STRINGY K-THEORY AND THE CHERN CHARACTER

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Abstract. In this paper we define a new orbifold K-theory $K_{orb}(\mathscr{X})$ for a Deligne-Mumford stack $\mathscr{X}$ with projective coarse moduli; and furthermore, for any smooth, projective variety X with an action of a finite group G we define a G-Frobenius algebra $\mathscr{K}(X,G)$, called the stringy K-theory of the G-variety X, whose associated Frobenius algebra of G-coinvariants is the orbifold K-theory of the quotient stack $[X/G]$. The algebra $K_{orb}(\mathscr{X})$ is linearly isomorphic to the "orbifold K-theory" of Adem-Ruan [AR], but carries a different product, generalizing the quantum K-theory of Givental [Gi] and Y. P. Lee [Le].

We prove there is a ring isomorphism $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to H(X,G)$, which we call the stringy Chern character, and a ring isomorphism $Ch_{orb} : K_{orb}(\mathscr{X}) \to H_{\mathrm{orb}}(\mathscr{X})$, which we call the orbifold Chern character, where $H(X,G)$ is the stringy cohomology ring of [FG, JKK], and $H_{\mathrm{orb}}(\mathscr{X})$ is the Chen-Ruan orbifold cohomology. We further show that $\mathscr{C}\mathbf{h}$ and Ch_{orb} respect all properties of a G-Frobenius (respectively Frobenius) algebra that do not involve the metric and that Grothendieck-Riemann-Roch holds for étale maps.

Our main result is a new, simple formula for the obstruction bundle, which allows one to completely exorcise complex curves from the definitions of the stringy Chow ring, stringy K-theory, orbifold K-theory, and Chen-Ruan orbifold cohomology. This new formula plays a key role in the proof that the stringy Chern character is a ring homomorphism and it also yields a simple proof of associativity and the trace axiom.

All of these results hold both in the algebro-geometric category and in the topological category for equivariant almost complex manifolds.

We conclude by showing that a K-theoretic version of Ruan's conjectures holds for the symmetric product of a complex projective surface with trivial first Chern class.

Contents

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1. Introduction

When X is a compact, complex manifold with an action of a finite group G, the stringy cohomology $\mathscr{H}(X,G)$ of X is a G-Frobenius algebra [FG, JKK]—an equivariant generalization of a Frobenius algebra—which contains the ordinary cohomology of X as a subalgebra. However, it is larger, in general, as it contains the so-called twisted sectors. Taking G-coinvariants, one obtains the Chen-Ruan orbifold cohomology $H_{\mathrm{orb}}([X/G])$ of the quotient stack $[X/G]$. As might be expected from Gromov-Witten theory, the product depends on an "obstruction bundle" described in terms of the cohomology of certain sheaves on admissible (or Galois) covers of $\mathbb{P}^1$.

The first main result of this paper is a new, simple formula for the obstruction bundle that avoids all use of complex curves, admissible covers, or moduli spaces. Instead, this formula expresses the obstruction bundle in terms of the representation of G on the tangent space TX. The formula also greatly simplifies the computation of both stringy and orbifold cohomology, and it allows us to give relatively simple proofs of associativity and the trace axiom.

As a simple corollary of this formula, we obtain a result originally due to Chen and Hu [CH] describing the obstruction bundle when G is Abelian.

The second main result of this paper is the introduction of a K-theoretic analog of stringy cohomology, the *stringy K-theory* $\mathscr{K}(X,G)$ of X, and the introduction of the *stringy Chern character* $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{H}(X,G)$. We prove that $\mathscr{K}(X,G)$ is a G-Frobenius algebra and that $\mathscr{C}\mathbf{h}$ is an isomorphism of G-commutative algebras, i.e., $\mathscr{C}\mathbf{h}$ preserves all of the properties of a G-Frobenius algebra except those involving the metric.

The third main result of this paper is the introduction of an *orbifold K-theory* $K_{orb}(\mathscr{X})$ (linearly isomorphic to that of [AR], but with a different, "quantum," product) for a general Deligne-Mumford stack $\mathscr{X}$ and the introduction of the *orbifold Chern character* $Ch_{orb} : K_{orb}(\mathscr{X}) \to H_{\mathrm{orb}}(\mathscr{X})$. We prove that Ch_{orb} is an isomorphism of *algebras*. These constructions and results hold even when the stack is not a global quotient by a finite group. But when $\mathscr{X} = [X/G]$ is a global quotient, the Frobenius algebra $K_{orb}([X/G])$ is isomorphic, as a Frobenius algebra, to the G-coinvariants $\overline{\mathscr{K}}(X,G)$ of $\mathscr{K}(X,G)$, and the orbifold Chern character Ch_{orb} is induced from the stringy Chern character $\mathscr{C}\mathbf{h}$.

1.1. Background and motivation.

We now describe part of our motivation for studying stringy K-theory. For convenience, we assume throughout this subsection that the coefficient ring is $\mathbb{C}$ rather than $\mathbb{Q}$.

Let Y be a projective, complex surface such that $c_1(Y) = 0$. For all n, consider Y^n with the symmetric group S_n acting by permuting its factors. The quotient orbifold $[Y^n/S_n]$ is called the symmetric product of Y. Let $Y^{[n]}$ denote the Hilbert scheme of n points on Y. The morphism $Y^{[n]} \to Y^n/S_n$ is a crepant resolution of singularities and is, furthermore, a hyper-Kähler resolution [Ru]. Fantechi and

Göttsche [FG] proved that there is a ring isomorphism $\psi' : H_{\mathrm{orb}}([Y^n/S_n]) \to H^\bullet(Y^{[n]})$, where $H^\bullet(Y^{[n]})$ is the ordinary cohomology ring (see also [Kau05, Ur]).*

The previous example is a verification, in a special case, of the following conjecture of Ruan [Ru], which was inspired by the work of string theorists studying topological string theory on orbifolds.

Conjecture 1.1 (Cohomological Hyper-Kähler Resolution Conjecture)**.** *Suppose that $\tilde{V} \to V$ is a hyper-Kähler resolution. The ordinary cohomology ring $H^\bullet(\tilde{V})$ of $\tilde{V}$ is isomorphic to the Chen-Ruan orbifold cohomology ring $H_{\mathrm{orb}}(V)$ of V.*

Let us return again to the example of the symmetric product. The algebra isomorphism $\psi' : H_{\mathrm{orb}}([Y^n/S_n]) \to H^\bullet(Y^{[n]})$ suggests that there should exist a K-theoretic analog $K_{orb}([Y^n/S_n])$ of $H_{\mathrm{orb}}([Y^n/S_n])$, a stringy Chern character isomorphism $Ch_{orb} : K_{orb}([Y^n/S_n]) \to H_{\mathrm{orb}}([Y^n/S_n])$, and an algebra isomorphism $\psi : K_{orb}([Y^n/S_n]) \to K(Y^{[n]})$, such that the following diagram commutes:

$$\begin{tikzcd} K_{orb}([Y^n/S_n]) \arrow[r, "Ch_{orb}"] \arrow[d, "\psi"'] & H_{\mathrm{orb}}([Y^n/S_n]) \arrow[d, "\psi'"'] \\ K(Y^{[n]}) \arrow[r, "\mathbf{ch}"] & H(Y^{[n]}). \end{tikzcd} \tag{1}$$

The previous discussion suggests the more general question of whether one can define a K-theoretic analogue of stringy cohomology and of Chen-Ruan orbifold cohomology in such a way that, for a global quotient, the ring of coinvariants of the former is equal to the latter. Such a generalization should also have a stringy Chern character that is a ring homomorphism and not just a linear isomorphism. Of course, since the usual Chern character does not preserve the metric for ordinary K-theory and ordinary cohomology, it is unreasonable to expect that a stringy Chern character would preserve the metric, but it should preserve all the other properties of the G-Frobenius algebras $\mathscr{K}(X,G)$ and $\mathscr{H}(X,G)$. That is, if we define a *G-commutative algebra* to be a G-Frobenius algebra but where all axioms involving the metric have been dropped, then the stringy Chern character ought to be an isomorphism of G-commutative algebras.

1.2. **Summary and discussion of main results.** Although most of our results are initially formulated and proved in the algebro-geometric category, with Chow rings and algebraic K-theory, they also hold in the topological category, with cohomology and topological K-theory (see Section 11) for almost complex manifolds with a G-equivariant almost complex structure. We show that these algebraic structures depend only upon the homotopy class of the G-equivariant almost complex structure. Our results can also be generalized to equivariant stable complex manifolds (see Remark 11.2).

As a G-graded G-module, the stringy K-theory $\mathscr{K}(X,G)$ of X is just the ordinary K-theory $K(I_G(X))$ of $I_G(X) := \coprod_{g\in G} X^g$, the *inertia variety* of X (not to be confused with the inertia orbifold, or inertia stack $\coprod_{(g)}[X^{(g)}/Z_G(g)]$, where the

*In fact, they proved that the isomorphism holds over $\mathbb{Q}$, provided that the multiplication on $H_{\mathrm{orb}}([Y^n/S_n])$ is twisted by signs. This sign change can be regarded as a kind of discrete torsion (see Section 11.3 for more details).

sum runs over conjugacy classes in G). The vector space $\mathscr{K}(X,G)$ is endowed with a metric in a manner similar to that in the stringy Chow ring $\mathscr{A}(X,G)$. The only nontrivial part in the definition of $\mathscr{K}(X,G)$ is the multiplication.

The definition of the multiplication is a generalization of Givental [Gi] and Lee's [Le] K-theoretic version of the Gromov-Witten theory of a smooth, projective variety X without a group action. Recall that they define a virtual structure sheaf on the moduli space of stable maps $\overline{\mathscr{M}}_{g,n}(X)$ and use it to define the correlators in the theory by analogy with the usual Gromov-Witten theory of X. On the other hand, the G-Frobenius algebra $\mathscr{A}(X,G)$ can be understood to be a construction analogous to Gromov-Witten theory on the moduli space $\overline{\mathscr{M}}^G_{g,n}(X,0)$, the moduli space of G stable maps of degree 0 into X [JKK] . The multiplication on $\mathscr{K}(X,G)$ is defined by combining these ideas, using a virtual structure sheaf $\mathscr{O}_{\mathrm{vir}} := \lambda_{-1}(\mathscr{R}^*)$ on the substack $\xi(X,0,\mathbf{m})$ of $\overline{\mathscr{M}}^G_{0,3}(X,0)$, where $\mathscr{R}$ is the obstruction bundle. The virtual structure sheaf plays a role in stringy K-theory similar to that of the top Chern class $c_{\mathrm{top}}(\mathscr{R})$ in stringy cohomology.

Our first main result is a simple and remarkable, new formula for the obstruction bundle $\mathscr{R}$ that is responsible for the multiplications on both $\mathscr{A}(X,G)$ and $\mathscr{K}(X,G)$. The formula uses certain sums of eigenbundles of the (right) G-action on TX, as follows. For any order r element $m \in G$, define $\mathscr{S}_m \in \mathscr{K}_m(X) := K(X^m)$ as

$$\mathscr{S}_m := \bigoplus_{k=1}^{r-1} \frac{k}{r} W_{m,k}, \tag{2}$$

where $W_{m,k}$ is the eigenspace of $TX|_{X^{m_i}}$ where m acts with eigenvalue $\exp(-2\pi k i/r)$. This $\mathscr{S}_m$ has virtual rank precisely equal to the age of m.

Theorem 1.2. *Let X be a smooth variety (not necessarily projective, or even proper) with an action of a finite group G.*

The obstruction bundle $\mathscr{R}(\mathbf{m})$ that is responsible for the multiplication in stringy K-theory and the stringy Chow ring can be expressed solely in terms of eigenspaces of the tangent bundle TX and its restrictions to various fixed point loci in X.

More precisely, if $\mathbf{m} = (m_1, m_2, m_3) \in G^3$ is such that $m_1 m_2 m_3 = 1$ and m_i has order r_i in G, then on the fixed point locus $X^{\mathbf{m}} := X^{m_1} \cap X^{m_2}$, the obstruction bundle $\mathscr{R}(\mathbf{m})$ satisfies the following equality in $K(X^{\mathbf{m}})$:

$$\mathscr{R}(\mathbf{m}) = TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathscr{S}_{m_i}|_{X^{\mathbf{m}}}, \tag{3}$$

We have a similar formula for the obstruction bundle in orbifold K-theory and orbifold Chow (see Theorem 9.2).

The formula for the obstruction bundle also allows us to prove that $\mathscr{K}(X,G)$ is a G-Frobenius algebra. But unfortunately, the ordinary Chern character $\mathscr{K}(X,G) \to \mathscr{A}(X,G)$ does not respect the stringy multiplication operations. We repair this problem by defining the *stringy Chern character* $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$ to be a deformation of the ordinary Chern character (see Equation (37)) involving the same element $\mathscr{S}_m$ that appeared in the formula for the obstruction bundle. Again using the formula of Theorem 1.2, we prove our second main result.

Theorem 1.3. *The stringy Chern character $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$, is an isomorphism of G-commutative algebras.*

We also prove naturality properties and the Grothendieck-Riemann-Roch theorem for $\mathscr{C}\mathbf{h}$ with respect to G-equivariant étale maps (See Theorem 7.3).

After proving these results about $\mathscr{K}(X,G)$, we introduce the *orbifold K-theory*, $K_{orb}(\mathscr{X})$, a Frobenius algebra associated to a smooth, Deligne-Mumford stack $\mathscr{X}$ with a projective coarse moduli in a manner analogous to orbifold cohomology [CR1] and orbifold Chow ring [AGV]. Moreover, we define a deformation Ch_{orb} of the ordinary Chern character, which we call the *orbifold Chern character*.

Our third main result is the following theorem.

Theorem 1.4. *The orbifold Chern character $Ch_{orb} : K_{orb}(\mathscr{X}) \to A^{\bullet}_{orb}(\mathscr{X})$ is an isomorphism of commutative algebras.*

When $\mathscr{X} = [X/G]$ is a global quotient of a smooth projective variety by a finite group, the Frobenius algebra $K_{orb}(\mathscr{X})$ is isomorphic to the Frobenius algebra of G-coinvariants $\overline{\mathscr{K}}(X,G)$ of $\mathscr{K}(X,G)$, and the orbifold Chern character is the same as the map induced on the G-coinvariants of $\mathscr{K}(X,G)$ by the stringy Chern character $\mathscr{C}\mathbf{h}$.

As we mentioned above, all these results are proved initially in the algebro-geometric category, but we prove in Section 11 that their analogues in the topological category also hold.

We then apply these results to the case of the symmetric product of a smooth, projective surface Y with trivial canonical bundle and verify that the Conjecture 1.1 holds in this case; that is $K_{orb}([Y^n/S_n])$ is isomorphic to $K(Y^{[n]})$.

1.3. **Directions for further research.** These results suggest many different directions for further research. The first is to generalize to the case where G is a Lie group and to higher-degree Gromov-Witten invariants. This will be explored elsewhere. It would also be interesting to study stringy generalizations of the usual algebraic structures of K-theory, e.g., the Adam's operations and λ-rings. Another interesting direction would be to study stringy generalizations of other K-theories, including algebraic K-theory and higher K-theory. It would also be very interesting find an analogous construction in orbifold conformal field theory, e.g., twisted vertex algebras and the chiral de Rham complex [FS]. Finally, it would be interesting to see if our results can shed light upon the relationship between Hochschild cohomology and orbifold cohomology [DE] in the context of deformation quantization.

1.4. **Notation and conventions.** Unless otherwise specified, we assume throughout the paper that all cohomology rings have coefficients in the rational numbers $\mathbb{Q}$. Also, unless otherwise specified, all groups are finite and all group actions are right actions.

The stack quotient of a variety X by G will be denoted $[X/G]$ and the coarse moduli space of this quotient will be denoted X/G.

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2. The ordinary Chow ring and K-theory of a variety

In this section, we briefly review some basic facts about the Chow ring, K-theory, and certain characteristic classes that we will need. Throughout this section, all varieties we consider will be smooth, projective varieties over $\mathbb{C}$.

Recall that a *Frobenius algebra* is a unital, commutative, associative algebra with an invariant metric. To each smooth, projective variety X, one can associate two Frobenius algebras, namely, the *Chow ring* $A^\bullet(X)$ *of* X, and *the K-theory* $K(X)$ *of* X. Furthermore, there is an isomorphism of unital, commutative, associative algebras $\mathbf{ch} : K(X) \to A^\bullet(X)$ called the *Chern character*. The Chern character does not, however, preserve the metric. We will now briefly review these constructions in order to fix notation and conventions, referring the interested reader to [Fu, FL] for more details.

2.1. The Chow ring.

The Chow ring of a smooth, projective variety X is additively a $\mathbb{Z}$-graded Abelian group $A^\bullet(X,\mathbb{Z}) = \bigoplus_{p=0}^{D} A^p(X,\mathbb{Z})$, where D is the dimension of X, and $A^p(X)$ is the group of finite formal sums of $(D-p)$-dimensional subvarieties of X, modulo rational equivalence.

In this paper we will always work with rational coefficients, and we write

$$A^\bullet(X) := A^\bullet(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The vector space $A^\bullet(X)$ is endowed with a commutative, associative multiplication which preserves the $\mathbb{Z}$-grading, arising from the intersection product, and possesses an identity element $\mathbf{1} := [X]$ in $A^0(X)$. The intersection product $A^p(X)\otimes A^q(X) \to A^{p+q}(X)$ is denoted by $v \otimes w \mapsto v \cup w$ for all p, q.

Given a proper morphism $f : X \to Y$ between two varieties, there is an induced pushforward morphism $f_* : A^\bullet(X) \to A^\bullet(Y)$. In particular, if Y is a point and $f : X \to Y$ is the obvious map, then one can define integration via the formula

$$\int_{[X]} v := f_*(v)$$

for all v in $A^\bullet(X)$. The integral vanishes unless v belongs to $A^D(X)$. Define a symmetric, nondegenerate bilinear form $\eta_A : A^\bullet(X) \otimes A^\bullet(X) \to \mathbb{Q}$ via $\eta_A(v,w) := \int_{[X]} v \cup w$.

Proposition 2.1. *Let $A^\bullet(X)$ be the Chow ring.*

1. *The tuple $(A^\bullet(X), \cup, \mathbf{1}, \eta_A)$ is a Frobenius algebra graded by $\mathbb{Z}$.*
2. *If $f : X \to Y$ is any morphism, then the associated pullback morphism $f^* : A^\bullet(Y) \to A^\bullet(X)$ is a homomorphism of Frobenius algebras graded by $\mathbb{Z}$.*
3. *(Projection formula) For any proper morphism $f : X \to Y$, if $\alpha \in A^\bullet(X)$ and $\beta \in A^\bullet(Y)$, we have*

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta.$$

2.2. K-theory.

$K(X;\mathbb{Z})$ is additively equal to the free Abelian group generated by isomorphism classes of (complex algebraic) vector bundles on X modulo the subgroup generated by

$$[E] \ominus [E'] \ominus [E''] \tag{4}$$

for each exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0. \tag{5}$$

Here $\ominus$ denotes subtraction and $\oplus$ denotes addition in the free Abelian group. We define

$$K(X) := K(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The multiplication operation, also denoted by $\otimes$, taking $K(X) \otimes K(X) \to K(X)$ is the usual tensor product $[E] \otimes [E'] \mapsto [E \otimes E']$ for all vector bundles E and E'. We denote the multiplicative identity by $\mathbf{1} := [\mathscr{O}_X]$.

Given a proper morphism $f : X \to Y$ between two smooth varieties, there is an induced pushforward morphism $f_* : K(X) \to K(Y)$ given by $f_*([E]) = \sum_{i=0}^{D}(-1)^i R^i f_* E$. In particular, if Y is a point and $f : X \to Y$ is the obvious map, then the Euler characteristic of $v \in K(X)$ is the pushforward

$$\chi(X, v) = f_*(v).$$

Define a symmetric, nondegenerate bilinear form $\eta_K : K(X) \otimes K(X) \to \mathbb{Q}$ via $\eta_K(v, w) := \chi(X, v \otimes w)$.

Proposition 2.2. *Let $K(X)$ be the K-theory of X.*

1. *The tuple $(K(X), \otimes, \mathbf{1}, \eta_K)$ is a Frobenius algebra.*
2. *If $f : X \to Y$ is any morphism, then the associated pullback morphism $f^* : K(Y) \to K(X)$ is a homomorphism of Frobenius algebras.*
3. *(projection formula) For any proper morphism $f : X \to Y$, if $\alpha \in K(X)$ and $\beta \in K(Y)$ we have*

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta$$

While $K(X)$ does not have a $\mathbb{Z}$-grading like $A^\bullet(X)$, it does have a *virtual rank (or augmentation)*. That is, for each connected component U of X, there is a surjective ring homomorphism $\mathbf{vr} : K(U) \to \mathbb{Q}$ which assigns to each vector bundle E on U its rank. In addition, $K(X)$ has a natural involution which takes a vector bundle $[E]$ to its dual $[E^*]$.

Another important property of K-theory is that it is a so-called λ*-ring*. That is, for every non-negative integer i, there is a map $\lambda^i : K(Y) \to K(Y)$ defined by $\lambda^i([E]) := [\bigwedge^i E]$, where $\bigwedge^i E$ is the i-th exterior power of the vector bundle E. In particular, $\lambda^0([E]) = \mathbf{1}$, and $\lambda^i([E]) = 0$ if i is greater than the rank of E.

These maps satisfy the relations

$$\lambda^k(\mathscr{F} \oplus \mathscr{F}') = \bigoplus_{i=0}^{k} \lambda^i(\mathscr{F})\lambda^{k-i}(\mathscr{F}')$$

for all $k = 0, 1, 2, \ldots$ and all $\mathscr{F}$, $\mathscr{F}'$ in $K(Y)$. These relations can be neatly stated in terms of the universal formal power series in t

$$\lambda_t(\mathscr{F}) := \bigoplus_{i=0}^{\infty} \lambda^i(\mathscr{F})t^i \tag{6}$$

by demanding that it satisfy the multiplicativity relation

$$\lambda_t(\mathscr{F} \oplus \mathscr{F}') = \lambda_t(\mathscr{F})\lambda_t(\mathscr{F}'). \tag{7}$$

If E is a rank-r vector bundle over X, then one can define

$$\lambda_{-1}([E]) := \bigoplus_{i=0}^{r}(-1)^i\lambda^i([E])$$

in $K(X)$, which will play an important role in this paper.

2.3. **Chern classes, Todd classes, and the Chern character.** The *Chern polynomial of* $\mathscr{F}$ in $K(X)$ is defined to be the universal formal power series in t

$$c_t(\mathscr{F}) := \sum_{i=0}^{\infty} c_i(\mathscr{F})t^i,$$

where $c_i(\mathscr{F})$, the *i-th Chern class of* $\mathscr{F}$, belongs to $A^i(X)$ for all i, and c_t and the c_i satisfy the following axioms:

(1) If $\mathscr{F} = [\mathscr{O}(D)]$ is a line bundle defined by a divisor D, then
$$c_t(\mathscr{F}) = \mathbf{1} + Dt.$$
(2) The Chern classes commute with pullback, i.e., if $f : X \to Y$ is any morphism, then $c_i(f^*\mathscr{F}) = f^*c_i(\mathscr{F})$ for all $\mathscr{F}$ in $K(X)$ and all i.
(3) If
$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$
is an exact sequence, then
$$c_t(\mathscr{F}) = c_t(\mathscr{F}')c_t(\mathscr{F}'').$$

In particular, $c_0(\mathscr{F}) = \mathbf{1}$ for all $\mathscr{F}$.

A fundamental tool is the *splitting principle*, which says that for any vector bundle E on X, there is a morphism $f : Y \to X$, such that $f^* : A^\bullet(X) \to A^\bullet(Y)$ is injective, and $f^*([E])$ splits (in K-theory) as a sum of line bundles:

$$f^*([E]) = [\mathscr{L}_1] \oplus \cdots \oplus [\mathscr{L}_r], \tag{8}$$

where $r = \mathbf{vr}([E])$. We define the *Chern roots* of $[E]$ to be $a_i := c_1(\mathscr{L}_i)$, and thus by Property (3) of the Chern polynomial we have

$$c_t([E]) = \prod_{i=1}^{r}(\mathbf{1} + a_i t). \tag{9}$$

Of course, the Chern roots depend on the choice of f, but any relations derived in this way among the Chern classes of $[E]$ will hold in $A^\bullet(X)$ regardless of the choice of f.

From the Chern classes, one can construct the *Chern character*

$$\mathbf{ch} : K(X) \to A^\bullet(X)$$

by associating to a rank-r vector bundle E over X the element

$$\mathbf{ch}([E]) := \sum_{i=1}^{r}\exp(a_i) = r + c_1([E]) + \frac{1}{2}(c_1^2([E]) - 2c_2([E])) + \cdots , \tag{10}$$

where $a_1, \ldots, a_r$ are the Chern roots of $[E]$.

For each connected component U of X, the virtual rank is the algebra homomorphism $\mathbf{vr} : K(U) \to \mathbb{Q}$, which is the composition of $\mathbf{ch} : K(U) \to A^\bullet(U)$ with the canonical projection $A^\bullet(U) \to A^0(U) \cong \mathbb{Q}$.

We have the following theorem.

Theorem 2.3. *The Chern character* $\mathbf{ch} : K(X) \to A^\bullet(X)$ *is an isomorphism of unital commutative, associative algebras. Furthermore, if* $f : X \to Y$ *is any morphism, then the following diagram commutes:*

$$\begin{tikzcd} K(Y) \arrow[r, "f^*"] \arrow[d, "\mathbf{ch}"'] & K(X) \arrow[d, "\mathbf{ch}"'] \\ A^\bullet(Y) \arrow[r, "f^*"] & A^\bullet(X). \end{tikzcd} \tag{11}$$

Remark 2.4. In general, the Chern character does not commute with pushforward. That is the content of the Grothendieck-Riemann-Roch theorem, which we will review shortly. Since the metrics of both $K(X)$ and $A^\bullet(X)$ are defined by pushforward, this means the Chern character does not respect the metrics.

To state the Grothendieck-Riemann-Roch theorem we will need the *Todd class*, $\mathbf{td} : K(X) \to A^\bullet(X)$, which is defined by imposing the multiplicativity condition

$$\mathbf{td}(\mathscr{F} \oplus \mathscr{F}') = \mathbf{td}(\mathscr{F})\mathbf{td}(\mathscr{F}')$$

for all $\mathscr{F}$, $\mathscr{F}'$ in $K(X)$, and by also demanding that if E is a rank r vector bundle on X, then

$$\mathbf{td}([E]) := \prod_{i=1}^{r} \phi(a_i),$$

where $a_i = 1, \dots, r$ are the Chern roots of $[E]$ and

$$\phi(t) := \frac{te^t}{e^t - 1}$$

is regarded as a element in $\mathbb{Q}[[t]]$. Therefore, $\mathbf{td}(\mathscr{F}) = \mathbf{1} + x$, where x belongs to $\bigoplus_{i=1}^{D} A^i(X)$.

Theorem 2.5 (Grothendieck-Riemann-Roch)**.** *For any proper morphism* $f : X \to Y$ *of non-singular varieties and any* $\mathscr{F} \in K(X)$*, we have*

$$\mathbf{ch}(f_*(\mathscr{F})) \cup \mathbf{td}(TY) = f_*(\mathbf{ch}(\mathscr{F}) \cup \mathbf{td}(TX)), \tag{12}$$

where TX *and* TY *are the tangent bundles of* X *and* Y*, respectively.*

A useful proposition which intertwines many of these structures is the following.

Proposition 2.6. *If* E *is a vector bundle of rank* r *over* X*, then the following identity holds in* $A^\bullet(X)$*:*

$$\mathbf{td}([E])\mathbf{ch}(\lambda_{-1}([E^*])) = c_{\mathrm{top}}([E]), \tag{13}$$

where $c_{\mathrm{top}}([E])$ *is the top Chern class* $c_r([E])$*.*

Notation 2.7. When E is a vector bundle over X, we will often write $c_t(E)$ instead of $c_t([E])$, and similarly for λ_t, $\mathbf{td}$ and $\mathbf{ch}$.

Remark 2.8. Since $K(X)$ is a $\mathbb{Q}$-vector space, we will need to make sense of expressions such as $\mathbf{td}(\frac{1}{n}[E])$, where n is a positive integer and E is a rank r vector bundle over X. Observe that

$$\mathbf{td}([E]) = \mathbf{td}\left(\bigoplus_{i=1}^{n} \frac{1}{n}[E]\right) = \left(\mathbf{td}(\frac{1}{n}[E])\right)^n.$$

Consider the formal power series $\Phi(t_1, \ldots, t_r)$ in $\mathbb{Q}[[t_1, \ldots, t_r]]$ defined by

$$\Phi(t_1, \ldots, t_r) := \prod_{i=1}^{r} \phi(t_i).$$

In particular, $\mathbf{td}([E]) = \Phi(a_1, \ldots, a_r)$. Since $\Phi(t_1, \ldots, t_r)$ is equal to $\mathbf{1}$ plus higher order terms, we can define $\Phi^{\frac{1}{n}}(t_1, \ldots, t_r)$ to be the unique formal power series in $\mathbb{Q}[[t_1, \ldots, t_r]]$ equal to $\mathbf{1}$ plus higher order terms such that

$$(\Phi^{\frac{1}{r}}(t_1, \ldots, t_r))^r = \Phi(t_1, \ldots, t_r).$$

We define

$$\mathbf{td}^{\frac{1}{r}}([E]) := \Phi^{\frac{1}{r}}(a_1, \ldots, a_r).$$

3. G-graded G-modules and G-(equivariant) Frobenius algebras

We begin by introducing some algebraic structures which we will need throughout the rest of the paper.

Definition 3.1. A finite-dimensional, G-graded vector space $\mathscr{H} := \bigoplus_{m \in G} \mathscr{H}_m$ which is endowed with the structure of a right G-module by isomorphisms $\rho(\gamma) : \mathscr{H} \xrightarrow{\sim} \mathscr{H}$ for all γ in G, is said to be a *G-graded G-module* if $\rho(\gamma)$ takes $\mathscr{H}_m$ to $\mathscr{H}_{\gamma^{-1}m\gamma}$ for all m in G.

A *G-invariant metric* on a G-graded G-module $\mathscr{H}$ is a symmetric, nondegenerate, bilinear form η on $\mathscr{H}$ which is G-invariant (under the diagonal G-action) and which respects the grading, i.e., for all v_{m_+} in $\mathscr{H}_{m_+}$ and v_{m_-} in $\mathscr{H}_{m_-}$ we have $\eta(v_{m_+}, v_{m_-}) = 0$ unless $m_+ m_- = 1$.

G-graded G-modules form a category whose objects are G-graded G-modules and whose morphisms are homomorphisms of G-modules which respect the G-grading. Furthermore, the dual of a G-graded G-module inherits the structure of a G-graded G-module.

Let us adopt the notation that v_m is a vector in $\mathscr{H}_m$ for any $m \in G$.

Definition 3.2. A tuple $((\mathscr{H}, \rho), \cdot, \mathbf{1}, \eta)$ is said to be a *G-(equivariant) Frobenius algebra* [Kau02, Kau03, Tu] provided that the following hold:

(1) (G-graded G-module) $(\mathscr{H}, \rho)$ is a G-graded G-module.
(2) (Self-invariance) For all γ in G, $\rho(\gamma) : \mathscr{H}_\gamma \to \mathscr{H}_\gamma$ is the identity map.
(3) (Metric) η is a symmetric, nondegenerate, bilinear form on $\mathscr{H}$ such that $\eta(v_{m_1}, v_{m_2})$ is nonzero only if $m_1 m_2 = 1$.
(4) (G-graded Multiplication) The binary product $(v_1, v_2) \mapsto v_1 \cdot v_2$, called the *multiplication* on $\mathscr{H}$, preserves the G-grading (i.e., the multiplication is a map $\mathscr{H}_{m_1} \otimes \mathscr{H}_{m_2} \to \mathscr{H}_{m_1 m_2}$) and is distributive over addition.
(5) (Associativity) The multiplication is associative; i.e.,

$$(v_1 \cdot v_2) \cdot v_3 = v_1 \cdot (v_2 \cdot v_3)$$

for all v_1, v_2, and v_3 in $\mathscr{H}$.
(6) (Braided Commutativity) The multiplication is invariant with respect to the braiding:

$$v_{m_1} \cdot v_{m_2} = (\rho(m_1^{-1}) v_{m_2}) \cdot v_{m_1}$$

for all $m_i \in G$ and all $v_{m_i} \in \mathscr{H}_{m_i}$ with $i = 1, 2$.

(7) (G-equivariance of the Multiplication)

$$(\rho(\gamma)v_1) \cdot (\rho(\gamma)v_2) = \rho(\gamma)(v_1 \cdot v_2)$$

for all γ in G and all $v_1, v_2 \in \mathscr{H}$.

(8) (G-invariance of the Metric)

$$\eta(\rho(\gamma)v_1, \rho(\gamma)v_2) = \eta(v_1, v_2)$$

for all γ in G and all $v_1, v_2 \in \mathscr{H}$.

(9) (Multiplicative Invariance of the Metric)

$$\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$$

for all $v_1, v_2, v_3 \in \mathscr{H}$.

(10) (G-invariant Identity) The element $\mathbf{1}$ in $\mathscr{H}_1$ is the identity element under the multiplication, and it satisfies

$$\rho(\gamma)\mathbf{1} = \mathbf{1}$$

for all γ in G.

(11) (Trace Axiom) For all a, b in G and v in $\mathscr{H}_{[a,b]}$, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$, if L_v denotes left multiplication by v, then we have

$$\mathrm{Tr}_{\mathscr{H}_a}(L_v \rho(b^{-1})) = \mathrm{Tr}_{\mathscr{H}_b}(\rho(a)L_v).$$

Remark 3.3. When G is the trivial group, a G-Frobenius algebra is nothing more than a Frobenius algebra. Given any G-Frobenius algebra $\mathscr{H}$, the subalgebra $\mathscr{H}_1$ is a Frobenius algebra with a G-action which preserves the multiplication, unit, and metric.

Remark 3.4. One can readily generalize the above definition to a G-Frobenius superalgebra by introducing an additional $\mathbb{Z}/2\mathbb{Z}$-grading and inserting signs in the usual manner.

We will also need a related, simpler algebraic gadget in our discussion of the Chern character.

Definition 3.5. A *G-commutative algebra* is a tuple $((\mathscr{H}, \rho), \cdot, \mathbf{1})$ satisfying all the axioms of a G-Frobenius algebra which do not involve the metric; namely, all but $(3), (8)$, and (9).

Remark 3.6. When G is the trivial group then a G-commutative algebra is nothing more than a commutative, associative algebra with unit.

Definition 3.7. Let $(\mathscr{H}, \rho)$ be a G-graded G-module. Let $\pi_G : \mathscr{H} \to \mathscr{H}$ be the averaging map

$$\pi_G(v) := \frac{1}{|G|} \sum_{\gamma \in G} \rho(\gamma)v$$

for all v in $\mathscr{H}$. Let $\overline{\mathscr{H}}$ be the image of π_G. The vector space $\overline{\mathscr{H}}$ is called the *space of G-coinvariants of $\mathscr{H}$,* and it inherits a grading by the set $\overline{G}$ of conjugacy classes of G:

$$\overline{\mathscr{H}} = \bigoplus_{\overline{\gamma} \in \overline{G}} \overline{\mathscr{H}}_{\overline{\gamma}}.$$

For any metric η on $\mathscr{H}$, we define $\overline{\eta}$ to be the restriction of the metric $\frac{1}{|G|}\eta$ to $\overline{\mathscr{H}}$.

The following Proposition is immediate.

Proposition 3.8. *The following properties hold:*

- *If the tuple* $((\mathscr{H},\rho),\cdot,\mathbf{1},\eta)$ *is a* G*-Frobenius algebra, then its* G*-coinvariants* $(\overline{\mathscr{H}},\cdot,\mathbf{1},\overline{\eta})$ *form a Frobenius algebra, where* $\cdot$ *is induced from* $\mathscr{H}$*.*
- *If the tuple* $((\mathscr{H},\rho),\cdot,\mathbf{1})$ *is a* G*-commutative algebra, then its* G*-coinvariants* $(\overline{\mathscr{H}},\cdot,\mathbf{1})$ *form a commutative, associative algebra with unit, where* $\cdot$ *is induced from* $\mathscr{H}$*.*

4. The stringy Chow ring of a variety with G-action

In this section, we review the definition of the stringy Chow ring $\mathscr{A}(X,G)$ of a smooth, projective variety X with the action of a group G. The G-Frobenius algebra structure on $\mathscr{A}(X,G)$ was constructed in [FG, JKK]. In addition to its G-grading, it has a grading by rational numbers (the usual $\mathbb{Z}$-grading shifted by the age). When G is the trivial group, then $\mathscr{A}(X,G)$ is isomorphic, as a Frobenius algebra, to the usual Chow ring $A^\bullet(X)$.

Let X be a smooth, projective variety with a right action $\rho : G \to \mathrm{Aut}(X)$ of a finite group G. The G-action induces a G-action on $X \times G$, where γ in G takes (x,m) to $(\rho(\gamma)x, \gamma^{-1}m\gamma)$. As a vector space, the stringy Chow ring is just the Chow ring of the *inertia variety*

$$I_G(X) := \coprod_{m\in G} X^m \subseteq X \times G,$$

where $X^m := \{(x,m)|\rho(m)x = x\}$ with its induced G action. Note that the inertia *variety* is not the same as the inertia *orbifold*, or inertia *stack*,

$$[I_G(X)/G] = \coprod_{(g)} [X^{(g)}/Z_G(g)]$$

of [CR1, AGV], which is the stack quotient of the inertia variety $I_G(X)$ by the action of G.

The irreducible components of $I_G(X)$ are smooth, and $I_G(X)$ has a natural G-action $\rho(\gamma) : X^m \to X^{\gamma^{-1}m\gamma}$. The inertia variety also has a G-equivariant *canonical involution*

$$\sigma : I_G(X) \to I_G(X),$$

which maps X^m to $X^{m^{-1}}$ via $(x,m) \mapsto (x,m^{-1})$ for all m in G.

Let

$$\mathscr{A}(X,G) := A^\bullet(I_G(X)) = \bigoplus_{m\in G} \mathscr{A}_m(X),$$

where $\mathscr{A}_m(X) := A^\bullet(X^m)$ for all m in G. The vector space $\mathscr{A}(X,G)$ inherits a right G-action $\rho : G \to \mathrm{Aut}(\mathscr{A}(X,G))$ from the action on $I_G(X)$, and there is a natural pairing $\eta_{\mathscr{A}}$ on $\mathscr{A}(X,G)$ defined by

$$\eta_{\mathscr{A}}(v_1,v_2) := \int_{[I_G(X)]} v_1 \cup \sigma^* v_2$$

for all v_1, v_2 in $\mathscr{A}(X,G)$. Let $\mathbf{1}$ be the unit element in $\mathscr{A}_1(X) = A^\bullet(X)$.

We also have several natural morphisms. Let $X^{\mathbf{m}} := X^{m_1} \cap X^{m_2} \cap X^{m_3}$ for all triples $\mathbf{m} := (m_1,m_2,m_3)$ in G^3 such that $m_1m_2m_3 = 1$, where X^{m_i} is regarded as a subvariety of X. We define

$$\mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i},$$

and

$$\mathbf{i}_{m_i} : X^{m_i} \to X$$

to be the canonical inclusion morphisms for all $i = 1, 2, 3$, and define

$$\check{\mathbf{e}}_{m_i} := \sigma \circ \mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i^{-1}}.$$

Let X have dimension D, q belong to X, and m belong to the isotropy subgroup of q in G. Denote the set of eigenvalues of the action of m on T_qX by $\{\exp(-2\pi i a_1), \dots, \exp(-2\pi i a_D)\}$, where for each $j = 1, \dots, D$ the rational number a_j belongs to the interval $[0, 1)$. The *age* $a(m, q)$ *of* m *at* q is defined to be $\sum_{j=1}^{D} a_j$.[†] Since $a(m, q)$ depends only upon the connected component $U \subseteq X^m$ containing q, we define the age $a(m, U)$ of m on U to be $a(m, q)$ for any q in U.

For all m in G, all connected components U of X^m, and all elements v_m in $A^p(U) \subseteq \mathscr{A}_m(X)$, for p the usual integral degree in the Chow ring, we define a $\mathbb{Q}$*-grading (or stringy grading)* $|v_m|_{str}$ on $\mathscr{A}_m(X)$ by

$$|v_m|_{str} := a(m, U) + p. \tag{14}$$

The key ingredient in defining the multiplication on the stringy Chow ring is the *obstruction bundle* $\mathscr{R}$, defined for each triple $\mathbf{m} = (m_1, m_2, m_3) \in G^3$ such that $m_1 m_2 m_3 = 1$, as follows. Let $\langle \mathbf{m} \rangle$ be the subgroup generated by the elements m_1, m_2, and m_3. There is a presentation of the fundamental group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$ as $\langle c_1, c_2, c_3 | c_1 c_2 c_3 = 1 \rangle$, where c_1, c_2 and c_3 are little loops around $p_1 = 0, p_2 = 1$, and $p_3 = \infty$, respectively. We define a natural homomorphism $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \langle \mathbf{m} \rangle$, taking c_i to m_i. This defines a principal $\langle \mathbf{m} \rangle$-bundle over $\mathbb{P}^1 - \{0, 1, \infty\}$ which extends to a smooth connected curve E. The curve E has an action of $\langle \mathbf{m} \rangle$ such that the quotient $E/\langle \mathbf{m} \rangle$ has genus zero, and the natural map $E \to E/\langle \mathbf{m} \rangle$ is branched at the three points p_1, p_2, p_3 with monodromy m_1, m_2, m_3, respectively.

Let $\pi : E \times X^{\mathbf{m}} \to X^{\mathbf{m}}$ be the second projection. We define the obstruction bundle $\mathscr{R}(\mathbf{m})$ on $X^{\mathbf{m}}$ to be

$$\mathscr{R}(\mathbf{m}) := R^1 \pi_*^{\langle \mathbf{m} \rangle} (\mathscr{O}_E \boxtimes TX|_{X^{\mathbf{m}}}). \tag{15}$$

One can check that the restriction of the obstruction bundle $\mathscr{R}(\mathbf{m}) \to X^{\mathbf{m}}$ to a connected component U of $X^{\mathbf{m}}$ has rank

$$a(m_1, U) + a(m_2, U) + a(m_3, U) - \operatorname{codim}(U \subseteq X). \tag{16}$$

For those familiar with quantum cohomology, this obstruction bundle is the analogue of the obstruction bundle for stable maps, but with additional accounting for the structure of the group action on X. That is, $c_{top}(\mathscr{R})$ is the virtual fundamental class on (distinguished components of) the moduli space of genus-zero, three-pointed G-stable maps into X. The base space $X^{\mathbf{m}}$ in the definition of the obstruction bundle is actually the distinguished component $\xi_{0,3}(X, 0, \mathbf{m}) \cong pt \times X^{\mathbf{m}}$ of $\overline{\mathscr{M}}^G_{0,3}(X, 0, \mathbf{m})$. The interested reader may refer to [JKK, §6] for more details.

Definition 4.1. The *stringy multiplication (or stringy product)* on $\mathscr{A}(X, G)$ is defined by

$$v_{m_1} \cdot v_{m_2} := \check{\mathbf{e}}_{m_3 *}(\mathbf{e}^*_{m_1} v_{m_1} \cup \mathbf{e}^*_{m_2} v_{m_2} \cup c_{\mathrm{top}}(\mathscr{R}(\mathbf{m}))) \tag{17}$$

[†]The signs of a_j and $a(m, q)$ differ from [FG] and [CR1] because our group actions are on the right.

for all v_{m_i} belonging to $\mathscr{A}_{m_i}(X)$, for $i = 1, 2$, and $\mathbf{m} := (m_1, m_2, m_3)$ in G^3 such that $m_1 m_2 m_3 = 1$. Otherwise, the product is defined to be zero.

Theorem 4.2. [FG, JKK] *Let X be a smooth, projective variety with an action of a finite group G.*

(1) *The tuple $((\mathscr{A}(X,G), \rho), \cdot, \mathbf{1}, \eta_{\mathscr{A}})$ is a G-Frobenius algebra.*
(2) $|\mathbf{1}|_{str} = 0$.
(3) *The multiplication respects the $\mathbb{Q}$-grading, i.e., for all homogeneous elements v_{m_i} in $\mathscr{A}_{m_i}(X)$, for $i = 1, 2$, we have*

$$|v_{m_1} \cdot v_{m_2}|_{str} = |v_{m_1}|_{str} + |v_{m_2}|_{str}.$$

(4) *The metric has a definite $\mathbb{Q}$-grading, i.e., for all homogeneous elements v_{m_i} in $\mathscr{A}_{m_i}(X)$, for $i = 1, 2$, we have*

$$\eta_{\mathscr{A}}(v_{m_1}, v_{m_2}) = 0 \tag{18}$$

unless $m_1 m_2 = 1$, and

$$|v_{m_1}|_{str} + |v_{m_2}|_{str} = \dim X. \tag{19}$$

Remark 4.3. Sometimes the $\mathbb{Q}$-grading just happens to be integral. If X is n dimensional and its canonical bundle K_X has a nowhere-vanishing section Ω, then for all m in G, we have

$$\rho(m)^*\Omega = \exp(2\pi i a(m))\Omega.$$

Thus, if G preserves Ω, then $a(m)$ must be an integer.

A special case is when X is $2n$ dimensional, possessing a (complex algebraic) symplectic form ω in $\bigwedge^2 T^*X$. This can arise if X happens to be a hyper-Kähler manifold. If, in addition, G preserves ω, then G preserves the nowhere vanishing section ω^n of K_X. In this case, for all m in G and for every connected component U of X^m, the associated age [Kal] is the integer

$$a(m, U) = \frac{1}{2}\operatorname{codim}(U \subseteq X).$$

5. The stringy K-theory of a variety with G-action

In this section, we introduce the stringy K-theory $\mathscr{K}(X,G)$ of a smooth projective variety X with an action of a finite group G. Like stringy Chow, $\mathscr{K}(X,G)$ is a G-Frobenius algebra which, when G is the trivial group, reduces to the ordinary K-theory $K(X)$. However, unlike the stringy Chow ring, $\mathscr{K}(X,G)$ lacks a $\mathbb{Q}$-grading. This should not be surprising as even ordinary K-theory lacks a grading by "dimension."

As a vector space, the *stringy K-theory* $\mathscr{K}(X,G)$ of a smooth, projective variety X with an action of a group G is defined to be

$$\mathscr{K}(X,G) := K(I_G(X)) = \bigoplus_{m \in G} \mathscr{K}_m(X),$$

where $\mathscr{K}_m(X) := K(X^m)$ for all m in G. Again, $\mathscr{K}(X,G)$ inherits a right G-action $\rho : G \to \operatorname{Aut}(\mathscr{K}(X,G))$ from the action on $I_G(X)$. We let $\eta_{\mathscr{K}}$ be the pairing on $\mathscr{K}(X,G)$ defined by

$$\eta_{\mathscr{K}}(v_1, v_2) := \chi(I_G(X), v_1 \otimes \sigma^* v_2)$$

for all v_1, v_2 in $\mathscr{K}(X,G)$. Let $\mathbf{1} := [\mathscr{O}_X]$ be the isomorphism class of the structure sheaf of X in $\mathscr{K}_1(X) = K(X)$.

Definition 5.1. The *multiplication (or product)* on $\mathscr{K}(X,G)$ is defined by

$$\mathscr{F}_{m_1} \cdot \mathscr{F}_{m_2} := \check{\mathbf{e}}_{m_3 *}(\mathbf{e}^*_{m_1}\mathscr{F}_{m_1} \otimes \mathbf{e}^*_{m_2}\mathscr{F}_{m_2} \otimes \lambda_{-1}(\mathscr{R}(\mathbf{m})^*)) \tag{20}$$

for all $\mathscr{F}_{m_i}$ belonging to $\mathscr{K}_{m_i}(X)$, for $i = 1,2$ and $\mathbf{m} := (m_1, m_2, m_3)$ in G^3 such that $m_1 m_2 m_3 = 1$. Otherwise, the product is defined to be zero.

Theorem 5.2. *Let X be a smooth, projective variety with an action of a finite group G. The tuple $((\mathscr{K}(X,G),\rho),\cdot,\mathbf{1},\eta_{\mathscr{K}})$ is a G-Frobenius algebra.*

Proof. The only nontrivial parts of this theorem are the associativity of the multiplication and the trace axiom. These are proved in Theorems 10.4 and 10.7. $\square$

Remark 5.3. As with stringy Chow, there is a natural way to think of this construction in terms of stable maps. Here the virtual fundamental class $c_{top}(\mathscr{R})$ on $\overline{\mathscr{M}}^G_{0,3}(X,0,\mathbf{m})$ has been replaced by a *virtual structure sheaf* $\mathscr{O}_{\mathrm{vir}} := \lambda_{-1}(\mathscr{R}^*)$.

Let us consider a case where the obstruction bundle $\mathscr{R}$ on $X^{\mathbf{m}}$ is trivial, namely, when $m_i = 1$ for some $i = 1,2,3$. If $m_1 = 1$ and $m_2 m_3 = 1$, then the stringy multiplication is given by the restriction to X^{m_3} of the ordinary multiplication in ordinary K-theory, i.e.,

$$\mathscr{F}_{m_1=1} \cdot \mathscr{F}_{m_2} = \mathscr{F}_{m_1}|_{X^{m_3}} \otimes \sigma^*\mathscr{F}_{m_2}. \tag{21}$$

A similar result holds if $m_2 = 1$ and $m_1 m_3 = 1$. In particular, this means that stringy multiplication on the untwisted sector $\mathscr{K}_1(X)$ coincides with the ordinary multiplication on $\mathscr{K}_1(X)$.

More interesting is the case where $m_3 = 1$ and $m_1 m_2 = 1$. In this case, we have

$$\mathscr{F}_{m_1} \cdot \mathscr{F}_{m_2} = \check{\mathbf{e}}_{m_3 *}(\mathbf{e}^*_{m_1}\mathscr{F}_{m_1} \otimes \mathbf{e}^*_{m_2}\mathscr{F}_{m_2}). \tag{22}$$

Even here, the stringy multiplication is nontrivial since the map $\check{\mathbf{e}}_{m_3}$ could be between varieties of different dimensions.

6. Explicit description of the obstruction bundle

In this section, we give an elementary description of the obstruction bundle $\mathscr{R}$ in terms of the representation of G on the tangent bundle TX. This follows from an explicit formula for $H^1(E,\mathscr{O}_E)$ as a G-module in terms of the regular representation of G.

We begin by introducing some notation. Let X be a smooth, projective variety with an action of a finite group G. Let m be an element in G of order r which generates the cyclic subgroup $\langle m\rangle$ in G. Let W_m either denote the normal bundle

$$N_m := \left(TX|_{X^m}\right)/TX^m$$

of X^m in X or the restriction $TX|_{X^m}$ of the tangent bundle. Since W_m is an $\langle m\rangle$-equivariant vector bundle over X^m, it has an eigenbundle decomposition

$$W_m = \bigoplus_{k=0}^{r-1} W_{m,k}, \tag{23}$$

where $W_{m,k}$ is the eigenbundle where the action of m has eigenvalue $\zeta^k = \exp(-2\pi i k/r)$. In particular, we have

$$N_{m,0} = 0. \tag{24}$$

For all m, define $\mathscr{S}_m$ in $K(X^m)$ as

$$\mathscr{S}_m := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{m,k}. \tag{25}$$

Remark 6.1. Clearly, $\mathscr{S}_m$ is the same whether W_m is N_m or $TX|_{X^m}$.

The G-equivariant involution $\sigma : X^m \to X^{m^{-1}}$ yields a G-equivariant isomorphism $\sigma^* : W_{m^{-1}} \to W_m$ for all m in G. If m acts by multiplication by ζ^k, then m^{-1} acts by ζ^{r-k}, so we have

$$\sigma^* W_{m^{-1},0} = W_{m,0} \tag{26}$$

and

$$\sigma^* W_{m^{-1},k} = W_{m,r-k} \tag{27}$$

for all $k \in \{1, \dots, r-1\}$. Consequently, the induced map $\sigma^* : K(X^{m^{-1}}) \to K(X^m)$ satisfies

$$\mathscr{S}_m \oplus \sigma^* \mathscr{S}_{m^{-1}} = N_m, \tag{28}$$

by Equations (24) and (27).

Remark 6.2. The virtual rank of $\mathscr{S}_m$ on a connected component U of X^m is precisely the age $a(m, U)$, and taking the virtual rank of both sides of Equation (28) yields the well-known equation

$$a(m, U) + a(m^{-1}, U) = \operatorname{codim}(U \subseteq X). \tag{29}$$

Hence, $\mathscr{S}_m$ could be regarded as a K-theoretic version of the age.

6.1. **The key formula.** The purpose of this section is to prove Theorem 1.2, that is,to establish Equation (3).

Theorem 1.2 is a consequence of Theorem 6.3, which we will prove first. The basic setup for Theorem 6.3 is as follows. Let E be a smooth algebraic curve of genus $\tilde{g}$, not necessarily connected, with a finite group G acting effectively on E. Assume that the quotient E/G has genus g. Denote the orbits where the action is not free by $p_1, \dots, p_n \in E/G$. A choice of base point $\widetilde{p} \in E$ induces a homomorphism of groups

$$\varphi_{\widetilde{p}} : \pi_1(E/G - \{p_1, \dots, p_n\}, p) \to G,$$

where p is the image of $\widetilde{p}$ in E/G (we assume $p \notin \{p_1, \dots, p_n\}$). Denote by H the image of $\varphi_{\widetilde{p}}$ in G. Note that the number α of connected components of E is the index $[G : H]$. There is a presentation of $\pi_1(E/G - \{p_1, \dots, p_n\}, p)$ of the form $\langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n | \prod_{i=1}^n c_i = \prod_{j=1}^g [a_j, b_j] \rangle$, where the c_i are loops around the points p_i. For each $i \in \{1, \dots, n\}$ we call the image $m_i := \varphi_{\widetilde{p}}(c_i) \in G$ of c_i the *monodromy* around p_i, and we denote the order of m_i by r_i. Of course, a different choice of $\widetilde{p}$ will change all of the m_i by simultaneous conjugation with an element of G.

Theorem 6.3. *Given the setup described above, and letting $\mathbb{C}[G]$ denote the group ring regarded as a G-module under (right) multiplication, we have the following equality in the representation ring of G,*

$$H^1(E; \mathscr{O}_E) = \mathbb{C}[H\backslash G] \oplus (g-1)\mathbb{C}[G] \oplus \bigoplus_{i=1}^n \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{|G|} \operatorname{Ind}_{\langle m_i \rangle}^G \mathbb{C}[G]_{m_i,k_i}, \tag{30}$$

where $\mathbb{C}[G]_{m_j,k_j}$ *is the eigenspace of the action of* m_j *with eigenvalue* $\exp(2\pi i k_j/r_j)$, *and* $\operatorname{Ind}_{\langle m_i\rangle}^G \mathbb{C}[G]_{m_i,k_i}$ *is the induced representation* $\mathbb{C}[G]_{m_i,k_i} \otimes_{\mathbb{C}[\langle m_i\rangle]} \mathbb{C}[G]$.

Proof. It suffices to check that these two virtual representations have the same virtual character. The trace of the action of an element $\gamma \in G$ on the right hand side is

$$\chi_{\mathbb{C}[H\backslash G]}(\gamma) + (g-1)|G|\delta_{\gamma,e} + \sum_{i=1}^{n}\sum_{k_i=0}^{r_i-1} \frac{k_i}{|G|}\chi_{\operatorname{Ind}_{\langle m_i\rangle}^G}\mathbb{C}[G]_{m_i,k_i}(\gamma).$$

It is well known (e.g., [FH, ex 3.19]) that, for a representation V of a subgroup $H < G$, we have

$$\chi_{\operatorname{Ind}_H^G V}(\gamma) = \frac{|G|}{|H|}\sum_{\sigma\in H\cap C(\gamma)} \frac{\chi_V(\sigma)}{|C(\gamma)|},$$

where $C(\gamma)$ is the conjugacy class of γ in G. In our case, with $H = \langle m_i\rangle$ of order r_i, and $V = \mathbb{C}[G]_{m_i,k_i}$ of dimension $\frac{|G|}{r_i}$, we have

$$\begin{aligned}\chi_{\operatorname{Ind}_H^G V}(\gamma) &= \frac{|G|}{r_i|C(\gamma)|}\sum_{m_i^l\in C(\gamma)} \chi_V(m_i^l)\\ &= \frac{|G|}{r_i|C(\gamma)|}\sum_{m_i^l\in C(\gamma)} \zeta_i^{lk_i} dimV\\ &= \frac{|G|^2}{r_i^2|C(\gamma)|}\sum_{m_i^l\in C(\gamma)} \zeta_i^{lk_i},\end{aligned}$$

where $\zeta_j = \exp(2\pi i/r_j)$, for each $j \in \{1,\dots,n\}$. Thus the trace of the right hand side of equation (30) becomes

$$\chi_{\mathbb{C}[H\backslash G]}(\gamma) + (g-1)|G|\delta_{\gamma,e} + \sum_{i-1}^{n}\sum_{k_i=0}^{r_i-1} \frac{k_i|G|}{r_i^2 C(\gamma)|}\sum_{m_i^l\in C(\gamma)} \zeta_i^{lk_i}.$$

If $\gamma = e$ is the identity element of G, we have

$$\begin{aligned}\operatorname{Tr}_{RHS}(e) &= \alpha + |G|(g-1) + \sum_{i=1}^{n}\sum_{k_i=0}^{r_i-1}\frac{k_i|G|}{r_i^2}\\ &= \alpha + |G|(g-1) + |G|\sum_{i=1}^{n}\frac{r_i-1}{2r_i}\\ &= \dim_{\mathbb{C}} H^1(E,\mathscr{O}_E), \qquad (31)\end{aligned}$$

where the last equality follows from the Riemann-Hurwitz formula and the fact that the genus of E/G is g.

If $\gamma \neq e$ then

$$\begin{aligned}
\mathrm{Tr}_{RHS}(\gamma) &= \chi_{\mathbb{C}[H\backslash G]}(\gamma) + \sum_{i=1}^{n} \frac{|G|}{r_i^2 |C(\gamma)|} \sum_{m_i^l \in C(\gamma)} \sum_{k_i=0}^{r_i-1} k_i \zeta_i^{lk_i} \\
&= \chi_{\mathbb{C}[H\backslash G]}(\gamma) + \sum_{i=1}^{n} \frac{|G|}{r_i^2 |C(\gamma)|} \sum_{m_i^l \in C(\gamma)} r_i \frac{\zeta_i^{-l}}{1-\zeta_i^{-l}} \\
&= \sum_{\substack{\sigma \in H\backslash G \\ \sigma\gamma=\sigma}} 1 + \sum_{i=1}^{n} \frac{|G|}{r_i |C(\gamma)|} \sum_{m_i^l \in C(\gamma)} \frac{\zeta_i^{-l}}{1-\zeta_i^{-l}}, \qquad (32)
\end{aligned}$$

where the last equality follows from standard results on induced representations [FH, 3.18].

This formula is related to fixed points of the action of γ on E as follows. The element γ can only fix points that lie over the p_i, for $i \in \{1, \dots, n\}$. If $\widetilde{p}_i$ is a point over p_i fixed by γ, then $\widetilde{p}_i$ has holonomy conjugate to m_i, and thus γ must be conjugate to m_i^l for some l. Conversely, if γ is conjugate to m_i^l for some l, then γ fixes all points $\widetilde{p}_i$ that lie over p_i, and γ acts on the tangent space $T_{\widetilde{p}_i}E$ by ζ_i^l.

If $\widetilde{p}_i$ and $\widetilde{p}'_i$ both are fixed by γ with action ζ_i^l on the tangent space, then $\widetilde{p}'_i = \widetilde{p}_i\sigma$ for some $\sigma \in G$, such that σ commutes with γ, but if $\sigma \in \langle m_i \rangle$, then $\widetilde{p}_i = \widetilde{p}'_i$. So the number of such points lying over p_i is exactly $\frac{|Z_G(\gamma)|}{|\langle m_i \rangle|} = \frac{|G|}{r_i|C(\gamma)|}$, where $Z_G(\gamma)$ is the centralizer of γ in G.

The term $\sum_{\substack{\sigma \in H\backslash G \\ \sigma\gamma=\sigma}} 1$ counts connected components of E which are mapped to themselves by γ; that is, it is the trace of γ for the natural representation of G on $H^0(E, \mathscr{O}_E)$. If we now denote by $d\gamma_{\widetilde{p}} = \zeta_i^l$ the action of γ on the tangent space $T_{\widetilde{p}}E$ at a fixed point $\widetilde{p} \in E$, the above argument shows that

$$\mathrm{Tr}_{RHS}(\gamma) = \chi_{H^0(E,\mathscr{O}_E)}(\gamma) + \sum_{\widetilde{p} \text{ fixed by } \gamma} \frac{(d\gamma_{\widetilde{p}})^{-1}}{1-(d\gamma_{\widetilde{p}})^{-1}}.$$

But the Eichler trace formula says precisely that this is the trace of the action of γ on $H^1(E, \mathscr{O}_E)$; that is, the traces of equation (30) agree (see [FK, §V.2.0] for E connected with $\tilde{g} > 1$, and [Sh, §17] in general). $\square$

Proof of Theorem 1.2. For any $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1m_2m_3 = 1$, the curve E in the definition of $\mathscr{R}(\mathbf{m})$ is connected and has an effective action of $G' := \langle m_1, m_2, m_3 \rangle$ with quotient $\mathbb{P}^1 = E/G'$ and three branch points p_1, p_2, p_3.

We have

$$\mathscr{R}(\mathbf{m}) = R^1\pi_*^{G'}(\mathscr{O}_E \boxtimes TX|_{X^{\mathbf{m}}}) \cong (H^1(E, \mathscr{O}_E) \otimes TX|_{X^{\mathbf{m}}})^{G'},$$

and by Theorem 6.3 this is

$$
\begin{aligned}
(H^1(E, \mathscr{O}_E) \otimes TX|_{X^{\mathbf{m}}})^{G'} &= \left(\left(\mathbb{C} \ominus \mathbb{C}[G'] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{|G'|} \operatorname{Ind}_{\langle m_i \rangle}^{G'} \mathbb{C}[G']_{m_i,k_i}\right) \otimes TX|_{X^{\mathbf{m}}}\right)^{G'} \\
&= TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{|G'|} \left(\operatorname{Ind}_{\langle m_i \rangle}^{G'} \mathbb{C}[G']_{m_i,k_i} \otimes TX|_{X^{\mathbf{m}}}\right)^{G'} \\
&= TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{r_i} \left(TX|_{X^{\mathbf{m}}}\right)_{m_i,k_i} \\
&= TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathscr{S}_{m_i}|_{X^{\mathbf{m}}}. \qquad (33)
\end{aligned}
$$

□

6.2. **The Abelian case.** It is instructive to consider the special case where G is an Abelian group. In this case, our analysis of the obstruction bundle $\mathscr{R}$ yields, as a simple corollary, a result originally due to Chen and Hu [CH].

Consider the obstruction bundle $\mathscr{R}$ over $X^{\mathbf{m}}$, where $\mathbf{m} = (m_1, m_2, m_3)$ in G^3 satisfies $m_1 m_2 m_3 = 1$. Let us assume without loss of generality that $G = \langle \mathbf{m} \rangle$. Since G is Abelian, one can simultaneously diagonalize the actions of m_i, for $i = 1, 2, 3$ on $\mathscr{R}$. If $W_{\mathbf{m}}$ denotes the normal bundle of $X^{\mathbf{m}}$ in X, then we have the simultaneous eigenbundle decomposition

$$W_{\mathbf{m}} = \bigoplus_{\mathbf{k}} W_{\mathbf{m},\mathbf{k}}, \qquad (34)$$

where the sum is over all $\mathbf{k} = (k_1, k_2, k_3)$ such that $k_i = 0, \ldots, r_i - 1$, and r_i is the order of m_i for all $i \in \{1, 2, 3\}$. The eigenbundle $W_{\mathbf{m},\mathbf{k}}$ of $W_{\mathbf{m}}$ is the bundle where for all $j \in \{1, 2, 3\}$ each m_j has an eigenvalue $\exp(-2\pi i k_j / r_j)$. The following proposition is an easy corollary of Theorem 1.2.

Proposition 6.4. [CH] *When G is Abelian, under the above assumptions, then*

$$\mathscr{R} = \bigoplus_{\mathbf{k}} W_{\mathbf{m},\mathbf{k}} \qquad (35)$$

in $K(X^{\mathbf{m}})$, where the sum is over triples $\mathbf{k} = (k_1, k_2, k_3)$, for $k_i = 0, \ldots, r_i - 1$ and $i = 1, 2, 3$, such that

$$\frac{k_1}{r_1} + \frac{k_2}{r_2} + \frac{k_3}{r_3} = 2. \qquad (36)$$

Proof. It is a straightforward exercise to verify that the right hand side of Equation (3) agrees with Equation (35) when G is Abelian. □

Remark 6.5. Chen and Hu use this characterization of the obstruction bundle to give a de Rham model for Chen-Ruan orbifold cohomology when the orbifold arises as the quotient of a variety by an Abelian group. It would be interesting to see how their constructions can be generalized to non-Abelian groups in light of Equation (3).

7. The stringy Chern character

In this section, we introduce a stringy generalization of the ordinary Chern character. For general G, the ordinary Chern character fails to be a ring homomorphism; however, this drawback can be overcome through the introduction of the appropriate correction terms to give what we call the *stringy Chern character* $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$. The map $\mathscr{C}\mathbf{h}$ is an isomorphism of G-commutative algebras for any smooth, projective variety X with an action of a finite group G. When G is the trivial group, $\mathscr{C}\mathbf{h}$ reduces to the ordinary Chern character from ordinary K-theory to the ordinary Chow ring of X.

Definition 7.1. Let X be a smooth, projective variety with an action of G. The *stringy Chern character* $\mathscr{C}\mathbf{h} : \mathscr{K}(X,G) \to \mathscr{A}(X,G)$ is the linear map defined by

$$\mathscr{C}\mathbf{h}(\mathscr{F}_m) := \mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}^{-1}(\mathscr{S}_m) \tag{37}$$

for all m in G and $\mathscr{F}_m$ in $\mathscr{K}_m(X)$, where $\mathscr{S}_m$ is defined in Equation (25), $\mathbf{td}$ is the Todd class, and $\mathbf{ch}$ is the ordinary Chern character.

We are now ready to prove Theorem 1.3, that is, that $\mathscr{C}\mathbf{h}$ is an isomorphism of G-commutative algebras.

Proof. First, we show that $\mathscr{C}\mathbf{h}$ is an isomorphism of G-graded G-modules. It is an isomorphism of G-graded vector spaces, since $\mathbf{td}$ is invertible (it is a series starting with $\mathbf{1}$). The equivariance under the G-action follows from naturality properties of $\mathbf{td}$, $\mathbf{ch}$, the cup product, and $\mathscr{S}_m$.

We now prove that $\mathscr{C}\mathbf{h}$ respects multiplication. We suppress the cup and tensor product symbols to avoid notational clutter. Let $\mathscr{F}_{m_i}$ belong to $\mathscr{K}_{m_i}(X)$, for $i = 1,2,3$, where $m_1m_2m_3 = 1$. Let $\mathbf{e}_{m_i}$ denote the inclusion $X^{\mathbf{m}} \to X^{m_i}$ and $\check{\mathbf{e}}_{m_i} := \sigma \circ \mathbf{e}_{m_i} : X^{\mathbf{m}} \to X^{m_i^{-1}}$ for all $i = 1,2,3$. We have

$$\begin{aligned}
\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1} \cdot \mathscr{F}_{m_2}) &= \mathbf{ch}(\mathscr{F}_{m_1} \cdot \mathscr{F}_{m_2})\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}}) \\
&= \mathbf{ch}(\check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathscr{F}_{m_1}\mathbf{e}^*_{m_2}\mathscr{F}_{m_2}\lambda_{-1}(\mathscr{R}^*)))\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}}) \\
&= \check{\mathbf{e}}_{m_3*}(\mathbf{ch}(\mathbf{e}^*_{m_1}\mathscr{F}_{m_1}\mathbf{e}^*_{m_2}\mathscr{F}_{m_2}\lambda_{-1}(\mathscr{R}^*))\mathbf{td}(T\check{\mathbf{e}}_{m_3}))\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}})) \\
&= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})\mathbf{ch}(\lambda_{-1}(\mathscr{R}^*))\mathbf{td}(T\check{\mathbf{e}}_{m_3}))\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}}) \\
&= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}^{-1}(\mathscr{R})\mathbf{td}(T\check{\mathbf{e}}_{m_3}))\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}}) \\
&= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\ominus\mathscr{R} \oplus T\check{\mathbf{e}}_{m_3}))\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}}) \\
&= \mathbf{e}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\ominus\mathscr{R} \oplus T\check{\mathbf{e}}_{m_3})\check{\mathbf{e}}^*_{m_3}\mathbf{td}(\ominus \mathscr{S}_{m_3^{-1}})) \\
&= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\ominus\mathscr{R} \oplus T\check{\mathbf{e}}_{m_3} \ominus \check{\mathbf{e}}^*_{m_3}\mathscr{S}_{m_3^{-1}})),
\end{aligned}$$

where the first two equalities follow from the definition of the multiplication and $\mathscr{C}\mathbf{h}$, the third from the Grothendieck-Riemann-Roch theorem, the fourth from the fact that the usual Chern character $\mathbf{ch}$ commutes with pull back and is a homomorphism with respect to the usual products in the Chow ring, and the fifth from Equation (13). The sixth and eighth equalities follow from multiplicativity of $\mathbf{td}$, and the seventh follows from the projection formula.

If we let $\mathscr{T} \in K(X^{\mathbf{m}})$ be

$$\mathscr{T} := \ominus\mathscr{R} \oplus TX^{\mathbf{m}} \ominus TX^{m_3^{-1}}\Big|_{X^{\mathbf{m}}} \ominus \check{\mathbf{e}}^*_{m_3}\mathscr{S}_{m_3^{-1}},$$

then by plugging in Equation (28), we obtain

$$\mathscr{T} = \ominus\mathscr{R} \oplus TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \mathscr{S}_{m_3}|_{X^{\mathbf{m}}}. \tag{38}$$

Therefore, we obtain the equality

$$\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1}\cdot\mathscr{F}_{m_2}) = \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\mathscr{T})). \tag{39}$$

Similarly, we see that

$$\begin{aligned}\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1})\cdot\mathscr{C}\mathbf{h}(\mathscr{F}_{m_2}) &= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathscr{C}\mathbf{h}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R}))\\ &= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}(\mathbf{ch}(\mathscr{F}_{m_1})e^*_{m_1}\mathbf{td}(\ominus\mathscr{S}_{m_1}))\mathbf{e}^*_{m_2}(\mathbf{ch}(\mathscr{F}_{m_2})\mathbf{td}(\ominus\mathscr{S}_{m_2}))c_{\mathrm{top}}(\mathscr{R}))\\ &= \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\ominus e^*_{m_1}\mathscr{S}_{m_1}\ominus e^*_{m_2}\mathscr{S}_{m_2})),\end{aligned}$$

where the first two equalities are by definition and the third is by multiplicativity of **td**. Thus, if

$$\mathscr{T}' := \ominus\,\mathscr{S}_{m_1}|_{X^{\mathbf{m}}} \ominus \mathscr{S}_{m_2}|_{X^{\mathbf{m}}}, \tag{40}$$

then

$$\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1})\cdot\mathscr{C}\mathbf{h}(\mathscr{F}_{m_2}) = \check{\mathbf{e}}_{m_3*}(\mathbf{e}^*_{m_1}\mathbf{ch}(\mathscr{F}_{m_1})\mathbf{e}^*_{m_2}\mathbf{ch}(\mathscr{F}_{m_2})c_{\mathrm{top}}(\mathscr{R})\mathbf{td}(\mathscr{T}')). \tag{41}$$

$\mathscr{C}\mathbf{h}$ is therefore an algebra homomorphism if and only if the right hand sides of Equations (39) and (41) are equal. A sufficient condition for this equality to hold is if $\mathscr{T} = \mathscr{T}'$, or equivalently, if Equation (3) holds.

It remains to show that $\mathscr{C}\mathbf{h}$ respects the trace axiom. Consider m_1, m_2 in G, v_{m_1} in $\mathscr{K}_{m_1}$, and $v_{[m_1,m_2]}$ in $\mathscr{K}_{[m_1,m_2]}$. Let L_w denote left multiplication by w. Since $\mathscr{C}\mathbf{h}$ is a ring isomorphism commuting with the G-action, we obtain

$$\mathscr{C}\mathbf{h}\circ L_{v_{[m_1,m_2]}}\circ\rho(m_2^{-1})\circ\mathscr{C}\mathbf{h}^{-1} = L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}\circ\rho(m_2^{-1}) \tag{42}$$

and

$$\mathscr{C}\mathbf{h}\circ\rho(m_1)\circ L_{v_{[m_1,m_2]}}\circ\mathscr{C}\mathbf{h}^{-1} = \rho(m_2)\circ L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}. \tag{43}$$

Taking the trace of both sides of Equation (42) and using the cyclicity of the trace, we obtain

$$\mathrm{Tr}_{\mathscr{K}_{m_1}(X)}(L_{v_{[m_1,m_2]}}\circ\rho(m_2^{-1})) = \mathrm{Tr}_{\mathscr{A}_{m_1}(X)}(L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}\circ\rho(m_2^{-1})). \tag{44}$$

Doing the same to Equation (43), we obtain

$$\mathrm{Tr}_{\mathscr{K}_{m_2}(X)}(\rho(m_1)\circ L_{v_{[m_1,m_2]}}) = \mathrm{Tr}_{\mathscr{A}_{m_2}(X)}(\rho(m_2)\circ L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}). \tag{45}$$

However, the left hand sides of Equations (44) and (45) are equal by the trace axiom on $\mathscr{K}(X,G)$. Thus, we obtain

$$\mathrm{Tr}_{\mathscr{A}_{m_1}(X)}(L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}\circ\rho(m_2^{-1})) = \mathrm{Tr}_{\mathscr{A}_{m_2}(X)}(\rho(m_2)\circ L_{\mathscr{C}\mathbf{h}(v_{[m_1,m_2]})}),$$

as desired.

□

Remark 7.2. It is instructive to consider the homomorphism property of $\mathscr{C}\mathbf{h}$ when the obstruction bundle $\mathscr{R}$ on $X^{\mathbf{m}}$ is trivial. When $m_1 = 1$ and $m_2m_3 = 1$, it is trivial to verify from Equation (22) that

$$\mathscr{C}\mathbf{h}(\mathscr{F}_{m_1}\cdot\mathscr{F}_{m_2}) = \mathscr{C}\mathbf{h}(\mathscr{F}_{m_1})\cdot\mathscr{C}\mathbf{h}(\mathscr{F}_{m_2}). \tag{46}$$

Indeed, Equation (46) continues to hold even if $\mathscr{C}\mathbf{h}$ were replaced by the ordinary Chern character **ch**. A similar result holds if $m_2 = 1$ and $m_1m_3 = 1$. However, when $m_1m_2 = 1$ and $m_3 = 1$, then Equation (46) would fail to hold if $\mathscr{C}\mathbf{h}$ were replaced by the ordinary Chern character $\mathbf{ch}$ because of the presence of the nontrivial pushforward map $\check{\mathbf{e}}_{m_3*}$ in Equation (22). This shows that the stringy corrections to the Chern character are necessary even when the obstruction bundle is trivial.

Finally, $\mathscr{C}\mathbf{h}$ satisfies the usual functorial properties with respect to equivariant étale morphisms.

Theorem 7.3. *Let $f : X \to Y$ be a G-equivariant, étale morphism between smooth, projective varieties X and Y with G-action. The following properties hold.*

1. *(Pullback) The pullback maps $f^* : \mathscr{A}(Y,G) \to \mathscr{A}(X,G)$ and $f^* : \mathscr{K}(Y,G) \to \mathscr{K}(X,G)$ are homomorphisms of G-Frobenius algebras.*
2. *(Naturality) The following diagram commutes.*

$$\begin{tikzcd} \mathscr{K}(Y,G) \arrow[r, "f^*"] \arrow[d, "\mathscr{C}\mathbf{h}"'] & \mathscr{K}(X,G) \arrow[d, "\mathscr{C}\mathbf{h}"'] \\ \mathscr{A}(Y,G) \arrow[r, "f^*"] & \mathscr{A}(X,G) \end{tikzcd} \tag{47}$$

3. *(Grothendieck-Riemann-Roch) For all m in G and $\mathscr{F}_m$ in $\mathscr{K}_m(X)$,*

$$f_*(\mathscr{C}\mathbf{h}(\mathscr{F}_m) \cup \mathbf{td}(TX^m)) = \mathscr{C}\mathbf{h}(f_*\mathscr{F}_m) \cup \mathbf{td}(TY^m). \tag{48}$$

Proof. The proof of part (1) is the same as in the case of stringy cohomology [FG]. Since f is G-equivariant and étale, f^*TY^m is isomorphic to TX^m. This induces the desired map between the associated obstruction bundles.

Part (2) follows from the naturality of the ordinary Chern character and the fact that if f is étale, then $f^*\mathscr{S}^Y_m = \mathscr{S}^X_m$, where $\mathscr{S}^X_m$ and $\mathscr{S}^Y_m$ are as defined in Equation (25) for X and Y, respectively.

Part (3) follows from these same considerations, since

$$\begin{aligned} f_*(\mathscr{C}\mathbf{h}(\mathscr{F}_m) \cup \mathbf{td}(TX^m)) &= f_*(\mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}(\ominus\mathscr{S}^X_m) \cup \mathbf{td}(TX^m)) \\ &= f_*(\mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}(TX^m) \cup \mathbf{td}(\ominus\mathscr{S}^X_m)) \\ &= f_*(\mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}(TX^m) \cup \mathbf{td}(\ominus f^*\mathscr{S}^Y_m)) \\ &= f_*(\mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}(TX^m) \cup f^*\mathbf{td}(\ominus\mathscr{S}^Y_m)) \\ &= f_*(\mathbf{ch}(\mathscr{F}_m) \cup \mathbf{td}(TX^m)) \cup \mathbf{td}(\ominus\mathscr{S}^Y_m) \\ &= \mathbf{ch}(f_*\mathscr{F}_m) \cup \mathbf{td}(TY^m) \cup \mathbf{td}(\ominus\mathscr{S}^Y_m) \\ &= \mathscr{C}\mathbf{h}(f_*\mathscr{F}_m) \cup \mathbf{td}(TY^m), \end{aligned}$$

where the projection formula was used in the fifth equality and the ordinary Grothendieck-Riemann-Roch theorem was used in the sixth.

□

8. Discrete torsion

At this point, we wish to make a short comment about discrete torsion. As discussed in [Kau04], any G-Frobenius algebra $\mathscr{H}$ can be twisted by a discrete torsion, which is an element $\alpha \in Z^2(G, \mathbb{Q}^*)$, to obtain a G-Frobenius algebra with twisted sectors of the same dimension.

This procedure allows us to "twist" the stringy Chow ring $\mathscr{A}(X,G)$ and the stringy K-theory $\mathscr{K}(X,G)$. If one twists both rings by the same element α, then the stringy Chern character $\mathscr{C}\mathbf{h}$ again provides an isomorphism of G-commutative algebras.

We briefly recall the main points of the construction of twisting by discrete torsion, omitting the proofs which all follow from rather straightforward computations. A reference for the proofs is [Kau04].

For $\alpha \in Z^2(G,\mathbb{Q}^*)$, let $\mathbb{Q}^\alpha[G]$ be the twisted group ring, i.e., $\mathbb{Q}^\alpha[G] = \bigoplus_{m\in G}\mathbb{Q}e_m$ with the multiplication $e_{m_1}\star e_{m_2} = \alpha(m_1,m_2)e_{m_1m_2}$.

Set $\epsilon(\gamma,m) := \alpha(\gamma^{-1},m)/\alpha(\gamma^{-1}m\gamma,\gamma^{-1})$ and define $\rho(\gamma)(e_m) = \epsilon(\gamma,m)e_{\gamma^{-1}m\gamma}$. Define a bi-linear form η by $\eta(e_{m_+},e_{m_-}) = 0$ unless $m_+m_- = 1$ and $\eta(e_m,e_{m^{-1}}) = \alpha(m,m^{-1})$. Lastly, let $\mathbf{1} = e_1$.

Lemma 8.1. *$((\mathbb{Q}^\alpha[G],\rho),\star,\mathbf{1},\eta)$ is a G-Frobenius algebra.*

Definition 8.2. We define the tensor product $\hat{\otimes}$ of two G-Frobenius algebras $((\mathscr{H},\varphi),\star,\mathbf{1},\eta)$ and $((\mathscr{H}',\varphi'),\star',\mathbf{1}',\eta')$ to be the G-Frobenius algebra $\mathscr{H}\hat{\otimes}\mathscr{H}' = \bigoplus_{m\in G}(\mathscr{H}\hat{\otimes}\mathscr{H}')_m$ with $(\mathscr{H}\hat{\otimes}\mathscr{H}')_m := \mathscr{H}_m\otimes_{\mathbb{Q}}\mathscr{H}'_m$, diagonal multiplication $\star\otimes\star'$, diagonal G-action $\rho\otimes\rho'$, the tensor product metric $\eta\otimes\eta'$, and unit $\mathbf{1}\otimes\mathbf{1}'$.

Proposition 8.3. *The tensor product of two G-Frobenius algebras is a G-Frobenius algebra.*

Definition 8.4. For a G-Frobenius algebra $\mathscr{H}$ and an element $\alpha\in Z^2(G,\mathbb{Q}^*)$, we set $\mathscr{H}^\alpha := \mathscr{H}\hat{\otimes}\mathbb{Q}^\alpha[G]$.

Notice that as vector spaces

$$\mathscr{H}^\alpha_m = \mathscr{H}_m\otimes_{\mathbb{Q}}\mathbb{Q}\simeq\mathscr{H}_m. \tag{49}$$

Lemma 8.5. *Using the identification $\mathscr{H}^\alpha_m\cong\mathscr{H}_m$, the G-Frobenius structures for $((\mathscr{H}^\alpha,\rho^\alpha),\star^\alpha,\mathbf{1}^\alpha,\eta^\alpha))$ are*

$$v_{m_1}\star^\alpha v_{m_2} := \alpha(m_1,m_2)v_{m_1}\star v_{m_2},$$
$$\rho^\alpha(\gamma)v_m := \epsilon(\gamma,m)\rho(\gamma)v_m,$$

and

$$\eta^\alpha(v_m,v_{m^{-1}}) := \alpha(m,m^{-1})\eta(v_m,v_{m^{-1}})$$

for all v_{m_i} in $\mathscr{H}^\alpha_{m_i}$, v_m in $\mathscr{H}^\alpha_m$, and $v_{m^{-1}}$ in $\mathscr{H}^\alpha_{m^{-1}}$.

Proposition 8.6. *The G-Frobenius algebras $\mathscr{H}$ and $\mathscr{H}^\alpha$ are isomorphic if and only if α is a coboundary, that is, $[\alpha] = 0\in H^2(G,\mathbb{Q}^*)$.*

Remark 8.7. The above twisting procedure is also well defined for G-commutative algebras.

Proposition 8.8. *If $\Phi:\mathscr{H}\to\mathscr{H}'$ is an isomorphism of G-Frobenius algebras (or of G-commutative algebras) then $\Phi\otimes id$ is an isomorphism between $\mathscr{H}^\alpha$ and $\mathscr{H}'^\alpha$.*

Corollary 8.9. *Let $\mathscr{C}\mathbf{h}:\mathscr{K}(X,G)\to\mathscr{A}(X,G)$ denote the stringy Chern character. For all $\alpha\in Z^2(G,\mathbb{Q}^*)$, the map $\mathscr{C}\mathbf{h}^\alpha = \mathscr{C}\mathbf{h}\otimes id:\mathscr{K}^\alpha(X)\to\mathscr{A}^\alpha(X)$ is an isomorphism of G-commutative algebras.*

9. The orbifold K-theory of a stack

In this section we show how our constructions and results carry over to the case of a smooth Deligne-Mumford stack $\mathscr{X}$ with projective coarse moduli space. We do not restrict ourselves to the case of a global quotient by a finite group. In particular, the results of this section apply to quotients of the form $[X/\mathscr{G}]$, where $\mathscr{G}$ is a Lie group acting on X with finite stabilizers.

We will give a construction of an orbifold K-theory $K_{orb}(\mathscr{X})$ and an orbifold Chern character $Ch_{orb} : K_{orb}(\mathscr{X}) \to H_{\mathrm{orb}}(\mathscr{X})$. The construction of $K_{orb}(\mathscr{X})$ generalizes Givental and Lee's quantum K-theory [Le] to orbifolds, just as Chen-Ruan [CR2] and Abramovich-Graber-Vistoli [AGV] generalized quantum cohomology to orbifolds. Of course, as a vector space, our construction agrees with the construction of Adem and Ruan [AR], but our orbifold product has the virtue that the orbifold Chern character Ch_{orb} is a *ring* isomorphism—not just an additive isomorphism.

Of course, when G is a finite group acting on X, taking G-coinvariants of our stringy K-theory gives a form of orbifold K-theory $\overline{\mathscr{K}}(\mathscr{X})$ for the stack quotient $\mathscr{X} = [X/G]$. Furthermore, the stringy Chern character $\mathscr{C}\mathbf{h}$ induces a map $\overline{\mathscr{C}\mathbf{h}} : \overline{\mathscr{K}}(X,G) \to \overline{\mathscr{A}}(X,G)$, which is an isomorphism of commutative, associative algebras with unit. This is a K-theoretic version of the construction of the orbifold Chow ring of a stack quotient from stringy cohomology as in [FG, JKK], but we will show that this orbifold K-theory can be defined on Deligne-Mumford stacks that are not global quotients by finite group and show that the orbifold K-theory of a stack quotient is independent of the presentation of the stack.

The main result of this section is Theorem 1.4; namely, that Ch_{orb} is a ring isomorphism, and furthermore, for global quotients $\mathscr{X} = [X/G]$, we have $K_{orb}(\mathscr{X}) = \overline{\mathscr{K}}(X,G)$ and Ch_{orb} is induced from $\mathscr{C}\mathbf{h}$.

Theorem 1.4 will follow from a formula for the orbifold obstruction bundle (Theorem 9.2), just as its counterparts for stringy K-theory (Theorems 5.2 and 1.3) follow from the formula for the obstruction bundle for varieties with finite group action (Theorem 1.2).

Recall that the stack $\overline{\mathscr{M}}_{0,n}(\mathscr{X},0)$ of degree-zero, genus-zero, n-pointed orbifold stable maps into $\mathscr{X}$ is a Deligne-Mumford stack [AGV]. We denote the universal curve over it by $\varpi : \mathscr{C}_{0,n} \to \overline{\mathscr{M}}_{0,n}(\mathscr{X},0)$, and the universal stable map by $\bar{f} : \mathscr{C}_{0,n} \to \mathscr{X}$. The evaluation maps from $\overline{\mathscr{M}}_{0,n}(\mathscr{X},0)$ do not map just to $\mathscr{X}$, but rather to the inertia stack

$$\widetilde{\mathscr{X}} := \coprod_{(g)} \mathscr{X}_{(g)},$$

where the indices run over conjugacy classes of local automorphisms, and $\mathscr{X}_{(g)} = \{(x,(g)) | g \in \mathrm{stab}(x)\}$. The evaluation maps $ev_i : \overline{\mathscr{M}}_{0,n}(\mathscr{X},0) \to \widetilde{\mathscr{X}}$ are given by $ev_i([\bar{f} : \mathscr{C} \to \mathscr{X}]) = (f(p_i),(g_{p_i})) \in \mathscr{X}_{(g_{p_i})}$, where p_i is the ith marked point (gerbe) of $\mathscr{C}$, and g_{p_i} is the image of the canonical generator of $\mathrm{stab}(p_i)$ in $\mathrm{stab}(f(p_i))$. Of course, this image is only defined up to conjugacy, since if $\mathscr{X}$ is locally presented as $[X/G]$ near a point $p_i \in \mathscr{X}$, then a representative $\widetilde{p}_i \in X$ of p_i can be replaced by another representative $\widetilde{p}_i\gamma$ for any $\gamma \in G$, which replaces g_{p_i} by $\gamma^{-1}g_{p_i}\gamma$. As in earlier sections, we also define $\sigma : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{X}}$ to be the canonical involution $(x,(g)) \mapsto (x,(g^{-1}))$ and $\check{ev}_i = \sigma \circ ev_i$.

Definition 9.1. As a vector space, the *orbifold K-theory* $K_{orb}(\mathscr{X})$ of $\mathscr{X}$ is the ordinary K-theory of $\widetilde{\mathscr{X}}$:

$$K_{orb}(\mathscr{X}) := K(\widetilde{\mathscr{X}}) = \bigoplus_{(g)} K_{(g)} := \bigoplus_{(g)} K(\mathscr{X}_{(g)}).$$

To define the product on $K_{orb}(\mathscr{X})$, we use the obvious obstruction bundle on the stack $\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)$, namely, $\widetilde{\mathscr{R}} := R^1\varpi_*(\bar{f}^*T\mathscr{X})$, and we define the *virtual bundle* $\mathscr{O}_{\mathrm{vir}}$ to be

$$\mathscr{O}_{\mathrm{vir}} := \lambda_{-1}(\widetilde{\mathscr{R}}^*).$$

The orbifold product of two bundles $\mathscr{F}$ and $\mathscr{F}'$ in $K_{orb}(\mathscr{X})$ is defined to be

$$\mathscr{F} * \mathscr{F}' := (\check{ev}_3)_*(ev_1^*(\mathscr{F}) \otimes ev_2^*(\mathscr{F}') \otimes \mathscr{O}_{\mathrm{vir}}).$$

There is a natural map $i : \widetilde{\mathscr{X}} \to \mathscr{X}$ given by forgetting the extra data of the conjugacy class (g), and i is locally a regular embedding. That is, for each (g), the space $\mathscr{X}_{(g)}$ embeds into $\mathscr{X}$ in a natural way. Because these orbifold stable maps have degree zero, the composition $J = i \circ ev_j : \overline{\mathscr{M}}_{0,3}(\mathscr{X},0) \to \mathscr{X}$ is the same for all $j \in \{1,2,3\}$. As in the stringy case, for each conjugacy class (g) with g of order r, the element g acts by r-th roots of unity on $W_{(g)} := T\mathscr{X}|_{\mathscr{X}^{(g)}}$. We define $W_{(g),k}$ to be the eigenbundle of $W_{(g)}$, where g acts by multiplication by $\zeta^k = \exp(-2\pi ik/r)$. Note that this eigenbundle is determined only by the conjugacy class (g) rather than by the particular representative g. Finally, we define

$$\mathscr{S}_{(g)} := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{(g),k} \in K_{(g)}.$$

This allows us to define $\mathscr{S} \in K_{orb}(\mathscr{X})$ as $\mathscr{S} = \bigoplus_{(g)} \mathscr{S}_{(g)}$. As in the stringy case, the following theorem holds.

Theorem 9.2. *In the K-theory of* $\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)$*, the following relation holds for the obstruction bundle* $\widetilde{\mathscr{R}}$*.*

$$\widetilde{\mathscr{R}} \cong T\overline{\mathscr{M}}_{0,3}(\mathscr{X},0) \ominus J^*T\mathscr{X} \oplus \bigoplus_{i=1}^{3} ev_i^*\mathscr{S}. \tag{50}$$

Proof. The idea of the proof is to use distinguished components of the stack of pointed admissible covers $\xi_{0,3}$ [JKK, §2.5.1] to produce an étale cover of the moduli stack $\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)$. On this cover, we can easily produce an isomorphism of equivariant bundles, but it is not unique—it is only determined up to conjugacy. However, the bundles we really want are the coinvariant bundles of these equivariant bundles, and the induced isomorphism is independent of conjugation. Thus, étale descent applies, and we obtain the desired isomorphism.

We first recall the definitions from [JKK, §2.5.1 and §6] of $\xi^G_{0,3}(\mathbf{m})$ and $\xi^G_{0,3}(X,\mathbf{m})$. As described in the definition of the obstruction bundle for stringy Chow and stringy K-theory, there is a presentation of the fundamental group $\pi_1(\mathbb{P}^1 - \{0,1,\infty\})$ as $\langle c_1, c_2, c_3 | c_1c_2c_3 = 1\rangle$, where c_1, c_2 and c_3 are little loops around $p_1 = 0, p_2 = 1$, and $p_3 = \infty$, respectively. For any finite group G, and any triple of elements $\mathbf{m} = (m_1, m_2, m_3)$ in G, such that $m_1m_2m_3 = 1$, we can define a natural homomorphism $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}) \to \langle\mathbf{m}\rangle \subseteq G$. This defines a principal $\langle\mathbf{m}\rangle$-bundle over $\mathbb{P}^1 - \{0,1,\infty\}$ which extends to a smooth, connected curve E with three distinguished points $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}_3$ lying over the points p_1 ,p_2, and p_3, respectively. It also defines a principal G-bundle that extends to a smooth (not necessarily connected) curve $\tilde{E}$, with E as a connected component of $\tilde{E}$. The groups $\langle \mathbf{m} \rangle$ and G act on the curves E and $\tilde{E}$, respectively, such that the quotients $E/\langle \mathbf{m} \rangle$ and $\tilde{E}/G$ have genus zero, and the natural map $E \to E/\langle \mathbf{m} \rangle = \mathbb{P}^1$ and $\tilde{E} \to \tilde{E}/G = \mathbb{P}^1$ are branched at the three points p_1, p_2, p_3 with monodromy m_1, m_2, m_3, respectively.

Definition 9.3. We define $\xi_{0,3}^{\langle \mathbf{m} \rangle}$ to be the connected component of the stack of 3-pointed admissible $\langle \overline{m} \rangle$-covers of genus zero that contains the admissible cover $(E, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$. Similarly, we define $\xi_{0,3}^{G}$ to be the connected component of the stack of 3-pointed admissible $\langle \overline{m} \rangle$-covers of genus zero that contains the admissible cover $(\tilde{E}, \widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$.

If X is any variety with a G-action, a degree-zero, 3-pointed G-stable map of genus zero is a G-equivariant morphism $\tilde{E} \to X$ from a 3-pointed admissible G-cover to X, such that the induced morphism $\tilde{E}/G \to X/G$ is a 3-pointed stable map of genus zero. We define $\xi_{0,3}^{G}(X, 0, \mathbf{m})$ to be the component of the stack of pointed G-stable maps whose underlying 3-pointed admissible G-covers $\tilde{E}$ correspond to points of $\xi_{0,3}^{G}(\mathbf{m})$.

It is easy to see that there is a canonical isomorphism $I : \xi_{0,3}^{\langle \mathbf{m} \rangle}(\mathbf{m}) \to \xi_{0,3}^{G}(\mathbf{m})$, and that $\xi_{0,3}^{G}(\mathbf{m})$ is the stack quotient $\mathscr{B}H = [pt/H_{\mathbf{m}}]$ of a point modulo the group $H_{\mathbf{m}} := \langle m_1 \rangle \cap \langle m_2 \rangle \cap \langle m_3 \rangle$ (see [JKK, Prop 2.20]). Moreover, in [JKK, Lemma 6.7] it is shown that $\xi_{0,3}^{G}(X, 0, \mathbf{m})$ is canonically isomorphic to $\xi_{0,3}^{G} \times X^{\mathbf{m}}$. Finally, we have a natural morphism $q : \xi_{0,3}(X, 0, \mathbf{m}) \to \overline{\mathscr{M}}_{0,3}([X/G], 0)$ given by sending a G-stable map $[f : E \to X]$ to an induced map of quotient stacks $[\bar{f} : [E/G] \to [X/G]]$. This map is easily seen to be étale.

Now we may begin the proof. First note that if U is an étale cover of $\mathscr{X}$ consisting of a disjoint union of smooth varieties X_α with finite groups G_α acting to make $q_\alpha : X_\alpha \to \mathscr{X}$ induce an isomorphism $[X_\alpha/G_\alpha]$ to a neighborhood in $\mathscr{X}$ (that is $\{X_\alpha, G_\alpha, q_\alpha)\}$ form a uniformizing system), we may construct an étale cover

$$p : \coprod_{\alpha, \mathbf{m}} X_\alpha^{\mathbf{m}} \to \coprod_{\alpha, \mathbf{m}} \xi_{0,3}^{G_\alpha} \times X_\alpha^{\mathbf{m}} = \coprod_{\alpha, \mathbf{m}} \xi_{0,3}^{G_\alpha}(X_\alpha, 0, \mathbf{m}) \to \overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0),$$

where for each α the $\mathbf{m}$ run through all triples in G_α whose product is 1, and the first morphism is induced by the obvious (étale) map $pt \times X_\alpha^{\mathbf{m}} \to [pt/H_{\mathbf{m}}] \times X_\alpha^{\mathbf{m}} = \xi_{0,3}^{G_\alpha} \times X_\alpha^{\mathbf{m}}$.

For each α and $\mathbf{m}$, the pullback $p^*\widetilde{\mathscr{R}}$ is easily seen to be the usual obstruction bundle $\mathscr{R}(\mathbf{m})$ on $X_\alpha^{\mathbf{m}}$, and the pullback $p^* \left(T\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0) \ominus J^* T\mathscr{X} \oplus \bigoplus_{i=1}^3 ev_i^* \mathscr{S} \right)$ is clearly equal to $TX_\alpha^{\mathbf{m}} \ominus TX_\alpha|_{X_\alpha^{\mathbf{m}}} \oplus \bigoplus_{i=1}^3 \mathscr{S}_{m_i}|_{X_\alpha^{\mathbf{m}}}$. But to prove the theorem, we will need to provide a canonical isomorphism between these bundles.

The fibered product $X_\alpha^{\mathbf{m}} \times_{\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)} X_\beta^{\mathbf{m}'}$ is non-empty if and only if the stack $\xi_{0,3}^{G_\alpha}(X_\alpha, 0, \mathbf{m}) \times_{\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)} \xi_{0,3}^{G_\beta}(X_\beta, 0, \mathbf{m}')$ is non-empty; and it is straightforward to see that this occurs only if there is a group G with injective homomorphisms $G \hookrightarrow G_\alpha$ and $G \hookrightarrow G_\beta$, such that the triple $\mathbf{m}'$ is diagonally (i.e., all three terms simultaneously) conjugate to $\mathbf{m}$ in G^3. Moreover, for each connected component of $\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0)$, there is a well-defined diagonal conjugacy class of such triples.

For each such conjugacy class, choose a representative $\mathbf{m}$ and let $K = \langle m_1, m_2, m_3 \rangle$ be the group generated by the triple. As described above, this triple determines a well-defined distinguished component $\xi^K_{0,3}(\mathbf{m})$ of the stack of three-pointed, admissible K-covers of genus zero.

Choose, once and for all, an isomorphism $\Phi_{\mathbf{m}}$ of K-representations giving the (virtual) equality of Equation (30) in Theorem 6.3. For any other triple $\mathbf{m}'$ in the same conjugacy class, there is a canonical isomorphism of groups $K' = \langle m'_1, m'_2, m'_3\rangle \xrightarrow{\sim} K$ taking $\mathbf{m}'$ to $\mathbf{m}$, and a canonical (equivariant) isomorphism of representations $H^1(E'; \mathscr{O}_{E'}) \cong H^1(E; \mathscr{O}_E)$, where $E \to C \to \xi^K_{0,3}(\mathbf{m})$ is the three-pointed admissible K-cover with holonomy $\mathbf{m}$, and $E' \to C' \to \xi^{K'}_{0,3}(\mathbf{m}')$ is the three-pointed admissible K'-cover with holonomy $\mathbf{m}'$. Similarly, we have canonical (equivariant) isomorphisms of the representations

(51)

$$\mathbb{C}\ominus\mathbb{C}[K]\oplus\bigoplus_{i=1}^{n}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|K|}\operatorname{Ind}^K_{\langle m_i\rangle}\mathbb{C}[K]_{m_i,k_i} \cong \mathbb{C}\ominus\mathbb{C}[K']\oplus\bigoplus_{i=1}^{n}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|K'|}\operatorname{Ind}^{K'}_{\langle m_i\rangle}\mathbb{C}[K']_{m_i,k_i}.$$

Thus $\Phi_{\mathbf{m}}$ induces an isomorphism $\Phi_{\mathbf{m}'}$ for each triple $\mathbf{m}'$ which is conjugate to $\mathbf{m}$.

If G is any group containing both K and K', with K' a conjugate (say by $\gamma \in G$) of K, then letting $\tilde{E} \to C \to \xi^G_{0,3}(\mathbf{m})$ denote the distinguished three-pointed G-cover with holonomy $\mathbf{m}$, and $\tilde{E}' \to C \to \xi^G_{0,3}(\mathbf{m}')$ denote the distinguished universal three-pointed G-cover with holonomy $\mathbf{m}'$, the group action $\rho(\gamma) : \xi^G_{0,3}(\mathbf{m}) \xrightarrow{\sim} \xi^G_{0,3}(\mathbf{m}')$ identifies the base (γ acts on E and E'). Furthermore, we have canonical isomorphisms of G-representations

$$H^1(\tilde{E}; \mathscr{O}_{\tilde{E}}) \cong \operatorname{Ind}^G_K(H^1(E; \mathscr{O}_E)) \tag{52}$$

and

$$H^1(\tilde{E}'; \mathscr{O}_{\tilde{E}'}) \cong \operatorname{Ind}^G_{K'}(H^1(E'; \mathscr{O}_{E'})). \tag{53}$$

As G-representations, $H^1(\tilde{E}; \mathscr{O}_{\tilde{E}})$ and $\rho(\gamma)^* H^1(\tilde{E}'; \mathscr{O}_{\tilde{E}'})$ are not identical, but rather are conjugate; that is, $H^1(\tilde{E}'; \mathscr{O}_{\tilde{E}'})$ is the representation of G arising from conjugating the action of G on $H^1(\tilde{E}; \mathscr{O}_{\tilde{E}})$ by γ. The same holds for the induced representations

$$\begin{aligned} &\mathbb{C}[K\backslash G] \ominus \mathbb{C}[G]\bigoplus_{i=1}^{3}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|G|}\operatorname{Ind}^G_{\langle m_i\rangle}\mathbb{C}[G]_{m_i,k_i} \\ &\cong \operatorname{Ind}^G_K(\mathbb{C} \ominus \mathbb{C}[K]\bigoplus_{i=1}^{3}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|K|}\operatorname{Ind}^K_{\langle m_i\rangle}\mathbb{C}[K]_{m_i,k_i}), \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\mathbb{C}[K'\backslash G] \ominus \mathbb{C}[G]\bigoplus_{i=1}^{3}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|G|}\operatorname{Ind}^G_{\langle m'_i\rangle}\mathbb{C}[G]_{m'_i,k_i} \\ &\cong \operatorname{Ind}^G_{K'}(\mathbb{C} \ominus \mathbb{C}[K']\bigoplus_{i=1}^{3}\bigoplus_{k_i=0}^{r_i-1}\frac{k_i}{|K'|}\operatorname{Ind}^{K'}_{\langle m'_i\rangle}\mathbb{C}[K']_{m'_i,k_i}). \end{aligned} \tag{55}$$

Finally, for an open subset V of any X_α with G acting on V, pulling pack by the action

$$\rho(\gamma) : V^{\mathbf{m}} \xrightarrow{\sim} V^{\mathbf{m}'}$$

makes the G-bundle $\rho(\gamma)^* f^*TV = \mathscr{O}_{\tilde{E}'} \boxtimes \mathrm{Ind}_{K'}^G TV|_{V^{\mathbf{m}'}}$ on $V^{\mathbf{m}}$ isomorphic to the conjugate by γ of the G-bundle $f^*TV = \mathscr{O}_{\tilde{E}} \boxtimes \mathrm{Ind}_K^G TV|_{V^{\mathbf{m}}}$. Thus the isomorphisms $\Phi_{\mathbf{m}'}$ and the induced isomorphisms

$$\begin{aligned}\tilde{\Phi}_{\mathbf{m}} : R^1\pi_*(f^*TV) &= H^1(\tilde{E}; \mathscr{O}_{\tilde{E}}) \otimes \mathrm{Ind}_K^G TV|_{V^{\mathbf{m}}} \\ &\xrightarrow{\sim} \left(\mathbb{C}[K\backslash G] \ominus \mathbb{C}[G] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{|G|} \mathrm{Ind}_{\langle m_i \rangle}^G \mathbb{C}[G]_{m_i,k_i} \right) \otimes \mathrm{Ind}_K^G TV|_{V^{\mathbf{m}}} \end{aligned} \tag{56}$$

on $\xi_{0,3}^G(\mathbf{m}) \times V^{\mathbf{m}}$ are determined up to conjugacy by an element in G.

However, a representation and any conjugate of that representation have canonically identified coinvariants, so the isomorphisms $\tilde{\Phi}_{\mathbf{m}}$ induce isomorphisms of the coinvariant bundles

$$\begin{aligned}\bar{\Phi}_{\mathbf{m}} : R^1\pi_*^G(f^*TV) &\xrightarrow{\sim} \left(\left(\mathbb{C}[K\backslash G] \ominus \mathbb{C}[G] \oplus \bigoplus_{i=1}^{3} \bigoplus_{k_i=0}^{r_i-1} \frac{k_i}{|G|} \mathrm{Ind}_{\langle m_i \rangle}^G \mathbb{C}[G]_{m_i,k_i} \right) \otimes TV|_{V^{\mathbf{m}}} \right)^G \\ &= TV^{\mathbf{m}} \ominus TV|_{V^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{3} \mathscr{S}_{m_i}|_{V^{\mathbf{m}}}, \end{aligned} \tag{57}$$

which are independent of conjugation.

In summary, we have chosen an explicit isomorphism

$$\overline{\Phi} : p^*\widetilde{\mathscr{R}} \xrightarrow{\sim} p^* \left(T\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0) \ominus J^*T\mathscr{X} \oplus \bigoplus_{i=1}^{3} ev^*\mathscr{S} \right)$$

on the étale cover $\coprod_{\alpha,\mathbf{m}} X_\alpha^{\mathbf{m}} \xrightarrow{p} \overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0)$, with the particular property that on the product

$$\coprod X_\alpha^{\mathbf{m}} \times_{\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)} \coprod X_\alpha^{\mathbf{m}} \underset{t}{\overset{s}{\rightrightarrows}} \coprod X_\alpha^{\mathbf{m}}$$

we have $s^*\overline{\Phi} = t^*\overline{\Phi}$. Thus by étale descent the isomorphism $\overline{\Phi}$ descends from the cover $\coprod X_{\alpha^{\mathbf{m}}}$ to the stack $\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0)$.

□

Lemma 9.4. *The ring of G-coinvariants $\overline{\mathscr{K}}(X, G)$ of the stringy K-theory of X is isomorphic to the orbifold K-theory $K_{orb}([X/G])$ of the quotient stack.*

Proof. First, we have the isomorphism

$$K_{orb}([X/G]) = \bigoplus_{(g)} K([X^{(g)}/Z_G(g)]) \cong \bigoplus_{(g)} K(X^g)^{Z_G(g)},$$

where the sum includes only one representative g from each conjugacy class $(g) \subset G$. For any triple $\mathbf{m} \in G^3$ with $m_1m_2m_3 = 1$, we have the following commutative diagram of morphisms for every $i \in \{1, 2, 3\}$.

$$\begin{array}{ccc} X^{\mathbf{m}} & \xrightarrow{e_i} & X^{m_i} \\ \downarrow p_{\mathbf{m}} & & \downarrow q_{m_i} \\ \overline{\mathscr{M}}_{0,3}([X/G], 0) & \xrightarrow{e_i} & [X^{(m_i)}/Z_G(m_i)] \end{array} \tag{58}$$

If $p : \coprod_{\mathbf{m}} X^{\mathbf{m}} \to \overline{\mathscr{M}}_{0,3}(X,0)$ is the obvious morphism induced by the $p_{\mathbf{m}}$, and $q : I_G(X) \to \coprod_{(m)}[X^{(m)}/Z_G(m)] = \widetilde{[X/G]}$ is the obvious morphism induced by the q_m, then for any conjugacy class (g) in G, the following diagram is Cartesian.

(59)
$$\begin{array}{ccc} \coprod\limits_{\substack{\mathbf{m} \\ m_3\in(g)}} X^{\mathbf{m}} & \xrightarrow{e_3} & \coprod\limits_{m_3\in(g)} X^{m_3} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \overline{\mathscr{M}}_{0,3}([X/G],0) & \xrightarrow{e_3} & [X^{(g)}/Z_G(g)] \end{array}$$

Moreover, for any $\alpha \in K(X^g)^{Z_G(g)} \subseteq K(X^g)$, the pullback $q^*(\alpha) \in \mathscr{K}(X,G)$ is G-invariant and is exactly
$$q^*(\alpha) = \sum_{h\in(g)} i_{g,h}\alpha,$$
where $i_{g,h}$ is the obvious map $K(X^g)^{Z_G(g)} \to K(X^h)^{Z_G(h)}$. It is easy to see that q^* is a linear isomorphism $q^* : K_{orb}([X/G]) \to \overline{\mathscr{K}}(X,G)$, so all that remains is to check that q^* is a ring homomorphism.

For for any triple $(g_1),(g_2),(g_3)$ of conjugacy classes in G, and for any $\alpha_i \in K(X^{(g_i)}/Z_G(g_i))$ for each $i \in \{1,2\}$ we have

(60)
$$\begin{aligned} q^*(\alpha_1\cdot\alpha_2) &= q^*(e_3)_*\left(e_1^*\alpha_1\otimes e_2^*\alpha_2\otimes\lambda_{-1}(\widetilde{\mathscr{R}}^*)\right) \\ &= (e_3)_*p^*\left(e_1^*\alpha_1\otimes e_2^*\alpha_2\otimes\lambda_{-1}(\widetilde{\mathscr{R}}^*)\right) \\ &= (e_3)_*\left(p^*e_1^*\alpha_1\otimes p^*e_2^*\alpha_2\otimes\lambda_{-1}(p^*\widetilde{\mathscr{R}}^*)\right) \\ &= (e_3)_*\left(e_1^*q^*\alpha_1\otimes e_2^*q^*\alpha_2\otimes\lambda_{-1}(\mathscr{R}^*)\right) \\ &= q^*\alpha_1\cdot q^*\alpha_2, \end{aligned}$$

where the first and last equalities are the definition of multiplication, the second follows from the fact that p and q are flat, and the fourth from the fact (shown in the proof of Theorem 9.2) that $p^*\widetilde{\mathscr{R}} = \mathscr{R}$ and commutativity of Diagram (58).

Thus q^* is a ring isomorphism $\square$

To finish the proof of Theorem 1.4 we must define the orbifold Chern character.

Definition 9.5. For any $\mathscr{F}_{(g)} \in K_{(g)}$ the *orbifold Chern character* $Ch_{orb} : K_{orb}(\mathscr{X}) \to A^\bullet_{orb}(\mathscr{X})$ is
$$Ch_{orb}(\mathscr{F}_{(g)}) = \mathbf{ch}(\mathscr{F}_{(g)}) \cup \mathbf{td}^{-1}(\mathscr{S}_{(g)}),$$
thus for any $\mathscr{F} \in K_{orb}(\mathscr{X})$ we have
$$Ch_{orb}(\mathscr{F}) = \mathbf{ch}(\mathscr{F}) \cup \mathbf{td}^{-1}(\mathscr{S}).$$

We can now adapt the proofs we have given in the stringy case to the orbifold case to finish the proof of Theorem 1.4. In particular, the proof of associativity given in Theorem 10.4 is easily adapted to give a proof of associativity for the orbifold product in $K_{orb}(\mathscr{X})$.

Similarly, using Theorem 9.2, one can easily adapt the proof of Theorem 1.3 to show that the orbifold Chern character Ch_{orb} is a ring homomorphism. Since the ordinary Chern character is a linear isomorphism $\mathbf{ch} : K(X_{(g)}) \xrightarrow{\sim} A(X_{(g)})$

for every conjugacy class (g), and since $\mathbf{td}(\mathscr{S})$ is invertible, we see that $Ch_{orb} : K_{orb}(\mathscr{X}) \to A^{\bullet}_{orb}(\mathscr{X})$ is an isomorphism of rings. Thus Theorem 1.4 holds.

We conclude this section with an orbifold version of Theorem 7.3.

Theorem 9.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be an étale morphism of smooth, Deligne-Mumford stacks with projective coarse moduli. The following properties hold.*

1. *(Pullback) The pullback maps $f^* : A^{\bullet}_{orb}(\mathscr{Y}) \to A^{\bullet}_{orb}(\mathscr{X})$ and $f^* : K_{orb}(\mathscr{Y}) \to K_{orb}(\mathscr{X})$ are homomorphisms of Frobenius algebras.*
2. *(Naturality) The following diagram commutes.*

$$\begin{tikzcd} K_{orb}(\mathscr{Y}) \arrow[r, "f^*"] \arrow[d, "Ch_{orb}"'] & K_{orb}(\mathscr{X}) \arrow[d, "Ch_{orb}"'] \\ A^{\bullet}_{orb}(\mathscr{Y}) \arrow[r, "f^*"] & A^{\bullet}_{orb}(\mathscr{X}) \end{tikzcd} \tag{61}$$

3. *(Grothendieck-Riemann-Roch) For all $\mathscr{F}$ in $A^{\bullet}_{orb}(\mathscr{X})$,*

$$f_*(Ch_{orb}(\mathscr{F}) \cup \mathbf{td}(T\mathscr{X})) = Ch_{orb}(f_*\mathscr{F}) \cup \mathbf{td}(T\mathscr{Y}). \tag{62}$$

The proof of this Theorem is a straightforward adaptation of the proof of its stringy counterpart, Theorem 7.3.

10. Associativity and the trace axiom revisited

In this section, we use the explicit formula for the obstruction bundle, Equation (3), to give an elementary proof of associativity and the trace axiom for both the stringy Chow ring and stringy K-theory. For both associativity and the trace axiom, these proofs avoid all uses of admissible G-covers and G-stable maps or their moduli. That is, the entire stringy multiplication on $\mathscr{K}(X,G)$ can be formulated by *defining* the obstruction bundle via Equation (3). In the case of associativity, our argument refines the proof in [FG]. In the case of the trace axiom, this proof is distinct from that in [JKK], although we indicate relations to the moduli-theoretic proof where appropriate.

10.1. **Associativity.** Let us recall some basic excess intersection theory. Consider smooth, projective varieties V, Y_1, Y_2, and Z which form the following Cartesian square

$$\begin{tikzcd} V \arrow[r, "i_1"] \arrow[d, "j_2"'] & Y_1 \arrow[d, "j_1"'] \\ Y_2 \arrow[r, "i_2"] & Z \end{tikzcd}, \tag{63}$$

where i_1, i_2 are regular embeddings and j_1, j_2 are morphisms of schemes.

Let $E(V, Y_1, Y_2) \to V$ be the *excess normal (vector) bundle*, which is the cokernel of the natural map $N_{V/Y_1} \to N_{Y_2/Z}|_V$, where N_{V/Y_1} and $N_{Y_2/Z}$ denote the normal bundles of V in Y_1 and Y_2 in Z, respectively. In $K(V)$ one thus obtains the equality

$$[E(Z, Y_1, Y_2)] = TZ|_V \ominus TY_1|_V \ominus TY_2|_V \oplus TV. \tag{64}$$

Under these hypotheses, the following theorem holds (see Theorems 1.3 and 1.4 in [FL, Chapter IV.1]).

Theorem 10.1. *For all $\mathscr{F}$ in $K(Y_2)$ and v in $A^\bullet(Y_2)$,*

$$j_1^* i_{2*}\mathscr{F} = i_{1*}(\lambda_{-1}(E(Z,Y_1,Y_2)^*) \otimes j_2^*\mathscr{F}) \tag{65}$$

and

$$j_1^* i_{2*} v = i_{1*}(c_{\mathrm{top}}(E(Z,Y_1,Y_2)) \cup j_2^* v). \tag{66}$$

The previous theorem gives rise to the following fact about $\mathscr{R}$.

Theorem 10.2. *Let $\mathbf{m} := (m_1,\ldots,m_4)$ in G^4 such that $m_1m_2m_3m_4 = 1$. Let $X^{\mathbf{m}}$ consist of those points in X which are fixed by m_i for all $i \in \{1,\ldots,4\}$. The following equation holds in $K(X^{\mathbf{m}})$:*

$$\begin{aligned}\mathscr{R}(m_1,m_2,(m_1m_2)^{-1})\big|_{X^{\mathbf{m}}} &\oplus \mathscr{R}(m_1m_2,m_3,m_4)|_{X^{\mathbf{m}}} \oplus E_{m_1,m_2}\\ &= \mathscr{R}(m_1,m_2m_3,m_4)|_{X^{\mathbf{m}}} \oplus \mathscr{R}(m_2,m_3,(m_2m_3)^{-1})\big|_{X^{\mathbf{m}}} \oplus E_{m_2,m_3},\end{aligned} \tag{67}$$

where

$$E_{m_1,m_2} := E(X^{m_1m_2}, X^{\langle m_1,m_2\rangle}, X^{\langle m_1m_2,m_3\rangle}) \tag{68}$$

and

$$E_{m_2,m_3} := E(X^{m_2m_3}, X^{\langle m_1,m_2m_3\rangle}, X^{\langle m_2,m_3\rangle}). \tag{69}$$

Furthermore, both sides of Equation (67) are equal in $K(X^{\mathbf{m}})$ to

$$TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^{4} \mathscr{S}_{m_i}|_{X^{\mathbf{m}}}. \tag{70}$$

Proof. Plug in the definitions of the excess normal bundles and the formula for the obstruction bundle $\mathscr{R}$ from Equation (3), then apply Equation (28) and simplify the result. One discovers that both the right hand and left hand sides of Equation (67) are both equal in $K(X^{\mathbf{m}})$ to Equation (70). □

Remark 10.3. For the reader familiar with the G-stable maps of [JKK], we note that the element $TX^{\mathbf{m}} \ominus TX|_{X^{\mathbf{m}}} \oplus \bigoplus_{i=1}^4 \mathscr{S}_{m_i}|_{X^{\mathbf{m}}}$ in Equation (70) may be interpreted as the fiber of the obstruction bundle over $\{q\} \times X^{\mathbf{m}}$ in $\xi_{0,4}(\mathbf{m}) \times X^{\mathbf{m}} = \xi_{0,4}(X,0,\mathbf{m})$, where q is any point in $\xi_{0,4}(\mathbf{m})$. This can be seen by an argument similar to that in the proof of [JKK, Prop 6.21].

Theorem 10.4. *Let X be a smooth, projective variety with an action of a finite group G. The multiplications in stringy K-theory $((\mathscr{K}(X,G),\rho),\cdot,\mathbf{1},\eta_{\mathscr{K}})$ and in the stringy Chow ring $((\mathscr{A}(X,G),\rho),\cdot,\mathbf{1},\eta_{\mathscr{A}})$ are both associative.*

Proof. Consider $\mathbf{m} = (m_1,m_2,m_3,m_4)$ in G^4 such that $m_1m_2m_3m_4 = 1$. If E_{m_1,m_2} and E_{m_2,m_3} are defined as in Equations (68) and (69), then the following equalities hold:

$$\begin{aligned}&c_{\mathrm{top}}(\mathscr{R}(m_1,m_2,(m_1m_2)^{-1}))\big|_{X^{\mathbf{m}}} \cup c_{\mathrm{top}}(\mathscr{R}(m_1m_2,m_3,m_4))|_{X^{\mathbf{m}}} \cup c_{\mathrm{top}}(E_{m_1,m_2}) =\\ &c_{\mathrm{top}}(\mathscr{R}(m_1,m_2m_3,m_4))|_{X^{\mathbf{m}}} \cup c_{\mathrm{top}}(\mathscr{R}(m_2,m_3,(m_2m_3)^{-1}))\big|_{X^{\mathbf{m}}} \cup c_{\mathrm{top}}(E_{m_2,m_3})\end{aligned} \tag{71}$$

and

(72)
$$\lambda_{-1}(\mathscr{R}(m_1, m_2, (m_1m_2)^{-1})^*)\big|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(\mathscr{R}(m_1m_2, m_3, m_4)^*)|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(E^*_{m_1,m_2}) =$$
$$\lambda_{-1}(\mathscr{R}(m_1, m_2m_3, m_4)^*)|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(\mathscr{R}(m_2, m_3, (m_2m_3)^{-1})^*)\big|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(E^*_{m_2,m_3}).$$

Equation (71) follows by taking the top Chern class of both sides of Equation (67) and then using multiplicativity of c_{top}. Equation (72) follows by taking the dual of Equation (67), applying λ_{-1}, and then using multiplicativity of λ_{-1}.

Associativity will follow from Equations (71) and (72) and the definitions of the multiplications as follows.

Let $m_+ = (m_1m_2)^{-1}$ and $m_- = (m_1m_2)$. Consider the following diagram:

$$\begin{tikzcd}
 & & X^{\mathbf{m}} \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & \\
 & X^{\langle m_1, m_2, m_+\rangle} \arrow[dl, "\mathbf{e}_{m_1}"'] \arrow[d, "\mathbf{e}_{m_2}"'] \arrow[dr, "\check{\mathbf{e}}_{m_+}"] & & X^{\langle m_-, m_3, m_4\rangle} \arrow[dl, "\mathbf{e}_{m_-}"'] \arrow[d, "\mathbf{e}_{m_3}"'] \arrow[dr, "\check{\mathbf{e}}_{m_4}"] & \\
X^{m_1} & X^{m_2} & X^{m_-} & X^{m_3} & X^{m_4},
\end{tikzcd}$$

where ϕ and ψ are the obvious inclusions. Note that the diamond in the middle is Cartesian and that the usual inclusions $\epsilon_i : X^{\mathbf{m}} \to X^{m_i}$ factor as

(73) $$\epsilon_1 = \mathbf{e}_{m_1} \circ \phi \qquad \epsilon_2 = \mathbf{e}_{m_2} \circ \phi$$

(74) $$\epsilon_3 = \mathbf{e}_{m_3} \circ \psi \qquad \epsilon_4 = \mathbf{e}_{m_4} \circ \psi.$$

And finally, we define

$$\check{\epsilon}_4 = \sigma \circ \epsilon_4 = \check{\mathbf{e}}_{m_4} \circ \psi.$$

For any $\mathscr{F}_1 \in \mathscr{K}_{m_1}$, $\mathscr{F}_2 \in \mathscr{K}_{m_2}$, $\mathscr{F}_3 \in \mathscr{K}_{m_3}$, we have

$$\begin{aligned}
(\mathscr{F}_1 \cdot \mathscr{F}_2) \cdot \mathscr{F}_3 &= (\check{\mathbf{e}}_{m_4})_* \Bigg(\mathbf{e}^*_{m_-} (\check{\mathbf{e}}_{m_+})_* \left[\mathbf{e}^*_{m_1} \mathscr{F}_1 \otimes \mathbf{e}^*_{m_2} \mathscr{F}_2 \otimes \lambda_{-1}(\mathscr{R}(m_1, m_2, m_+)^*) \right] \\
&\qquad \otimes \mathbf{e}^*_{m_3} \mathscr{F}_3 \otimes \lambda_{-1}(\mathscr{R}(m_-, m_3, m_4)) \Bigg) \\
&= (\check{\mathbf{e}}_{m_4})_* \Bigg(\psi_* \Big(\phi^* \left[\mathbf{e}^*_{m_1} \mathscr{F}_1 \otimes \mathbf{e}^*_{m_2} \mathscr{F}_2 \otimes \lambda_{-1}(\mathscr{R}(m_1, m_2, m_+)^*) \right] \\
&\qquad \otimes \lambda_{-1}(E^*_{m_1,m_2}) \Big) \otimes \mathbf{e}^*_{m_3} \mathscr{F}_3 \otimes \lambda_{-1}(\mathscr{R}(m_-, m_3, m_4)) \Bigg) \\
&= (\check{\mathbf{e}}_{m_4})_* \Bigg(\psi_* \Big(\phi^* \mathbf{e}^*_{m_1} \mathscr{F}_1 \otimes \phi^* \mathbf{e}^*_{m_2} \mathscr{F}_2 \otimes \phi^* \left(\lambda_{-1}(\mathscr{R}(m_1, m_2, m_+)^*) \right) \\
&\qquad \otimes \lambda_{-1}(E^*_{m_1,m_2}) \otimes \psi^* \mathbf{e}^*_{m_3} \mathscr{F}_3 \otimes \psi^* (\lambda_{-1}(\mathscr{R}(m_-, m_3, m_4)^*)) \Big) \Bigg)
\end{aligned}$$

$$= (\check{\epsilon}_4)_* \Bigg(\epsilon_1^* \mathscr{F}_1 \otimes \epsilon_2^* \mathscr{F}_2 \otimes \phi^* \lambda_{-1}(\mathscr{R}(m_1, m_2, m_+)^*) \otimes \lambda_{-1} E^*_{m_1,m_2}$$

$$\otimes \epsilon_3^* \mathscr{F}_3 \otimes \psi^*(\lambda_{-1}(\mathscr{R}(m_-, m_3, m_4)^*)) \Bigg)$$

$$\begin{aligned} &= (\check{\epsilon}_4)_* \Bigg(\epsilon_1^* \mathscr{F}_1 \otimes \epsilon_2^* \mathscr{F}_2 \otimes \epsilon_3^* \mathscr{F}_3 \otimes \lambda_{-1}(\mathscr{R}(m_1, m_2, m_+)^*)|_{X^{\mathbf{m}}} \\ &\qquad \otimes \lambda_{-1}(\mathscr{R}(m_-, m_3, m_4)^*)|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(E^*_{m_1,m_2}) \Bigg), \end{aligned} \tag{75}$$

where the first equality is the definition, the second equality follows from Theorem 10.1, the third equality follows from the projection formula, and the fourth and fifth equalities follow from Equations (73) and (74) and the definitions of ψ and ϕ.

A similar argument shows that the product $\mathscr{F}_1 \cdot (\mathscr{F}_2 \cdot \mathscr{F}_3)$ is given by

$$\begin{aligned} \mathscr{F}_1 \cdot (\mathscr{F}_2 \cdot \mathscr{F}_3) &= (\check{\epsilon}_4)_* \Big(\epsilon_1^* \mathscr{F}_1 \otimes \epsilon_2^* \mathscr{F}_2 \otimes \epsilon_3^* \mathscr{F}_3 \otimes \lambda_{-1}(\mathscr{R}(m_1, m_2 m_3, m_4)^*)|_{X^{\mathbf{m}}} \\ &\qquad \otimes \lambda_{-1}(\mathscr{R}(m_2, m_3, (m_2 m_3)^{-1})^*)\big|_{X^{\mathbf{m}}} \otimes \lambda_{-1}(E^*_{m_2,m_3}) \Big) \end{aligned} \tag{76}$$

But by equation (72) these two expressions (75) and (76) are equal, so associativity holds. $\square$

10.2. **The trace axiom.** We now prove the trace axiom in a similar way. Throughout this section, we fix elements a and b in G and let $m_1 := [a, b]$. We will also find it useful to define $\tilde{a} := aba^{-1}$ and $\tilde{b} := a^{-1}$.

Let $\mathbf{m}' := (m'_1, m'_2, m'_3) := ([a, b], bab^{-1}, a^{-1})$ and $\mathbf{m}'' := (m''_1, m''_2, m''_3) := ([a, b], b, ab^{-1}a^{-1})$. Observe that we have the identity $\mathbf{m}'' = ([\tilde{a}, \tilde{b}], \tilde{b}\tilde{a}\tilde{b}^{-1}, \tilde{a}^{-1})$.

Let $H := \langle a, b \rangle = \langle \tilde{a}, \tilde{b} \rangle$. Let $H' := \langle \mathbf{m}' \rangle$ and $H'' := \langle \mathbf{m}'' \rangle$. Both H' and H'' are subgroups of H. Let $\mathscr{R}(\mathbf{m}')$ denote the obstruction bundle on $X^{H'}$ and $\mathscr{R}(\mathbf{m}'')$ denote the obstruction bundle on $X^{H''}$, as in Equation (15).

Consider the commutative diagram

$$\begin{tikzcd} X^H \arrow[r, "{j'}_2"] \arrow[d, "{j'}_1"'] & X^{H'} \arrow[d, "{\Delta'}_2"] \\ X^a \arrow[r, "{\Delta'}_1"] & X^{bab^{-1}} \times X^{a^{-1}}. \end{tikzcd} \tag{77}$$

Here ${j'}_1$ and ${j'}_2$ are the obvious inclusion morphisms, ${\Delta'}_2$ is the diagonal map, and ${\Delta'}_1$ is the composition of the morphisms

$$X^a \xrightarrow{\Delta} X^a \times X^a \xrightarrow{\rho(b^{-1}) \times \sigma} X^{bab^{-1}} \times X^{a^{-1}},$$

where Δ is the diagonal map and $\rho(b^{-1})$ is the induced group action on the tangent bundle. Let $\mathscr{E}'$ be the excess intersection bundle $E(X^{bab^{-1}} \times X^{a^{-1}}, X^{H'}, X^a)$.

Similarly, consider the commutative diagram

$$\begin{tikzcd} X^H \arrow[r, "{j''}_2"] \arrow[d, "{j''}_1"'] & X^{H''} \arrow[d, "{\Delta''}_2"'] \\ X^{\tilde a} \arrow[r, "{\Delta''}_1"] & X^{\tilde b \tilde a \tilde b^{-1}} \times X^{\tilde a^{-1}} \end{tikzcd} , \tag{78}$$

where ${j''}_1$ and ${j''}_2$ are the obvious inclusion morphisms, ${\Delta''}_2$ is the diagonal map, and ${\Delta''}_1$ is the composition of the morphisms

$$X^{\tilde a} \xrightarrow{\Delta} X^{\tilde a} \times X^{\tilde a} \xrightarrow{\rho(\tilde b^{-1})\times\sigma} X^{\tilde b \tilde a \tilde b^{-1}} \times X^{\tilde a^{-1}},$$

where Δ is the diagonal map and $\rho(\tilde b^{-1})$ is the induced action of $\rho(\tilde b^{-1})$ on the tangent bundle. Let $\mathscr{E}''$ be the excess intersection bundle $E(X^{\tilde b \tilde a \tilde b^{-1}} \times X^{\tilde a^{-1}}, X^{H''}, X^{\tilde a})$.

Theorem 10.5. *The following equality holds in $K(X^H)$:*

$$j'^*_2 \mathscr{R}(\mathbf{m}') \oplus \mathscr{E}' = j''^*_2 \mathscr{R}(\mathbf{m}'') \oplus \mathscr{E}''. \tag{79}$$

Furthermore, both sides are equal to

$$TX^H \oplus \mathscr{S}_{m_1}|_{X^H} . \tag{80}$$

Proof. All equalities in this proof are understood to be in $K(X^H)$. Observe that

$$\begin{aligned} j'^*_2 \Delta'^*_2 T(X^{bab^{-1}} \times X^{a^{-1}}) &= TX^{bab^{-1}}\Big|_{X^H} \oplus TX^{a^{-1}}\Big|_{X^H} \\ &= \rho(b^{-1})(TX^a)\big|_{X^H} \oplus \sigma^* TX^a|_{X^H} \\ &= TX^a|_{X^H} \oplus TX^a|_{X^H} , \end{aligned}$$

where the third equality follows from the fact that $\rho(b^{-1}) \times \sigma$ are isomorphisms. Plugging this into the definition of the excess intersection bundle yields

$$\mathscr{E}' = TX^H \oplus TX^a|_{X^H} \oplus TX^a|_{X^H} \ominus TX^a|_{X^H} \ominus TX^{H'}\Big|_{X^H} ,$$

which simplifies to

$$\mathscr{E}' = TX^H \oplus TX^a|_{X^H} \ominus TX^{H'}\Big|_{X^H} . \tag{81}$$

On the other hand, Equation (3) yields the equality

$$\mathscr{R}(\mathbf{m}') = TX^{H'}\Big|_{X^H} \ominus TX|_{X^H} \oplus \mathscr{S}_{m_1}|_{X^H} \oplus \mathscr{S}_{bab^{-1}}|_{X^H} \oplus \mathscr{S}_{a^{-1}}|_{X^H} .$$

Together with the equality

$$\mathscr{S}_{bab^{-1}}|_{X^H} = \rho(b^{-1})(\mathscr{S}_a)\big|_{X^H}$$

and Equation (28), we obtain

$$\mathscr{R}(\mathbf{m}') = TX^{H'}\Big|_{X^H} \ominus TX^a|_{X^H} \oplus \mathscr{S}_{m_1}|_{X^H} . \tag{82}$$

Combining Equations (81) and (82) yields the identity

$$j'^*_2 \mathscr{R}(\mathbf{m}') \oplus \mathscr{E}' = TX^H \oplus \mathscr{S}_{m_1}|_{X^H} . \tag{83}$$

One now does precisely the same calculation, applied to the morphisms in the diagram (78), and one obtains

$$j''^*_2 \mathscr{R}(\mathbf{m}'') \oplus \mathscr{E}'' = TX^H \oplus \mathscr{S}_{m_1}|_{X^H} . \tag{84}$$

□

Remark 10.6. For the reader familiar with G-stable maps, we note that the element $TX^H \oplus \mathscr{S}_{m_1}|_{X^H}$ from Equation (80) is the restriction of the obstruction bundle over $\xi_{1,1}(m_1, a, b) \times X^H$ to $\{q\} \times X^H$, where q is any point in $\xi_{1,1}(m_1, a, b)$. The details of this are given in [JKK, Prop. 6.21].

Theorem 10.7. *If X is a smooth, projective variety with an action of a finite group G, then both stringy K-theory $((\mathscr{K}(X,G),\rho),\cdot,\mathbf{1},\eta)$ and the stringy Chow ring $((\mathscr{A}(X,G),\rho),\cdot,\mathbf{1},\eta)$ satisfy the trace axiom.*

Proof. We will prove the trace axiom in the case of $\mathscr{K}(X,G)$. The proof in the case of $\mathscr{A}(X,G)$ is analogous, just as it is in the proof of associativity.

We begin by fixing some notation. Let $\mathbf{1}_{X^a}$ denote the trivial bundle $\mathscr{O}_{X^a}$. Let $\{\mathscr{F}_{\alpha[a]}\}$ be a basis for $\mathscr{K}_a(X)$ with $\alpha[a] = 1, \ldots, d_a$, where d_a is the dimension of $\mathscr{K}_a(X)$, and $\eta^{\alpha[a]\beta[a^{-1}]}$ is the inverse of the metric $\eta_{\mathscr{K}}$ restricted to $\mathscr{K}_a(X) \oplus \mathscr{K}_{a^{-1}}(X)$.

We now observe that

$$\Delta'^*_2 \Delta'_{1*} \mathbf{1}_{X^a} = \eta^{\alpha[a]\beta[a^{-1}]} \left(\rho(b^{-1})\mathscr{F}_{\alpha[a]}\right)\big|_{X^{H'}} \otimes \mathscr{F}_{\beta[a^{-1}]}\big|_{X^{H'}} \tag{85}$$

and

$$\Delta''^*_2 \Delta''_{1*} \mathbf{1}_{X^{\tilde{a}}} = \rho(\tilde{b}^{-1})\mathscr{F}_{\alpha[\tilde{a}]}\Big|_{X^{H''}} \otimes \mathscr{F}_{\alpha[\tilde{a}^{-1}]}\big|_{X^{H''}} \eta^{\alpha[\tilde{a}]\alpha[\tilde{a}^{-1}]}. \tag{86}$$

Now, consider the following diagram

$$\begin{tikzcd} X^H \arrow[r, "j'_2"] \arrow[d, "j''_2"'] & X^{H'} \arrow[d, "\epsilon'_{m_1}"] \\ X^{H''} \arrow[r, "\epsilon''_{m_1}"] & X^{m_1}. \end{tikzcd} \tag{87}$$

We see that

$$\begin{aligned} & \operatorname{Tr}_{\mathscr{K}_a(X)}(L_{v_{m_1}} \circ \rho(b^{-1})) \\ = \; & \chi\left(X^{H'}, \lambda_{-1}(\mathscr{R}(\mathbf{m}')) \otimes \epsilon'^*_{m_1} v_{m_1} \otimes \left(\rho(b^{-1})\mathscr{F}_{\alpha[a]}\right)\big|_{X^{H'}} \otimes \mathscr{F}_{\alpha[a^{-1}]}\big|_{X^{H'}} \eta^{\alpha[a]\alpha[a^{-1}]}\right) \\ = \; & \chi\left(X^{H'}, \epsilon'^*_{m_1} v_{m_1} \otimes \lambda_{-1}(\mathscr{R}(\mathbf{m}')^*) \otimes \Delta'^*_2 \Delta'_{1*} \mathbf{1}_{X^a}\right) \\ = \; & \chi\left(X^{H'}, \epsilon'^*_{m_1} v_{m_1} \otimes \lambda_{-1}(\mathscr{R}(\mathbf{m}')^*) \otimes j'_{2*}\lambda_{-1}(\mathscr{E}')\right) \\ = \; & \chi\left(X^{H'}, j'_{2*}(j'^*_2 \epsilon'^*_{m_1} v_{m_1} \otimes j'^*_2 \lambda_{-1}(\mathscr{R}(\mathbf{m}')^*) \otimes \lambda_{-1}(\mathscr{E}')\right), \end{aligned}$$

where the first equality follows from the definition of trace, the second from Equation (85), the third from Theorem (10.1), and the fourth from the projection formula. Using the functoriality of pushforward, we conclude that

$$\operatorname{Tr}_{\mathscr{K}_a(X)}(L_{v_{m_1}} \circ \rho(b^{-1})) = \chi\left(X^H, v_{m_1}|_{X^H} \otimes \lambda_{-1}(j'^*_2 \mathscr{R}(\mathbf{m}') \oplus \mathscr{E}')\right). \tag{88}$$

Similarly,

$$\begin{aligned}
& \mathrm{Tr}_{\mathscr{K}_b(X)}\left(\rho(a)\circ L_{v_{m_1}}\right) \\
= \;& \mathrm{Tr}_{\mathscr{K}_{\tilde b\tilde a\tilde b^{-1}}(X)}\left(\rho(\tilde b^{-1})\circ L_{v_{m_1}}\right) \\
= \;& \chi(X^{H''}, \lambda_{-1}\left(\mathscr{R}(\mathbf{m}''))\otimes {\epsilon''}^*_{m_1} v_{m_1}\otimes(\rho(\tilde b^{-1})\ \mathscr{F}_{\alpha[\tilde a]}\big|_{X^{H''}})\otimes \mathscr{F}_{\alpha[\tilde a^{-1}]}\big|_{X^{H''}}\,\eta^{\alpha[\tilde a]\alpha[\tilde a^{-1}]}\right) \\
= \;& \chi\left(X^{H''}, {\epsilon''}^*_{m_1} v_{m_1}\otimes\lambda_{-1}(\mathscr{R}(\mathbf{m}'')^*)\otimes {\Delta''}^*_2 {\Delta''}_{1*}\mathbf{1}_{X^{\tilde a}}\right) \\
= \;& \chi\left(X^{H''}, {\epsilon''}^*_{m_1} v_{m_1}\otimes\lambda_{-1}(\mathscr{R}(\mathbf{m}'')^*)\otimes {j''}_{2*}\lambda_{-1}(\mathscr{E}'')\right),
\end{aligned}$$

and we ultimately obtain,

$$\mathrm{Tr}_{\mathscr{K}_b(X)}\left(\rho(a)\circ L_{v_{m_1}}\right) = \chi\left(X^H,\, v_{m_1}|_{X^H}\otimes\lambda_{-1}({j''}^*_2\mathscr{R}(\mathbf{m}'')\oplus\mathscr{E}'')\right). \tag{89}$$

Combining Equations (88), (89), and (79) yields the desired result. □

11. Stringy topological K-theory and stringy cohomology

All of the results in the previous sections have their counterparts in the topological category.

11.1. **Ordinary topological K-theory and cohomology.** Throughout this section, unless otherwise stated, G is a finite group acting on a compact, almost complex manifold X preserving the almost complex structure.

Furthermore, let $H^\bullet(X)$ be the rational cohomology of X. It is a Frobenius superalgebra: a Frobenius algebra with a multiplication that is graded commutative.

Topological K-theory $K_{\mathrm{top}}(X) := K_{\mathrm{top}}(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is also a Frobenius superalgebra with the $\mathbb{Z}/2\mathbb{Z}$-grading:

$$K_{\mathrm{top}}(X) = K^0_{\mathrm{top}}(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\oplus K^1_{\mathrm{top}}(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}.$$

Here $K^0_{\mathrm{top}}(X;\mathbb{Z})$ is defined exactly as $K(X;\mathbb{Z})$ but in the topological category. That is, $K^0_{\mathrm{top}}(X;\mathbb{Z})$ is additively generated by isomorphism classes of complex topological vector bundles over X modulo the relation of Equation (4) whenever Equation (5) holds. The odd part $K^1_{\mathrm{top}}(X;\mathbb{Z})$ is defined to be $K^0_{\mathrm{top}}(X\times\mathbb{R};\mathbb{Z})$. Equivalently, we may take $K^1_{\mathrm{top}}(X;\mathbb{Z})$ to be the kernel of the restriction map $i^*: K^0_{\mathrm{top}}(X\times S^1)\to K^0_{\mathrm{top}}(X\times pt)$ induced in K-theory from the inclusion of a point $i: X\times pt\to X\times S^1$.

Associated to a differentiable proper map of almost complex manifolds $f: X\to Y$, there is the induced pushforward morphism $f_*: K_{\mathrm{top}}(X)\to K_{\mathrm{top}}(Y)$ (see [Kar, IV 5.24] and [AH, Sec. 4]). In particular, if Y is a point and $f: X\to Y$ is the obvious map, we again define the Euler characteristic $\chi(X,\mathscr{F}) := f_*\mathscr{F}$. And associated to any continuous $f: X\to Y$, there is a pullback homomorphism $f^*: K_{\mathrm{top}}(Y)\to K_{\mathrm{top}}(X)$ [Kar, II.1.12].

For any compact, almost complex manifolds X and Y, there are natural morphisms

$$\nu: K^n_{\mathrm{top}}(X)\otimes K^m_{\mathrm{top}}(Y)\to K^0_{\mathrm{top}}(X\times Y\times\mathbb{R}^{n+m}).$$

Bott periodicity says that if Y is a point, there is an isomorphism

$$\beta: K^0_{\mathrm{top}}(X)\xrightarrow{\sim} K^0_{\mathrm{top}}(X\times\mathbb{R}^2)$$

[Kar, III.1.3], which is natural with respect to both pullback and pushforward. Therefore, for any compact, almost complex manifold X, composition of ν with pullback along the diagonal map $\Delta : X \to X \times X$ gives a multiplication

$$\mu : K^n_{\mathrm{top}}(X) \otimes K^m_{\mathrm{top}}(X) \to K^0_{\mathrm{top}}(X \times \mathbb{R}^{n+m}) \subseteq K_{\mathrm{top}}(X)$$

if $n + m \leq 1$ and

$$\mu : K^1_{\mathrm{top}}(X) \otimes K^1_{\mathrm{top}}(X) \longrightarrow K^0_{\mathrm{top}}(X \times \mathbb{R}^2) \xrightarrow{\beta^{-1}} K_{\mathrm{top}}(X),$$

if $n = m = 1$. Here β^{-1} is the inverse of the Bott isomorphism. We will write $\mathscr{F}_1 \otimes \mathscr{F}_2$ to denote $\mu(\mathscr{F}_1, \mathscr{F}_2)$. This product makes $K_{\mathrm{top}}(X)$ into a commutative, associative superalgebra [Kar, II.5.1 and II.5.27].

We can now define a metric on $K_{\mathrm{top}}(X)$ by

$$\eta_{K_{\mathrm{top}}}(\mathscr{F}_1, \mathscr{F}_2) := \chi(X, \mathscr{F}_1 \otimes \mathscr{F}_2),$$

and we define $\mathbf{1} := \mathscr{O}_X$. It is straightforward to check that $(K_{\mathrm{top}}(X), \otimes, \mathbf{1}, \eta_{K_{\mathrm{top}}})$ is a Frobenius superalgebra. Moreover, the projection formula holds for proper, differentiable maps with a compact target [Kar, IV.5.24].

The Frobenius superalgebra of topological K-theory satisfies the usual naturality properties with respect to pullback, is also a λ-ring [Kar, §7.2], and satisfies the splitting principle [Kar, Thm IV.2.15].

For all i, the i-th Chern class $c_i(\mathscr{F})$ associated to any $\mathscr{F}$ in $K^0_{\mathrm{top}}(X)$ belongs to $H^{2i}(X)$, and so $H^{2p}(X)$ may be regarded as the analogue of the Chow group $A^p(X)$. The associated Chern polynomial c_t satisfies the usual multiplicativity and naturality properties, and the Chern character $\mathbf{ch} : K_{\mathrm{top}}(X) \to H^\bullet(X)$, defined by Equation (10), is an isomorphism of commutative, associative superalgebras [Kar, Thm. V.3.25]. The Todd classes are defined from the ordinary Chern classes as before. In addition, Proposition (2.6) holds in topological K-theory since it follows from the splitting principle, the Chern character isomorphism, and the λ-ring properties [FH, Prop. I.5.3].

Finally, the Grothendieck-Riemann-Roch formula (see [Kar, Cor V.4.18] or [AH, Thm. 4.1]) and the excess intersection formula (Theorem 10.1) hold (see [Qu, Prop 3.3], which is written for cobordism, but the proof works as well for topological K-theory).

Remark 11.1. Let X be a compact, G-manifold with a smoothly varying one parameter family of G-equivariant almost complex structures $J_t : TX \to TX$ for all t, say, in the interval $[0, 1]$. Because of the homotopy invariance of characteristic classes, the resulting G-Frobenius algebras $\mathscr{H}(X; G)$ and $\mathscr{K}(X; G)$, and the stringy Chern character are all independent of t. Therefore, these stringy algebraic structures depend only upon the homotopy class of the G-equivariant almost complex structure on the G-manifold X.

In particular, when X is a compact symplectic manifold with an action of G preserving the symplectic structure, since there exists a unique up to homotopy, G-equivariant almost complex structure compatible with the symplectic form [GGK, Ex. D.12], these stringy algebraic structures are invariants of the symplectic manifold with G-action.

Remark 11.2. While we are primarily interested in G-equivariant almost complex manifolds in this section, our constructions generalize in a straightforward way to the case where X is a compact manifold with an oriented, G-equivariant stable complex structure (see [GGK, App. D]). The key point [GHK] is that a G-equivariant stable complex structure induces an almost complex structure on the normal bundle to any submanifold X^H consisting of points fixed by H for any subgroup H of G. Furthermore, both $\mathscr{S}_m$ (see Remark 6.1) and the right hand side of Equation (3) only depend upon such normal bundles.

11.2. **Stringy topological K-theory and stringy cohomology.** Let X be a compact, almost complex manifold with an action of a finite group $\rho : G \to \mathrm{Aut}(X)$ preserving the almost complex structure.

Fantechi and Göttsche's [FG] stringy cohomology $\mathscr{H}(X,G)$ of X, is given by

$$\mathscr{H}(X,G) := \bigoplus_{m\in G} \mathscr{H}_m(X),$$

where $\mathscr{H}_m(X) := H^\bullet(X^m)$, and the definition of the multiplication is still given by Equation (17), and similarly for the metric and identity element. However, the $\mathbb{Q}$-grading here is not quite that defined by Equation (14), but is defined instead by the equation

$$|v_m|_{str} := 2a(v_m) + |v_m|, \tag{90}$$

where $|v_m| := p$ when v_m belongs to $H^p(X^m)$ and $a(v_m) := a(m,U)$.

Furthermore, Theorem (4.2) holds, provided that $\mathscr{A}(X,G)$ is everywhere replaced by $\mathscr{H}(X,G)$, dimension $\dim X$ is understood to be the dimension of X as a real manifold, and "G-Frobenius algebra" is replaced by "G-Frobenius superalgebra."

Stringy topological K-theory $\mathscr{K}^{\mathrm{top}}(X,G) := \bigoplus_{m\in G} \mathscr{K}_m^{\mathrm{top}}(X)$ is defined additively by $\mathscr{K}_m^{\mathrm{top}}(X) := K_{\mathrm{top}}(X^m)$ for all m in G. The stringy multiplication, metric, and identity element are defined just as in the case of $\mathscr{K}(X,G)$. This is compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading because the obstruction bundle $\mathscr{R}$ is an element of $K^0_{\mathrm{top}}(X^{\mathbf{m}})$.

Since the Eichler trace formula holds for all compact Riemann surfaces, our formula (3) for the obstruction bundle, and indeed the entire analysis in Sections 6 and 10, holds in topological K-theory. Consequently, $((\mathscr{K}^{\mathrm{top}}(X,G),\rho),\cdot,\mathbf{1},\eta)$ is a G-Frobenius superalgebra.

Furthermore, the stringy Chern character $\mathscr{C}\mathbf{h} : \mathscr{K}^{\mathrm{top}}(X,G) \to \mathscr{H}^\bullet(X,G)$ is still defined by Equation (37). The rest of the analysis in Section 7 holds, provided that $\mathscr{A}(X,G)$ is everywhere replaced by $\mathscr{H}(X,G)$ and K-theory is everywhere replaced by topological K-theory. Therefore, $\mathscr{C}\mathbf{h} : \mathscr{K}^{\mathrm{top}}(X,G) \to \mathscr{H}^\bullet(X,G)$ is an isomorphism of G-commutative superalgebras.

Finally, the analysis in Section 9 holds after replacing Chow groups by cohomology everywhere. In particular, $\overline{\mathscr{K}}_{\mathrm{top}}(X,G)$ is isomorphic to $K_{\mathrm{orb}}([X/G])$, the stringy topological K-theory of $[X/G]$, while $\overline{\mathscr{H}}(X,G)$ is isomorphic [FG] to the stringy (or Chen-Ruan orbifold) cohomology $H^\bullet_{orb}([X/G])$. Therefore, the stringy Chern character $\overline{\mathscr{C}\mathbf{h}} : \overline{\mathscr{K}}_{\mathrm{top}}(X,G) \to \overline{\mathscr{H}}(X,G)$ gives a ring isomorphism $Ch_{orb} : K_{\mathrm{orb}}(X) \to H_{\mathrm{orb}}(X)$.

11.3. **The symmetric product and crepant resolutions.** One of the most interesting examples of stringy K-theory and cohomology is the symmetric product. Let $X := Y^n$, where Y is a complex manifold of complex dimension d with the symmetric group S_n acting on Y^n by permuting its factors. In this case, for any

$m \in S_n$ it is easy to see that the age $a(m)$ is related to the length of the permutation $l(m)$:

$$a(m) = l(m)d/2.$$

Consequently, by Equation (90), the $\mathbb{Q}$-grading on $\mathscr{H}(X, G)$ is, in fact, a grading by (possibly odd) integers.

Consider stringy topological K-theory $\mathscr{K}^{\mathrm{top}}(Y^n, S_n)$ of the S_n-variety Y^n. Choose the 2-cocycle (discrete torsion) α in $Z^2(S_n, \mathbb{Q}^*)$

$$\alpha(m_1, m_2) := (-1)^{\varepsilon(m_1, m_2)},$$

where ε is defined by

$$\varepsilon(m_1, m_2) := \frac{1}{2}(l(m_1) + l(m_2) - l(m_1 m_2)).$$

It is straightforward to verify that $\varepsilon(m_1, m_2)$ is an integer. Now, twist the S_n-Frobenius algebra $\mathscr{K}^{\mathrm{top}}(Y^n, S_n)$ by α, as in Section 8, to yield a new S_n-Frobenius algebra $((\mathscr{K}^{\mathrm{top}}(Y^n, S_n), \rho), \star, \mathbf{1}, \eta^{\alpha})$ which we will denote by $\mathbf{K}^{\mathrm{top}}(Y^n, S_n)$. Notice that the G-action is unchanged by the twist, but the twisted multiplication $\mathscr{K}^{\mathrm{top}}(Y^n, S_n)$ is given by the formula

$$v_{m_1} \star v_{m_2} := \alpha(m_1, m_2) v_{m_1} \cdot v_{m_2}, \tag{91}$$

where $\cdot$ denotes the stringy multiplication in $\mathscr{K}^{\mathrm{top}}(Y^n, S_n)$.

Twisting the multiplication on the stringy cohomology of Y^n in the same fashion, we obtain the S_n-Frobenius algebra $((\mathscr{H}(Y^n, S_n), \rho), \star, \mathbf{1}, \eta^{\alpha})$, which we will denote by $\mathbf{H}(Y^n, S_n)$. By the obvious topological analogue of Corollary 8.9, the stringy Chern character $\mathscr{C}\mathbf{h} : \mathbf{K}^{\mathrm{top}}(Y^n, S_n) \to \mathbf{H}(Y^n, S_n)$ is an isomorphism of S_n-commutative algebras. After taking S_n-coinvariants, we obtain a ring isomorphism

$$Ch_{orb} : \mathbf{K}^{\mathrm{top}}_{\mathrm{orb}}([Y^n/S_n]) \to \mathbf{H}_{\mathrm{orb}}([Y^n/S_n]),$$

where $\mathbf{K}^{\mathrm{top}}_{\mathrm{orb}}([Y^n/S_n])$ is $K_{\mathrm{orb}}([Y^n/G])$, but with the twisted multiplication, and similarly for $\mathbf{H}_{\mathrm{orb}}([Y^n/S_n])$.

What makes these particular twisted rings interesting is the following theorem.

Theorem 11.3. *Let Y be a complex, projective surface such that $c_1(Y) = 0$. Consider Y^n with S_n acting by permutation of its factors. If $Y^{[n]}$ denotes the Hilbert scheme of n points in Y, then $\mathbf{K}^{\mathrm{top}}_{\mathrm{orb}}([Y^n/S_n])$ is isomorphic as a Frobenius algebra to $K_{\mathrm{top}}(Y^{[n]})$.*

Proof. We define ψ so that the following diagram commutes

$$\begin{tikzcd} \mathbf{K}^{\mathrm{top}}_{\mathrm{orb}}([Y^n/S_n]) \arrow[r, "\mathscr{C}\mathbf{h}_{\mathrm{orb}}"] \arrow[d, "\psi"'] & \mathbf{H}_{\mathrm{orb}}([Y^n/S_n]) \arrow[d, "\psi'"'] \\ K_{\mathrm{top}}(Y^{[n]}) \arrow[r, "\mathbf{ch}"] & H(Y^{[n]}) \end{tikzcd}, \tag{92}$$

where ψ' is the ring isomorphism Ψ^{-1} in [FG, Thm 3.10]. This uniquely defines ψ, since $\mathbf{ch}$ and $\mathscr{C}\mathbf{h}_{\mathrm{orb}}$ are ring isomorphisms.

The homomorphism ψ also preserves the metrics because of the Hirzebruch-Riemann-Roch Theorem and the fact that ψ' preserves the metrics. $\square$

Remark 11.4. The rings $\mathbf{K}^{\mathrm{top}}_{\mathrm{orb}}([Y^n/S_n]) \otimes_{\mathbb{Q}} \mathbb{C}$ and $K_{orb}([Y^n/S_n]) \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic (see [Ru]). Since $Y^{[n]} \to Y^n/S_n$ in the previous theorem is a crepant (and hyper-Kähler) resolution, this is an example of a K-theoretic version of Conjecture 1.1. Our result is nontrivial precisely because of the nontrivial definition of multiplication on $K_{\mathrm{orb}}([Y^n/S_n])$ and the stringy Chern character.

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