Onsager's Algebra, Loop Algebra and Chiral Potts Model

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Introduction

In this note, I would like to draw attention to an interesting algebra coming from solvable systems in statistical mechanics, and also its connection with "rapidity" variable of the high genus Rieman surface in chiral Potts model [2] [4] [5] [12]. This algebra appeared in the Onsager's original paper [11] on the solution of planar Ising model in zero magnetic field. Simplifications of Onsager's work were found afterwards by using fermion algebra, a different method with the original one. However renewed interests on Onsager's algebra arise in recent years. Dolan and Grady [6] considered Hamiltonians H of the form

$$H = A_0 + k' A_1$$

with a parameter k' and given operators A_0, A_1. They showed that a pair of operators A_0 and A_1 which are "self dual" in a certain sense, and satisfy the Dolan-Grady (DG) conditions, namely

$$[A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0] \qquad [A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1]$$

generate a infinite sequence of commuting operators for H. This sequence is in fact the generators of Onsager's algebra, and the Ising model was the only realization given by them. Subsequently, von Gehlen and Rittenberg [10] found that certain $\mathbf{Z}_N$ symmetric Hamiltonian spin chains have Ising-like spectra and the operator pair appeared in the Hamiltonian also satisfy DG conditions. Recently, a new exactly solvable two-dimension lattice model-the chiral Potts model-has been found [1] [2] [3] [5]. A special case is the superintegrable one which has much Ising-like structure for the solution even though it is an N-state model. The corresponding quantum spin chain is the one investigated by von Gehlen and Rittenberg, and again by Albertini et al . The conclusions they obtained rely heavily on the numerical computation. Recently B. Davies [8] [9] gave a mathematical treatment on the study of Onsager's algebra just from DG conditions, and also obtained its irreducible representations. It seems as if in mathematical literature, no reference can be found. However, the Davies' arguments still lack rigorousness in some respects from the mathematical point of view. It is one purpose of this note, to put Davies' treatments on a more firm mathematical footing. We find a realization of Onsager's algebra inside the loop algebra $L(sl_2(\mathbf{C}))$ of $sl_2(\mathbf{C})$ is easier for the study of representations of Onsager's algebra since we have a better understanding of representations of $L(sl_2(\mathbf{C}))$ in mathematical literature, e.g. [6]. By the structure of irreducible representations of Onsager's algebra, one obtains the spectrum of a Hamiltonian of superintegrable chiral spin chain which depends on an "extra" parameter. We shall examine the relation between this parameter and the "rapidity" variable appeared in the solution of Yang-Baxter equation in chiral Potts model [2] [3] [4]. In particular, for ground-state sector, the spectrum obtained by Baxter [3] is described by zeros of a certain polynomial whose variable is related to the "rapidity" curve. These zeros are the data of representations of Onsager's algebra. This simply implies certain representations of $sl_2(\mathbf{C})$ being assigned to given marked points on the "rapidity" curve. The significance of this interesting phenomenon is still under investigation.

The plan of this paper is as follows. In section 1, we follows Davies' argument [9] to derive the important relations between generators of Onsager's algebra from the DG conditions (Theorem 1 below). Using these relations Onsager's algebra can be identified with the fixed subalgebra of an involution of $L(sl_2(\mathbf{C}))$. In section 2, we describe the basic construction of representations of Onsager's algebra through that of $L(sl_2(\mathbf{C}))$, and in Section 3 they are shown to generate all the irreducible representations of Onsager's algebra as in [8]. In Section 4, we study the spectrum of the quantum $N-$state chiral Potts spin chain in superintegrable case. Using the description of irreducible representations of Onsager's algebra, the eigenvalues of Hamiltonian are expressed in an explicit form involving a parameter, whose relation with "rapidity" curve of chiral Potts model is also discussed there.

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Notation

In this note we shall make much use of the adjoint representation of a Lie algebra and write

$$ad_A B = [A, B]$$

for elements A, B in the Lie algebra. The Jacobi identity can be expressed as

$$ad_A[B, C] = [ad_A B, C] + [B, ad_A C] .$$

We shall always denote

$\mathfrak{g} = sl_2(\mathbf{C})$,

$E, F, H =$ the generators of $\mathfrak{g}$ with

$$[E, F] = H,\ [H, E] = 2E,\ [H, F] = -2F,$$

$\mathcal{P} = \{\text{non}-\text{zero integral dominate weight of } \mathfrak{g}\}$,

$V(\lambda)$ = the irreducible $\mathfrak{g}$ – module with highest weight $\gamma \in \mathcal{P}$,
$L(\mathfrak{g})$ = the loop algebra $\mathbf{C}[t, t^{-1}] \bigotimes_{\mathbf{C}} \mathfrak{g}$ with bracket $[\ ]_0$ given by

$$[f \otimes x, g \otimes y]_0 = fg \otimes [x, y]$$

for f, g $\in \mathbf{C}[t, t^{-1}]$ and x, y$\in\mathfrak{g}$, (We often think of $L(\mathfrak{g})$ as the space of algebraic maps $\mathbf{C}^* \rightarrow \mathfrak{g}$).

1. *Onsager's algebra , Loop Algebra of* $sl_2(\mathbf{C})$

The following notion is defined in [7] [9] :

Definition Two elements A, B in a Lie algebra satisfy Dolan-Grady (DG) conditions if

$$ad_A^3 B = 16\, ad_A B\,, \quad ad_B^3 A = 16\, ad_B A\,.$$

Lemma 1 Let $\mathfrak{L}$ be a Lie algebra. For $A, B \in \mathfrak{L}$, define the elements α, β of $\mathfrak{L}$ by

$$\alpha = A + \frac{1}{8} ad_B ad_A B\,, \quad \beta = B + \frac{1}{8} ad_A ad_B A\,.$$

Then A, B satisfy DG conditions if and only if A, B, α, β satisfy the conditions

$$[\beta, A] = [A, B] = [B, \alpha]\,.$$

Proof. As

$$B + \beta = 2B - \frac{1}{8} ad_A^2 B\,,$$

the equality of $[\beta, A] = [A, B]$ is equivalent to $ad_A^3 B = 16\, ad_A B$. Similarly,

$$[A, B] = [B, \beta] \Leftrightarrow ad_B^3 A = 16\, ad_B A\,. \qquad \text{q.e.d.}$$

Definition Onsager algebra $\mathfrak{A}$ is the universal complex Lie algebra generated by two elements A_0, A_1 satisfying DG conditions.

Let G_1 be the commutator

$$[A_1, A_0] = 4G_1\,.$$

and define an infinite sequence of elements $A_m, G_m\, (m \in \mathbf{Z})$ in $\mathfrak{A}$ by

$$A_{m+1} - A_{m-1} = \frac{1}{2} ad_{G_1} A_m\,, \quad G_m = \frac{1}{4} ad_{A_m} A_0\,. \qquad (1)$$

We shall follow Davies' argument [9] to show the following relations for the generators A_m, G_m of $\mathfrak{A}$.

Theorem 1 A_m, G_m satisfy the identities:

$$[A_l, A_m] = 4G_{l-m} \quad , \qquad (2_a)$$
$$[G_l, A_m] = 2A_{l+m} - 2A_{m-l} \quad , \qquad (2_b)$$
$$[G_l, G_m] = 0 \quad . \qquad (2_c)$$

We shall proceed the proof of the above theorem in a number of steps. First we note that the relation (2_a) will imply the following relations:

$$A_{n+1} - A_{-n-1} = -\tfrac{1}{8} ad_{A_0} ad_{A_1} A_{-n}, \quad n \geq 0, \qquad (3_a)$$
$$A_{n+1} - A_{-n+1} = \tfrac{1}{8} ad_{A_1} ad_{A_0} A_n, \qquad n \geq 1. \qquad (3_b)$$

The cases $(3_a)_{n=0}$ and $(3_b)_{n=1}$ are obvious by (1). The general cases follows from induction by using the following lemma.

Lemma 2 Let $n \geq 1$. (i) If $[A_0, A_n] = [A_{-n}, A_0]$, then

$$(3_a)_n \Leftrightarrow (3_b)_n \quad .$$

(ii) If $[A_1, A_{n+1}] = [A_{-n+1}, A_1]$, then

$$(3_a)_{n-1} \Leftrightarrow (3_b)_{n+1} \quad .$$

Proof. Applying $\frac{1}{8} ad_{A_1}$ to the equality $[A_0, A_n] = [A_{-n}, A_0]$, we have

$$\frac{1}{8} ad_{A_1} ad_{A_0} A_n = -\frac{1}{8} ad_{A_0} ad_{A_1} A_{-n} - \frac{1}{2} ad_{G_1} A_{-n}$$
$$= -\frac{1}{8} ad_{A_0} ad_{A_1} A_{-n} - (A_{-n+1} - A_{-n-1}),$$

hence (i). Similarly we obtain (ii) by applying $\frac{1}{8} ad_{A_0}$ to $[A_1, A_{n+1}] = [A_{-n+1}, A_1]$. q.e.d.

First we reduce Theorem 1 to the only one relation (2_a):

Lemma 3

$$(2_a) \Rightarrow (2_b) \text{ and } (2_c).$$

Proof: Step I. We show

$$(2_a) \Rightarrow (2_b) .$$

By (2_a) (3_a) (3_b), we have for $l \geq 1$,

$$[G_l,\ A_0] = -\frac{1}{4} ad_{A_0} [A_1,\ A_{-l+1}] = 2(A_l - A_{-l}),$$
$$[G_l,\ A_1] = -\frac{1}{4} ad_{A_1} [A_l,\ A_0] = 2(A_{l+1} - A_{-l+1}).$$

Therefore (2_b) holds for $(l,m) = (l,0),(l,1)$. Applying ad_{G_m} on (2_a) for $(l,m) = (l,0)$, we have

$$4[G_m, G_l] = [ad_{G_m} A_l,\ A_0] + [A_l,\ ad_{G_m} A_0] \quad . \quad (4)$$

By (1) and (4) for $(m,l) = (1,l)$,

$$\begin{aligned} 4[G_1, G_l] &= 2[A_{l+1} - A_{l-1}, A_0] + 2[A_l, A_1 - A_{-1}] \\ &= 8(G_{l+1} - G_{l-1}) + 8(G_{l-1} - G_{l+1}) = 0 \quad . \quad (\text{by } (2_a)) \end{aligned}$$

Hence (2_c) holds for $l = 1$ or $m = 1$. Applying the relations

$$A_{n+2} = A_n + \frac{1}{2} ad_{G_1} A_{n+1}$$

repeatedly, we obtain the expression:

$$A_{n+l} = g_l(ad_{G_1})A_n + h_l(ad_{G_1})A_{n+1} \quad \text{for } l \geq 0 \, ,$$

where $g_l(x)$ and $h_l(x)$ are polynomials generated by the recursion

$$\begin{aligned} g_{l+1} &= \tfrac{x}{2} g_l + g_{l-1} \, , \quad g_0 = 1 \, , g_1 = 0, \\ h_{l+1} &= \tfrac{x}{2} h_l + h_{l-1} \, , \quad h_0 = 0 \, , h_1 = 1. \end{aligned}$$

Then

$$\begin{aligned} [G_l, A_m] &= ad_{G_l}(g_m(ad_{G_1})A_0 + h_m(ad_{G_1})A_1) \\ &= g_m(ad_{G_1}) ad_{G_l} A_0 + h_m(ad_{G_1}) ad_{G_l} A_1 \\ &= 2g_m(ad_{G_1})(A_l - A_{-l}) + 2h_m(ad_{G_1})(A_{l+1} - A_{1-l}) \\ &= 2A_{l+m} - 2A_{m-l} \, . \end{aligned}$$

Hence we obtain (2_b).

Step II.

$$(2_a) \text{ and } (2_b) \Rightarrow (2_c) \, .$$

We have

$$\begin{aligned} 4[G_m, G_l] &= [ad_{G_m} A_l,\ A_0] + [A_l,\ ad_{G_m} A_0] && (\text{by } (4)) \\ &= 2[A_{l+m} - A_{l-m}, A_0] + 2[A_l, A_m - A_{-m}] && (\text{by } (2_b)) \\ &= 8(G_{l+m} - G_{l-m}) + 8(G_{l-m} - G_{l+m}) = 0 \quad . && (\text{by } (2_a)) \end{aligned}$$

q.e.d.

The Lemma 4–6 below are for the proof of (2_a).

Lemma 4 If A_0, A_1 satisfy DG conditions, then

$$[A_{-2},\ A_0] = [A_{-1},\ A_1] = [A_0,\ A_2] = [A_1,\ A_3], \quad (5)$$

$$[A_{-3},\ A_0] = [A_{-2},\ A_1] = [A_{-1},\ A_2] = [A_0,\ A_3] = [A_1,\ A_4]. \quad (6)$$

Proof. By Lemma 1, we have

$$[A_{-1},\, A_0] = [A_0,\, A_1] = [A_1,\, A_2].$$

By Lemma 2, the relations $(3_a)_{n=1}$, $(3_b)_{n=2}$ hold. Applying ad_{G_1} on the above equalities, we have

$$[A_{-1},\, A_1] - [A_{-2},\, A_0] = [A_0,\, A_2] - [A_{-1},\, A_1] = [A_1,\, A_3] - [A_0,\, A_2].$$

Acting $ad_{A_1} ad_{A_0}$ on $[A_1,\, A_0]$, we have

$$[A_1,\, [A_0, [A_1,\, A_0]]] = [[A_1, [A_0,\, A_1]],\, A_0] = [4ad_{G_1} A_1, A_0] = 8[A_2, A_0].$$

Since the left hand side equals to

$$[A_1,\, [-4G_1, A_0]] = 8[A_1, A_{-1}],$$

we obtain $[A_0,\, A_2] = [A_{-1},\, A_1]$, hence the equalities (5). By Lemma 2, the relation $(3_a)_{n=2}$ holds.

In order to obtain (6), we must have four equations. Applying ad_{G_1} on $[A_{-1},\, A_1] = [A_0,\, A_2]$ in (5) and cancelling the commutators whose indices differ by 1, we have

$$2[A_{-1}, A_2] = [A_{-2}, A_1] + [A_0, A_3]. \qquad (7)$$

Act $ad_{A_0} ad_{A_1}$ on $[A_{-2},\, A_0] = [A_{-1},\, A_1]$ in (5). The left hand side becomes

$$\begin{aligned} &[ad_{A_0} ad_{A_1} A_{-2}\,, A_0] - 4[A_0,\, ad_{G_1} A_{-2}] \\ &= -8[A_3 - A_{-3}, A_0] - 8[A_0, A_{-1} - A_{-3}] \qquad (\text{by } (3_a)_{n=2}) \\ &= 8([A_0, A_3] + [A_{-1}, A_0]) \quad , \end{aligned}$$

while the right hand side,

$$\begin{aligned} &[ad_{A_0} ad_{A_1} A_{-1}, A_1] + [ad_{A_1} A_{-1}, ad_{A_0} A_1] \\ &= -8[A_2 - A_{-2}, A_1] + 4ad_{G_1}[A_1, A_{-1}] \qquad (\text{by } (3_a)_{n=1}) \\ &= 8([A_1, A_2 - A_{-2}] + [A_2 - A_0, A_{-1}] + [A_1, A_0 - A_{-2}]) \\ &= 8(2[A_{-2}, A_1] - [A_{-1}, A_2] + [A_{-1}, A_0]). \end{aligned}$$

This gives

$$2[A_{-2}, A_1] = [A_{-1}, A_2] + [A_0, A_3]. \qquad (8)$$

By (7) (8),

$$[A_{-2}, A_1] = [A_{-1}, A_2] = [A_0, A_3].$$

From this it follows $ad_{G_1}[A_{-1}, A_1] = 0$. Now consider

$$
\begin{aligned}
&ad_{A_0}ad_{G_1}ad_{A_0}[A_{-1}, A_1] = [ad_{A_0}G_1, [A_0, [A_{-1}, A_1]]] + [G_1, ad^2_{A_0}[A_{-1}, A_1]] \\
&= -[ad^2_{A_0}G_1, [A_{-1}, A_1]] + [A_0, [ad_{A_0}G_1, [A_{-1}, A_1]]] - ad_{G_1}ad^2_{A_0}ad_{A_1}A_{-1} \\
&= [4ad^3_{A_0}A_1, [A_{-1}, A_1]] - [A_0, [G_1, ad_{A_0}[A_{-1}, A_1]]] \\
&\quad +8ad_{G_1}ad_{A_0}(A_2 - A_{-2}) \qquad \text{(by } (3_a)_{n=1}) \\
&= [-16G_1, [A_{-1}, A_1]] - ad_{A_0}ad_{G_1}ad_{A_0}[A_{-1}, A_1] + 0 \qquad \text{(by (5))} \\
&= -ad_{A_0}ad_{G_1}ad_{A_0}[A_{-1}, A_1],
\end{aligned}
$$

hence $ad_{A_0}ad_{G_1}ad_{A_0}[A_{-1}, A_1] = 0$. Then by $(3_a)_{n=1}$,

$$
\begin{aligned}
0 = \frac{1}{8}ad_{A_0}[G_1, A_2 - A_{-2}] = 2[A_0, A_3 - A_1 - A_{-1} + A_{-3}] = 2[A_0, A_3 + A_{-3}], \\
[A_0, A_3] = [A_{-3}, A_0].
\end{aligned}
$$

Hence we obtain the first three equalities of (6). From $[A_{-1}, A_2] = [A_0, A_3]$, there follows $ad_{G_1}[A_0, A_2] = 0$. As the computation in the previous case,

$$
\begin{aligned}
&ad_{A_1}ad_{G_1}ad_{A_1}[A_0, A_2] = [ad_{A_1}G_1, [A_1, [A_0, A_2]]] + [G_1, ad^2_{A_1}[A_0, A_2]] \\
&= -[ad^2_{A_1}G_1, [A_0, A_2]] + [A_1, [ad_{A_1}G_1, [A_0, A_2]]] - ad_{G_1}ad^2_{A_1}ad_{A_0}A_2 \\
&= [-16G_1, [A_0, A_2]] - ad_{A_1}ad_{G_1}ad_{A_1}[A_0, A_2] \\
&\quad +8ad_{G_1}ad_{A_1}(A_3 - A_{-1}) \qquad \text{(by } (3_b)_{n=2}) \\
&= -ad_{A_1}ad_{G_1}ad_{A_1}[A_0, A_2] \ ,
\end{aligned}
$$

hence

$$
\begin{aligned}
0 &= \frac{1}{8}ad_{A_1}ad_{G_1}ad_{A_1}[A_0, A_2] = ad_{A_1}[G_1, A_3 - A_{-1}] \\
&= 2[A_1, A_4 - A_2 - A_0 + A_{-2}] = 2[A_1, A_4 + A_{-2}], \\
&[A_1, A_4] = [A_{-2}, A_1].
\end{aligned}
$$

Therefore we obtain (6). q.e.d.

Lemma 5 Every adjacent pair A_m, A_{m+1} satisfies DG conditions and $[A_m, A_{m+1}] = 4G_1$ for all m.

Proof. Acting ad_{G_1} on the equalities $[A_0, A_2] = [A_1, A_3]$, $[A_{-2}, A_0] = [A_{-1}, A_1]$ in (5) and using Lemma 1 and 4 to eliminate $[A_l, A_m]$ with $l - m = 1, 3$, we have

$$[A_1, A_2] = [A_2, A_3], \ [A_{-2}, A_{-1}] = [A_{-1}, A_0].$$

Applying Lemma 1, we find DG conditions for m=1, —1, and also $G_1 = [A_1, A_2] = [A_{-1}, A_0]$. By induction, the conclusions are obvious. q.e.d.

By Lemma 5, A_m, A_{m+1} satisfy DG conditions with $[A_{m+1}, A_m] = 4G_1$, for all m. Therefore for a positive integer N, the following statements are equivalent:

$$[A_l, A_m] = [A_{l+1}, A_{m+1}] \text{ for } m - l = N$$
$$\Leftrightarrow [A_l, A_m] = [A_{l+1}, A_{m+1}] \text{ for } m - l = N,\ 0 \le m \le N.$$

Hence the proof of (2_a) is reduced to the following Lemma.

Lemma 6 If A_0, A_1 satisfy DG conditions, then for any positive integer N and all values l, m satisfying $m = 0, \ldots, N$, $m - l = N$, we have

$$[A_l, A_m] = [A_{l+1}, A_{m+1}].$$

Proof. By Lemma 4 , the results holds for $N = 1, 2, 3$. For the general N, we shall prove by induction, i.e. for $N \ge 4$, the conclusion of all positive integers $\le (N-1)$ will imply the one for N. By Lemma 2, the relations $(3_a)_n$ $(3_b)_n$ for $1 \le n \le (N-1)$ hold. Let $m = 0, \ldots, N-1$, $m - l = N-1$. Acting $2ad_{G_1}$ on the equation $[A_l, A_m] = [A_{l+1}, A_{m+1}]$, and cancelling the terms of indices different by $(N-2)$, we get

$$[A_{l-1}, A_m] - [A_l, A_{m+1}] = [A_l, A_{m+1}] - [A_{l+1}, A_{m+2}].$$

Setting $l' = l - 1$, we obtain N equations

$$[A_{l'}, A_m] - [A_{l'+1}, A_{m+1}] = [A_{l'+1}, A_{m+1}] - [A_{l'+2}, A_{m+2}],$$
$$m = 0, \ldots, (N-1),\ m - l' = N.$$

To find one more equation, we consider the case of even and odd N separately. For $N = 2m$, acting $ad_{A_1} ad_{A_0}$ on the equation $[A_{-2m+2}, A_1] = [A_{-m+1}, A_m]$, then using the induction assumption and $(3_a)_{n=(2m-2),(m-1)}, (3_b)_{n=m}$, we get

$$\begin{aligned}
&8[A_1, A_{2m-1}] + 8[A_{-2m+1}, A_1] \\
&= [ad_{A_1} ad_{A_0} A_{-m+1}, A_m] + [ad_{A_0} A_{-m+1}, ad_{A_1} A_m] \\
&\quad + [ad_{A_1} A_{-m+1}, ad_{A_0} A_m] + [A_{-m+1}, ad_{A_1} ad_{A_0} A_m] \\
&= [4ad_{G_1} A_{-m+1} + ad_{A_0} ad_{A_1} A_{-m+1}, A_m] + 0 + 0 + [A_{-m+1}, ad_{A_1} ad_{A_0} A_m] \\
&= 8[A_{-m+2} - A_{-m}, A_m] - 8[A_m - A_{-m}, A_m] + 8[A_{-m+1}, A_{m+1} - A_{-m+1}] \\
&= 8[A_{-m+2}, A_m] + 8[A_{-m+1}, A_{m+1}],
\end{aligned}$$

hence

$$[A_{-2m+1}, A_1] = [A_{-m+1}, A_{m+1}].$$

Therefore we obtain the result for even N.

For $N = 2m+1$, we act $ad_{A_0}ad_{A_1}$ and $ad_{A_1}ad_{A_0}$ on the equations $[A_{-2m}, A_0] = [A_{-m}, A_m]$, $[A_{-2m+1}, A_1] = [A_{-m}, A_m]$ respectively. As before, we get

$$\begin{aligned} &8[A_{-2m+1}, A_0] + 8[A_0, A_{2m+1}] \\ &= [ad_{A_0}ad_{A_1}A_{-m}, A_m] + [ad_{A_1}A_{-m}, ad_{A_0}A_m] \\ &+[ad_{A_0}A_{-m}, ad_{A_1}A_m] + [A_{-m}, ad_{A_0}ad_{A_1}A_m] \\ &= 8[A_{-m-1}, A_m] + 8[A_{-m}, A_{m-1}] \\ &+[ad_{A_1}A_{-m}, ad_{A_0}A_m] + [ad_{A_0}A_{-m}, ad_{A_1}A_m], \end{aligned}$$

and

$$\begin{aligned} &8[A_1, A_{2m}] + 8[A_{-2m}, A_1] \\ &= [ad_{A_1}ad_{A_0}A_{-m}, A_m] + [ad_{A_0}A_{-m}, ad_{A_1}A_m] \\ &+[ad_{A_1}A_{-m}, ad_{A_0}A_m] + [A_{-m}, ad_{A_1}ad_{A_0}A_m] \\ &= 8[A_{-m+1}, A_m] + 8[A_{-m}, A_{m+1}] \\ &+[ad_{A_0}A_{-m}, ad_{A_1}A_m] + [ad_{A_1}A_{-m}, ad_{A_0}A_m]. \end{aligned}$$

Subtract the second from the first of the above equations and drop equalities belonging to the induction assumption. We have the extra relation

$$[A_0, A_{2m+1}] - [A_{-2m}, A_1] = [A_{-m-1}, A_m] - [A_{-m}, A_{m+1}],$$

from which the result follows. q.e.d.

From the above Lemma 3–6, we obtain Theorem 1. Now we can easily embed Onsager's algebra into the loop algebra $L(\mathfrak{g})$.

Proposition 1 Onsager's algebra $\mathfrak{A}$ is isomorphic to the Lie-subalgebra of $L(\mathfrak{g})$ generated by elements $2t^mE + 2t^{-m}F$ and $(t^m - t^{-m})H$, $m \in \mathbf{Z}$.

Proof. Denote

$$\begin{aligned} \overline{A_m} &= 2t^mE + 2t^{-m}F \\ \overline{G_m} &= (t^m - t^{-m})H \quad \text{for } m \in \mathbf{Z}. \end{aligned}$$

Then $\overline{A_m}$, $\overline{G_m}$ satisfy the relations $(2_a), (2_b)$ and (2_c), hence there exists a Lie-homomorphism

$$\varphi : \mathfrak{A} \to L(\mathfrak{g})$$

with $\varphi(A_m) = \overline{A_m}$, $\varphi(G_m) = \overline{G_m}$. It is easy to see that $\overline{A_m}, \overline{G_m}$ $(m \in \mathbf{Z})$ are linearly independent elements in $L(\mathfrak{g})$. Then the result follows immediately. q.e.d.

Denote the Lie-involutions

$$\begin{aligned} &\theta : \mathfrak{g} \to \mathfrak{g}, \quad \theta(\mathrm{E}) = \mathrm{F}, \theta(\mathrm{F}) = \mathrm{E}, \theta(\mathrm{H}) = -\mathrm{H}, \\ &\theta_0 : L(\mathfrak{g}) \to L(\mathfrak{g}), \quad \theta_0(\mathrm{f}(t) \otimes \mathrm{x}) = \mathrm{f}(t^{-1}) \otimes \theta(\mathrm{x}). \end{aligned}$$

By the above Proposition, we may regard $\mathfrak{A}$ as the Lie-subalgebra of $L(\mathfrak{g})$ fixed by θ_0. For the rest of this note we shall always make this identification and write

$$A_m = 2t^m E + 2t^{-m} F$$
$$G_m = \left(t^m - t^{-m}\right) H \qquad \text{for } m \in \mathbf{Z}\,.$$

2. *Evaluation map of Onsager's algebra*

Let $\bigoplus_n \mathfrak{g}$ be the direct sum of n copies of $\mathfrak{g}$. The generators E, F, H of $\mathfrak{g}$ in the i–th factor of $\bigoplus_n \mathfrak{g}$ are denoted by E_i, F_i, H_i. Then E_i, F_i, H_i $(1 \leq i \leq n)$ form a base of $\bigoplus_n \mathfrak{g}$. For $\mathbf{a} = (a_1, ..., a_n) \in (\mathbf{C}^*)^n$, the homomorphism

$$L(\mathfrak{g}) \to \bigoplus_n \mathfrak{g} \quad , f \rightsquigarrow (f(a_1), \ldots, f(a_n)),$$

is a Lie algebra homomorphism. Its restriction on $\mathfrak{A}$ is called the evaluation of $\mathfrak{A}$ at $\mathbf{a}$ and denoted by

$$e_{\mathbf{a}} : \mathfrak{A} \to \bigoplus_n \mathfrak{g}\,.$$

We have

$$e_{\mathbf{a}}(A_m) = 2 \sum_{i=1}^{n} \left(a_i^m E_i + a_i^{-m} F_i\right) \quad , \qquad (9)$$
$$e_{\mathbf{a}}(G_m) = \sum_{i=1}^{n} \left(a_i^m - a_i^{-m}\right) H_i \quad , \quad m \in \mathbf{Z}. \quad (10)$$

We shall study the representations of $\mathfrak{A}$ via the above evaluation maps. Introduce the notation

$\mathcal{C} = (\mathbf{C} - \{0, \pm 1\}) / \sim$, here $\sim$ is defined by identifying a with a^{-1}.

Note that $\mathcal{C}$ is bijective to the complex plane minus two points. For $a \in \mathbf{C} - \{0, \pm 1\}$, $\overline{a}$ denote the element of $\mathcal{C}$ determined by a.

Lemma 7 For $\mathbf{a} = (a_1, ..., a_n) \in (\mathbf{C}^*)^n$, $e_{\mathbf{a}}$ is a surjective map if and only if the conditions

$$a_i^2 \neq 1, \ \overline{a_i} \neq \overline{a_j} \quad \text{for } i \neq j,$$

hold.

Proof: Denote

$$r = max\{\ i \in \mathbf{Z}_{\geq 0} | e_{\mathbf{a}}(A_0), ..., e_{\mathbf{a}}(A_i) \text{ are linear independent}\},$$
$$\mathcal{A} = < e(A_j),\ 1 \leq j \leq r >_{\mathbf{C}}.$$

$$\text{Claim}: \quad \mathcal{A} = < e_{\mathbf{a}}(A_m),\ m \in \mathbf{Z} >_{\mathbf{C}} .$$

There is a polynomial P(t) of degree $r+1$

$$\mathrm{P(t)} = \mathrm{p}_0 + \mathrm{p}_1 \mathrm{t} + \ldots + \mathrm{p}_{r+1} \mathrm{t}^{r+1}$$

such that

$$\mathrm{p}_0 e_{\mathbf{a}}(A_0) + \mathrm{p}_1 e_{\mathbf{a}}(A_1) + \ldots + \mathrm{p}_{r+1} e_{\mathbf{a}}(A_{r+1}) = 0,$$

which means

$$\mathrm{P}(a_j) = \mathrm{P}\left(a_j^{-1}\right) = 0 \qquad \text{for } 1 \leq j \leq n .$$

Note that the constant term p_0 can not be 0, otherwise

$$\mathrm{P(t)} = \mathrm{tQ(t)}, \quad \mathrm{Q}(a_j) = \mathrm{Q}\left(a_j^{-1}\right) = 0 \quad \text{for all } j,$$

which contradicts the linear independence of $e_{\mathbf{a}}(A_1), \ldots, e_{\mathbf{a}}(A_r)$. By $\mathrm{p}_{r+1} \neq 0$, we have $e_{\mathbf{a}}(A_{r+1}) \in \mathcal{A}$. For $s \in \mathbf{N}$,

$$a_j^{s}\mathrm{P}(a_j) = a_j^{-s}\mathrm{P}\left(a_j^{-1}\right) = 0 \qquad \text{for } 1 \leq j \leq n,$$

which implies

$$e_{\mathbf{a}}(A_{r+1+s}) \in < e_{\mathbf{a}}(A_s), \ldots, e_{\mathbf{a}}(A_{r+s}) >_{\mathbf{C}} .$$

By induction, $e_{\mathbf{a}}(A_m) \in \mathcal{A}$ for $m \geq 0$. Similarly,

$$a_j^{-s}\mathrm{P}(a_j) = a_j^{s}\mathrm{P}\left(a_j^{-1}\right) = 0 \ \text{ for } 1 \leq j \leq n,\ s > 0,$$

By $\mathrm{p}_0 \neq 0$,$e_{\mathbf{a}}(A_{-s}) \in < e_{\mathbf{a}}(A_{-s+1}), \ldots, e_{\mathbf{a}}(A_{-s+1+r}) >_{\mathbf{C}}$. Hence $e_{\mathbf{a}}(A_m) \in \mathcal{A}$ for m<0, therefore

$$\mathcal{A} = < e_{\mathbf{a}}(A_m),\ m \in \mathbf{Z} >_{\mathbf{C}} .$$

As

$$\begin{aligned} e_{\mathbf{a}}(A_m) &\in \sum_{i=1}^{n}(\mathbf{C}E_i + \mathbf{C}F_i), \\ e_{\mathbf{a}}(G_m) &\in \sum_{i=1}^{n} \mathbf{C}H_i \qquad\qquad \text{for } m \in \mathbf{Z}, \end{aligned}$$

the surjectivity of the Lie-homomorphism $e_{\mathbf{a}}$ is equivalent to the equality

$$\mathcal{A} = \sum_{i=1}^{n} (\mathrm{C}F_{i}+\mathrm{C}F_{i}) .$$

It is easy to see the following equivalent conditions hold,

$$\mathcal{A} = \sum_{i=1}^{n} (\mathrm{C}F_{i}+\mathrm{C}F_{i})$$

$$\Leftrightarrow det \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & \dots & a_n & a_1^{-1} & \dots & a_n^{-1} \\ \cdot & \dots & \cdot & \dots & \dots & \cdot \\ a_1^i & \dots & a_n^i & a_1^{-i} & \dots & a_n^{-i} \\ \cdot & \dots & \dots & \dots & \dots & \cdot \\ a_1^r & \dots & a_n^r & a_1^{-r} & \dots & a_n^{-r} \end{pmatrix} \neq 0 \text{ with } r = 2n-1 ,$$

$$\Leftrightarrow a_j^2 \neq 1, \ a_j \neq a_k^{\pm 1} \ \text{ for } j \neq k.$$

There follows the conclusion. q.e.d.

Remark The above polynomial

$$\mathrm{P(t)} = \mathrm{p}_0 + \mathrm{p}_1 \mathrm{t} + \dots + \mathrm{p}_{r+1} \mathrm{t}^{r+1}$$

has the degree $r+1 = 2n$ with $2n$ distinct roots a_j, a_j^{-1} $(j = 1, \dots, n)$. This polynomial P(t) (up to a scalar) is characterized by one of the following equivalent conditions:

(i) P(t) is the polynomial with the smallest degree such that

$$\mathrm{p}_0 e_{\mathbf{a}}(A_0) + \mathrm{p}_1 e_{\mathbf{a}}(A_1) + \dots + \mathrm{p}_{r+1} e_{\mathbf{a}}(A_{r+1}) = 0 .$$

(ii) P(t) is the polynomial with the smallest degree such that

$$\mathrm{p}_0 e_{\mathbf{a}}(A_m) + \mathrm{p}_1 e_{\mathbf{a}}(A_{m+1}) + \dots + \mathrm{p}_{r+1} e_{\mathbf{a}}(A_{m+r+1}) = 0 \quad \text{for } m \in \mathbf{Z} .$$

For non-trivial integrable dominate weights μ_i in $\mathcal{P}$, $1 \leq i \leq n$, $\bigotimes_{i=1}^{n} V(\mu_i)$ is an irreducible $\bigoplus_{n} \mathfrak{g}$—module under action

$$(\xi_1, \dots, \xi_n)v = \sum_{i} (1 \otimes .. \otimes \xi_i \otimes .. \otimes 1)v .$$

It defines the $\mathfrak{A}$—module $V_{\mathbf{a}}(\mu) = \bigotimes_{i} V(\mu_i)$ via $e_{\mathbf{a}}$ for a= $(a_1, \dots, a_n) \in (\mathrm{C}^*)^n$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{P}^n$. By Lemma 7, $V_{\mathbf{a}}(\mu)$ is an irreducible $\mathfrak{A}$—module when $\overline{a_i}$ $(1 \leq i \leq n)$ are distinct elements in $\mathcal{C}$.

Lemma 8 Let $\mathrm{a}=(a_1,...,a_n)$, $\mathrm{a}'=(a'_1,...,a'_n)$ be elements in $(\mathrm{C}^*)^n$ with $a_i^2 = a_i'^2$ for all i. Then $V_{\mathrm{a}}(\mu) \simeq V_{\mathrm{a}'}(\mu)$ as $\mathfrak{A}$—modules for $\mu \in \mathcal{P}^n$.

Proof. Renumbering the indices, we may assume $a_i = a_i'^{-1}$ for $i \le m$, $a_i = a'_i$ for $i > m$. For each $i \le m$, there is an automorphism φ_i of $V(\mu_i)$ such that

$$\varphi_i(\xi v) = \theta(\xi)\varphi_i(v) \quad \text{for } \xi \in \mathfrak{g}, \quad v \in V(\mu_i)\,.$$

Let

$$\Phi : \bigotimes_{i=1}^{n} V(\mu_i) \to \bigotimes_{i=1}^{n} V(\mu_i)\,,$$

$$\Theta : \bigoplus_n \mathfrak{g} \to \bigoplus_n \mathfrak{g}$$

be the isomorphisms defined by $\Phi = \varphi_1 \otimes .. \otimes \varphi_m \otimes id. \otimes \ldots \otimes id.$, $\Theta = \underbrace{\theta \oplus .. \oplus \theta}_{m} \oplus id. \oplus \ldots \oplus id.$. It is easy to see that Θ is a Lie-isomorphism and $\Theta \cdot e_{\mathrm{a}} = e_{\mathrm{a}'}$. By the relation

$$\Phi(Xw) = \Theta(X)\Phi(w) \quad \text{for } X \in \bigoplus_n \mathfrak{g},\ w \in \bigotimes_{i=1}^{n} V(\mu_i)\,,$$

$V_{\mathrm{a}}(\mu) \simeq V_{\mathrm{a}'}(\mu)$ as $\mathfrak{A}$—modules. q.e.d.

For $\mathrm{a}=(a_1,...,a_n) \in (\mathrm{C}^*)^n$ with $a_i^2 \neq 1$, the structure of $\mathfrak{A}$—module $V_{\mathrm{a}}(\mu)$ depends only on the elements $\overline{a_i}$ in C and we shall write

$$V_{\bar{\mathrm{a}}}(\mu) = V_{\mathrm{a}}(\mu) \quad \text{here } \bar{\mathrm{a}} = (\overline{a_1},...,\overline{a_n}) \in C^n\ .$$

Denote

$$(C^n)' = \{\bar{\mathrm{a}} = (\overline{a_1},\ldots,\overline{a_n}) \in C^n \mid \overline{a_i} \neq \overline{a_j} \ \text{ for } \ i \neq j\}.$$

Proposition 2 For $(\mu,\bar{\mathrm{a}}) \in \mathcal{P}^n \times (C^n)'$, $(\mu',\overline{\mathrm{a}'}) \in \mathcal{P}^l \times (C^l)'$,

$$V_{\bar{\mathrm{a}}}(\mu) \simeq V_{\overline{\mathrm{a}'}}(\mu')$$

as $\mathfrak{A}$—modules if and only if $n = l$ and for some permutation σ of $\{1,...,n\}$, $\mu_i = \mu'_{\sigma(i)}$, $\overline{a_i} = \overline{a'_{\sigma(i)}}$ for $i = 1,..,n$.

Proof. When $V_{\bar{\mathrm{a}}}(\mu)$ is isomorphic to $V_{\overline{\mathrm{a}'}}(\mu')$, the polynomials P(t), P′(t) corresponding to $V_{\mathrm{a}}(\mu)$ and $V_{\mathrm{a}'}(\mu')$ are the same by the remark of Lemma 7. Then $n = l$ and for some permutation σ of $\{1,...,n\}$, $a_i^{\pm 1} = a'_{\sigma(i)}$, $i = 1,\ldots,n$. It follows $\mu_i = \mu'_{\sigma(i)}$. The converse statement is obvious. q.e.d.

3. *Representation of Onsager's algebra*

In this section we shall give the classification of all the irreducible finite dimensional representations of $\mathfrak{A}$. Let V be a such $\mathfrak{A}$—module with $\dim_{\mathrm{C}} V > 1$.

Denote $\overline{A_m}$, $\overline{G_m}$ the linear transformations of V corresponding to the $\mathfrak{A}$−action of A_m, G_m for $m \in \mathbf{Z}$. Since the addition of a constant multiple of the identity matrix to either (or both) to $\overline{A_0}$, $\overline{A_1}$ makes no difference to DG conditions , the traces of $\overline{A_0}$, $\overline{A_1}$ can be reduced to 0. Hence for the discussion of this section we shall always assume

$$\overline{A_m},\ \overline{G_m} \in sl(V) \ \text{ for } m \in \mathbf{Z}\,.$$

The $\overline{A_m}$'s generate a finite dimensional subspace of $sl(V)$. For some positive integer r, there is a non-trivial linear relation of $\{\overline{A_m} | m = 0, ..., r+1\}$, namely

$$\mathrm{p}_0\overline{A_0} + \mathrm{p}_1\overline{A_1} + ... + \mathrm{p}_{r+1}\overline{A_{r+1}} = 0 \ \ , \ \ \mathrm{p_j} \in \mathbf{C}\,.$$

Claim: We have the more general recurrence relations of length $r+1$:

$$\sum_{j=1}^{r+1} \mathrm{p_j}\overline{A_{j+s}} = 0\,, \qquad (11)$$

$$\sum_{j=1}^{r+1} \mathrm{p_j}\overline{G_{j+s}} = 0 \qquad (12)$$

for $s \in \mathbf{Z}$.

The second equality simply follows from (2_b). This implies

$$\left[\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+1}},\ \overline{A_0}\right] = \left[\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+1}},\ \overline{A_1}\right] = 0\,.$$

By Schur's Lemma,

$$\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+1}} = 0\,.$$

By (2_b),

$$ad_{G_1}\left(\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+s}}\right) = 2\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+s+1}} - 2\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{j+s-1}} \quad \text{for } s \in \mathbf{Z}\,.$$

Therefore the equalities (11) hold for s+1, s-2 whenever it holds for s-1 and s. Hence the conclusions follow from the case s=0, 1.

We now assume the length $r+1$ of the above recurrence relation is the minimal one. By (2_b), we also have

$$\sum_{j=0}^{r+1} \mathrm{p_j}\overline{A_{-j+s}} = 0\ . \qquad (13)$$

The minimal property of $r+1$ implies either $p_j = p_{r+1-j}$ for all j or $p_j = -p_{r+1-j}$ for all j. Define the polynomial

$$P(t) = p_0 + p_1 t + \ldots + p_{r+1} t^{r+1}.$$

Then

$$P(t) = \pm t^{r+1} P\left(\frac{1}{t}\right). \qquad (14)$$

Consequently the zeros of P(t) not equal to ± 1 occur in reciprocal pairs. This polynomial is characterized by the following property:

Definition The polynomial P(t) satisfying (11) (12) (14) with the minimal degree is called the minimal polynomial of the irreducible finite dimensional $\mathfrak{A}$−module V.

Remark. The minimal polynomial of $V_{\overline{n}}(\mu)$ is the one appeared in the proof of Lemma 7.

Label the zeros (with multiplicity) of P(t)

$$a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1},\ b_1, \ldots, b_{n'} \qquad (15)$$

in such way that

$$a_i^2 \neq 1,\ \ b_j^2 = 1 \quad \text{for all} \quad i, j \ ,$$

and for some finite sequences $\{k_s\}$, $\{k_t'\}$,

$$1 = k_1 < k_2 < \ldots < k_d \leq n,\ \ 1 = k_1' < k_2' < \ldots < k_{d'}' \leq n' \ , \qquad (16)$$

and

$$\begin{aligned} a_i = a_j \quad &\Leftrightarrow \quad k_s \leq i, j < k_{s+1} \quad \text{for some } s, \\ b_i = b_j \quad &\Leftrightarrow \quad k_t' \leq i, j < k_{t+1}' \quad \text{for some } t. \end{aligned}$$

The linear subspace of $sl(V)$ spanned by $\overline{A_m}$ $(m \in \mathbf{Z})$ has the dimension $r+1$ $(= 2n + n')$ with a base $\overline{E_i}, \overline{F_i}, e_j$ $(i = 1, \ldots, n,\ j = 1, \ldots, n')$ which are determined by the relations

$$\overline{A_m} = 2 \sum_{i=1}^{n} \left(c_{m,i} \overline{E_i} + c_{-m,i} \overline{F_i}\right) + 2 \sum_{j=1}^{n'} d_{m,j} e_j$$

$$\text{with } c_{m,i} = \left(\frac{d}{dt}\right)^{i-k_s} (t^m)_{|t=a_i}\ ,\ \ d_{m,j} = \left(\frac{d}{dt}\right)^{j-k_t'} (t^m)_{|t=b_j}\ ,$$

$$\text{here } m \in \mathbf{Z},\quad k_s \leq i < k_{s+1}\ ,\quad k_t' \leq j < k_{t+1}'.$$

Then $c_{m,i}\,, d_{m,j}$ $(m \in \mathbf{Z},\ 1 \le i \le n,\ 1 \le j \le n')$ satisfy the assumption of the following lemma.

Lemma 9 Let $n,\ n'$ be non-negative integers with $2n + n' = r + 1$. Let $\overline{E_i}\,, \overline{F_i}\,, e_j$ be elements in $sl(V)$ and $c_{m,i}\,, d_{m,j}$ $(m \in \mathbf{Z},\ 1 \le i \le n,\ 1 \le j \le n')$ complex numbers with $d_{m,j} = d_{-m,j}$, such that the following conditions hold:

$$\text{(i)}\quad \overline{A_m} = 2\sum_{i=1}^{n} \left(c_{m,i}\overline{E_i} + c_{-m,i}\overline{F_i}\right) + 2\sum_{j=1}^{n'} d_{m,j}e_j \quad \text{for } m \in \mathbf{Z}\ ,$$

(ii) For $l \in \mathbf{Z}$,

$$\begin{pmatrix} c_{l,1} & \cdots & c_{l,n} & c_{-l,1} & \cdots & c_{-l,n} & d_{l,1} & \cdots & d_{l,n'} \\ & \vdots & & & \cdots & & & \vdots & \\ c_{l+r,1} & \cdots & c_{l+r,n} & c_{-l-r,1} & \cdots & c_{-l-r,n} & d_{l+r,1} & \cdots & d_{l+r,n'} \end{pmatrix}$$
$$= \begin{pmatrix} c_{0,1} & \cdots & c_{0,n} & c_{0,1} & \cdots & c_{0,n} & d_{0,1} & \cdots & d_{0,n'} \\ & \vdots & & & \cdots & & & \vdots & \\ c_{r,1} & \cdots & c_{r,n} & c_{-r,1} & \cdots & c_{-r,n} & d_{r,1} & \cdots & d_{r,n'} \end{pmatrix} U_l$$

for some upper triangular matrix U_l. Then $e_j\ , 1 \le j \le n'$, commute with all the $\overline{A_m}$.

Proof. It suffices to show that for $1 \le j \le n'$,

$$[X,\ e_j] = 0 \quad \text{for} \quad X = \overline{E_i}\,, \overline{F_i}, e_k\ .$$

By (i), it follows

$$4\overline{G_m} = \left[\overline{A_m}\,, \overline{A_0}\right] = 2\sum_{i=1}^{n} \left[c_{m,i}\overline{E_i} + c_{-m,i}\overline{F_i}\,, \overline{A_0}\right] + 2\sum_{j=1}^{n'} d_{m,j}\left[e_j\,, \overline{A_0}\right]$$
$$= \sum_{i=1}^{n} (c_{m,i} + c_{-m,i})\left[\overline{E_i} + \overline{F_i}\,,\ \overline{A_0}\right] + (c_{m,i} - c_{-m,i})\left[\overline{E_i} - \overline{F_i}\,,\ \overline{A_0}\right]$$
$$+2\sum_{j=1}^{n'} d_{m,j}\left[e_j\,, \overline{A_0}\right]\ .$$

By $d_{m,j} = d_{-m,j}$ and $\overline{G_m} = -\overline{G_{-m}}$, we have

$$\overline{G_m} = \sum_{i=1}^{n} \left(c_{m,i}\overline{H_i} - c_{-m,i}\overline{H_i}\right) + \sum_{j=1}^{n'} d_{m,j}h_j$$
$$\text{with } 4\overline{H_i} = \left[\overline{E_i} - \overline{F_i}\,, \overline{A_0}\right],\ \ h_j = 0.$$

Let $(\alpha_{s,t})$ denote the inverse of the matrix

$$2\begin{pmatrix} c_{0,1} & \cdots & c_{0,n} & c_{0,1} & \cdots & c_{0,n} & d_{0,1} & \cdots & d_{0,n'} \\ & \vdots & & & \cdots & & & \vdots & \\ c_{r,1} & \cdots & c_{r,n} & c_{-r,1} & \cdots & c_{-r,n} & d_{r,1} & \cdots & d_{r,n'} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \overline{A_0} \\ \overline{A_1} \\ \vdots \\ \overline{A_r} \end{pmatrix} = 2\begin{pmatrix} c_{0,1} & \cdots & c_{0,n} & c_{0,1} & \cdots & c_{0,n} & d_{0,1} & \cdots & d_{0,n'} \\ c_{1,1} & \cdots & c_{1,n} & c_{-1,1} & \cdots & c_{-1,n} & d_{1,1} & \cdots & d_{1,n'} \\ & \vdots & & & \cdots & & & \vdots & \\ c_{r,1} & \cdots & c_{r,n} & c_{-r,1} & \cdots & c_{-r,n} & d_{r,1} & \cdots & d_{r,n'} \end{pmatrix} \begin{pmatrix} \overline{E_1} \\ \vdots \\ \overline{E_n} \\ \overline{F_1} \\ \vdots \\ \overline{F_n} \\ e_1 \\ \vdots \\ e_{n'} \end{pmatrix},$$

$$\begin{pmatrix} \overline{G_l} \\ \overline{G_{l+1}} \\ \vdots \\ \overline{G_{l+r}} \end{pmatrix} = \begin{pmatrix} c_{l,1} & \cdots & c_{l,n} & c_{-l,1} & \cdots & c_{-l,n} & d_{l,1} & \cdots & d_{l,n'} \\ c_{l+1,1} & \cdots & c_{l+1,n} & c_{-l-1,1} & \cdots & c_{-l-1,n} & d_{l+1,1} & \cdots & d_{l+1,n'} \\ & \vdots & & & \cdots & & & \vdots & \\ c_{l+r,1} & \cdots & c_{l+r,n} & c_{-l-r,1} & \cdots & c_{-l-r,n} & d_{l+r,1} & \cdots & d_{l+r,n'} \end{pmatrix} \begin{pmatrix} \overline{H_1} \\ \vdots \\ \overline{H_n} \\ -\overline{H_1} \\ \vdots \\ -\overline{H_n} \\ h_1 \\ \vdots \\ h_{n'} \end{pmatrix},$$

hence

$$(\alpha_{s,t})\begin{pmatrix} \overline{A_0} \\ \overline{A_1} \\ \vdots \\ \overline{A_r} \end{pmatrix} = \begin{pmatrix} \overline{E_1} \\ \vdots \\ \overline{E_n} \\ \overline{F_1} \\ \vdots \\ \overline{F_n} \\ e_1 \\ \vdots \\ e_{n'} \end{pmatrix}, \qquad (17)$$

$$2(\alpha_{s,t})\begin{pmatrix} \overline{G_l} \\ \overline{G_{l+1}} \\ \vdots \\ \overline{G_{l+r}} \end{pmatrix} = U_l \begin{pmatrix} \overline{H_1} \\ \vdots \\ \overline{H_n} \\ -\overline{H_1} \\ \vdots \\ -\overline{H_n} \\ h_1 \\ \vdots \\ h_{n'} \end{pmatrix}. \qquad (18)$$

For $X = \overline{E_i}$, $\overline{F_i}$ or e_k, by $h_j = 0$ and the upper triangular property of the matrix U_l, we have

$$\begin{aligned} [X,\, e_j] &= \sum_{l,k=0}^{r} \alpha_{s,l}\alpha_{2n+j,k}\left[\overline{A_l},\, \overline{A_k}\right] = -4\sum_{l,k=0}^{r} \alpha_{s,l}\alpha_{2n+j,k}\overline{G_{k-l}} \\ &= -2\sum_{l=0}^{r}\alpha_{s,l}\left(2\sum_{k=0}^{r}\alpha_{2n+j,k}\overline{G_{k-l}}\right) = 0\,. \end{aligned}$$

q.e.d.

The above lemma implies that if $n' > 0$, the linear subspace of $sl(V)$ spanned by $\overline{A_m}$ $(m \in \mathbf{Z})$ has a non-trivial center, which contradicts the irreducibility of the $\mathfrak{A}-$module V. Hence ± 1 can not be a zero of P(t), and all the zeros of P(t) are now labelled by (15) with $n' = 0$.

Proposition 3 Let V be a non-trivial irreducible representation of $\mathfrak{A}$. If the minimal polynomial of V has only simple zeros, then V is isomorphic to $V_{\overline{\mathrm{a}}}(\mu)$ for some $(\mu, \overline{\mathrm{a}}) \in \mathcal{P}^n \times (\mathcal{C}^n)'$.

Proof. Let P(t) be the minimal polynomial and $r+1$ be the degree. Label its zeros in pairs, i.e. the zeros are

$$a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$$

with $r+1 = 2n$ and $a_i^2 \neq \pm 1$ for all i.. As before, let

$$\begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,r} \\ \vdots & & \cdots & \vdots \\ \vdots & & \cdots & \vdots \\ \alpha_{2n,0} & \alpha_{2n,1} & \cdots & \alpha_{2n,r} \end{pmatrix}$$

be the inverse of the non-singular matrix

$$2\begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_n & a_1^{-1} & \cdots & a_n^{-1} \\ . & \cdots & . & . & \cdots & . \\ a_1^i & \cdots & a_n^i & a_1^{-i} & \cdots & a_n^{-i} \\ . & \cdots & . & . & \cdots & . \\ a_1^r & \cdots & a_n^r & a_1^{-r} & \cdots & a_n^{-r} \end{pmatrix},$$

and $\overline{E_i}, \overline{F_i}$, $1 \leq i \leq n$, be the elements of $sl(V)$ defined by

$$\overline{E_i} = \sum_{j=0}^{r} \alpha_{i,j} \overline{A_j}\ , \quad \overline{F_i} = \sum_{j=0}^{r} \alpha_{n+i,j} \overline{A_j}\ . \qquad (19)$$

Then

$$\overline{A_m} = 2\sum_{i=1}^{n} (a_i^m \overline{E_i} + a_i^{-m} \overline{F_i})\ , \quad \text{for } m \in \mathbf{Z}\ . \qquad (20)$$

By (17) (18), it follows that for $i = 1, \ldots, n$, $l \in \mathbf{Z}$,

$$\begin{aligned} a_i^l \overline{E_i} &= \sum_{j=0}^{r} \alpha_{i,j} \overline{A_{j+l}}\ , & a_i^{-l} \overline{F_i} &= \sum_{j=0}^{r} \alpha_{n+i,j} \overline{A_{j+l}}\ , \\ a_i^l \overline{H_i} &= 2\sum_{j=0}^{r} \alpha_{i,j} \overline{G_{j+l}}\ , & -a_i^{-l} \overline{H_i} &= 2\sum_{j=0}^{r} \alpha_{n+i,j} \overline{G_{j+l}}\ . \end{aligned} \qquad (21)$$

Claim: $\overline{E_i}, \overline{F_i}, \overline{H_i}$ $(i = 1, \ldots, n)$ satisfy the relations

$$\begin{gathered}[\overline{E_i}, \overline{E_j}] = [\overline{F_i}, \overline{F_j}] = [\overline{H_i}, \overline{H_j}] = 0\,, \\ [\overline{E_i}, \overline{F_j}] = \delta_{i,j}\overline{H_j}\,, \quad [\overline{H_i}, \overline{E_j}] = 2\delta_{i,j}\overline{E_j}\,, \quad [\overline{H_i}, \overline{F_j}] = -2\delta_{i,j}\overline{F_j}\,.\end{gathered} \tag{22}$$

By the direct computation and using Theorem 1, we have

$$[\overline{E_i}, \overline{E_j}] = \sum_{l,k=0}^{r} \alpha_{il}\alpha_{jk}[\overline{A_l}, \overline{A_k}] = 4\sum_{l,k=0}^{r} \alpha_{il}\alpha_{jk}\overline{G_{l-k}} = -2\sum_{l=0}^{r} \alpha_{il}a_j^{-l}\overline{H_j} = 0\,,$$

$$[\overline{H_i}, \overline{H_j}] = 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{j,k}[\overline{G_l}, \overline{G_k}] = 0\,,$$

$$\begin{aligned}[\overline{E_i}, \overline{F_j}] &= 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{n+j,k}[\overline{A_l}, \overline{A_k}] = 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{n+j,k}\overline{G_{l-k}} \\ &= 2\sum_{l=0}^{r} \alpha_{i,l}a_j^l\overline{H_j} = \delta_{i,j}\overline{H_j}\,,\end{aligned}$$

$$[\overline{H_i}, \overline{E_j}] = 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{j,k}[\overline{G_l}, \overline{A_k}] = 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{j,k}(\overline{A_{k+l}} - \overline{A_{k-l}})$$

$$\begin{aligned}[\overline{H_i}, \overline{E_j}] &= 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{j,k}[\overline{G_l}, \overline{A_k}] = 4\sum_{l,k=0}^{r} \alpha_{i,l}\alpha_{j,k}(\overline{A_{k+l}} - \overline{A_{k-l}}) \\ &= 4\sum_{l=0}^{r} \alpha_{i,l}\left(a_j^l\overline{E_j} - a_j^{-l}\overline{F_j}\right) = 2\delta_{i,j}\overline{E_j}\ .\end{aligned}$$

The same argument for $[\overline{F_i}, \overline{F_j}] = 0$ and $[\overline{H_i}, \overline{F_j}] = -2\delta_{i,j}\overline{F_j}$. Hence $\mathfrak{A}$—action of V can be factored through a representation of $\bigoplus_n \mathfrak{g}$ via $e_{\mathbf{a}}$

$$\begin{array}{ccc}\mathfrak{A} & \rightarrow & sl(V) \\ & e_{\mathbf{a}} \searrow & \uparrow \\ & & \bigoplus_n \mathfrak{g}\end{array}\ .$$

The irreducibility of $\mathfrak{A}$—module V implies V also irreducible as $\bigoplus_n \mathfrak{g}$ module, hence corresponds to some $\mu \in \mathcal{P}^n$. Therefore V is isomorphic to $V_{\mathbf{a}}(\mu)$ as $\mathfrak{A}$—modules. q.e.d.

We are going to show the minimal polynomial P(t) of an irreducible representation of $\mathfrak{A}$ always satisfies the assumption of the above Proposition.

Proposition 4 P(t) has no multiple zeros.

Proof. Step I. Claim: $\pm\sqrt{-1}$ are the only possible multiple zeros of P(t), and in this situation it has the multiplicity 2.

Say a_1 is a multiple zero, i.e. the difference $\delta := k_2 - k_1$ in (16) is greater than 1. Let $(\alpha_{s,t})$ be the matrix in the proof of Lemma 9. Now the formula (17) (18) give

$$\overline{E_{\delta-1}} = \sum_{s=0}^{r} \alpha_{\delta-1,s}\overline{A_s}\,,$$

$$\overline{H_\delta} = 2\sum_{s=0}^{r} \alpha_{\delta,s}\overline{G_s}\,, \qquad -\overline{H_\delta} = 2\sum_{s=0}^{r} \alpha_{n+\delta,s}\overline{G_s}\,.$$

The conditions (i) (ii) of Lemma 9 and the formula (17) (18) imply

$$(\alpha_{s,t})\begin{pmatrix}\overline{A_l}\\ \overline{A_{l+1}}\\ \vdots\\ \overline{A_{l+r}}\end{pmatrix}=U_l\begin{pmatrix}\overline{E_1}\\ \vdots\\ \overline{E_n}\\ \overline{F_1}\\ \vdots\\ \overline{F_n}\end{pmatrix}, \tag{23}$$

$$2(\alpha_{s,t})\begin{pmatrix}\overline{G_l}\\ \overline{G_{l+1}}\\ \vdots\\ \overline{G_{l+r}}\end{pmatrix}=U_l\begin{pmatrix}\overline{H_1}\\ \vdots\\ \overline{H_n}\\ -\overline{H_1}\\ \vdots\\ -\overline{H_n}\end{pmatrix}. \tag{24}$$

The entries $U_{l;s,t}$ of the above upper triangular matrix U_l can be calculated from Leibniz's formula

$$(\varphi\psi)^{(n)}=\sum_{\gamma=0}^{n}\binom{n}{\gamma}\varphi^{(\gamma)}\psi^{(n-\gamma)}$$

with $\varphi(x)=x^l,\ \psi(x)=x^s$. In particular,

$$U_{l;1,t}=\begin{cases}\frac{d^{t-1}}{dx^{t-1}}(x^l)_{|x=a_1} & \text{for } 1\le t\le\delta,\\ 0 & \text{otherwise.}\end{cases}$$

$$U_{l;\delta-1,t}=\begin{cases}a_1^l & \text{for } t=\delta-1,\\ (\delta-1)la_1^{l-1} & \text{for } t=\delta,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\overline{E_1},\ldots,\overline{E_n},\overline{F_1},\ldots,\overline{F_n}$ are linearly independent. We have

$$\begin{aligned}&[\overline{H_2},\overline{E_{\delta-1}}]=2\sum_{s,t=0}^{r}\alpha_{2,s}\alpha_{\delta-1,t}[\overline{G_s},\overline{A_t}]=4\sum_{s,t=0}^{r}\alpha_{2,s}\alpha_{\delta-1,t}(\overline{A_{t+s}}-\overline{A_{t-s}})\\ &=4\sum_{s=0}^{r}\alpha_{2,s}\{a_1^s\overline{E_{\delta-1}}+(\delta-1)sa_1^{s-1}\overline{E_\delta}-a_1^{-s}\overline{E_{\delta-1}}-(\delta-1)(-s)a_1^{-s+1}(a_1^{-2}\overline{E_\delta})\}\\ &=2(\delta-1)\overline{E_\delta}\ ,\end{aligned}$$

and

$$\begin{aligned}&[-\overline{H_2},\overline{E_{\delta-1}}]=2\sum_{s,t=0}^{r}\alpha_{n+2,s}\alpha_{\delta-1,t}[\overline{G_s},\overline{A_t}]=4\sum_{s,t=0}^{r}\alpha_{n+2,s}\alpha_{\delta-1,t}(\overline{A_{t+s}}-\overline{A_{t-s}})\\ &=4\sum_{s=0}^{r}\alpha_{n+2,s}\{a_1^s\overline{E_{\delta-1}}+(\delta-1)sa_1^{s-1}\overline{E_\delta}-a_1^{-s}\overline{E_{\delta-1}}-(\delta-1)(-s)a_1^{-s+1}(a_1^{-2}\overline{E_\delta})\}\\ &=2(\delta-1)a_1^{-2}\overline{E_\delta}\ ,\end{aligned}$$

hence $a_1^2 = -1$. We also have

$$\begin{aligned}[\overline{H_2}, \overline{E_1}] &= 4 \sum_{s,t=0}^{r} \alpha_{2,s}\alpha_{1,t}(\overline{A_{t+s}} - \overline{A_{t-s}}) \\ &= 4 \sum_{s=0}^{r} \alpha_{2,s} \left\{ \sum_{k=1}^{n} \frac{d^{k-1}}{dx^{k-1}}(x^s)_{|x=a_1} \overline{E_k} - \sum_{k=1}^{n} \frac{d^{k-1}}{dx^{k-1}}(x^{-s})_{|x=a_1} \overline{E_k} \right\} = 2\overline{E_2} \ ,\end{aligned}$$

$$\begin{aligned}[-\overline{H_2}, \overline{E_1}] &= 4 \sum_{s,t=0}^{r} \alpha_{n+2,s}\alpha_{1,t}(\overline{A_{t+s}} - \overline{A_{t-s}}) \\ &= 4 \sum_{s=0}^{r} \alpha_{n+2,s} \left\{ \sum_{k=1}^{n} \frac{d^{k-1}}{dx^{k-1}}(x^s)_{|x=a_1} \overline{E_k} - \sum_{k=1}^{n} \frac{d^{k-1}}{dx^{k-1}}(x^{-s})_{|x=a_1} \overline{E_k} \right\} \\ &= 2a_1^{-2}\overline{E_2} + \Delta\end{aligned}$$

here $\Delta = \sum_{k=3}^{\delta} \Delta_k \overline{E_k}$ and $\Delta_3 = 3a_1^{-3}$ in case $\delta \geq 3$, which contradicts the linear independence of $\overline{E_k}$'s. Therefore we obtain $a_1^2 = -1, \ \ \delta = 2$.

Step II. We may assume now

$$P(t) = (t^2+1)^2 Q(t),$$

here Q(t) has no multiple zeros. Consider first the special case when

$$P(t) = (t^2+1)^2 ,$$

or equivalently, the zeros of P(t) are $a, \ a, \ a^{-1}, a^{-1}$ with $a = \sqrt{-1}$. By (23) (24), for $l \in \mathbf{Z}$,

$$(\alpha_{s,t}) \begin{pmatrix} \overline{A_l} \\ \overline{A_{l+1}} \\ \overline{A_{l+2}} \\ \overline{A_{l+3}} \end{pmatrix} = U_l \begin{pmatrix} \overline{E_1} \\ \overline{E_2} \\ \overline{F_1} \\ \overline{F_2} \end{pmatrix} , \qquad (25)$$

$$2(\alpha_{s,t}) \begin{pmatrix} \overline{G_l} \\ \overline{G_{l+1}} \\ \overline{G_{l+2}} \\ \overline{G_{l+3}} \end{pmatrix} = U_l \begin{pmatrix} \overline{H_1} \\ \overline{H_2} \\ -\overline{H_1} \\ -\overline{H_2} \end{pmatrix} , \qquad (26)$$

with

$$(\alpha_{s,t})^{-1} = 2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ a & 1 & a^{-1} & 1 \\ a^2 & 2a & a^{-2} & 2a^{-1} \\ a^3 & 3a^2 & a^{-3} & 3a^{-2} \end{pmatrix} , \qquad (27)$$

and

$$U_l = \begin{pmatrix} a^l & la^{l-1} & 0 & 0 \\ 0 & a^l & 0 & 0 \\ 0 & 0 & a^{-l} & la^{-l+1} \\ 0 & 0 & 0 & a^{-l} \end{pmatrix}. \qquad (28)$$

By computation, using the formulas (25)-(28), we have

$$[\overline{E_1}, \overline{E_2}] = -4 \sum_{s,t=0}^{3} \alpha_{1,s}\alpha_{2,t}\overline{G_{t-s}} = -2\sum_{s=0}^{3} \alpha_{1,s}a^{-s}\overline{H_2} = 0\,,$$
$$[\overline{E_1}, \overline{F_1}] = -4 \sum_{s,t=0}^{3} \alpha_{1,s}\alpha_{3,t}\overline{G_{t-s}} = -2 \sum_{s=0}^{3} \alpha_{1,s}(-a^s\overline{H_1} + sa^{s+1}\overline{H_2}) = \overline{H_1}\,,$$
$$[\overline{E_1}, \overline{F_2}] = -4 \sum_{s,t=0}^{3} \alpha_{1,s}\alpha_{4,t}\overline{G_{t-s}} = -2 \sum_{s=0}^{3} \alpha_{1,s}(-a^s\overline{H_2}) = \overline{H_2}\,,$$

$$[\overline{E_2}, \overline{F_2}] = -4 \sum_{s,t=0}^{3} \alpha_{2,s}\alpha_{4,t}\overline{G_{t-s}} = -2\sum_{s=0}^{3} \alpha_{2,s}a^{-s}\overline{H_2} = 0\,,$$

$$\begin{aligned}[\overline{H_1}, \overline{E_1}] &= 4 \sum_{s,t=0}^{3} \alpha_{1,s}\alpha_{1,t}(\overline{A_{t+s}} - \overline{A_{t-s}}) \\ &= 4\sum_{s=0}^{3} \alpha_{1,s}(a^s\overline{E_1} + sa^{s-1}\overline{E_2} - a^{-s}\overline{E_1} + sa^{-s-1}\overline{E_2}) = 2\overline{E_1}\,,\end{aligned}$$
$$[\overline{H_1}, \overline{E_2}] = 4 \sum_{s,t=0}^{3} \alpha_{1,s}\alpha_{2,t}(\overline{A_{t+s}} - \overline{A_{t-s}}) = 4 \sum_{s=0}^{3} \alpha_{1,s}(a^s\overline{E_2} - a^{-s}\overline{E_2}) = 2\overline{E_2}\,,$$

Similarly,

$$[\overline{F_1}, \overline{F_2}] = 0\,,\ [\overline{F_1}, \overline{E_2}] = -\overline{H_2}\,, [\overline{H_1}, \overline{F_1}] = -2\overline{F_1}\ , [\overline{H_1}, \overline{F_2}] = -2\overline{F_2}\,,$$
$$[\overline{H_2}, \overline{F_1}] = 2a^2\overline{F_2} = -2\overline{F_2}\ ,\ [\overline{H_2}, \overline{F_2}] = 0.$$

The above relations of $\overline{E_i}$, $\overline{F_i}$, $\overline{H_i}$, $i = 1,2$, are also satisfied by the Lie algebra $\mathfrak{g}[t]/t^2 := \mathfrak{g} \otimes (\mathbb{C}[t]/t^2\mathbb{C}[t])$ (with the obvious Lie-structure) via the correspondence λ :

$$\begin{aligned} E\otimes 1 &\to \overline{E_1}, & F\otimes 1 &\to \overline{F_1}, & H\otimes 1 &\to \overline{H_1}\,, \\ E\otimes t &\to \overline{E_2}, & F\otimes t &\to \overline{F_2}, & H\otimes t &\to \overline{H_2}\,. \end{aligned}$$

Then the representation of $\mathfrak{A}$ on V is factored through a representation of $\mathfrak{g}[t]/t^2$:

$$\begin{array}{ccc} \mathfrak{A} & \to & sl(V) \\ & f \searrow & \uparrow \lambda \\ & & \mathfrak{g}[t]/t^2 \end{array}\ ,$$

here f is defined by

$$f(A_0) = 2(E \otimes 1 + F \otimes 1)\ ,$$
$$f(A_1) = 2\sqrt{-1}(E \otimes 1 - F \otimes 1) + 2(E \otimes t + F \otimes t)\ .$$

The representation λ of $\mathfrak{g}[t]/t^2$ is irreducible. As $t\mathfrak{g}[t]/t^2$ is a Lie-ideal of $\mathfrak{g}[t]/t^2$, the irreducibility of λ implies $V = \left(t\mathfrak{g}[t]/t^2\right)v_0$ for some vector v_0 of V. Since $E \otimes t$ annihilates $t\mathfrak{g}[t]/t^2$, $\overline{E_2}$ is the zero map of V, a contradiction to the linear independence of $\overline{E_i}$'s.

For the general case, write

$$\mathrm{P(t)} = (\mathrm{t}^2 + 1)^2 \mathrm{Q(t)},$$

with Q(x) no multiple zeros. With the same argument as the previous case together with the one given in Proposition 3, we can show that the $\mathfrak{A}$—module V can be factored through a representation of $\left(\mathfrak{g}[t]/t^2\right) \oplus \left(\bigoplus_n \mathfrak{g}\right)$:

$$\begin{array}{ccc} \mathfrak{A} & \rightarrow & sl(V) \\ (f, e_{\mathbf{a}}) \searrow & & \uparrow \lambda \\ & & (\mathfrak{g}[t]/t^2) \oplus \left(\bigoplus_n \mathfrak{g}\right) \end{array}$$

for some integer n and a$\in (\mathrm{C}^*)^n$. By the irreducibility of the representation λ, we can derive a contradiction just as before. q.e.d.

By Proposition 2– 4, we now have the following conclusion:

Theorem 2 Every non-trivial irreducible finite dimensional $\mathfrak{A}$−module is isomorphic to $V_{\overline{\mathbf{a}}}(\mu)$ for some $(\mu, \overline{\mathbf{a}}) \in \mathcal{P}^n \times (\mathcal{C}^n)'$. All such modules are parametrized by $\left(\coprod_{n \in \mathbb{N}} \mathcal{P}^n \times (\mathcal{C}^n)'\right) / (mod.\ permutation)$.

4. *Superintegrable chiral Potts model*

The Hamiltonian of superintegrable N−state chiral Potts spin chain has the form

$$H(k') = H_0 + k' H_1 \qquad (29)$$

with a temperature-like variable k'. For a row of L sites, H_0 and H_1 are Hermitian operators acting on the vector space which is the direct product of L copies of $\mathbf{C}^N$: $\mathcal{V} = \mathbf{C}^N \otimes \ldots \otimes \mathbf{C}^N$, and they are defined by

$$H_0 = -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \left(1 - \omega^{-n}\right)^{-1} X_j^n\ ,$$
$$H_1 = -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \left(1 - \omega^{-n}\right)^{-1} Z_j^n Z_{j+1}^{N-n}\ ,$$

here

$$\begin{aligned} \omega &= e^{2\pi i/N} \\ X_j &= I_N \otimes \ldots \otimes X^{jth} \otimes \ldots \otimes I_N \\ Z_j &= I_N \otimes \ldots \otimes Z^{jth} \otimes \ldots \otimes I_N , \end{aligned}$$

I_N is the $N \times N$ identity matrix, the elements of the $N \times N$ matrices Z and X are

$$\begin{aligned} X_{l,m} &= \delta_{l,m+1} \pmod N \\ Z_{l,m} &= \delta_{l,m}\omega^{l-1} \quad . \end{aligned}$$

$H(k')$ is Hermitian for real value k' and has real eigenvalues. We are going to study the behavior of the continuous dependence of its eigenvalues. Note that the operators H_0 and H_1 satisfy

$$[H_1,[H_1,[H_1,H_0]]] = 4N^2[H_1,H_0] \qquad [H_0,[H_0,[H_0,H_1]]] = 4N^2[H_0,H_1] .$$

Then the operators A_0 and A_1 defined by

$$A_0 = -2N^{-1}H_0 \qquad A_1 = -2N^{-1}H_1 , \qquad (30)$$

satisfy DG conditions.

<u>Theorem 3</u> The eigenvalues of operator $H(k')$ in (29) are of the form

$$\lambda(k') = (\alpha + k'\beta) + 2N\sum_{j=1}^{n} m_j\sqrt{1 + k'^2 - 2k'\cos\theta_j}$$

$$\text{here } \ k', \alpha, \beta, \theta_j \in \mathbf{R} \ , \ m_j = -s_j, -s_j+1, \ldots, s_j \quad \left(s_j \in \frac{1}{2}\mathbf{N}\right) .$$

Proof. By the Hermitian property of H_0, H_1, we have a decomposition of the representation space $\mathcal{V}$ into irreducible subrepresentations $\mathcal{W}$. It needs only to have the expression for the eigenvalues of $H(k')$ on an irreducible subrepresentation $\mathcal{W}$. When the dimension of $\mathcal{W}$ equals to one, the eigenvalue is simply equal to $\alpha + k'\beta$. We now assume $dim_{\mathbf{C}}\mathcal{W} \geq 2$. Let A_0 and A_1 be defined in (30). Then $\mathcal{W}$ is an irreducible $\mathfrak{A}-$module. Let α, β be the real numbers such that

$$A'_0 = A_0 + 2N^{-1}\alpha , \ A'_1 = A_1 + 2N^{-1}\beta \in sl(\mathcal{W}).$$

By Theorem 2, we may assume

$$\mathcal{W} = V_{\bar{n}}(\mu)$$

for some $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{P}^n$, $\overline{\mathrm{a}} = (\overline{a_1}, \ldots, \overline{a_n}) \in (C^n)'$. A'_0 and A'_1 act on $V_{\overline{\mathrm{a}}}(\mu) = \bigotimes_j V(\mu_j)$ by

$$\varphi(A'_0) = 2\sum_{j=1}^{n}(E_j + F_j)$$

$$\varphi(A'_1) = 2\sum_{j=1}^{n}\{-\cos\theta_j(E_j + F_j) - i\sin\theta_j(E_j - F_j)\}$$

here $e^{-i\theta_j} = -a_j$. Denote

$$dim_{\mathbf{C}} V(\mu_j) = 2s_j + 1\,,\; s_j \in \frac{1}{2}\mathbf{N}\,,$$
$$(J_x)_j = \frac{1}{2}(E_j + F_j),\; (J_y)_j = \frac{-i}{2}(E_j - F_j),\; (J_z)_j = \frac{1}{2}H_j$$

where $(J_x)_j, (J_y)_j, (J_z)_j$ are the usual irreducible matrix representations of angular momentum of dimension $(2s_j + 1)$. Then

$$\varphi(H(k')) = \alpha + k'\beta - 2N\sum_{j=1}^{n}\left\{(1 - k'\cos\theta_j)(J_x)_j + k'\sin\theta_j(J_y)_j\right\}$$

with $\theta_j \in \mathbf{R}$. Each term in the sum is of the form

$$-2N(1 - k'\cos\theta_j)(J_x)_j - 2Nk'\sin\theta_j(J_y)_j\,.$$

After rotating the xy plane, it is transformed into

$$2N\sqrt{1 + k'^2 - 2k'\cos\theta_j}\,(J_x)_j$$

which has the eigenvalue

$$2Nm_j\sqrt{1 + k'^2 - 2k'\cos\theta_j} \quad, m_j = -s_j, -s_j + 1, \ldots, s_j\,.$$

Therefore we obtain the result. q.e.d.

Associated to the chiral Potts chain, there are the spatial operator with eigenvalues $e^{2\pi iP/L}$ $(P = 0, 1, \ldots, L-1)$, and spin shift operator $\left(:= \prod_{j=1}^{L} X_j\right)$ with eigenvalues $e^{2\pi iQ/N}(Q = 0, 1, \ldots, N-1)$. H_0 and H_1 commute with these operators. So the representation space $\mathcal{V}$ decomposes into different subrepresentations of H_0, H_1 labelled by a pair of integers (P, Q). The DG conditions associated to H_0, H_1 provide a further, hidden, symmetry. As von Gehlen and Rittenberg

[10] observed, in the representation where X_j of (29) is in the diagonal form, by the relations

$$\sum_{m=1}^{N-1} \frac{\omega^k}{1-\omega^{-m}} = \frac{N-1}{2} - k \quad \text{for} \quad k = 0, 1, \ldots, (N-1),$$

$-\frac{1}{2}H_0$ is the usual representation for the z component of angular momentum for the N dimension irreducible representation of SU(2). For a chain of length L, the maximum and minimum eigenvalues of H_0 differ by $2(N-1)L$, while in any sector of given Q, the eigenvalues differ by multiple of $2N$. So the different number of distinct eigenvalues of H_0 in a Q sector is given by

$$n = \left[\frac{NL - L - Q}{N}\right] \tag{31}$$

where [ ] stands for integer part. For small k', the eigenvalues of H are close to those of H_0. By the analyses we have before, there are at least N distinct irreducible representations of $\mathfrak{A}$ in order to reproduce the spectrum of H in one sector for a pair (P, Q). This integer n refers to the largest sector which must necessarily contain the ground state. This is the sector found by Baxter [3]. He gave the closed-form solution of the characteristic polynomial $f(z^N)$ for the ground-state sectors $(P = 0)$

$$f\left(z^N\right) = N^L z^{-Q} \sum_{j=0}^{N-1} \omega^{(Q+L)j} \left(\frac{z^N - 1}{z - \omega^j}\right)^L \tag{32}$$

(equation (16) of [3]). The RHS is invariant when one substitutes z by ωz, hence a polynomial of z^N. It is easy to see that the degree of $f(Z)$ equals to the integer n of (31). Let $\left(\overline{t_1}\right)^{-N}, \left(\overline{t_2}\right)^{-N}, \ldots, \left(\overline{t_n}\right)^{-N}$ be the zeros of $f(Z)$. Define

$$\cos\theta_j = \frac{\overline{t_j}^N + 1}{\overline{t_j}^N - 1}, \quad e^{-i\theta_j} = -a_j \ , \tag{33}$$

equivalently

$$\left(\overline{t_j}\right)^{\frac{N}{2}} = -i \cot\frac{\theta}{2} = \frac{1 - a_j}{1 + a_j} \ .$$

By [3] and Theorem 3, the representation $V_{\bar{a}}(\mu)$ of $\mathfrak{A}$ associated to this sector is given by

$$\begin{aligned} \mathrm{a} &= (a_1, \ldots, a_n) \qquad a_j : \text{defined by (32)} \ , \\ \mu &= (\mu_1, \ldots, \mu_n) \qquad \mu_j : \text{spin } \frac{1}{2} \text{ representation of } SU(2). \end{aligned}$$

The algebraic curve where "rapidity" variables of chiral Potts $N-$state model lie is the hyperelliptic curves defined by

$$t^N = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{1-k'^2} \qquad k'^2 \neq 0,1 \qquad (34)$$

[3] [12] (here we use Baxter's notation). The variables t and $\bar{t}$ (for $\overline{t_j}$) are related by

$$\bar{t} = \left(\frac{1+k'}{1-k'}\right)^{1/N} t \ .$$

Now the eigenvalues for Hamiltonian H have the expression

$$\begin{aligned} &\alpha + k'\beta \pm \sqrt{1 + k'^2 - 2k' \cos\theta_j} \\ &= \alpha + k'\beta \pm (1-k')\frac{1+\lambda_j}{1-\lambda_j} \end{aligned}$$

here λ_j is obtained by (33) by letting $t = \left(\frac{1-k'}{1+k'}\right)^{1/N}\overline{t_j}$. In conclusion we have observed that the spectrum of ground-state sector is determined by zeros of the polynomial (32) with parameter identified with t variable of the "rapidity" curve (33). The corresponding representation of Onsager's algebra is obtained by attaching spin 1/2 representation of SU(2) to each zero and their λ values give the spectrum of the spin chain Hamiltonian.

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