phi_f \in R(\mathcal{W})$. It is immediate that such a $\phi = \phi_f$ satisfies the conditions of the proposition. *q.e.d.*

The following proposition should be viewed as a deformation of classical Schubert calculus [BGG, De].

Proposition 4 *1. The space $\{f \in R(\mathcal{W}) \mid f|\Delta_x = 0$ if $x \neq w_\circ\}$ is a free right S-module of rank one, generated by a homogeneous element $\varphi \in R(\mathcal{W})$ of degree $l(w_\circ)$.*

2. The $\partial_x \varphi$ with $x \in \mathcal{W}$ form a basis of $R(\mathcal{W})$ when considered as a right S-module.

Proof[Proposition]: Let ϕ be as in proposition 3. Then $\partial_{w_\circ}\phi$ is not zero by lemma 5 and of degree zero, hence a scalar. The $\partial_x \phi$ are linearily independent for the right S-action, again by lemma 5, and they generate the right S-module $R(\mathcal{W})$, by equation $(*)$.

To establish the proposition, we prove first

Lemma 6 *For any $y \in \mathcal{W}$, the images in $R(\leq y)$ of the $\partial_x \phi, xw_\circ \leq y$ form a basis of this right S-module.*

Proof: This follows from three obvious facts: First $R(\leq y)$ is a quotient of $R(\mathcal{W})$, second $\partial_x \phi$ vanishes on Δ_z unless $xw_\circ \leq z$ and third $R(\leq y)$ is generically free of rank $|\{z \leq y\}|$ as a right S-module. *q.e.d.*

Now if $f \in R(\mathcal{W})$ vanishes on all Δ_x except $\Delta_{w_\circ}$, it is clear that $f = \phi h$ for suitable $h \in S$ (so in particular we can put $\varphi = \phi$.) *q.e.d.[Proposition]*

Finally we get at

Proof[Proposition 2]: Recall that in the proposition $\mathcal{W}$ was assumed to be a dihedral group. If $x > sx$ the proposition follows from lemma 1. If $x = e$ it follows from lemma 2. So suppose $x < sx, x \neq e$. Then $A - sA = \{x, rx\}$ with $r \in \mathcal{W}$ a reflection. Consider the subspace $\Delta_x + \Delta_{rx} \subset V \times V$. This is a hyperplane. Let $\beta \in S \otimes S$ be its equation. This is well defined up to a scalar.

Lemma 7 *$R(A)$ is generated as an object of S^s-mod-S by (the images of) β and* 1.

Proof: $R(A)$ is generated as an object of $S - mod - S$ by 1. So as an object of $S^s - mod - S$ by $\alpha = \alpha \otimes 1$ and 1. Since $deg\beta = 1$, the only thing we have to show is that β does not lie in the image of $(V^*)^s \otimes 1 + 1 \otimes V^*$. Now the orthogonal complement of the latter subspace of $(V \oplus V)^*$ is the line $V^- \oplus 0 \subset V \oplus V$. But this is not contained in $\Delta_x + \Delta_{rx}$. *q.e.d.*

Now consider the S-subbimodule of $R(A)$ generated by β. Certainly $\beta|\Delta_x = \beta|\Delta_{rx} = 0$. Remark that $\beta|\Delta_y \neq 0$ for all y other than x, rx. Indeed it is easy to see that $\Delta_y + \Delta_z = V \times V \Leftrightarrow \Delta_y \cap \Delta_z = 0 \Leftrightarrow V^{y^{-1}z} = 0 \Leftrightarrow y^{-1}z$ is neither the identity nor a reflection $\Leftrightarrow y \neq z$ but $det(y) = det(z)$. So $\Delta_y + \Delta_x + \Delta_{rx} = V \times V$ for any y other than x, rx and thus $\beta|\Delta_y \neq 0$ for all those y. Thus the S-subbimodule of R(A) generated by β has to be isomorphic to $R(A \cap sA)(1)$.

Let $M \subset R(A)$ be the subobject in $S^s - mod - S$ generated by β and consider the short exact sequence

$$S \otimes_{S^s} M \hookrightarrow S \otimes_{S^s} R(A) \twoheadrightarrow coker.$$

Using lemma 1 and glancing at its proof, $S \otimes_{S^s} M \cong R(A \cap sA)(1)$. Using lemma 1 again the $S \otimes S$-action on our three bimodules factors over $R(A \cup sA)$. Using lemma 7, *coker* is a cyclic $R(A \cup sA)$-module. Using lemma 6 to count dimensions in each degree, we see that even $coker \cong R(A \cup sA)$. Thus the sequence splits. *q.e.d.[Proposition 2]*

3 Deformation of projectives in category $\mathcal{O}$

3.1 Preliminaries concerning differential operators

Recall the notations $\mathbf{g} \supset \mathbf{b} \supset \mathrm{h}$, $S = S(\mathrm{h})$, $\mathrm{U} \supset Z$. We always write $\otimes_{\mathbf{C}} = \otimes$ and set $\mathbf{g}_R = \mathbf{g} \otimes R$ for any commutative C-algebra R. In this subsection only let $M = \mathrm{U} \otimes_{\mathrm{b}} S \in \mathbf{g}_S - mod$ be the "Verma sheaf on h^*". We want to prove:

Theorem 8 *Let $E, F \in \mathbf{g} - mod$ be of finite dimension.*

1. *$Hom_{\mathbf{g}_S}(E \otimes M, F \otimes M)$ is a free S-module of rank $dim(E \otimes F^*)^{\mathrm{h}}$.*

2. *For any commutative S-algebra S' the canonical homomorphism* $Hom_{\mathbf{g}_S}(E \otimes M, F \otimes M) \otimes_S S' \to Hom_{\mathbf{g}_{S'}}(E \otimes M \otimes_S S', F \otimes M \otimes_S S')$ *is an isomorphism.*

Proof[Theorem]: For any $\mathbf{g}_S$-module N let $End_S N$ be a $\mathbf{g}$-module via the adjoint action and let $\mathcal{E}nd_S N \subset End_S N$ be the subspace of $ad\mathbf{g}$-finite endomorphisms. We deduce the theorem from the following

Proposition 5 *For any commutative S-algebra S' the multiplication induces an isomorphism* $\mathbf{U} \otimes_Z S' \to \mathcal{E}nd_{S'}(\mathbf{U} \otimes_{\mathbf{b}} S')$.

Remark: If we put $S' = S/(ker\lambda)$ with $\lambda \in \mathbf{h}^*$, we obtain Joseph's description of the $ad\mathbf{g}$-finite endomorphisms of a Verma module ([Ja], 7.25).

Before we prove the proposition, let us deduce the theorem. Indeed, we get

$$\begin{aligned} Hom_{\mathbf{g}_S}(E \otimes M, F \otimes M) &= Hom_{\mathbf{g}}(E \otimes F^*, End_S M) \\ &= Hom_{\mathbf{g}}(E \otimes F^*, \mathbf{U} \otimes_Z S) \end{aligned}$$

and thus 1.) follows directly from Kostant's description of $\mathbf{U}$ as an Z-$ad\mathbf{g}$-module. I leave to the reader the (similar) proof that the proposition implies 2.). *q.e.d.[Theorem]*

Proof[Proposition]: First check that the multiplication $\mathbf{U} \otimes S' \to End_{S'}(\mathbf{U} \otimes_{\mathbf{b}} S')$ factors over $\mathbf{U} \otimes_Z S'$. We may assume $S' = S$ and then this follows from the definition of the Harish-Chandra homomorphism $\xi : Z \to S$.

To prove that our map is an isomorphism we need some geometry. Let $G \supset B \supset H$ be connected algebraic groups with Lie algebras $\mathbf{g} \supset \mathbf{b} \supset \mathbf{h}$ and consider the principal H-bundle $\pi : G/(B,B) \to G/B$, with the right H-action given by $g(B,B)t = gt(B,B) \forall t \in H$. Denote it by $\pi : Y \to X$. This H-bundle is G-equivariant. Let $\mathcal{D}_Y$ be the sheaf of algebraic differential operators on Y.

Now the "relative enveloping algebra" $\mathcal{U} = (\pi_* \mathcal{D}_Y)^H$ is an sheaf of S-algebras on X (see [BB2]) and the operator representation $\mathbf{U} \to \Gamma(Y, \mathcal{D}_Y)$ gives us an algebra homomorphism $\mathbf{U} \to \Gamma(X, \mathcal{U})$. More generally, $\mathcal{U}' = \mathcal{U} \otimes_S S'$ is an sheaf of S'-algebras on X and we get an algebra homomorphism $\alpha : \mathbf{U} \otimes S' \to \Gamma(X, \mathcal{U}')$.

By local considerations the geometric stalk $\mathcal{U}'/\mathcal{U}'\mathbf{m}_e$ of $\mathcal{U}'$ at $e = B \in G/B = X$ is a faithful module over $\Gamma(X, \mathcal{U}')$. The constant differential operator 1 on Y leads to $v \in \mathcal{U}'/\mathcal{U}'\mathbf{m}_e$. Restrict $\mathcal{U}'/\mathcal{U}'\mathbf{m}_e$ to $\mathbf{U} \otimes S'$ via α. Universal properties give us a $\mathbf{U} \otimes S'$-morphism $\mathbf{U} \otimes_{\mathbf{b}} S' \to \mathcal{U}'/\mathcal{U}'\mathbf{m}_e$ such that $1 \otimes 1 \mapsto v$. This can be checked to be an isomorphism. We deduce that α factors through $\beta : \mathbf{U} \otimes_Z S' \to \Gamma(X, \mathcal{U}')$.

To prove the proposition, consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{U} \otimes_Z S' & \to & \mathcal{E}nd_{S'}(\mathbf{U} \otimes_{\mathbf{b}} S') \\ \beta \downarrow & & \| \\ \Gamma(X, \mathcal{U}') & \xrightarrow{\gamma} & \mathcal{E}nd_{S'}(\mathcal{U}'/\mathcal{U}'\mathrm{m}_e) \end{array}$$

with the horizontal arrows given by multiplication. We need only show that β and γ are isomorphisms.

We start out with β. In case $S = S'$ it is well known to be an isomorphism, see e.g. [So2]. To prove the general case we just need to know that $\Gamma(X, \mathcal{U} \otimes_S S') = \Gamma(X, \mathcal{U}) \otimes_S S'$. Now for any S-module N the cohomology groups $H^\nu(X, \mathcal{U} \otimes_S N)$ vanish for $\nu > 0$, the argument being the same as in the special case $N = S$ treated in [So2]. Thus the two functors $N \mapsto \Gamma(X, \mathcal{U} \otimes_S N)$ and $N \mapsto \Gamma(X, \mathcal{U}) \otimes_S N$ are both exact. Since the natural transformation from the second to the first is obviously an isomorphism in case $N = S$, it has to be an isomorphism in general. So indeed β is an isomorphism.

Finally we have to show that γ is an isomorphism. The argument is a new version of [So2], 3.4. Consider the commutative diagram

$$\begin{array}{ccl} \Gamma(X, \mathcal{U}') & \ni & D \mapsto \phi_D(g) = (g^{-1}D)(v) \\ & & \searrow \\ \gamma \downarrow & & \{\phi : G \to \mathcal{U}'/\mathcal{U}'\mathrm{m}_e \mid \phi \text{ regular}, \phi(gb) = b^{-1}\phi(g) \forall g \in G, b \in B\} \\ & & \nearrow \\ \mathcal{E}nd_{S'}(\mathcal{U}'/\mathcal{U}'\mathrm{m}_e) & \ni & f \mapsto \phi_f(g) = (g^{-1}f)(v) \end{array}$$

Here we have quietly integrated the adjoint $\mathbf{g}$-action on $\mathcal{E}nd_{S'}(\mathcal{U}'/\mathcal{U}'\mathrm{m}_e)$ to a G-action - to do this without thinking, assume G simply connected.

Certainly $\mathbf{g}$ acts on $\mathcal{U}'/\mathcal{U}'\mathrm{m}_e = \mathbf{U} \otimes_{\mathbf{b}} S'$. On the other hand, B acts on $\mathcal{U}'/\mathcal{U}'\mathrm{m}_e$ as this is the stalk of the G-equivariant right $\mathcal{O}_X$-module $\mathcal{U}'$ at the point $e \in X$ fixed by B. This action is given by $b(u \otimes s) = (Ad(b)(u)) \otimes s$, as the reader may check himself. However on $\mathcal{E}nd_{S'}(\mathcal{U}'/\mathcal{U}'\mathrm{m}_e)$ the differential of the B-action coincides with the restriction of the $\mathbf{g}$-action to $\mathbf{b}$. This proves that ϕ_f has the correct transformation property.

We have already seen that γ is injective. For any G-equivariant right $\mathcal{O}_X$-module $\mathcal{U}'$ the map $D \mapsto \phi_D$ is a bijection. To force γ to be bijective, it suffices to show that $f \mapsto \phi_f$ is injective. But $\phi_f \equiv 0 \Rightarrow (uf)(v) = (u\phi_f)(e) = 0$ for all $u \in \mathbf{U}$. If $f \neq 0$ there is some $u \in \mathbf{U}$ such that $f(uv) \neq 0$, since f is S'-linear. Assume this u to be of minimum possible degree. Then $(\check{u}f)(v) = f(uv) \neq 0$ and this contradiction proves injectivity of $f \mapsto \phi_f$. *q.e.d.[Proposition]*

3.2 Deformation of projectives

Consider the classical category $\mathcal{O} = \mathcal{O}(\mathbf{g}, \mathbf{b}) = \{M \in \mathbf{g} - mof \mid M$ is locally finite over $\mathbf{b}$ and semisimple over $\mathbf{h}\}$. For any $\lambda \in \mathbf{h}^*$ consider the Verma module $M(\lambda) = \mathbf{U} \otimes_{\mathbf{b}} \mathbf{C}_\lambda$, its unique irreducible quotient $L(\lambda)$ and the projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{O}$. Let $\mathbf{h}^* \supset R \supset R^+ \supset \Delta$ be the dual of $\mathbf{h}$, the roots of $\mathbf{g}$, the roots of $\mathbf{b}$ and the simple roots. Let $P(R) \subset \mathbf{h}^*$ be the weight lattice.

Under the action of $\mathbf{h}$ our category $\mathcal{O}$ decomposes into $\mathcal{O} = \bigoplus \mathcal{O}_\Lambda$ where Λ runs over all shifted weight lattices $\Lambda \in \mathbf{h}^*/P(R)$. Let $\rho \in \mathbf{h}^*$ be the half sum of positive roots. Set $\Lambda^+ = \{\lambda \in \Lambda \mid <\lambda + \rho, \check{\alpha}> \notin \{-1, -2, \ldots\} \forall \alpha \in R^+\}$. Under the action of $Z \subset \mathbf{U}$ and using once more the action of $\mathbf{h}$, the $\mathcal{O}_\Lambda$ decompose further into $\mathcal{O}_\Lambda = \bigoplus \mathcal{O}_\lambda$ where λ runs over Λ^+.

The $\mathcal{O}_\lambda$ cannot be decomposed further. Let the dot action of $\mathcal{W}$ on $\mathbf{h}^*$ be defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$. The simple objects of $\mathcal{O}_\lambda$ are precisely the $L(\mu)$ with $\mu \in (\mathcal{W} \cdot \lambda) \cap (\lambda + \mathbf{Z}R)$. For any $\lambda, \mu \in \Lambda^+$ there is a translation functor $\theta_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ (see [Ja]).

Now let $T = S_{(0)}$ be the local ring of $SpecS$ at $0 \in \mathbf{h}^* \subset SpecS$. We are going to define for any $\Lambda \in \mathbf{h}^*/P(R)$ a full additive subcategory $\mathcal{D}_\Lambda \subseteq \mathbf{g}_T - mof$ along with a decomposition $\mathcal{D}_\Lambda = \bigoplus_{\lambda \in \Lambda^+} \mathcal{D}_\lambda$ and translation functors $\theta_\lambda^\mu : \mathcal{D}_\lambda \to \mathcal{D}_\mu$ for any two $\lambda, \mu \in \Lambda^+$. I think of the objects of $\mathcal{D}_\Lambda$ (resp. $\mathcal{D}_\lambda$) as deformations of projectives in $\mathcal{O}_\Lambda$ (resp. $\mathcal{O}_\lambda$).

Let us give the definitions! I will be rather short, since most of the material is treated in [So3]. First define for any $\lambda \in \mathbf{h}^*$ the "deformed Verma" $M_\lambda \in \mathbf{g}_T - mof$ by $M_\lambda = \mathbf{U} \otimes_{\mathbf{b}} (\mathbf{C}_\lambda \otimes T)$. Here $\mathbf{b}$ acts on $\mathbf{C}_\lambda$ as usual, on T via $\mathbf{b} \to \mathbf{h} \to S \to T$ and on $(\mathbf{C}_\lambda \otimes T)$ via the tensor product action. Then let $\mathcal{D}(\lambda) \subset \mathbf{g}_T - mof$ be the full subcategory consisting of all direct summands of modules of the form $E \otimes M_\lambda$ with $E \in \mathbf{g} - mod^e$.

Now $\mathcal{D}(\lambda)$ decomposes under the action of $Z \otimes T$. Namely, for any maximal ideal $\chi \subset Z$ denote by $\mathcal{D}_\chi(\lambda)$ the category of all $M \in \mathcal{D}(\lambda)$ such that $suppM \subset Spec(Z \otimes T)$ has $(\chi, \mathbf{m})$ as its unique closed point, where $\mathbf{m} \subset T$

is the maximal ideal. On closed points the Harish-Chandra homomorphism gives $\xi : \mathbf{h}^* \to MaxZ$. In this notation $\mathcal{D}(\lambda) = \bigoplus_{\chi\in\xi(\Lambda)} \mathcal{D}_\chi(\lambda)$ with $\Lambda = \lambda + P(R)$.

On the other hand $\mathcal{D}(\lambda)$ decomposes under the action of $\mathbf{h} \otimes T$. Namely, for any $\mu \in \mathbf{h}^*$ and $M \in \mathbf{g}_T - mod$ define the μ-eigenspace $M^\mu \subset M$ by $M^\mu = \{v \in M \mid Xv = (X + \mu(X))v \forall X \in \mathbf{h}\}$ where the left hand side multiplication is to be understood with $X \in \mathbf{h} \subset \mathbf{g}$, but the right hand side with $X + \mu(X) \in S \subset T$. Then $M^\mu \subset M$ is a T-submodule, and for all $c \in \mathbf{h}^*/\mathbf{Z}R$ the subspace $M^c = \bigoplus_{\mu\in c} M^\mu \subset M$ is a $\mathbf{g}_T$-submodule of M.

Now let again $\lambda \in \mathbf{h}^*$ be arbitrary and set $\Lambda = \lambda + P(R)$. One verifies that any $M \in \mathcal{D}(\lambda)$ decomposes as T-module into $M = \bigoplus_{\mu\in\Lambda} M^\mu$ and as $\mathbf{g}_T$-module into $M = \bigoplus_{c\in\Lambda/\mathbf{Z}R} M^c$. This gives even a decomposition of categories $\mathcal{D}(\lambda) = \bigoplus_{c\in\Lambda/\mathbf{Z}R} \mathcal{D}^c(\lambda)$.

For any $\mu \in \Lambda$ denote by $\bar{\mu}$ its image in $\Lambda/\mathbf{Z}R$ and set $\mathcal{D}_\mu(\lambda) = \mathcal{D}^{\bar{\mu}}(\lambda) \cap \mathcal{D}_{\xi(\mu)}(\lambda)$. With these notations we have the decomposition $\mathcal{D}(\lambda) = \bigoplus_{\mu\in\Lambda^+} \mathcal{D}_\mu(\lambda)$.

Let $pr_\mu : \mathcal{D}(\lambda) \to \mathcal{D}_\mu(\lambda)$ be the projection functors along this decomposition. For any two $\mu, \nu \in \Lambda^+$ we define the translation functor $\theta_\mu^\nu : \mathcal{D}_\mu(\lambda) \to \mathcal{D}_\nu(\lambda)$ by $\theta_\mu^\nu M = pr_\nu(E \otimes M)$ where $E \in \mathbf{g} - mod$ is finite dimensional with extremal weight $\nu - \mu$. We have the adjointness $(\theta_\mu^\nu, \theta_\nu^\mu)$.

If both $\lambda, \mu \in \mathbf{h}^*$ are dominant regular and $\lambda + P(R) = \mu + P(R)$ then $\theta_\lambda^\mu M_\lambda = M_\mu$ so in particular $\mathcal{D}(\lambda) = \mathcal{D}(\mu)$. For $\Lambda \in \mathbf{h}^*/P(R)$ we put $\mathcal{D}_\Lambda = \mathcal{D}(\lambda)$ with $\lambda \in \Lambda$ any dominant regular element, and for $\mu \in \Lambda^+$ we set $\mathcal{D}_\mu = \mathcal{D}_\mu(\lambda)$. These definitions do not depend on the choice of λ and the translation functors $\theta_\mu^\nu : \mathcal{D}_\mu \to \mathcal{D}_\nu$ are well defined.

Now I want to explain the useful properties which make it worthwhile to define these deformations.

Theorem 9 *1. For any two $M, N \in \mathcal{D}_\Lambda$ the space $Hom_{\mathbf{g}_T}(M, N)$ is a free T-module of finite rank.*

2. For any commutative T-algebra T' the canonical map $Hom_{\mathbf{g}_T}(M, N) \otimes_T T' \to Hom_{\mathbf{g}_{T'}}(M \otimes_T T', N \otimes_T T')$ is an isomorphism.

Proof: This follows from the definitions and theorem 8. *q.e.d.*

Theorem 10 *1. The specialization $\otimes_T \mathbf{C} : \mathcal{D}_\Lambda \to \mathcal{O}_\Lambda$ gives a bijection between objects of $\mathcal{D}_\Lambda$ and projectives in $\mathcal{O}_\Lambda$ (both considered up to isomorphism).*

2. *The translation functors commute with specialization.*

Proof: 2.) is clear. 1.) is proved in [So3]. In some sense it is a refined and disguised version of the "classification of projective functors" theorem from [BG]. *q.e.d.*
More generally, let $I \subset T$ be an ideal of finite codimension. Let I as well denote $I \cap S$. On any $M \in \mathbf{g}-mod$ which is locally finite over $\mathbf{h}$ the nilpotent part of the $\mathbf{h}$-action gives rise to a morphism $S \to End_{\mathbf{g}} M$. Let $\mathcal{O}^I$ consist of all locally $\mathbf{b}$-finite $M \in \mathbf{g} - mof$ such that this S-action factors over S/I. We have analoguously to category $\mathcal{O}$ (the case $I = \mathbf{m}$) decompositions $\mathcal{O}^I = \bigoplus \mathcal{O}^I_\Lambda$, $\mathcal{O}^I_\Lambda = \bigoplus \mathcal{O}^I_\lambda$ and translations $\theta^\mu_\lambda : \mathcal{O}^I_\lambda \to \mathcal{O}^I_\mu$.

Theorem 11 1. *The I-specialization $\otimes_T T/I : \mathcal{D}_\Lambda \to \mathcal{O}^I_\Lambda$ gives a bijection between objects of $\mathcal{D}_\Lambda$ and projectives in $\mathcal{O}^I_\Lambda$ (both considered up to isomorphism).*

2. *The translation functors commute with I-specialization.*

Proof: 2.) is clear. For 1.) remark that $M_\lambda \otimes_T T/I$ is projective in $\mathcal{O}^I$, for all dominant $\lambda \in \mathbf{h}^*$. Thus for all $M \in \mathcal{D}_\Lambda$ the object $M \otimes_T T/I$ is projective in $\mathcal{O}^I_\Lambda$. Then the statement follows from the preceding theorem. *q.e.d.*
On the other hand the situation over the generic point is easy. Put $Q = QuotT$. Certainly $\mathbf{h}^* \subset \mathbf{h}^*_Q$. Consider in $\mathbf{h}^*_Q$ also the "tautological weight" τ whose restriction to $\mathbf{h} \subset \mathbf{h}_Q$ is given as the identity to $\mathbf{h} \subset S \subset Q$. For $\Lambda \in \mathbf{h}^*/P(R)$ the category $\mathcal{O}_{\Lambda+\tau} = \mathcal{O}_{\Lambda+\tau}(\mathbf{g}_Q, \mathbf{b}_Q)$ decomposes as $\mathcal{O}_{\Lambda+\tau} = \bigoplus_{\lambda\in\Lambda} \mathcal{O}_{\lambda+\tau}$, and the summands are semisimple with only one simple object, namely the irreducible Verma module $M(\lambda+\tau)$ over $\mathbf{g}_Q$.

Theorem 12 1. *Specialization to the generic point is a functor $\otimes_T Q : \mathcal{D}_\Lambda \to \mathcal{O}_{\Lambda+\tau}$ and maps $\mathcal{D}_\lambda$ to $\bigoplus_\mu \mathcal{O}_{\mu+\tau}$ where μ runs over $(\mathbf{Z}R+\lambda) \cap (\mathcal{W}\cdot\lambda)$.*

2. *Under $\otimes_T Q$ the translation $\theta^{\lambda'}_\lambda : \mathcal{D}_\lambda \to \mathcal{D}_{\lambda'}$ decomposes into the matrix of functors $(T^{\mu'}_\mu)$ for $\mu \in (\mathbf{Z}R+\lambda) \cap (\mathcal{W}\cdot\lambda)$, $\mu' \in (\mathbf{Z}R+\lambda') \cap (\mathcal{W}\cdot\lambda')$ with $T^{\mu'}_\mu = \theta^{\mu'+\tau}_{\mu+\tau}$ (resp. $T^{\mu'}_\mu = 0$) if there exists (resp. doesn't exist) $w \in \mathcal{W}$ such that $w\cdot\lambda = \mu$, $w\cdot\lambda' = \mu'$.*

Proof: Left to the reader.

3.3 Endomorphisms of deformed antidominant projectives

To save energy and indices, let us henceforth restrict our attention to the integral case. Let $w_\circ \in \mathcal{W}$ be the longest element. For $\lambda \in P(R)^+$ let $P_\lambda \in \mathcal{D}_\lambda$ be the deformation of the antidominant projective $P(w_\circ \cdot \lambda) \in \mathcal{O}_\lambda$. Set $\mathcal{W}_\lambda = \{w \in \mathcal{W} \mid w \cdot \lambda = \lambda\}$. Let $h_\lambda : Z \otimes T \to T \otimes_{T^\mathcal{W}} T$ be the composition $Z \otimes T \xrightarrow{\xi \otimes id} S \otimes T \xrightarrow{(+\lambda) \otimes id} S \otimes T \to T \otimes_{T^\mathcal{W}} T$, where $(+\lambda) : S \to S$ denotes the comorphism of $(+\lambda) : \mathbf{h}^* \to \mathbf{h}^*$.

Theorem 13 *Assume $\lambda \in P(R)^+$. Then the multiplication $Z \otimes T \to End P_\lambda$ is a surjection, $h_\lambda : Z \otimes T \to T \otimes_{T^\mathcal{W}} T$ has image $T^{\mathcal{W}_\lambda} \otimes_{T^\mathcal{W}} T$ and both these maps have the same kernel. So $T^{\mathcal{W}_\lambda} \otimes_{T^\mathcal{W}} T = End P_\lambda$ canonically.*

Now let $\lambda, \mu \in P(R)^+$ and assume $\mathcal{W}_\mu \subset \mathcal{W}_\lambda$. Then certainly $\theta_\lambda^\mu P_\lambda \cong P_\mu$.

Theorem 14 *We have a commutative diagram*

$$\begin{array}{ccc} T^{\mathcal{W}_\lambda} \otimes_{T^\mathcal{W}} T & \to & End P_\lambda \\ \downarrow & & \theta_\lambda^\mu \downarrow \\ T^{\mathcal{W}_\mu} \otimes_{T^\mathcal{W}} T & \to & End P_\mu . \end{array}$$

where the left vertical arrow is just the inclusion.

Now let us give the proofs.

Proof[Theorem 13]: For regular λ this is just a step in the proof of the Endomorphismensatz of [So3], although it is not explicitely stated there. We can however argue in the opposite direction as well. From theorem 12 we obtain a commutative diagram

$$\begin{array}{ccc} Z \otimes T & \to & End_{\mathbf{g}_T} P_\lambda \\ \downarrow & & \downarrow \\ Z \otimes Q & \to & End_{\mathbf{g}_Q}(\oplus M(\mu + \tau)). \end{array}$$

If we read it carefully, it proves that $ker h_\lambda$ annihilates P_λ. By some invariant theory $im h_\lambda = T^{\mathcal{W}_\lambda} \otimes_{T^\mathcal{W}} T$. Thus a map $T^{\mathcal{W}_\lambda} \otimes_{T^\mathcal{W}} T \to End P_\lambda$. Since it induces isomorphisms on the generic point and the closed point of $Spec T$, the latter by the Endomorphismensatz of [So3], it has to be an isomorphism. *q.e.d.[Theorem 13]*

Proof[Theorem 14]: It certainly suffices to check commutativity over the generic point, i.e. after applying $\otimes_T Q$. But then this follows from some thinking and theorem 12. *q.e.d.[Theorem 14]*

3.4 Homomorphisms between deformed projectives

Let us fix $\lambda \in P(R)^+$. Let us abreviate $T^{W_\lambda} = T^\lambda$. Remember $P_\lambda \in \mathcal{D}_\lambda$ and the surjection $T^\lambda \otimes T \twoheadrightarrow End P_\lambda$. Thus we have a functor $\mathbf{V} = \mathbf{V}_\lambda = Hom_{\mathfrak{g}_T}(P_\lambda, \ ) : \mathfrak{g}_T - mod \to T^\lambda - mod - T$.

Theorem 15 *For any two $M, N \in \mathcal{D}_\lambda$ the canonical map*

$$\mathbf{V} : Hom_{\mathfrak{g}_T}(M, N) \to Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N)$$

is an isomorphism.

We will start out proving approximations to this theorem. Remark first that for any commutative T-algebra T' and $M \in \mathcal{D}_\lambda$ we have canonically

$$\begin{aligned} \mathbf{V}(M \otimes_T T') &= Hom_{\mathfrak{g}_T}(P_\lambda, M \otimes_T T') \\ &= Hom_{\mathfrak{g}_{T'}}(P_\lambda \otimes_T T', M \otimes_T T') \\ &= Hom_{\mathfrak{g}_T}(P_\lambda, M) \otimes_T T' \text{ by theorem 9} \\ &= (\mathbf{V}M) \otimes_T T'. \end{aligned}$$

Choose now an ideal $I \subset T$ of finite codimension and set $\mathcal{D}_\lambda^I = \mathcal{D}_\lambda \otimes_T T/I \subset \mathfrak{g}_T - mod$. By theorem 11 the category $\mathcal{D}_\lambda^I$ consists just of the projective objects in $\mathcal{O}_\lambda^I$ and certainly $Hom_{\mathfrak{g}_T}(M, N) = Hom_{\mathfrak{g}}(M, N) \forall M, N \in \mathcal{D}_\lambda^I$. We show as a first approximation to our theorem:

Proposition 6 *For any $M, N \in \mathcal{D}_\lambda^I$ the canonical map*

$$\mathbf{V} : Hom_{\mathfrak{g}}(M, N) \to Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N)$$

is an isomorphism.

Proof: We make an induction on the codimension of I. For $I = \mathfrak{m}$ the proposition reduces to the "structure theorem" of [So3]. So suppose $I \subset J \subset T$ are two different ideals, $J/I \cong \mathbf{C}$ and the theorem is known for J already.

Since N is free over T/I, there is a short exact sequence

$$E = \{N \otimes_T J/I \hookrightarrow N \twoheadrightarrow N \otimes_T T/J\}$$

in $\mathcal{O}_\lambda^I$. Since M is projective in $\mathcal{O}_\lambda^I$, the sequence $Hom_{\mathfrak{g}}(M, E)$ is exact as well. On the other hand the preceding remarks show that $\mathbf{V}E$ is the sequence

$$\mathbf{V}N \otimes_T J/I \hookrightarrow \mathbf{V}N \twoheadrightarrow \mathbf{V}N \otimes_T T/J.$$

By theorem 9 the right T-module $\mathbf{V}N$ is free over T/I, thus $\mathbf{V}E$ is also exact. So $Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}E)$ is left exact.

Consider the obvious map of sequences

$$Hom_{\mathbf{g}}(M, E) \to Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}E).$$

It is an isomorphism on both ends, by the structure theorem and the induction hypothesis. It is an injection in the middle, since N has a Verma flag, so $socN$ is a direct sum of copies of the irreducible Verma module. We conclude by a diagram chase that our map of sequences is also an isomorphism in the middle. *q.e.d.*

Let $\mathcal{O}_\lambda^\infty \subset \mathbf{g}-mod$ be the full subcategory of modules of finite lenght with all composition factors in $\mathcal{O}_\lambda$. In other words, $\mathcal{O}_\lambda^\infty = \bigcup \mathcal{O}_\lambda^I$. The nilpotent part of the $\mathbf{h}$-action on objects of $\mathcal{O}_\lambda^\infty$ gives rise to an S-action which extends to a T-action. Thus $\mathcal{O}_\lambda^\infty$ embeds as a full subcategory in $\mathbf{g}_T - mod$.

Corollary 1 *Let $Q \in \mathcal{O}_\lambda^I$ be projective. Then for any $M \in \mathcal{O}_\lambda^\infty$ the canonical map*

$$\mathbf{V} : Hom_{\mathbf{g}}(M, Q) \to Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}Q)$$

is an isomorphism.

Proof: For $M \in \mathcal{O}_\lambda^I$ projective this is the proposition. For $M \in \mathcal{O}_\lambda^I$ arbitrary use a projective resolution. For $M \in \mathcal{O}_\lambda^\infty$ arbitrary one restricts to M/IM. *q.e.d.*

Proof[Theorem]: In the following discussion we will concentrate on the right T-module structures of all our objects. At the generic point of $SpecT$ our map $\mathbf{V}$ of the theorem is an isomorphism, since there by theorem 12 all objects of $\mathcal{D}_\lambda$ decompose into sums of irreducible Verma modules. Thus $\mathbf{V}$ is injective and its cokernel $coker\mathbf{V}$ is torsion.

Now consider for any ideal $I \subset T$ of finite codimension the commutative diagram

$$\begin{array}{ccc} Hom_{\mathbf{g}_T}(M, N) \otimes_T T/I & \to & Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N) \otimes_T T/I \\ \downarrow & & \downarrow \\ Hom_{\mathbf{g}_T}(M \otimes_T T/I, N \otimes_T T/I) & \to & Hom_{T^\lambda \otimes T}(\mathbf{V}M \otimes_T T/I, \mathbf{V}N \otimes_T T/I). \end{array}$$

The left vertical and lower horizontal are already known to be isomorphisms, by theorem 8 and the preceding proposition. Thus $\mathbf{V} : Hom_{\mathbf{g}_T}(M, N) \to$

$Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N)$ induces a split injection on the completions at $\mathfrak{m} \in SpecT$. Now completion is exact on noetherian T-modules, thus $(coker\mathbf{V})^\wedge = coker(\mathbf{V}^\wedge)$. This is a submodule of $Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N)^\wedge$ via the splitting and is torsion over T since $coker\mathbf{V}$ is. But $Hom_{T^\lambda \otimes T}(\mathbf{V}M, \mathbf{V}N)$ clearly is torsion free as a T-module, thus its completion is torsion free over T as well. These statements together show $coker\mathbf{V} = 0$, i.e. $\mathbf{V}$ is an isomorphism. *q.e.d.[Theorem]*

3.5 Relation with translations

For any $\lambda \in P(R)^+$ let us denote $C^\lambda = EndP_\lambda = T^{\mathcal{W}_\lambda} \otimes_{T^{\mathcal{W}}} T$. We thus have the functor $\mathbf{V}_\lambda : \mathcal{D}_\lambda \to C^\lambda - mod$. Now suppose $\mu \in P(R)^+$ as well and $\mathcal{W}_\mu \subset \mathcal{W}_\lambda$. Let $res_\mu^\lambda : C^\mu - mod \to C^\lambda - mod$ be the restriction.

Theorem 16 *The following diagrams commute:*

$$\begin{array}{ccc} \mathcal{D}_\mu & \to & C^\mu - mod \\ \theta_\mu^\lambda \downarrow & & \downarrow res_\mu^\lambda \\ \mathcal{D}_\lambda & \to & C^\lambda - mod \end{array} \qquad \begin{array}{ccc} \mathcal{D}_\lambda & \to & C^\lambda - mod \\ \theta_\lambda^\mu \downarrow & & \downarrow Hom_{C^\lambda}(C^\mu, \ ) \\ \mathcal{D}_\mu & \to & C^\mu - mod \end{array}$$

It is useful to have in mind as well:

Proposition 7 *There is an equivalence of functors*

$$C^\mu \otimes_{C^\lambda} \cong Hom_{C^\lambda}(C^\mu, \ ) : C^\lambda - mod \to C^\mu - mod.$$

Proof[Theorem]: Certainly $\theta_\lambda^\mu P_\lambda \cong P_\mu$ and by theorem 14 the induced map on endomorphisms is just the inclusion $C^\lambda \to C^\mu$. Thus for any $Q \in \mathcal{D}_\mu$ we have

$$\begin{aligned} \mathbf{V}\theta_\mu^\lambda Q &= Hom(P_\lambda, \theta_\mu^\lambda Q) \\ &= Hom(\theta_\lambda^\mu P_\lambda, Q) \\ &= Hom(P_\mu, Q) \\ &= res_\mu^\lambda(\mathbf{V}Q), \end{aligned}$$

and the first diagram commutes.

In particular $\mathbf{V}\theta_\mu^\lambda P_\mu \cong C^\mu$ as C^λ-module and also as C^μ-module, where the latter action comes from the C^μ-action on P_μ. Thus

$$\begin{aligned} \mathbf{V}\theta_\lambda^\mu Q &= Hom(P_\mu, \theta_\lambda^\mu Q) \\ &= Hom(\theta_\mu^\lambda P_\mu, Q) \\ &= Hom_{C^\lambda}(\mathbf{V}\theta_\mu^\lambda P_\mu, \mathbf{V}Q) \\ &= Hom_{C^\lambda}(C^\mu, \mathbf{V}Q). \end{aligned}$$

q.e.d.[Theorem]
Proof[Proposition]: Both functors are exact and strongly additive, thus we need only check $C^\mu \cong Hom_{C^\lambda}(C^\mu, C^\lambda)$ as C^μ-modules. A silly but quick way to see this is to put $Q = P_\lambda$ in the preceding sequence of equations. *q.e.d.[Proposition]*

For any $s \in \mathcal{S}$ we have the wall crossing functor $\theta_s : \mathcal{D}_0 \to \mathcal{D}_0$ defined by $\theta_s = \theta_\lambda^0 \theta_0^\lambda$ where $\lambda \in P(R)^+$ has stabilizer $\mathcal{W}_\lambda = \{e, s\}$. Remark that $EndP_0 = C^0 = T \otimes_{T^\mathcal{W}} T = S \otimes_{T^\mathcal{W}} T$. Thus we may interpret $\mathbf{V}$ as a functor $\mathbf{V} : \mathcal{D}_0 \to S - mod - T$. Certainly $\mathbf{V}M_0 = T$. Furthermore

Lemma 8 $\mathbf{V}\theta_s \cong S \otimes_{S^s} \mathbf{V} : \mathcal{D}_0 \to S - mod - T.$

Proof: This follows from the above theorem and proposition. *q.e.d.*

4 Hecke algebras and bimodules, revisited

In this section $\mathcal{W}$ is always a Weyl group. It acts on h by the reflection representation and we set $S = S(\mathrm{h})$.

4.1 Some results on bimodules

Let $\mathbf{B}_\alpha, \mathbf{B}_\beta \in S - Mod - S$ be both of the form $S \otimes_{S^s} S \ldots \otimes_{S^t} S$ for suitable $s, \ldots, t \in \mathcal{S}$ depending on α, β.

Proposition 8 *$Hom_{S \otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta)$ is a free right (and left) S-module of finite rank.*

Proof: By lemma 8 of the preceding subsection there are objects $P_\alpha, P_\beta \in \mathcal{D}_0$ such that $\mathbf{V}P_\alpha \cong \mathbf{B}_\alpha \otimes_S T, \mathbf{V}P_\beta \cong \mathbf{B}_\beta \otimes_S T$. But

$$\begin{aligned} Hom_{\mathbf{g}_T}(P_\alpha, P_\beta) &= Hom_{S \otimes T}(\mathbf{V}P_\alpha, \mathbf{V}P_\beta) \\ &= Hom_{S \otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta) \otimes_S T \end{aligned}$$

is a free right T-module of finite rank by theorem 9. Since $Hom_{S \otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta)$ is graded and finitely generated, this proves the proposition. *q.e.d.*

Proposition 9 *For any commutative (not necessarily graded) S-algebra S' the canonical map $Hom_{S \otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta) \otimes_S S' \to Hom_{S \otimes S'}(\mathbf{B}_\alpha \otimes_S S', \mathbf{B}_\beta \otimes_S S')$ is an isomorphism.*

Proof: We first show this for $S' = S^0 = \mathbf{C}$. Indeed

$$\begin{aligned} Hom_{S\otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta)\otimes_S S^0 &= Hom_{S\otimes T}(\mathbf{B}_\alpha\otimes_S T, \mathbf{B}_\beta\otimes_S T)\otimes_T S^0 \\ &= Hom_{\mathbf{g}_T}(P_\alpha, P_\beta)\otimes_T S^0 \text{ by theorem 15} \\ &= Hom_{\mathbf{g}}(P_\alpha\otimes_T S^0, P_\beta\otimes_T S^0) \text{ by theorem 9} \\ &= Hom_{S\otimes S^0}(\mathbf{B}_\alpha\otimes_T S^0, \mathbf{B}_\beta\otimes_T S^0) \text{ by proposition 6.} \end{aligned}$$

Now we want to deduce the case of arbitrary S'. We need

Lemma 9 *Let $f : H' \to H$ be a morphism of graded free S-modules of finite rank and suppose the specialized map $H'\otimes_S S^0 \to H\otimes_S S^0$ is an injection. Then f is a split injection.*

Proof[Lemma]: First specialize to the generic point. Put $Q = QuotS$. Then $dim_Q(cokerf)\otimes_S Q \geq rk(H) - rk(H')$. By the assumptions $dim_{\mathbf{C}}(cokerf)\otimes_S S^0 = rk(H) - rk(H')$. But for general reasons

$$dim_Q(cokerf)\otimes_S Q \leq dim_{\mathbf{C}}(cokerf)\otimes_S S^0$$

and since $cokerf$ is graded equality implies it is free. Hence $cokerf$ is free over S and $dim_Q(cokerf)\otimes_S Q = dim_Q(H\otimes_S Q) - dim_Q(H'\otimes_S Q)$. This in turn implies that f induces an injection $H'\otimes_S Q \to H\otimes_S Q$, and since H' is torsion free f has to be an injection itself. But $cokerf$ is free, thus f is split. *q.e.d.[Lemma]*

Using this lemma, we show

Lemma 10 *Let $H' \to H \to H''$ be a complex of graded free S-modules of finite rank and suppose the specialized complex $H'\otimes_S S^0 \to H\otimes_S S^0 \to H''\otimes_S S^0$ left exact. Then the complex itself is left exact and split, i.e. isomorphic to a complex $H' \to H'\oplus H_1' \to H_1'\oplus H_1''$ with the obvious maps.*

Proof: Apply the preceding lemma twice. *q.e.d.*

Now we prove the proposition for S' arbitrary. Indeed, we just have to show that for all $M \in S - mod$ the canonical map

$$can : Hom_{S\otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta)\otimes_S M \to Hom_{S\otimes S}(\mathbf{B}_\alpha, \mathbf{B}_\beta\otimes_S M)$$

is an isomorphism. Let $F = \{(S\otimes S)^m \to (S\otimes S)^n \to \mathbf{B}_\alpha\}$ be a graded free resolution of $\mathbf{B}_\alpha$. We get a morphism of sequences

$$can : Hom_{S\otimes S}(F, \mathbf{B}_\beta)\otimes_S M \to Hom_{S\otimes S}(F, \mathbf{B}_\beta\otimes_S M).$$

By the case $S' = S^0$ which we did already, this is an isomorphism for $M = S^0$. In particular $Hom_{S\otimes S}(F, \mathbf{B}_\beta) \otimes_S S^0$ is left exact. Thus by the preceding lemma $Hom_{S\otimes S}(F, \mathbf{B}_\beta)$ is left exact and split as a sequence of right S-modules, thus $Hom_{S\otimes S}(F, \mathbf{B}_\beta) \otimes_S M$ is left exact for any M. This in turn shows that *can* is always an isomorphism. *q.e.d.*

4.2 Realization of the Kazhdan-Lusztig basis via bimodules

We now prove theorem 2 from the introduction. Basically we showed part 4.) in the preceding subsection. In addition to this information we have to use theorem 7 and the results of [So3].

Proof[Theorem 2]: 1.) We establish the existence of the $\mathbf{B}_x$. This is done by an induction on the lenght of x, the case $x = e$ being trivial. Recall the coinvariants $C = S/(S^+)^{\mathcal{W}} S$. In [So3] we defined certain $B_y \in C - Mod^e - C$ and $D_{y^{-1}} = B_y \otimes_C \mathbf{C} \in C - Mod^e$. It is clear from the definitions that $B_y \cong \mathbf{B}_y \otimes_S C$ and $D_{y^{-1}} \cong \mathbf{B}_y \otimes_S S^0$ if $\mathbf{B}_y$ happens to exist. I simplify notation and put $\mathbf{D}_y = D_{y^{-1}}$.

Now suppose $\mathbf{B}_x$ is already constructed and $sx > x$ for some $s \in \mathcal{S}$. Then $\mathbf{C}'_s \mathbf{C}'_x = \mathbf{C}'_{sx} + \sum_{y<x} n(s,x,y)\mathbf{C}'_y$ with $n(s,x,y)$ suitable integers ≥ 0. Consider the graded ring $End^\bullet_{S\otimes S}(S(-1) \otimes_{S^s} \mathbf{B}_x)$. We have

$$\begin{aligned} End^\bullet_{S\otimes S}(S(-1) \otimes_{S^s} \mathbf{B}_x) \otimes_S S^0 &= End^\bullet_S(S(-1) \otimes_{S^s} \mathbf{B}_x \otimes_S S^0) \\ &= End^\bullet_S(S(-1) \otimes_{S^s} \mathbf{D}_x). \end{aligned}$$

By [So3] we know that $S(-1) \otimes_{S^s} \mathbf{D}_x \cong \mathbf{D}_{sx} \oplus \bigoplus_{y<x} n(s,x,y)\mathbf{D}_y$. By the Erweiterungssatz of [So3] the endomorphisms of this object live only in degrees ≥ 0. Thus the same is true for $End^\bullet_{S\otimes S}(S(-1) \otimes_{S^s} \mathbf{B}_x)$ and in degree zero we get a ring isomorphism $End^0_{S\otimes S}(S(-1) \otimes_{S^s} \mathbf{B}_x) \rightarrow End^0_S(S(-1) \otimes_{S^s} \mathbf{D}_x)$. Let p be the projection onto $\mathbf{D}_{sx}$ on the right hand side, and denote its preimage by p as well. This idempotent induces a decomposition $S \otimes_{S^s} \mathbf{B}_x = imp \oplus kerp$ such that $imp \otimes_S S^0 \cong \mathbf{D}_{sx}$ and $kerp \otimes_S S^0 \cong \bigoplus_{y<x} n(s,x,y)\mathbf{D}_y$. Now if $M, N \in S - Molf - S$ are such that $< M >, < N > \in \mathcal{E}(\mathcal{H})$ we know that $M \otimes_S S^0 \cong N \otimes_S S^0$ implies $M \cong N$, say since under the action of $\mathcal{H}$ on $< C - Mod^e >$ the annihilator in $\mathcal{H}$ of $< \mathbf{C} >$ is zero. In particular $kerp \cong \bigoplus_{y<x} n(s,x,y)\mathbf{B}_y$, and it follows that $\mathcal{E}(\mathbf{C}'_{sx}) =< imp >$. Thus $\mathbf{B}_{sx} = imp$ does the job.

2.) By construction $\mathbf{B}_x \otimes_S S^0 \cong \mathbf{D}_x$ and $\mathbf{D}_x$ is indecomposable.

4.) is clear from the preceding subsection.

3.) Remark that by 4.) $End^{\bullet}_{S\otimes S}(\bigoplus_x \mathbf{B}_x) \otimes_S S^0 = End^{\bullet}_S(\bigoplus_x \mathbf{D}_x)$. By the Erweiterungssatz the latter ring lives only in positive degrees and its degree zero part is the span of the projections along the direct sum. From this 3.) follows immediately. *q.e.d.[Theorem 2]*

4.3 Deformations of projectives, revisited

The $L_x = L(x^{-1} \cdot 0) \in \mathcal{O}_0$ for $x \in \mathcal{W}$ represent the simple objects of this category. For any ideal $I \subset T$ of finite codimension let $P^I_x \in \mathcal{O}^I$ be the projective cover of L_x in $\mathcal{O}^I$.

Proposition 10 *We have* $\mathbf{V}P^I_x \cong \mathbf{B}_x \otimes_S T/I$ *for all* $x \in \mathcal{W}$.

Proof: Let $P^{\mathcal{D}}_x \in \mathcal{D}_0$ be the deformation of the above projective. We proceed by induction on the lenght of x, the case $x = e$ being trivial. Suppose the theorem is established for x and $sx > x$ for $s \in \mathcal{S}$. Certainly $\theta_s P^{\mathcal{D}}_x \cong P^{\mathcal{D}}_{sx} \oplus \bigoplus_{y<x} n(s,x,y) P^{\mathcal{D}}_y$ with the above notations, by the Kazhdan-Lusztig conjectures and theorem 10. On the other hand $S \otimes_{S^s} \mathbf{B}_x \cong \mathbf{B}_{sx} \oplus \bigoplus_{y<x} n(s,x,y)\mathbf{B}_y$ when we forget about grading. If we apply $\mathbf{V}$ to the first equation and $\otimes_S T$ to the second, the left hand sides are isomorphic by lemma 8. Thus the right hand sides are isomorphic as well. If we then apply $\otimes_T T/I$ to them and use the induction hypothesis, indeed $\mathbf{V}P^I_{sx} = \mathbf{V}P^{\mathcal{D}}_{sx} \otimes_T T/I = \mathbf{B}_{sx} \otimes_T T/I$. *q.e.d.*

5 Harish-Chandra bimodules

5.1 Construction and uniqueness of V

Recall from the introduction the category $\mathcal{H}$ of Harish-Chandra bimodules with generalized trivial central character from both sides. We want to establish the existence and unicity of an exact functor $\mathbf{V} : \mathcal{H} \to \mathbf{C}-mod$ such that $\mathbf{V}$ annihilates all irreducibles except the irreducible principle series $L \in \mathcal{H}$ and $dim\mathbf{V}L = 1$. This should be clear for general reasons. In our special situation we can procede as follows: Choose projective covers P^n in $\mathcal{H}^{(Z^+)^n}$

of L, choose surjections $P^{n+1} \twoheadrightarrow P^n$ and set $\mathbf{V}X = \lim Hom_{\mathcal{H}}(P^n, X)$ for $X \in \mathcal{H}$. There is our functor.

If $\mathbf{V}'$ is another one, choose $v \in \mathbf{V}'L$, a compatible system of surjections $P^n \twoheadrightarrow L$ and a compatible system of preimages $v^n \in \mathbf{V}'P^n$ of v. Then the maps $\mathbf{V}X = \lim Hom_{\mathcal{H}}(P^n, X) \to \mathbf{V}'X$ given by $\{f^n\} \mapsto (\mathbf{V}'f^n)(v^n)$ for $n \gg 0$ are easily seen to define an equivalence of functors.

5.2 Homomorphisms to projective objects

To establish certain properties of $\mathbf{V}$ we give another construction. Recall the category $\mathcal{O}_0^\infty$ from subsection 3.4. In [Sol] I construct an equivalence $\mathcal{H} = \mathcal{O}_0^\infty$. This commutes with the left Z-actions on these categories. On the other hand $\xi : Z \to S$ induces an isomorphism $Z^\wedge = S^\wedge$ of the completions at Z^+ (resp. S^+) of Z (resp. S) and this way the right $Z^\wedge$-action on $\mathcal{H}$ corresponds to the $S^\wedge$-action on $\mathcal{O}_0^\infty$ given by the nilpotent part of the h-action.

Now remember our deformed antidominant projective $P_0 \in \mathcal{D}_0$ with $End_{\mathfrak{g}_T} P_0 = T \otimes_{T^W} T$ and the functor $\mathbf{V}_0 = Hom_{\mathfrak{g}_T}(P_0, \ ) : \mathfrak{g}_T - mod \to T \otimes_{T^W} T - mod$. Consider the composition $\mathcal{H} = \mathcal{O}_0^\infty \to T \otimes_{T^W} T - mod^e \to T - mod^e - T \to Z - mod^e - Z$, the last arrow given by restriction via $\xi : Z \to T$. This functor has the characterizing properties, so we just constructed our old $\mathbf{V} : \mathcal{H} \to Z - mod - Z$ in a rather akward way. However we get directly for any Z^+-primary ideal $I \subset Z$:

Theorem 17 *Let $Q \in \mathcal{H}^I$ be projective. Then for any $M \in \mathcal{H}$ the functor $\mathbf{V}$ induces an isomorphism $Hom_{\mathcal{H}}(M, Q) \to Hom_{Z \otimes Z}(\mathbf{V}M, \mathbf{V}Q)$.*

Proof: Translate corollary 1 from subsection 3.4. *q.e.d.*

Theorem 18 *Let P_x^I be the indecomposable projectives of $\mathcal{H}^I$, suitably parametrized by $x \in \mathcal{W}$. Then $\mathbf{V}P_x^I \cong \mathbf{B}_x / \mathbf{B}_x I_S$ as Z-bimodules, where $I_S \subset S$ denotes the S^+-primary part of $\xi(I)S$.*

Proof: Translate proposition 10 from subsectionDpr. *q.e.d.*

Proposition 11 $\mathcal{H}/ker\mathbf{V} = T \otimes_{T^W} T - mod^e$.

Proof: Clear. *q.e.d.*

5.3 The functor V commutes with tensor products

Let $GKdim : \mathcal{H} \to \{-\infty, 0, 1, 2, \ldots\}$ be the Gelfand-Kirillov dimension.

Lemma 11 *$GKdim X \geq GKdim X \otimes_{\mathbf{U}} Y \leq GKdim Y$ for all $X, Y \in \mathcal{H}$.*

Proof: [Ja], 10.3. *q.e.d.*

Let $I \subset Z$ be a Z^+-primary ideal. All projectives of $\mathcal{H}^I$ are direct summands of $\mathbf{U}$-bimodules of the form $E \otimes \mathbf{U}/I\mathbf{U}$ for $E \in \mathfrak{g} - mod^e$. Here the left $\mathfrak{g}$-action is the tensor product action, but the right $\mathfrak{g}$-action is just the action on the second factor. So all projectives of $\mathcal{H}^I$ (resp. ${}^I\mathcal{H}$) are projective as right (resp. left) $\mathbf{U}/I\mathbf{U}$-modules. Consider the bifunctor $\otimes_{\mathbf{U}} = \otimes_{\mathbf{U}/I\mathbf{U}} : \mathcal{H}^I \times {}^I\mathcal{H} \to \mathcal{H}$. We note Tor^i its higher derived functors. They depend on I. The Tor^i can be computed using a projective resolution in either variable. Thus the preceding lemma generalizes to

Lemma 12 *$GKdim X \geq GKdim Tor^i(X, Y) \leq GKdim Y$ for all $X \in \mathcal{H}^I$, $Y \in {}^I\mathcal{H}$, $i \geq 0$.*

Proof: Already given. *q.e.d.*

Now consider the irreducible principal series $L = soc(\mathbf{U}/Z^+\mathbf{U})$. The short exact sequence $L \hookrightarrow \mathbf{U}/Z^+\mathbf{U} \twoheadrightarrow coker$ gives us an exact sequence

$$Tor^1(coker, L) \to L \otimes_{\mathbf{U}} L \to L \twoheadrightarrow coker \otimes_{\mathbf{U}} L$$

and applying $\mathbf{V}$ to it, we see

Lemma 13 *The composition $L \otimes_{\mathbf{U}} L \to \mathbf{U}/Z^+\mathbf{U} \otimes_{\mathbf{U}} L = L$ induces an isomorphism $\mathbf{V}(L \otimes_{\mathbf{U}} L) \cong \mathbf{V}(L)$.*

Proof: Already given. *q.e.d.*

Remember the projective system P^n from subsection 5.1 giving rise to $\mathbf{V}$. Choose a nonzero map $P^1 \to \mathbf{U}/Z^+\mathbf{U}$. Using universal properties choose a map $\phi_1 : P^1 \to P^1 \otimes_{\mathbf{U}} P^1$ such that the diagram

$$\begin{array}{ccc} P^1 & \to & P^1 \otimes_{\mathbf{U}} P^1 \\ \downarrow & & \downarrow \\ \mathbf{U}/Z^+\mathbf{U} & = & \mathbf{U}/Z^+\mathbf{U} \otimes_{\mathbf{U}} \mathbf{U}/Z^+\mathbf{U} \end{array}$$

commutes. Using universal properties again, choose inductively maps ϕ_n : $P^n \to P^n \otimes_{\mathbf{U}} P^n$ for all n such that

$$\begin{array}{ccc} P^n & \to & P^n \otimes_{\mathbf{U}} P^n \\ \downarrow & & \downarrow \\ P^{n-1} & \to & P^{n-1} \otimes_{\mathbf{U}} P^{n-1} \end{array}$$

commutes. These choices give us a natural transformation $\phi : \mathbf{V}(X) \otimes \mathbf{V}(Y) \to \mathbf{V}(X \otimes_{\mathbf{U}} Y)$ between functors $\mathcal{H} \times \mathcal{H} \to \mathbf{C} - mod$, by setting $\phi(\{f_n\} \otimes \{g_n\}) = \{\phi_n \circ (f_n \otimes g_n)\}$.

By naturality this induces a natural transformation $\phi : \mathbf{V}(X) \otimes_Z \mathbf{V}(Y) \to \mathbf{V}(X \otimes_{\mathbf{U}} Y)$ between functors $\mathcal{H} \times \mathcal{H} \to Z - mod - Z$.

Proposition 12 *For all $X, Y \in \mathcal{H}$ this map $\phi_{X,Y} : \mathbf{V}(X) \otimes_Z \mathbf{V}(Y) \to \mathbf{V}(X \otimes_{\mathbf{U}} Y)$ is an isomorphism.*

Proof: First we show surjectivity. Let $X'' \to X \twoheadrightarrow X'$ be right exact. If for some Y both $\phi_{X',Y}$ and $\phi_{X'',Y}$ are surjections, then $\phi_{X,Y}$ is surjective as well by a diagram chase. With the same argument on the other side, we are reduced to show $\phi_{X,Y}$ is surjective for simple X, Y. This in turn is clear from lemma 11 if $(X, Y) \neq (L, L)$ and from lemma 18 if $(X, Y) = (L, L)$. So indeed $\phi_{X,Y}$ is always a surjection.

To prove bijectivity, we may without restriction assume $X \in \mathcal{H}^I$ projective and $Y \in {}^I\mathcal{H}$ for some Z^+-primary ideal $I \subset Z$. For X projective in $\mathcal{H}^I$ we know it is projective in $mod - \mathbf{U}/I\mathbf{U}$. We also know $\mathbf{V}(X)$ is a free right Z/I-module, by theorem 18. Thus for projective $X \in \mathcal{H}^I$ both the functors $\mathbf{V}(X \otimes_{\mathbf{U}} Y)$ and $\mathbf{V}(X) \otimes_Z \mathbf{V}(Y)$ are exact for $Y \in {}^I\mathcal{H}$. We just have to show equality of dimensions for simple Y. For Y simple, $Y \neq L$ both sides vanish and there is no problem. To show equality for $Y = L$ then is equivalent to showing equality for $Y = \mathbf{U}/I\mathbf{U}$. In this case it is clear. *q.e.d.*

5.4 Some extensions of perverse sheaves

For any complex algebraic variety X let $D(X)$ be the bounded derived category of the category of mixed Hodge modules [Sa] on X and let $\mathcal{D}(X)$ be as in the introduction. Let now $X = \bigcup_{w \in \mathcal{W}} X_w$ be stratified, $\mathcal{W}$ some finite set. Suppose (1) the strata are irreducible and smooth and (2) their cohomology is pure. Let $\mathcal{C}_w \in D(X_w)$ be the intersection cohomology complex, i.e. the constant variation $\mathbf{R}$ placed in degree $-dim_{\mathbf{C}}X_w$. Let $\mathcal{L}_w \in D(X)$ be its middle extension. Both these objects are pure of weight $dim_{\mathbf{C}}X_w$. Suppose (3) that for any inclusion $i : X_v \to X$ the object $i^*\mathcal{L}_w \in D(X_v)$ is pure of the same weight and further that (4) $i^*\mathcal{L}_w \cong \bigoplus_\nu n^\nu_{v,w}\mathcal{C}_v[\nu]$ in $\mathcal{D}(X_v)$.

Proposition 13 *Suppose the stratified space X satisfies (1) through (4). Then (1) the hypercohomology induces an injection*

$$Hom^\bullet_{\mathcal{D}}(\mathcal{L}_v, \mathcal{L}_w) \to Hom^\bullet_{\mathbf{C}}(\mathbf{H}\mathcal{L}_v, \mathbf{H}\mathcal{L}_w)$$

and (2)

$$dim_{\mathbf{C}}Ext^\nu_{\mathcal{D}}(\mathcal{L}_v, \mathcal{L}_w) = \sum_{a+i+j=\nu, z\in\mathcal{W}} n^i_{v,z} n^j_{w,z} dim_{\mathbf{C}}H^a(X_z).$$

Proof: Certainly $Ext^\nu_{\mathcal{D}}(\mathcal{L}_v, \mathcal{L}_w) = \mathbf{H}^\nu\mathcal{RH}om(\mathcal{L}_v, \mathcal{L}_w)$. Now this hypercohomology is the limit of a spectral sequence with E_1-term

$$E_1^{p,q} = \mathbf{H}^{p+q} i^!_p \mathcal{RH}om(\mathcal{L}_v, \mathcal{L}_w)$$

where i_p denotes the inclusion of the union X_p of all strata of codimension p. But $i^!_p\mathcal{RH}om(\mathcal{L}_v, \mathcal{L}_w) = \mathcal{RH}om(i^*_p\mathcal{L}_v, i^!_p\mathcal{L}_w)$ and thus by our purity assumptions the spectral sequence degenerates at the E_1-term. This proves the formula 2.).

To prove statement 1.) remark that also $\mathbf{H}^\nu\mathcal{L}_v$ is the limit of a spectral sequence $E_1^{p,q} = \mathbf{H}^{p+q}i^!_p\mathcal{L}_v$ which also degenerates at this term for reasons of purity. We just have to show that any nonzero morphism $f : \mathcal{L}_v \to \mathcal{L}_w[\nu]$ in $\mathcal{D}(X)$ induces a nonzero morphism $i^!_p f : i^!_p\mathcal{L}_v \to i^!_p\mathcal{L}_w[\nu]$ for some p. But let u_p be the inclusion of $X_{\geq p} = \bigcup_{q\geq p} X_q$ into X. Now $X_{\geq p} = X_p \cup X_{\geq p+1}$ is a decomposition into an open and a closed subset. We denote by u and i the inclusions. Then we have a distinguished triangle $(i_! i^!, id, u_* u^*)u^!_p = (i_* u^!_{p+1}, u^!_p, u_* i^!_p)$ which says that $i^!_p f = 0$ and $u^!_{p+1} = 0$ imply $u^!_p f = 0$. So if all $i^!_p f = 0$ then $u^!_0 f = f = 0$. *q.e.d.*

Proof[Theorem 6]: The preceding proposition applies. Part 1.) shows the canonical map of the theorem to be an injection. Part 2.) computes the dimension of the left hand side of the canonical map. Using the second remark following theorem 6 together with remark 5 to theorem 2 allows us to compute the dimension of the right hand side of the canonical map. They turn out to be equal. *q.e.d.[Theorem 6]*

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