# Analytic continuation of solutions to integral equations and localization of singularities

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MPI / 94-93

# Analytic continuation of solutions to integral equations and localization of singularities

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August 23, 1994

**Abstract**

The paper contains the investigation of singularities of analytic continuation of solutions to integral equations of Fredholm type. The results are applied to the localization of singularities for continuation of solutions to boundary value problems outside their initial domain of definition.

## 1. Introduction

The present paper is concerned to the theory of continuation of solutions to elliptic differential equations. This theory, originated by the classical works by H. A. Schwarz [1] and G. Herglotz [2] and others, is now very intensively

*Supported by Max-Plank-Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis", Potsdam University and by Chair of Nonlinear Dynamic Systems and Control Processes, Moscow State University.

developed. The remarkable papers by H. S. Shapiro and D. Khavinson (see [3], [4]) and their collaborators and students as well as the powerful methods of complex analysis worked out recently by the authors (see [5], [6], [7]) allowed to obtain the essential progress in this theme. In particular, the practically complete results in the localization of singularities for balayage problems were obtained [8] and the investigation of singularities for continuation of solutions to two-dimensional boundary value problems based on the general reflection formula is given (see [9], [10]). These results have found an important application in a sery of the applied problems of electrodynamics (see [11], [12]).

We remark that, up to the present time, there were no general and effective methods of investigation (localization) of singularities of solutions to boundary value problems for elliptic equations on manifolds of the arbitrary dimension. The aim of the present paper is to fill this blank. Namely, we reduce the boundary value problem to a Fredholm integral equation using the potential theory. Then we investigate the analytic continuation of solutions to this integral equation.

An attempt of using integral equations for analytic continuation of solutions to boundary value problems was done by R. Millar (see [13], [14]). Unfortunately, his method of investigation of integral equations does not give an effective method of calculation of singularities of the continued solution.

In the present paper we propose an algorithm of computation of singularities for continuations of solutions to Fredholm integral equation. This method gives, in particular, the solution of the problem of localization of singularities for boundary value problems in spaces of arbitrary dimension.

## 2 Statement of the problem and preliminary considerations

The aim of this section and the subsequent two ones is to investigate the analytic continuations of solutions to integral equations of the form

$$u(x) - \lambda \int\limits_M K(x,y)u(y)dy = f(x) \qquad (1)$$

where $\lambda$ is a complex parameter, $x$ is a point on some smooth compact manifold $M$ and the kernel $K(x,y)$ is an integrable function on $M \times M$ (more exactly, $K(x,y)$ is a function with respect to the first variable $x$ and a density of form of the maximal degree with respect to the second variable $y$).

Let us formulate the exact requirements on the objects included into equation (1) under which we shall perform the further investigations.

First of all, we suppose that the manifold $M$ is a real part of some complex manifold $M_{\mathbf{C}}$. This means that $M$ is a submanifold of $M_{\mathbf{C}}$ : $M \subset M_{\mathbf{C}}$ and that in a neighbourhood of any point $x \in M$ there exists a (complex) coordinate system[1] $x = (x^1, \ldots, x^n)$ on $M_{\mathbf{C}}$ (where $n$ is the complex dimension of the manifold $M_{\mathbf{C}}$) such that the equations of the submanifold $M$ read

$$\operatorname{Im}(x^j) = 0,\ j = 1, \ldots, n.$$

Moreover, we suppose also that the manifold $M$ can be embedded into the Cartesian complex space $\mathbf{C}^N$ in such a way that it is an intersection of some analytic set in compactification $\mathbf{CP}^N$ of $\mathbf{C}^N$ and the space $\mathbf{C}^N$ itself.

Later on, we suppose that the kernel $K(x,y)$ extends up to an analytic function on $M_{\mathbf{C}} \times M_{\mathbf{C}}$ which can have ramification only on an analytic subset $\Sigma^K \subset M_{\mathbf{C}} \times M_{\mathbf{C}}$. To begin with, we suppose also for simplicity that all singularities of the kernel are bounded, that is,

$$|K(x,y)| \leq C$$

in a neighbourhood of any point of singularity of $K(x,y)$ with some positive constant $C$ (for generalization to the case of integrable kernels see Remark 3.1 below).

At last, we suppose that the function $f(x)$ in (1) can be extended up to the (ramified, in general) function on the manifold $M_{\mathbf{C}}$.

Thus, the first main aim of this paper can be formulated as follows:

*investigate if the extension of the solution $u(x)$ to equation (1) up to a (ramifying) analytic function on $M_{\mathbf{C}}$ is possible and to give the algorithm of computing the singularity set $\Sigma^u$ of such an extension.*

[1] Here and below we denote the real objects and their complexifications by one and the same letter. So, $(x^1, \ldots, x^n)$ can denote, depending of the context, the real coordinates on $M$ or the complex coordinates on $M_{\mathbf{C}}$. Similar, the kernel of the integral equation and its analytic continuation will be both denoted by $K(x,y)$ (see below). In what follows this does not lead to misunderstanding.

Before examining the possibility of an analytic extension, let us determine the singularity set of the extension. We shall do this by the method of successive approximations provided that the number $\lambda$ is small enough. It is well-known that, for sufficiently small values of $\lambda$ the solution to equation (1) can be calculated in the form of the series

$$u(x)=\sum_{k=0}^{\infty}\lambda^k u_k(x) \tag{2}$$

where

$$\begin{aligned} u_0(x) &= f(x), \\ u_{k+1}(x) &= \int\limits_M K(x,y)\,u_k(y)\,dy,\ k=0,1,\dots . \end{aligned} \tag{3}$$

To extend series (2) to the complex manifold $M_{\mathbf{C}}$ one has to replace the integration over $M$ in formula (3) by the integration over the homology class

$$h(x)\in H_n(M_{\mathbf{C}}\setminus(\Sigma_x^K\cap\Sigma^{u_k})) \tag{4}$$

which is realized by the real manifold $M$ for real values of $x$. Here by $\Sigma_x^K$ we denote the intersection

$$\Sigma_x^K=\Sigma^K\cap\{x=\text{const}\}\,.$$

Evidently, the series (2) will still converge in the complex manifold $M_{\mathbf{C}}$ after such a modification for sufficiently small values of $\lambda$.

Thus, for extending the solution $u(x)$ to the complex manifold $M_{\mathbf{C}}$ it is sufficient to extend the homology class (4) to the corresponding values of $x\in M_{\mathbf{C}}$. This can be done with the help of the Thom theorem (see [15], [7]) in the following way.

Consider the projection of the pair

$$(M_{\mathbf{C}}\times M_{\mathbf{C}}\,,\Sigma^K\cup\pi_2^{-1}(\Sigma^{u_k}))\xrightarrow{\pi_1}M_{\mathbf{C}} \tag{5}$$

where $\pi_i: M_{\mathbf{C}}\times M_{\mathbf{C}}\longrightarrow M_{\mathbf{C}},\ i=1,2$ is the projection on the $i$-th factor. Suppose that

$$\Sigma^K\cup\pi_2^{-1}(\Sigma^{u_k})=\bigcup_{j=1}^{N}A_j$$

is a stratification of the union $\Sigma^K \cup \pi_2^{-1}(\Sigma^{u_k})$. Then, by the Thom theorem, the projection (5) determines a locally trivial fibered pair outside projections of points $(x, y) \in A_j$ where the mapping of tangent spaces

$$\left(\pi_1|_{A_j}\right)_* : T_{(x,y)} A_j \longrightarrow T_x M_{\mathbf{C}}$$

is not epimorphic. The collection of such points is called the *Landau manifold* corresponding to projection (5) [16] and we denote it by $\mathcal{L}(\Sigma^{u_k})$. Thus, we determined a mapping $\mathcal{L}$ which takes an analytic set $\Sigma^{u_k}$ into the analytic set $\mathcal{L}(\Sigma^{u_k})$. Now the singularity set of function (2) can be described as follows.

First of all, it is evident that the singularity set of the function $u_0(x)$ coincides with the singularity set of $f(x)$. Then the singularity set of the function $u_1(x)$ can be determined as

$$\Sigma^{u_1} = \mathcal{L}\left(\Sigma^f\right), \tag{6}$$

and, generally,

$$\Sigma^{u_{k+1}} = \mathcal{L}\left(\Sigma^{u_k}\right) \tag{7}$$

for $k = 0, 1, 2, \ldots$. Hence, one can expect that the solution to (1) given by (2) will be regular outside the union

$$\Sigma^u = \bigcup_{k=1}^{\infty} \Sigma^{u_k} \tag{8}$$

where the sets $\Sigma^{u_k}$ are determined by the recurrent procedure (6), (7). One can see that the operator $\mathcal{L}$ defined above is the operator which describes the *propagation of singularities* of solutions to equation (1) just like the Hamilton flow describes the propagation of singularities of solutions to partial differential equations.

Certainly, in this section we had shown only that the singularity set of the solution $u$ contains (at least for small values of $\lambda$) the set $\Sigma^u$. In the subsequent sections we shall show that there exists the extension of solution to equation (1) to the whole manifold $M_{\mathbf{C}}$ except for points of (8) not only for small values of $\lambda$ but for all $\lambda$ which are not eigenvalues of equation (1). Thus, we shall prove that, inversely, the set $\Sigma^u$ contains the singularity set of solution. We shall also present the more geometrical description of the mapping $\mathcal{L}$.

Finally, in the last two sections of this paper we shall consider the application of the introduced theory to the problem of localization of singularities of solutions to elliptic boundary value problems.

# 3 Analytic continuation of solutions

In this section we shall prove the existence of the analytic continuation of solutions to equation (1) as a (ramifying) analytic function with the singularity set described in the previous section. The main tool for the proof will be the representation of solution to the integral equation via Fredholm determinants. Let us briefly recall the corresponding statements (see, for example, R. Courant and D. Hilbert [17], p.121).

The function

$$D(\lambda) = d_0 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{k!} d_k \tag{9}$$

where $d_0 = 1$ and

$$d_k = \int_M \cdots \int_M \begin{vmatrix} K(x_1, x_1) & K(x_2, x_1) & \dots & K(x_k, x_1) \\ K(x_1, x_2) & K(x_2, x_2) & \dots & K(x_k, x_1) \\ \dots & \dots & \dots & \dots \\ K(x_1, x_k) & K(x_2, x_k) & \dots & K(x_k, x_k) \end{vmatrix} dx_1 \dots dx_k, \tag{10}$$

$k = 1, 2, \dots$ is called *a Fredholm denominator* of equation (1).

To write down the explicit formula for solution of equation (1) we need also the function

$$D(x, y, \lambda) = d_0(x, y) + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{k!} d_k(x, y) \tag{11}$$

where $d_0(x, y) = K(x, y)$ and the rest of the functions $d_k(x, y)$ are:

$$d_k(x, y) = \int_M \cdots \int_M \begin{vmatrix} K(x, y) & K(z_1, y) & \dots & K(z_k, y) \\ K(x, z_1) & K(z_1, z_1) & \dots & K(z_k, z_1) \\ \dots & \dots & \dots & \dots \\ K(x, z_k) & K(z_1, z_k) & \dots & K(z_k, z_k) \end{vmatrix} dz_1 \dots dz_k\ , \tag{12}$$

$k = 1, 2, \dots$. It is well-known that, under the conditions of the previous section, the series (9) and (11) converge for any values of $\lambda$ and the following statement holds.

**Theorem 3.1** *Let $\lambda$ be a complex number such that $D(\lambda) \neq 0$. Then for any $f(x)$ the solution to equation (1) is given by the formula*

$$u(x) = \lambda \int\limits_M R(x,y,\lambda) f(y)\, dy + f(x)$$

*where the function $R(x,y,\lambda)$ is equal to*

$$R(x,y,\lambda) = \frac{D(x,y,\lambda)}{D(\lambda)}. \tag{13}$$

The function $R(x,y,\lambda)$ is called *a resolvent kernel* of equation (1).

**Remark 3.1** Theorem 3.1 remains valid also for kernels having *integrable singularities*. However, in this case the definitions of the numbers $d_k$ and the functions $d_k(x,y)$ must be modified. Namely, for this case $d_k$ and $d_k(x,y)$ must be determined by the same formulas (10) and (12) but one must replace all terms of the form $K(z_i, z_i)$ by zero.

*Proof.* It suffices to verify that the resolvent kernel given by (13) satisfies the integral equation

$$R(x,y,\lambda) = K(x,y) + \lambda \int\limits_M K(x,z) R(z,y,\lambda)\, dz.$$

It can be easily seen that for $R(x,y,\lambda)$ to be a solution to this equation the functions $d_k(x,y)$ must satisfy the following recurrent relations

$$\begin{aligned} d_0(x,y) &= K(x,y), \\ d_{k+1}(x,y) &= K(x,y)\, d_{k+1} + (k+1) \int\limits_M K(x,z) d_k(z,y)\, dz \end{aligned}$$

k=0,1,.... However, the first relation follows directly from the definitions (11), (12) and the second can be obtained by the decomposition of the determinants (12) with respect to the first its row. This proves the stated affirmation.

Now it is evident that in order to extend the solution to equation (1) to the complex domain it suffices to carry out such an extension for the resolvent kernel $R(x,y,\lambda)$ of this equation. However, the extension of the function $D(x,y,\lambda)$ can be given by formula (11) if the functions $d_k(x,y)$ are determined by the following recurrent procedure[2]

$$\begin{aligned} d_0(x,y) &= K(x,y), \\ d_{k+1}(x,y) &= K(x,y)\,d_{k+1} + (k+1) \int\limits_{h(x,y)} K(x,z)d_k(z,y)\,dz \end{aligned} \tag{14}$$

where $h(x,y)$ is an extension of the homology class of $M$ which can be constructed with the help of the Thom theorem similar to the considerations of the previous Section. The singularity set $\Sigma^R \subset M_{\mathbf{C}} \times M_{\mathbf{C}}$ of the function $R(x,y,\lambda)$ can be determined by formulas (6) - (8) of the previous section for the right-hand part $f(x)$ being equal to $K(x,y)$.

It is also easy to see that the functions $d_k(x,y)$ satisfying (14) can be computed in the explicit way with the help of the formula

$$d_k(x,y) = \int\limits_{H_k(x,y)} \begin{vmatrix} K(x,y) & K(z_1,y) & \dots & K(z_k,y) \\ K(x,z_1) & K(z_1,z_1) & \dots & K(z_k,z_1) \\ \dots & \dots & \dots & \dots \\ K(x,z_k) & K(z_1,z_k) & \dots & K(z_k,z_k) \end{vmatrix} dz_1 \dots dz_k \tag{15}$$

where $H_k(x,y)$ is an skew product of the homology classes $h$ in variables $(z_1,\dots,z_k)$, $z_j \in M_{\mathbf{C}}$, $j = 1,\dots,k$.

The following statement takes place.

**Theorem 3.2** *The extension of the resolvent kernel $R(x,y,\lambda)$ of equation* (1) *to the complex domain $(M_{\mathbf{C}} \times M_{\mathbf{C}}) \setminus \Sigma^R$ is given by the formula* (13) *if the functions $d_k(x,y)$ are given by* (15).

*Proof.* To prove the statement of the Theorem, it suffices to prove the convergence of the series (11) for any $(x,y) \in (M_{\mathbf{C}} \times M_{\mathbf{C}}) \setminus \Sigma^R$.

Let $\Omega$ be the arbitrary compact domain in $M_{\mathbf{C}} \times M_{\mathbf{C}}$. It is easy to see that the minimal length $l(h(x,y))$ of a contour representing the homology

[2]We remark that the functions $d_k(x,y)$ determined by relation (12) satisfy recurrent system (14) with $h(x)$ replaced by $M$.

class $h(x,y)$ is semi-continuous from the above. Hence, for some positive constant $C$ we have

$$l(h(x,y)) \leq C$$

for any point $(x,y) \in \Omega$. From this it also follows that there exists a compact subset $\Omega_1 \subset M_{\mathbf{C}}$ such that some representative of the homology class $H_k(x,y)$ lyes in

$$\Omega_1 \times \ldots \times \Omega_1$$

(k factors). Let now

$$C_1 = \sup_{x \in \Omega_1 \times \Omega_1} |K(x,y)| .$$

Then, due to the usual estimates of the determinant, the functions (15) satisfy the estimate

$$|d_k(x,y)| \leq k^{\frac{k}{2}} C_1^k C^k$$

and, hence, each term of the series (11) can be estimated by the constant

$$\frac{|\lambda|^k}{k!} k^{\frac{k}{2}} C_1^k C^k .$$

The latter estimate completes the proof.

As a consequence of this theorem we obtain the following result.

**Theorem 3.3** *Under the conditions of Section 2 for any complex $\lambda$ such that $D(\lambda) \neq 0$ there exists the analytic extension of solutions to equation (1) to $M_{\mathbf{C}} \setminus \Sigma^u$ where $\Sigma^u$ is given by formula (8).*

*Proof.* The analytic continuation of the solution to equation (1) is given by the formula

$$u(x) = \int\limits_{h(x)} R(x,y,\lambda) f(y)\, dy + f(x).$$

Hence, due to expression (11) for the resolvent kernel $R(x,y,\lambda)$, to complete the proof of this Theorem it is sufficient to verify that the singularities of the integrals

$$I_k(x) = \int\limits_{h(x)} d_k(x,y)\, f(y)\, dy$$

coincide with the singularities of the functions $u_k(x)$ determined by formulas (3). This fact, however, can be verified with the help of the Thom theorem similar to the considerations of the previous Section. One must just take into account the recurrent relations (14) for the functions $d_k(x, y)$. We leave to the reader the corresponding standard considerations.

# 4 Propagation of singularities

In this section we shall give the *geometrical* treatment of the mapping $\mathcal{L}$ determined in Section 2. We recall that for each analytic set $\Sigma \subset M_{\mathbf{C}}$ the set $\mathcal{L}(\Sigma)$ is defined as follows.

Consider the projection of the pair

$$(M_{\mathbf{C}} \times M_{\mathbf{C}}\,, \Sigma^K \cup \pi_2^{-1}(\Sigma)) \xrightarrow{\pi_1} M_{\mathbf{C}} \tag{16}$$

where $\pi_i : M_{\mathbf{C}} \times M_{\mathbf{C}} \longrightarrow M_{\mathbf{C}}$ is the projection on the $i$-th factor. Then by $\mathcal{L}(\Sigma)$ we denote the Landau manifold corresponding to this projection. Thus, we determined a mapping $\mathcal{L}$ which takes an analytic set $\Sigma$ into the analytic set $\mathcal{L}(\Sigma)$.

First of all we emphasize that the set $\mathcal{L}(\Sigma)$ is not empty even in the case when the initial set $\Sigma$ is empty. Hence, to begin the description of the mapping $\mathcal{L}$ we must describe the set $\mathcal{L}(\emptyset)$ which is a subset of $\mathcal{L}(\Sigma)$ for any analytic set $\Sigma$. To do this we consider the projection

$$(M_{\mathbf{C}} \times M_{\mathbf{C}}\,, \Sigma^K) \xrightarrow{\pi_1} M_{\mathbf{C}} \tag{17}$$

which is a specification of projection (16) to the case when $\Sigma = \emptyset$.

The set $\mathcal{L}(\emptyset)$ is now simply the Landau manifold of projection (17) (the geometrical description of the Landau manifold for a projection was presented in Section 2).

Now we denote by $A_j(x)$, $j = 1, 2, ..., N$ the strata of the analytic set

$$\Sigma_x^K = \Sigma^K \cap \{x = \text{const}\}\,.$$

We suppose that these strata are (open) analytic manifolds analytically dependent on $x$ in $M_{\mathbf{C}} \setminus \mathcal{L}(\emptyset)$.

Now we are able to present a geometrical description of the set $\mathcal{L}(\Sigma)$ for any analitic set $\Sigma$. To do this we consider a stratification

$$\Sigma = \bigcup_{k=1}^{M} B_k$$

where $B_k$ are strata of this stratification. For any $j = 1, ..., N$, $k = 1, ..., M$ we denote by $\mathcal{L}_j(B_k)$ the collection of points of $M_{\mathbf{C}}$ such that the manifold $A_j(x)$ is tangent to the stratum $B_k$ at some its point. Further, we put

$$\mathcal{L}(B_k) = \bigcup_{j=1}^{N} \mathcal{L}_j(B_k).$$

It is evident that projection (16) is locally trivial outside the union

$$\mathcal{L}(\emptyset) \cup \left( \bigcup_{k=1}^{M} \mathcal{L}(B_k) \right).$$

Thus, the latter union can be treated as the set $\mathcal{L}(\Sigma)$:

$$\mathcal{L}(\Sigma) = \mathcal{L}(\emptyset) \cup \left( \bigcup_{k=1}^{M} \mathcal{L}(B_k) \right). \tag{18}$$

To conclude this section we shall make some remarks. First of all, it is clear that the mapping $\mathcal{L}$ for integral equations is similar to the shift along the Hamiltonian vector field for differential equations in the following sense. For differential equations the singularity set of solution is invariant with respect to this shift. In other words, if some point is a point of singularity of a solution, then any shift of this point along the trajectories of the corresponding Hamiltonian vector field is also a singular point of this solution (of course, for strict considerations one has to pass to the phase space $T^*M$). Similar to this situation if some analytic set $\Sigma'$ is a subset of the singularity set of a solution to integral equation (1), then the 'shift' $\mathcal{L}(\Sigma)$ of this set 'along the mapping $\mathcal{L}$' is also a subset of the singularity set of this solution. Moreover, the mapping $\mathcal{L}$ can be 'decomposed' to the family of mappings $\mathcal{L}_j$ parametrized by strata $A_j$ of the analytic set $\Sigma_x^K$ similar to the fact that for differential equations the propagation of singularities is determined by each strata of the characteristic set of the corresponding operator. Certainly, this concluding considerations do not have a strict character, but they can help to understang the role of the introduced notions in the theory of analytic continuation of solutions to integral equations.

# 5 Application to the boundary value problem

In this section we apply the above technique to the *localization of singularities* of continuation of solutions to the Helmholtz equation in the three-dimentional space. To be definite, we consider the interior Neumann boundary value problem for the Helmholtz equation[3]

$$\begin{cases} \Delta u + k^2 u = 0, \ x \in \mathcal{D}, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma} = v \end{cases} \tag{19}$$

where $\mathcal{D}$ is a compact domain in $\mathbf{R}^3$ , $\Gamma = \partial\mathcal{D}$ is the boundary of the domain $\mathcal{D}$ which is supposed to be algebraic. The latter affirmation means that the equation of the boundary $\Gamma$ is given by

$$P(x) = 0 \tag{20}$$

where $P(x)$ is a polynomial in the variables $(x^1, x^2, x^3) \in \mathbf{R}^3$ with non-vanishing gradient on its (real) zeroes. We suppose also that the following condition is valid.

**Condition 5.1** The number $-k^2$ is not an eigenvalue of the Neumann problem for the Laplace operator in the domain $\mathcal{D}$.

We remark that, due to the explicit formulas for solution of the *Cauchy problem* in the complex domain (see [18], [7]), in order to find the singularities of the analitic continuation of the solution it is sufficient to construct the analytic continuation of the Cauchy data for equation (19) or, at least, to

[3]The reader can easily see the natural framework of applicability of the above theory. We remark here only that the only thing we need is that the equation involved to the considered boundary value problem has a fundamental solution.

localize singularities of this continuation. By the Cauchy data we mean, as usual, the values of the solution and of its normal derivative on the boundary. Since the value of the normal derivative is already given by the Neumann conditions of problem (19), we must localize singularities of the second Cauchy data, namely, of the restriction $u|_\Gamma$ of the solution to the boundary.

First of all, we shall derive the integral equation for the second Cauchy data for boundary value problem (19). To do this, we write down the Green's formula for the solution $u(x)$ of problem (19) and for the fundamental solution

$$G(x,y) = \frac{e^{ikr}}{4\pi r},$$

where $r$ is a distance between points $x$ and $y$ of the space $\mathbf{R}^3$:

$$r = \sqrt{(x^1-y^1)^2 + (x^2-y^2)^2 + (x^3-y^3)^2}.$$

This formula reads

$$u(x) = \int\limits_\Gamma \left( u(y)\frac{\partial G}{\partial n_y}(x,y) - G(x,y)\frac{\partial u}{\partial n_y}(y) \right) ds_y$$

for any $x \in \mathcal{D}$ where $ds_y$ is a plane element of the surface $\Gamma$.

If the point $x$ tends to some point lying on the boundary $\Gamma$ then, due to the well-known properties of the potential, we obtain the relation

$$u(x) = \frac{1}{2}\int\limits_\Gamma u(y)\frac{\partial G}{\partial n_y}(x,y)\, ds_y - \int\limits_\Gamma G(x,y)\frac{\partial u}{\partial n_y}(y) ds_y, \quad x \in \Gamma.$$

The latter formula allows us to derive an integral equation for the second Cauchy data on the manifold $\Gamma$:

$$u(\tilde{x}) = \int\limits_\Gamma K(\tilde{x},\tilde{y})u(\tilde{y})\, ds_{\tilde{y}} + f(\tilde{x}) \tag{21}$$

where $\tilde{x}$ and $\Gamma$, $\tilde{y}$ are points of $\Gamma$ and $K(\tilde{x},\tilde{y})$ and $f(\tilde{x})$ are given by the relations

$$K(\tilde{x},\tilde{y}) = \frac{1}{2}\,\frac{\partial G}{\partial n_y}(x,y)\bigg|_\Gamma, \tag{22}$$

$$f(\tilde{x}) = -\int\limits_{\Gamma} G(x,y)\frac{\partial u}{\partial n_y}(y)ds_y. \tag{23}$$

As it is known from the potential theory, the kernel (22) has a weak singularity at $x = y$ and we can apply the results of the preceding sections. However, we can also work with continuous kernels. To do this it is sufficient to pass to the second iteration of equation (21).

If Condition 5.1 is fulfilled, then equation (21) is equivalent to the following equation

$$u(\tilde{x}) = \int\limits_{\Gamma} K^{(2)}(\tilde{x},\tilde{y})u(\tilde{y})\,ds_{\tilde{y}} + f^{(2)}(\tilde{x}) \tag{24}$$

where $K^{(2)}(\tilde{x},\tilde{y})$ is a second iterated kernel for (22) and the function $f^{(2)}(\tilde{x})$ is given by the relation

$$f^{(2)}(\tilde{x}) = \int\limits_{\Gamma} K^{(1)}(\tilde{x},\tilde{y})f(\tilde{y})\,ds_{\tilde{y}} + \int\limits_{\Gamma} K(\tilde{x},\tilde{y})f(\tilde{y})\,ds_{\tilde{y}} + f(\tilde{x}).$$

Here $K^{(1)}(\tilde{x},\tilde{y})$ is the first iterated kernel for (22):

$$K^{(1)}(\tilde{x},\tilde{y}) = \int\limits_{\Gamma} K(\tilde{x},\tilde{z})K(\tilde{z},\tilde{y})\,ds_{\tilde{z}}.$$

Equation (24) can be compexified with the help of the formula

$$ds_{\tilde{y}} = \omega(\tilde{y}) = \left.\frac{\frac{\partial P}{\partial x^1}dx^2 \wedge dx^3 + \frac{\partial P}{\partial x^2}dx^3 \wedge dx^1 + \frac{\partial P}{\partial x^3}dx^1 \wedge dx^2}{\sqrt{\left(\frac{\partial P}{\partial x^1}\right)^2 + \left(\frac{\partial P}{\partial x^2}\right)^2 + \left(\frac{\partial P}{\partial x^3}\right)^2}}\right|_{\Gamma}.$$

The form $\omega(y)$ can be treated as the form on the complexification $\Gamma_{\mathbf{C}}$ of the manifold $\Gamma$ which is determined as an algebraic submanifold in $\mathbf{C}^3$ with the equation (20). The complexification has the form

$$u(\tilde{x}) = \int\limits_{h(\tilde{x})} K^{(2)}(\tilde{x},\tilde{y})u(\tilde{y})\omega(\tilde{y}) + f^{(2)}(\tilde{x}).$$

This equation will be the basic one for the construction of the extension of the second Cauchy data to the complex manifold $\Gamma_{\mathbf{C}}$. For simplicity we shall consider the case when the Neumann data $v(x)$ is an entire function on $\mathbf{C}^3$. The function (23) can be extended to the complex manifold $\Gamma_{\mathbf{C}}$ with the help of the formula

$$f(\widetilde{x}) = -\int\limits_{h(\widetilde{x})} G(x,y)\frac{\partial u}{\partial n_y}(y)\,\omega(y).$$

The considerations similar to those in the previous Section allows us to claim that the function $f(\widetilde{x})$ has singularities exactly at characteristic points of the manifold $\Gamma_{\mathbf{C}}$. This fact can be verified with the help of the Thom theorem if we take into account that the singularity set of the kernel $K(\widetilde{x},\widetilde{y})$ lyes in the intersection of the characteristic cone

$$\left(x^1-y^1\right)^2+\left(x^2-y^2\right)^2+\left(x^3-y^3\right)^2=0 \qquad (25)$$

and the manifold $\Gamma_{\mathbf{C}}$.

Thus, we had shown that the technique developed in the previous sections is applicable to the investigation of the singularity set of the analytic continuation of solutions to boundary value problem (19). To formulate the corresponding affirmation we must only to describe the mapping $\mathcal{L}$ introduced in the previous section in terms of the Hamilton flow corresponding to the Laplace operator.

First of all, we note that the set $\mathcal{L}(\emptyset)$ determined as the Landau manifold of projection (17) in the considered case exactly coincides with the set of characteristic points of the boundary $\Gamma$ since this set consists of poins at which the characteristic cone (25) is tangent to $\Gamma$. Later on, outside the set $\mathcal{L}(\emptyset)$ the intersection of the characteristic cone (25) with $\Gamma$ has exactly two strata: the vertex of the cone $A_1(x)$ and the stratum $A_2(x)$ consisting of all other points. To describe the set $\mathcal{L}(\Sigma)$ for an arbitrary stratified set

$$\Sigma = \bigcup_{j=1}^{M} B_j$$

one must determine in what cases the strata $A_k(x)$ are tangent to the strata $B_j$.

It is easy to see that the stratum $A_1(x)$ is tangent to any stratum $B_j$ if and only if the point $x$ belongs to this stratum. Let us derive the conditions under which the stratum $A_2(x)$ is tangent to $B_j$. To do this we suppose that the parametric equations of $B_j$ are

$$y^j = y^j(\alpha),\ j = 1,2,3$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$ are local coordinates on $B_j$. Then the stratum $A_2(x)$ is tangent to $B_j$ if and only if the function

$$\left(x^1 - y^1(\alpha)\right)^2 + \left(x^2 - y^2(\alpha)\right)^2 + \left(x^3 - y^3(\alpha)\right)^2$$

has a critical point. The corresponding equations are

$$\left(x^1 - y^1(\alpha)\right) \frac{\partial y^1(\alpha)}{\partial \alpha_j} + \left(x^2 - y^2(\alpha)\right) \frac{\partial y^2(\alpha)}{\partial \alpha_j} + \left(x^3 - y^3(\alpha)\right) \frac{\partial y^3(\alpha)}{\partial \alpha_j} = 0, \quad (26)$$

$j = 1, \dots, k$. Denoting by

$$p_k = x^k - y^k(\alpha),\ k = 1,2,3 \quad (27)$$

we see that the condition $y(\alpha) \in A_2(x)$ yields

$$p_1^2 + p_2^2 + p_3^2 = 0$$

and the relation (26) shows that $p = (p_1, p_2, p_3)$ is a conormal vector to $B_j$ at the point $y(\alpha)$. Later on, relation (27) means that the point $x$ lyes on the characteristic ray emanated from the point $y(\alpha)$ along the characteristic covector $p$. Thus, the set of points of $\mathcal{L}(\Sigma)$ originated due to the tangency between $A_2(x)$ and $B_j$ is the intersection of the characteristic conoid of $B_j$ and the boundary $\Gamma$.

Thus, we have proved the following affirmation.

**Theorem 5.1** *Let $\Sigma$ be an analytic subset of $\Gamma_C$ such that it includes all the characteristic points and the intersection between the characteristic conoid of $\Sigma$ and $\Gamma_C$ lies inside $\Sigma$ (this means that $\Sigma$ is invariant with respect to the mapping $\mathcal{L}$). Then there exists an analytic continuation of the restriction $u|_\Gamma$ to the set $\Gamma_C \setminus \Sigma$. Later on, the singularities of the analytic continuation of $u$ to $\mathbf{C}^3$ lye inside the characteristic conoid of $\Sigma$.*

This Theorem (or its generalizations) allows one to compute the singularities of continuation of solutions to boundary value problems outside their initial domain of definition, at least in the case when the singularity set of the fundamental solution for the operator included in the considered boundary value problem is known.

The following affirmation, which is due to D. Khavinson and H. S. Shapiro [19] (see also [20]) is a direct consequence of the theorem stated above.

**Corollary 5.1** *If the equation of the boundary $\Gamma$ is given by a second-order polynomial in variables $(x^1, \dots, x^n)$, then the solution of any interior boundary value problem for the Laplace operator (or, more generally, for the operator with the Laplace operator in the principal part) prolongates without singularities to the exterior of the domain $\mathcal{D}$.*

Indeed, the proof of this affirmation follows from the two facts.

First, the intersection between any characteristic ray emanated from a characteristic point $x_0 \in \Gamma$ and the manifold $\Gamma$ itself contains only the initial characteristic point. Actually, this characteristic ray is a straight line which is tangent to $\Gamma$ at the point $x_0$. Since $\Gamma$ is a quadratic surface, any srtaight line which is tangent to it has no intersection points with $\Gamma$ except for the point of tangency.

In other words, one can claim that in the considered case the equality

$$\mathcal{L}(\operatorname{char}\Gamma) = \operatorname{char}\Gamma$$

and, hence, the Cauchy data are holomorphic on $\Gamma \setminus \operatorname{char}\Gamma$.

Second, it is easy to see that under the above conditions the intersection between the characteristic conoid of $\Gamma$ and the real space lies inside the domain $\mathcal{D}$. Therefore, the continuation of solution to the exterior of the domain $\mathcal{D}$ has no singularities.

# 6 Example

To illustrate the calculations of singularities arising in investigations of boundary value problems, we consider the following Dirichlet problem for the Laplace operator in the space $\mathbf{R}^2$ (see [21]):

$$\begin{cases} \Delta u = 0, \\ u|_\Gamma = v \end{cases}$$

with the entire function $v$, where the equation of the boundary $\Gamma$ is

$$x^4 + y^4 = 1.$$

The characteristic set for the Laplace equation is given by:

$$p^2 + q^2 = 0,$$

and, hence, the equations for the characteristic points of the boundary are:

$$\begin{cases} x^4 + y^4 = 1, \\ x^6 + y^6 = 0. \end{cases} \tag{28}$$

The set of characteristic points we denote by char$\Gamma$. The latter system has the following solutions:

$$\left(\frac{1}{\sqrt[4]{2}}, \pm\frac{i}{\sqrt[4]{2}}\right), \left(-\frac{1}{\sqrt[4]{2}}, \pm\frac{i}{\sqrt[4]{2}}\right), \left(\pm\frac{i}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right), \left(\pm\frac{i}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}\right). \tag{29}$$

and

$$\left(\begin{array}{l} x = \pm e^{i\frac{\pi}{12}} \\ y = \pm e^{-i\frac{\pi}{12}} \end{array}\right), \left(\begin{array}{l} x = \pm e^{i\frac{5\pi}{12}} \\ y = \pm e^{-i\frac{5\pi}{12}} \end{array}\right), \left(\begin{array}{l} x = \pm e^{-i\frac{\pi}{12}} \\ y = \pm e^{i\frac{\pi}{12}} \end{array}\right), \left(\begin{array}{l} x = \pm e^{-i\frac{5\pi}{12}} \\ y = \pm e^{i\frac{5\pi}{12}} \end{array}\right). \tag{30}$$

Now let us calculate the set $\mathcal{L}(\text{char}\Gamma)$. The points of this set originated by roots (29), (30) are intersections of characteristical rays emanated from these roots and the surface $\Gamma$. Let us consider the root

$$\left(x = \frac{1}{\sqrt[4]{2}}, y = i\frac{1}{\sqrt[4]{2}}\right) \tag{31}$$

One of the characteristic rays emanated from this point is

$$\left(x - \frac{1}{\sqrt[4]{2}}\right) + i\left(y - i\frac{1}{\sqrt[4]{2}}\right) = 0, \text{ or } x + iy = 0, \tag{32}$$

and the intersection of this ray with $\Gamma$ is given by the system of equations

$$\begin{cases} x^4 + y^4 = 1, \\ x = -iy. \end{cases}$$

Thus, we see that ray (32) intersects $\Gamma$ at points of char$\Gamma$.

The other ray emanated from point (31) is

$$\left(x - \frac{1}{\sqrt[4]{2}}\right) - i\left(y - i\frac{1}{\sqrt[4]{2}}\right) = 0. \tag{33}$$

Deriving $x$ from the second equation we obtain

$$x - iy = \sqrt[4]{8}.$$

Substituting this expression to the first equation of the last system we obtain the equation for $y$:

$$\left[iy + \sqrt[4]{8}\right]^4 + y^4 = 1,$$

or

$$2y^4 - 4i\sqrt[4]{8}y^3 - 62\sqrt{2}y^2 + 16i\sqrt[4]{2}y + 7 = 0. \tag{34}$$

One of the roots of equation (34) is equal to $y = i\frac{1}{\sqrt[4]{2}}$. This follows from the fact that ray (33) passes through point (31). Therefore, the computation of the mapping $\mathcal{L}$ is reduced, in the considered case, to solving algebraic equations of the third order. We shall not carry out this procedure for all roots (29), (30) because it is rather a long task; the presented calculations illustrate the general principle well enough.

Thus, we see that, unlike the case of "balayage" problem, each characteristic point $(x_0, y_0)$ of the boundary originates not a single singularity of continuation of solution outside the previous domain of definition but an infinit sequence of such singularities. These singularities can be obtained as the intersections of the characteristic rays emanated from the points

$$\mathcal{L}(x_0, y_0),\ \mathcal{L}^2(x_0, y_0),\ \ldots$$

with the real plane. The explicit calculation of these singularities is not a complicated, but rather long and we omit this calculation.

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*Potsdam 1994*