THE COBAR CONSTRUCTION AS A HOPF ALGEBRA AND THE LIE DIFFERENTIAL

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Abstract. We show that the integral cobar construction ΩC_*X of a 1-reduced simplicial set X has the structure of a homotopy Hopf algebra. This leads to a rational differential Lie algebra $\mathcal{L}(X)$ given by a differential $d_{\mathcal{L}}$ on the free Lie algebra $L(s^{-1}\tilde{C}_*X)$. The universal enveloping algebra $U\mathcal{L}(X)$ is naturally isomorphic to $\Omega C_*X \otimes \mathbb{Q}$.

Let X be a 1-reduced simplicial set and let $s^{-1}\tilde{C}_*X$ be the desuspension of the reduced chain complex $\tilde{C}_*X$ of X. The cobar construction

$$\Omega C_*X = (T(s^{-1}\tilde{C}_*X), d_\Omega)$$

is given by a differential d_Ω on the tensor algebra $T(s^{-1}\tilde{C}_*X)$ defined over $\mathbb{Z}$. The homology of ΩC_*X coincides with the homology of the loop space $\Omega|X|$. A formula for d_Ω was given by Adams [Ad]. We show that the cobar construction ΩC_*X has over $\mathbb{Z}$ the additional natural structure of a "homotopy Hopf algebra" consisting of a coassociative diagonal

$$\psi : \Omega(C_*X) \to \Omega(C_*X) \otimes \Omega(C_*X)$$

and a derivation homotopy $E : \mathcal{T}\psi \simeq \psi$. Explicit formulas for ψ and E are given in (2.9). The result is based on the geometric cobar construction in [B]. Using the fundamental theory of Anick [A] we derive from the data $(\Omega C_*X, \psi, E)$ a differential Lie algebra

$$\mathcal{L}(X) = (L(s^{-1}\tilde{C}_*X), d_{\mathcal{L}})$$

which is defined by a differential $d_{\mathcal{L}}$ on the free rational Lie algebra $L(s^{-1}\tilde{C}_*X)$. The homology of $\mathcal{L}(X)$ coincides with the rational homotopy groups of $\Omega|X|$.

Moreover we describe a cocommutative diagonal

$$\phi : (\Omega C_*X) \otimes \mathbb{Q} \to (\Omega C_*X) \otimes (\Omega C_*X) \otimes \mathbb{Q}$$

together with a derivation homotopy $H : \psi \otimes \mathbb{Q} \simeq \phi$ and a natural isomorphism of differential Hopf algebras

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$$\beta : U\mathcal{L}(X) \cong (\Omega C_* X \otimes \mathbb{Q}, \phi)$$

Formulas for ϕ, H and β are obtained by an induction. A map $f : X \to Y$ between 1-reduced simplicial sets induces the map $F_* = L(s^{-1}\tilde{C}_* f) : L(s^{-1}\tilde{C}_* X) \to L(s^{-1}\tilde{C}_* Y)$ between free Lie algebras which is compatible with the differential $d_\mathcal{L}$. Hence $X \mapsto \mathcal{L}(X)$ yields a canonical functor from 1-reduced simplicial sets to free differential Lie algebras. This functor induces an isomorphism of rational homotopy categories. In fact, $\mathcal{L}$ is up to homotopy naturally equivalent to the functor $\lambda = NP\hat{Q}G$ of Quillen [Q] which is the composite of four intricate functors. We describe in section 5 inductively a sequence of elements $\beta(2), \beta(3), \dots$ which determine the differential $d_\mathcal{L}$ and we compute the quadratic part of $d_\mathcal{L}$. A complete formula for $d_\mathcal{L}$, however, is highly intricate since it can be considered as a kind of "simplicial form" of the classical Hausdorff formula; see (4.9) (4).

We cannot expect a simple form of the differential $d_\mathcal{L}$ since the homology of $\mathcal{L}(X)$ yields the rational homotopy groups of X; that is

$$\pi_{*+1}(X) \otimes \mathbb{Q} = H_*\mathcal{L}(X) = \text{kernel}\, d_\mathcal{L}/\text{image}\, d_\mathcal{L}$$

This supplements the result of Kan [K] who gave a combinatorial description of $\pi_*(X)$ in terms of the free simplicial group GX. Hence $\pi_*(X)$ requires computations in free groups while the formula for $\pi_*(X) \otimes \mathbb{Q}$ requires only the computation of the homology of a chain complex of rational vector spaces. We point out that $\mathcal{L}(X)$ is a chain complex of finite type if X has only finitely many non degenerate simplices. Hence in this case $d_\mathcal{L}$ yields a *finite* formula for $\pi_n(\| X \|) \otimes \mathbb{Q}$. Such a formula was not yet achieved in the literature.

For example using the Sullivan method one obtains $\pi_*(X) \otimes \mathbb{Q}$ only implicitly by computing a minimal model of the De Rham algebra $A(X)$; see Bousfield-Gugenheim [BG]. Here $A(X)$ is not of finite type if X has only finitely many non degenerate simplices so that $A(X)$ does not give finite formula for $\pi_n(\| X \|) \otimes \mathbb{Q}$. In this case the minimal model M_X of $A(X)$ is of finite type and hence also gives a finite formula for $\pi_n(\| X \|) \otimes \mathbb{Q}$. There is, however, no explicit formula for M_X in the literature and M_X is not natural in X.

On the other hand also the model $\lambda(X)$ of Quillen is considerably larger than $\mathcal{L}(X)$ above and is only defined as the normalization of a simplicial object; see (4.9) (4).

§ 1 Algebras, coalgebras and Hopf algebras

We fix in this section the main algebraic definitions we shall use; compare for example [MM, HMS, A, BL]. Let R be a commutative ring with unity 1 which we use as ring of coefficients. A graded module is a sequence $V = \{V_n, n \in \mathbb{Z}\}$ of R-modules V_n. The ring R is the graded module which is R in degree 0 and which is trivial otherwise. A map $f : V \to W$ of degree $|f| = r$ between graded modules is a sequence of R-linear maps $f_n : V_n \to W_{n+r}, n \in \mathbb{Z}$. Let $s^r V$ be the suspension of Y defined by $(s^r V)_n = V_{n-r}$ and let $s^r : V \to s^r V$ be the map of degree r obtained by the identity $V_n = (sV)_{n+r}$. We define the tensor product $V \otimes W$ by $(V \otimes W)_n = \oplus\{V_i \otimes V_j, i + j = n\}$. The interchange isomorphism

$$\mathcal{T} : V \otimes W \cong W \otimes V$$

carries $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$. Here we write $|x| = n$ if $x \in V_n$ and $|x|$ is termed the degree of $x \in V$. For maps $f : V \to V'$ and $g : W \to W'$ we obtain $f \otimes g : V \otimes W \to V' \otimes W'$ by

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$

A differential module V is a graded module together with a map $d : V \to V$ of degree -1 satisfying $dd = 0$. The tensor product of differential modules V, W is the graded module $V \otimes W$ with the differential $d \otimes 1 + 1 \otimes d$; that is $d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} x \otimes dy$. The maps $f : V \to W$ between differential modules are maps of degree 0 compatible with the differential, that is $df = fd$. We say that V is positive if $V_n = 0$ for $n < 0$. The positive differential modules and maps define a category $\mathbf{DM}^+$ which by the tensor product and the unit object R is a monoidal category termed $(\mathbf{DM}^+, \otimes, R)$. A homology isomorphism is a map $f : V \to W$ in $\mathbf{DM}^+$ which induces an isomorphism in homology $HV \cong HW$ where $HV = \text{kernel}\,(d)/\text{image}\,(d)$.

(1.1) Definition. A monoid in $\mathbf{DM}^+$ is a positive differential module A together with an associative multiplication $\mu : A \otimes A \to A$, $a \otimes b \mapsto a \cdot b$, and a unit $1 : R \to A$. In particular R is such a monoid. A differential algebra A is a monoid $(A, d, \mu, 1)$ in $\mathbf{DM}^+$ together with a map $\varepsilon : A \to R$ of monoids termed the augmentation of A. Let $\bar{A} = ker(\varepsilon) \in \mathbf{DM}^+$ be the augmentation ideal. Then μ defines the map $\bar{\mu} : \bar{A} \otimes \bar{A} \to \bar{A}$ in $\mathbf{DM}^+$. We say that A is connected if $1 : R \cong A_0$ is an isomorphism. A map $f : A \to B$ between differential algebras is a map $\mathbf{DM}^+$ compatible with the multiplication, unit and augmentation. Let $\mathbf{DA}$ be the category of differential algebras and such maps. For $A, B \in \mathbf{DA}$ we obtain $A \otimes B \in \mathbf{DA}$ by the tensor product of the underlying differential modules. The multiplication of $A \otimes B$ is defined by $(\mu \otimes \mu)(1 \otimes T \otimes 1)$; that is

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} (a \cdot c) \otimes (b \cdot d)$$

and $1 = 1 \otimes 1$ and $\varepsilon = \varepsilon \otimes \varepsilon$ are unit and augmentation of $A \otimes B$. Therefore $(\mathbf{DA}, \otimes, R)$ is again a monoidal category.

The next definition of coalgebras is dual to the definition of algebras in (1.1).

(1.2) Definition. A comonoid in $\mathbf{DM}^+$ is a positive differential module C together with an associative comultiplication (diagonal) $\Delta : C \to C \otimes C$ and a counit $1 : C \to R$. In particular R is such a comonoid. A differential coalgebra C is a comonoid $(C, d, \Delta, 1)$ in $\mathbf{DM}^+$ together with a map $\eta : R \to C$ of comonoids termed the coaugmentation of C. Let $\tilde{C} = coker\,(\eta)$ be the cokernel of η in $\mathbf{DM}^+$ and let $\tilde{\Delta} : \tilde{C} \to \tilde{C} \otimes \tilde{C}$ be the map in $\mathbf{DM}^+$ induced by Δ. The map $\tilde{\Delta}$ is called the reduced diagonal of C. We say that C is connected if $1 : C_0 \cong R$ is an isomorphism. A map $f : C \to D$ between differential coalgebras is a map in $\mathbf{DM}^+$ compatible with the comultiplication, counit and coaugmentation. Let $\mathbf{DC}$ be the category of differential colagebras and such maps.

Let V be a positive graded module. Then we obtain the tensor algebra $T(V)$ which as graded module is the direct sum in $\mathbf{DM}^+$

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \tag{1.3}$$

where $V^{\otimes 0} = R$ and $V^{\otimes n} = V \otimes \ldots \otimes V$ (n-fold). The canonical multiplication μ as $T(V)$ is defined by

$$\mu(a_1 \otimes \ldots \otimes a_r, a_{r+1} \otimes \ldots \otimes a_n) = a_1 \otimes \ldots \otimes a_n.$$

Let $i^n : V^{\otimes n} \to T(V)$ be the inclusion. Then the unit $1 : R \to T(V)$ is given by i^0 and the augmentation $\varepsilon : T(V) \to R$ is defined by $\varepsilon i^0 = 1, \varepsilon i^n = 0$ for $n \geq 1$. Let E be a graded set with $E_n = \phi$ (= empty set) for $n < 0$. Then E generates the free graded R-module $V = \mathrm{span}_R(E)$ and the tensor algebra

$$T(V) = T(\mathrm{span}_R(E)) = R < E > \tag{1.4}$$

is termed the free graded R-algebra generated by E. We call a differential algebra A free if the underlying graded algebra of A admits an isomorphism $A \cong T(V) = R < E >$ of augmented algebras where V is a free graded R-module. Note that A need not be a free object in the category **DA**.

(1.5) Definition. Let C be a differential coalgebra which is positive and connected. Then the cobar construction $\Omega(C)$ due to Adams [Ad] (see also [Mo]) is the differential algebra

$$\Omega(C) = (T(s^{-1}\tilde{C}), d_\Omega)$$

with the differential d_Ω determined by the restriction

$$d_\Omega i^1 = -i^1(s^{-1}ds) + i^2(s^{-1} \otimes s^{-1})\tilde{\Delta}s.$$

If $C_1 = 0$ then $\Omega(C)$ is connected. Moreover if C is free as a graded R-module then $\Omega(C)$ is free. Let $\mathbf{DC}_0$ be the full subcategory of **DC** consisting of connected differential coalgebras. The cobar construction is a functor $\Omega : \mathbf{DC}_0 \to \mathbf{DA}$. Clearly a map $f : C \to D$ in $\mathbf{DC}_0$ is carried to $\Omega(f) = T(s^{-1}\tilde{f})$ where $\tilde{f} : \tilde{C} \to \tilde{D}$ is induced by f.

(1.6) Definition. A differential Hopf algebra is a differential algebra A together with an associative comultiplication (diagonal)

$$\Delta : A \to A \otimes A$$

which is a map in **DA** such that (A, Δ) is a comonoid in the monoidal catgeory $(\mathbf{DA}, \otimes)$. Moreover the counit of (A, Δ) coincides with the augmentation $\varepsilon : A \to R$ and the unit $1 : R \to A$ is a coaugmentation of (A, Δ) so that (A, Δ) is a coalgebra in the sense of (1.2). The Hopf algebra is cocommutative if $T\Delta = \Delta$. Let **HA** be the category of differential Hopf algebras. Morphisms are maps in **DA** which are compatible with Δ; that is $(f \otimes f)\Delta = \Delta f$.

We shall use the following notion of homotopy for maps in **DA**.

(1.7) Definition. Let $f, g : A \to B$ be maps between differential algebras in **DA**. A map $F : A \to B$ of degree r between the underlying graded modules is an (f, g)-derivation if it satisfies

$$F(xy) = F(x)g(y) + (-1)^{r|x|}f(x)F(y)$$

A <u>derivation homotopy</u> from f to g is an (f,g) -derivation F of degree $+1$ satisfying

$$dF + Fd = f - g$$

We write $F : f \simeq g$. The relation of derivation homotopy is an equivalence relation if A is free and connected.

Anick in § 2 [A] uses a more general notion of homotopy but in this paper we need only derivation homotopies. In particular such homotopies are used in the following definition of homotopy Hopf algebras.

(1.8) Definition. A <u>homotopy Hopf algebra</u> (A, ψ, G_1, G_2) is a connected differential algebra A together with a map $\psi : A \to A \otimes A$ in $\mathbf{DA}$ such that the augmentation $\varepsilon : A \to R$ is a counit for ψ (that is $(\varepsilon \otimes 1)\psi = 1$ and $(1 \otimes \varepsilon)\psi = 1$) and

$$\begin{aligned} &G_1 : \mathcal{T}\psi \simeq \psi \\ &G_2 : (\psi \otimes 1)\psi \simeq (1 \otimes \psi)\psi \end{aligned}$$

are derivation homotopies. A map $f : (A, \psi, G_1, G_2) \to (A', \psi', G_1', G_2')$ between homotopy Hopf algebras is a map $f : A \to A'$ in $\mathbf{DA}$ which commutes with diagonals and homotopies; that is $(f \otimes f)\psi = \psi' f$, $(f \otimes f)G_1 = G_1' f, (f \otimes f \otimes f)G_2 = G_2' f$. Let $\mathbf{hHA}_0$ be the category of such homotopy Hopf algebras and maps.

One has the canonical functors

$$\mathbf{CHA}_0 \subset \mathbf{hHA}_0 \xrightarrow{\pi} \mathbf{DA}_0 \tag{1.9}$$

Here $\mathbf{CHA}_0 \subset \mathbf{HA}$ is the full subcategory of connected cocommuttaive objects and $\mathbf{DA}_0 \subset \mathbf{DA}$ is the full subcategory of connected objects. The inclusion carries (A, Δ) to $(A, \Delta, G_1 = 0, G_2 = 0)$ and the forgetful functor π carries (A, ψ, G_1, G_2) to A. We point out that any homotopy Hopf algebra above yields canonically a "Hopf algebra up to homotopy" in the sense of Anick; see 4.1 in [A].

§ 2 The cobar construction on a simplicial set

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and let $\mathbf{Set}$ be the category of sets. The simplicial category Δ is the subcategory of $\mathbf{Set}$ with objects the finite sets

$$\{0 \dots n\} = \{t \in \mathbb{N}, 0 \leq t \leq n\} \quad \text{with} \quad n \in \mathbb{N}. \tag{2.1}$$

Morphisms in Δ are the functions $g : \{0 \dots m\} \to \{0 \dots n\}$ which are order preserving; that is $i \leq j$ implies $g(i) \leq g(j)$. A <u>simplicial set</u> X is a contravariant functor $X : \Delta \to \mathbf{Set}$. A map $X \to Y$ between simplicial sets is a natural transformation of such functors. Let $\mathbf{sSet}$ be the category of simplicial sets and maps between them. We write for $X \in \mathbf{sSet}$ also $X_n = X(\{0 \dots n\})$ so that X determines a graded set $\{X_n\}$ also denoted by X. For a subset

$$a = \{a_0 < a_1 \dots < a_r\} \subset \{0 \dots n\}$$

we obtain the unique order preserving injective function $i_a : \{0 \dots r\} \to \{0 \dots n\}$ with image$(i_a) = a$. We define for $\sigma \in X_n$ the element

$$\sigma(a) = \sigma(a_0, a_1, \dots, a_r) = i_a^*(\sigma) \in X_r \tag{2.2}$$

The subset $\partial_i a = a - \{a_i\}$ with $i = 0, \dots, r$ is termed a face of a. This yields the faces of $\sigma \in X_n$ by

$$d_i(\sigma) = \sigma(\partial_i\{0 \dots n\}) \in X_{n-1}$$

for $i = 0, 1, \dots, n$. An element $\sigma \in X_n$ is degenerate if there exists a surjective function $g : \{0 \dots n\} \to \{0 \dots m\}$ in Δ with $m < n$ such that $\sigma \in$ image $(g^* : X_m \to X_n)$. Let $(DX)_n$ be the subset of X_n consisting of degenerate elements. Hence $DX \subset X$ is an inclusion of graded sets. We obtain the normalized chain complex of X by the quotient of graded R-modules

$$C_*X = span_R(X)/span_R(DX) \tag{2.3}$$

This is a positive differential module with differential

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma)$$

We say that X is r-reduced if $X_0 = X_1 = \dots = X_r = \{*\}$ consists of a single point $*$. If X is 0-reduced we obtain the Alexander-Whitney diagonal

$$\begin{aligned} &\Delta : C_*X \to C_*X \otimes C_*X \\ &\Delta(\sigma) = \sum_{i=0}^{n} \sigma(0, \dots, i) \otimes \sigma(i, \dots, n) \end{aligned} \tag{2.4}$$

Here $X_0 = \{*\}$ implies that $C_0X = \mathbb{Z}$ and one can check that $C_*X = (C_*X, d, \Delta)$ is a connected differential coalgebra. Moreover C_*X as a graded R-module is freely generated by the graded set $EX = X - DX$ of non-degenerate elements in X. The cobar construction $\Omega(C_*X)$ on the coalgebra C_*X is the free algebra

$$\Omega(C_*X) = T(s^{-1}\tilde{C}_*(X)) = R\langle s^{-1}\tilde{E}X\rangle$$

generated by the desuspension of the graded set $\tilde{E}X = EX - \{*\}$. The differential $d = d_\Omega$ is defined for $\sigma \in X_n$ by

$$\begin{aligned} d(s^{-1}\sigma) = &- \sum_{i=0}^{n} (-1)^i s^{-1}(d_i\sigma) \\ &+ \sum_{i=1}^{n-1} (-1)^i s^{-1}\sigma(0, \dots, i) \otimes s^{-1}\sigma(i, \dots, n) \end{aligned} \tag{2.5}$$

This is the explicit formula derived from (1.4) and (2.4). Any degenerate element in the formula represents the zero element. The cobar construction yields the functor

$$\Omega C_* : \mathbf{sSet}_1 \to \mathbf{DA}_0 \tag{2.6}$$

Here $\mathbf{sSet}_1$ is the category of 1-reduced simplicial sets. Adam's result in [Ad] shows that for $X \in \mathbf{sSet}_1$ there is a natural isomorphism of algebras

$$H_*(\Omega C_* X) \cong H_*(\Omega|X|, R) \tag{2.7}$$

where $\Omega|X|$ is the loop space of the realization $|X|$ and where $H_*(-, R)$ denotes the singular homology with coefficients in R.

We are going to describe the natural structure of a homotopy Hopf algebra for $\Omega C_* X$. For this we introduce the following notation. The number of elements of a subset $a \subset \mathbb{N}$ is denoted by $\sharp(a)$. Let

$$N = \{a; a \subset \mathbb{N}, 2 \leq \sharp(a) < \infty\} \tag{2.8}$$

be the set of finite subsets of $\mathbb{N}$ with at least 2 elements. We define the degree of $a \in N$ by $|a| = \sharp(a) - 2$. Moreover we define for $a, b \in N$

$$a \leq b \Leftrightarrow a \subset b \quad \text{and} \quad Min(a) = Min(b) \quad \text{and} \quad Max(a) = Max(b) \tag{1}$$

where Min and Max denote the minimum and maximum respectively. Let N^* be the free monoid generated by the set N. The elements of N^* are words

$$w = a^1 a^2 \dots a^k \quad \text{with} \quad a^i \in N \quad \text{for} \quad i = 1, \dots, k \tag{2}$$

Here $k = l(w)$ is termed the length of W and the degree of w is the sum $|w| = |a^1| + \dots + |a^k|$ of degrees. The product wv of words $v, w \in N^*$ is given by concatenation. For $n, m \in \mathbb{N}$ with $n < m$ we obtain the interval

$$\{n \dots m\} = \{t \in \mathbb{N}, n \leq t \leq m\}$$

Moreover each $b \in N$ determines by its total ordering $b = \{b_0 < \dots < b_r\}$ with $r = |b| + 1$ the word of intervals

$$\bar{b} = \{b_0 \dots b_1\}\{b_1 \dots b_2\} \dots \{b_{r-1} \dots b_r\} \tag{3}$$

This gives us the function $N \to N^*$ which carries b to $\bar{b}$. We define for $b \in N$ and w in (2) the words

$$w|_b = (a^1 \cap b)(a^2 \cap b) \dots (a^k \cap b) \quad \text{and} \tag{4}$$

$$(5) \qquad \begin{cases} b \cup w = (b \cup a^1) a^2 a^3 \dots a^k \\ w \cup b = a^1 a^2 \dots a^{k-1} (a^k \cup b). \end{cases}$$

We shall use the word $w \mid_b$ in (3.5) (3) where the letters of $w \mid_b$ are again elements in N. Here $a \cap b$ and $a \cup b$ denote the intersection, resp. union of the sets $a, b \subset \mathbb{N}$. Moreover the <u>carrier</u> of a word $w = a^1 \dots a^k \in N^*$ is the union of letters of w; that is,

$$(6) \qquad car(w) = a^1 \cup \dots \cup a^k \in N.$$

The <u>sign</u> function $\tau : N^* \to \{+1, -1\}$ is defined by $\tau(a) = 1$ for $a \in N$ and

$$(7) \qquad \tau(wv) = (-1)^{l(w)|v|} \tau(w) \cdot \tau(v) \quad \text{for} \quad w, v \in N^*$$

or equivalently by $\tau(w) = (-1)^t$ with

$$t = \sum_{i=1}^{k} (i-1)|a^i|.$$

(2.9) Definition. Let X be a 1-reduced simplicial set and let $\sigma \in X_n, n \geq 2$. For $w \in N^*$ with $car(w) \subset \{0 \dots n\}$ let

$$(1) \qquad (w)_\sigma = s^{-1}\sigma(a^{i_1}) \otimes \dots \otimes s^{-1}\sigma(a^{i_s}) \in T(s^{-1}\tilde{C}_* X)$$

where $i_1 < \dots < i_s$ are exactly those indices of the letters $a^1, \dots, a^k$ in $w = a^1 \dots a^k$ for which $|a^{i_1}|, \dots, |a^{i_s}| > 0$. If $|a^i| = 0$ for all $i = 1, \dots, k$ or equivalently if $|w| = 0$ we set $(w)_\sigma = 1$ where 1 is the unit in the tensor algebra $T(s^{-1}\tilde{C}_* X)$. The degree of the element $(w)_\sigma$ in the tensor algebra coincides with the degree of the word w in (2.8). For $\sigma \in X_n, n \geq 2$, we have $s^{-1}\sigma = (\{0 \dots n\})_\sigma$ and the differential d in the cobar construction $\Omega C_* X$ is given by the formula (see (2.5))

$$\begin{aligned} d(s^{-1}\sigma) = & - \sum_{i=0}^{n} (-1)^i (\partial_i \{0 \dots n\})_\sigma \\ & + \sum_{i=2}^{n-2} (-1)^i (\{0 \dots i\}\{i \dots n\})_\sigma \end{aligned}$$

We define a diagonal

$$(2) \qquad \psi : \Omega(C_* X) \to \Omega(C_* X) \otimes \Omega C_* X$$

for $\sigma \in X_n$ by

$$\psi(s^{-1}\sigma) = \sum_{f \leq \{0 \ldots n\}} \tau(\bar{f}) \cdot (\bar{f})_\sigma \otimes (f)_\sigma \tag{3}$$

We define a derivation homotopy $E : \mathcal{T}\psi \simeq \psi$ by

$$E(s^{-1}\sigma) = \sum_{i=1}^{n-1} \sum_{\substack{f \leq \{0 \ldots i\} \\ g \leq \{i \ldots n\}}} \tau \cdot (f \cup \bar{g})_\sigma \otimes (\bar{f} \cup g)_\sigma \tag{4}$$

where the sign $\tau \in \{+1, -1\}$ is

$$\tau = \tau(\bar{f})\tau(\bar{g}) \cdot (-1)^{(|f|+i)(|g|+n)} \tag{5}$$

In (3) and (4) we use the notation in (2.8) (1) ... (7).

(2.10) Theorem. *Let X be a 1-reduced simplicial set. Then the cobar construction $\Omega C_* X$ is a well defined homotopy Hopf algebra with diagonal ψ defined in (2.9). This diagonal is coassociative and the augmentation $\varepsilon : \Omega C_* X \to R$ is a counit for ψ. Moreover E in (2.9) is a derivation homotopy $E : \mathcal{T}\psi \simeq \psi$. Both ψ and E are natural in $X \in \mathbf{sSet}_1$ so that we get the commutative diagram of functors*

$$\begin{tikzcd} \mathbf{sSet}_1 \arrow[rr, "\Omega C_*"] \arrow[dr, "(\Omega C_*)_{\psi,E}"'] & & \mathbf{DA}_0 \\ & \mathbf{hHA}_0 \arrow[ur, "\pi"'] & \end{tikzcd}$$

with $(\Omega C_*)_{\psi,E}(X) = (\Omega C_*(X), \psi, G_1 = E, G_2 = 0)$; see (1.8).

We prove this result together with theorem (3.5) below.

Remark. We obtained the diagonal ψ of the cobar construction $\Omega C_* X$ in (2.9) already in [B] so that only the explicit formula for the derivation homotopy E is new. Majewski [M] independently showed that $\Omega C_* X$ has the natural structure of a homotopy Hopf algebra; his diagonal however is not associative and he does not describe explicit formulas. The diagonal ψ in (2.9) was used for the double cobar construction and double bar construction in $[B']$. Gerstenhaber-Voronov [GV] showed that the diagonal ψ corresponds to a "homotopy Gerstenhaber-algebra structure" on the cochain complex $C^* X$ of the simplicial set X; see corollary 6 in [GV]. Moreover Getzler-Jones [GJ] showed that this structure induced by ψ is a special "B_∞ -algebra structure" on $C^* X$. Therefore it is likely that theorem (2.10) and the explicit formula for the homotopy E has implication for the theory of homotopy Gerstenhaber algebras and B_∞ -algebras respectively.

§ 3 Cubical chains

For the proof of theorem (2.10) we need the cubical chain algebra of a topological monoid and the geometric cobar construction introduced in [B]. We begin by introducing classical formalism concerning cubes and cubical chains; see [HW, AH, A].

Let $CU_*(Y)$ denote the cubical singular chain complex of a topological space Y with coefficients in R. To construct $CU_*(Y)$, take the R-module generated freely by all the singular n-cubes (i.e. continuous maps $\sigma : I^n \to Y$, where I^n is the unit cube), and divide out by the R-submodule generated by degenerate cubes. (Here σ is degenerate if it factors through any of the n standard projections $I^n \to I^{n-1}$.) As is well known CU_* is naturally chain equivalent with the functor CS_* of simplicial singular chains and $H_n(Y; R) = H_n CU_*(Y)$ is the singular homology of Y. For a CW-complex Y let $C_*(Y)$ be the cellular chain complex with $C_n(Y) = H_n(Y^n, Y^{n-1}, R)$ where Y^n is the n-skeleton of Y. The set of n-cells of Y is a basis in the free module $C_n(Y)$.

The unit interval $I = I^1$ is a CW-complex with two 0-cells $e' = \{0\}, e'' = \{1\}$ and one 1-cell e. Let $\{e, e', e''\}^*$ be the free graded monoid generated by e, e', e'' with degree $|e| = 1, |e'| = |e''| = 0$ and let

(3.1) $$I(\infty) = R\langle e, e', e''\rangle = span_R\{e, e', e''\}^*$$

be the corresponding free graded algebra. Moreover $I(\infty)$ is a free differential algebra with differential

(1) $$d(e) = e'' - e'$$

A second grading of $I(\infty)$ is given by word length l with $l(e) = l(e') = l(e'') = 1$. We denote a word of length n in $I(\infty)$ by

(2) $$(a, b)_n = e_1 \dots e_n$$

where $a, b \subset \{1 \dots n\}$ are disjoint subsets with $e_i = e$ for $i \in b, e_i = e''$ for $i \in a$ and $e_i = e'$ for $i \in \{1 \dots n\} - a \cup b$. Let $I(n)$ be the span of all words of length n in $I(\infty)$. Then $I(n)$ is a differential submodule of $I(\infty)$ and we get maps of differential modules

(3) $$I(n) \cong C_*(I^n) \subset CU_*(I^n)$$

Here $C_*(I^n)$ is the cellular chain complex of the cube I^n and the isomorphism carries the generator (2) to the corresponding cell $e_1 \times \dots \times e_n$ of dimension $\sharp b$. The inclusion in (3) carries this cell to the face $I^{\sharp b} \subset I^n$ which is the closure of $e_1 \times \dots \times e_n$. The differential $d(\sigma)$ of the singular cube $(\sigma : I^n \to Y) \in CU_*(Y)$ is

(4) $$d(\sigma) = CU_*(\sigma)(de^n)$$

where $e^n = (\phi, \{1 \dots n\})_n$ is the n-dimensional cell in I^n. There is a unique map in **DA**

$$(5) \qquad \psi_I : I(\infty) \to I(\infty) \otimes I(\infty)$$

given by $\psi_I(e') = e' \otimes e'$, $\psi_I(e'') = e'' \otimes e''$ and $\psi_I(e) = e' \otimes e + e \otimes e''$. The map ψ_I restricted to $I(n)$ is induced by a cellular map $I^n \to I^n \times I^n$ which is homotopic to the diagonal. In fact ψ_I is coassociative and we can define an explicit derivation homotopy $E_I : \mathcal{T}\psi_I \simeq \psi_I$ by setting

$$(6) \qquad E_I(e') = E_I(e'') = 0 \quad \text{and} \quad E_I(e) = e \otimes e.$$

One readily checks that $dE_I(e) + E_I d(e) = \mathcal{T}\psi_I(e) - \psi_I(e)$.

(3.2) Definition. Let $CU'_*(Y)$ be the first Eilenberg subcomplex; i.e. the subcomplex of $CU_*(Y)$ generated by all singular cubes which send the 0-skeleton of I^n to the base point $*$ of Y. Assume that Y is a topological monoid with multiplication $\mu : Y \times Y \to Y$ and unit $* \in Y$. We obtain an algebra structure of $CU'_*(Y)$ by

$$(1) \qquad \sigma \cdot \sigma' = \mu(\sigma \times \sigma') : I^{n+m} = I^n \times I^m \to Y \times Y \to Y$$

for the singular cubes σ, σ' in $CU'_*(Y)$. We define a diagonal ψ on $CU'_*(Y)$ by

$$(2) \qquad \psi(\sigma) = (CU'_*(\sigma) \otimes CU'_*(\sigma))(\psi_I e^n)$$

and we define a derivation homotopy $E : T\psi \simeq \psi$ by

$$(3) \qquad E(\sigma) = (CU'_*(\sigma) \otimes CH'_*(\sigma))(E_I e^n)$$

Here we use ψ_I and E_I defined in (3.1) (5), (6).

Let $\mathbf{Mon}_0$ be the category of path connected topological monoids.

(3.3) Proposition. *For $Y \in \mathbf{Mon}_0$ the singular cubical chain complex $CU'_*(Y)$ is a well defined homotopy Hopf algebra with diagonal ψ defined in (3.2). This diagonal is associative and the augmentation $\varepsilon : CU'_*(Y) \to R$ is a counit of ψ. Moreover E in (3.2) is a derivation homotopy $E : \mathcal{T}\psi \simeq \psi$. Both ψ and E are natural in $Y \in \mathbf{Mon}_0$ so that we get the commutative diagram of functors*

$$\begin{tikzcd} \mathbf{Mon}_0 \arrow[rr, "CU'_*"] \arrow[dr, "(CU'_*)_{\psi,E}"'] & & \mathbf{DA}_0 \\ & \mathbf{hHA}_0 \arrow[ur, "\pi"'] & \end{tikzcd}$$

with $(CU'_*)_{\psi,E}(X) = (CU'_*(X), \psi, G_1 = E, G_2 = 0)$; see (1.8).

This result is essentially well known; compare for example [HW] and section 7 [A]. In [B] we define the geometric cobar construction which is a functor

$$(3.4) \qquad \underline{\Omega} : \mathbf{sSet}_1 \to \mathbf{Mon}_0$$

where $\underline{\Omega}X$ is a CW-complex with $\underline{\Omega}X \simeq \Omega|X|$. The cells of $\underline{\Omega}X$ have canonical characteristic maps $I^n \to \underline{\Omega}X$ defined on a cube. This defines the inclusion

$$C_*(\underline{\Omega}X) \subset CU'_*(\underline{\Omega}X)$$

where $C_*(\underline{\Omega}X)$ is the cellular chain complex of $\underline{\Omega}X$. Moreover we have by [B] the natural isomorphism

$$\Omega(C_*X) \cong C_*(\underline{\Omega}X)$$

of differential algebras. Now (3.3) and (2.10) fit together by the following result.

(3.5) Theorem. *Let $X \in \mathbf{sSet}_1$. Then the inclusion of differential algebras*

$$I : \Omega(C_*X) = C_*(\underline{\Omega}X) \subset CU'_*(\underline{\Omega}X)$$

is compatible with the diagonals ψ and derivation homotopies E defined in (2.9) and (3.2) respectively. Hence one has an inclusion

$$i : (\Omega C_*)_{\psi,E}(X) \subset (CU'_*)_{\psi,E}(\underline{\Omega}X)$$

in the category $\mathbf{hHA}_0$ which induces an isomorphism in homology and which is natural in X. Here $(\Omega C_*)_{\psi,E}$ and $(CU'_*)_{\psi,E}$ are the functors in (2.10) and (3.3).

Proof of (3.5) and (2.10). Let $n \geq 2$. For each non-degenerate element $\sigma \in X_n$ we have a cell e_σ in $\underline{\Omega}X$ together with a characteristic map, $m = n - 1$,

$$(1) \qquad \bar\sigma : I^m \to \underline{\Omega}X$$

of e_σ. Moreover the following diagram of differential modules commutes:

$$(2) \qquad \begin{array}{ccc} C_*I^m & \xrightarrow{C_*(\bar\sigma)} & C_*\underline{\Omega}X \\ \| & & \| \\ I(m) & \xrightarrow{\sigma_\sharp} & \Omega(C_*X) \end{array}$$

Here $\sigma_\sharp$ carries the generator $(a,b) = (a,b)_m \in I(m)$ to the element (see (2.9))

$$(3) \qquad \sigma_\sharp(a,b) = (\bar c|_d)_\sigma$$

where $c = \{0\dots n\} - a \cup b$ and $d = \{0\dots n\} - a$. Recall that $\bar C$ is the word of intervals defined by c as in (2.8) (3) and that $\bar c|_d$ is defined by (2.8) (4). The commutativity of (2) is a consequence of III.3.5 in [B]. One can check that $\sigma_\sharp$ is compatible with differentials in (3.1) and (2.5); that is $d\sigma_\sharp = \sigma_\sharp d$. For the proof of (3.5) it now suffices to show that the following equations hold:

$$\psi \sigma_\sharp = (\sigma_\sharp \otimes \sigma_\sharp)\psi_I \tag{4}$$

$$E\sigma_\sharp = (\sigma_\sharp \otimes \sigma_\sharp)E_I \tag{5}$$

Moreover it is sufficient to show these equations only on $e^m \in I(m)$. For this we have

$$\begin{aligned} \psi_I(e^m) &= (e' \otimes e + e \otimes e'')^m \\ &= \sum \tau_{a,b}(\phi, a) \otimes (a, b) \end{aligned} \tag{6}$$

Here the sum is taken over all partitions (a, b) of $\{1 \dots m\}$ and $\tau_{a,b}$ is the sign of the permutation $(a_1 < \dots < a_r, b_1 < \dots < b_s)$ with $r = \sharp a, s = \sharp b$. One can check that this sign coincides with

$$\tau_{a,b} = \tau(\bar{f}), \quad \text{see (2.8) (7),} \tag{7}$$

with $f = b \cup \{0, n\} \in N$ and $\bar{f} = \{0 \dots b_1\}\{b_1 \dots b_2\} \dots \{b_s \dots n\} \in N^*$. Moreover using the definition of $\sigma_\sharp$ in (3) we get

$$\sigma_\sharp(\phi, a) = (\bar{f})_\sigma \tag{8}$$

$$\sigma_\sharp(a, b) = (f)_\sigma \tag{9}$$

Thus we derive from (6) ... (9)

$$(\sigma_\sharp \otimes \sigma_\sharp)\psi_I(e^m) = \sum_{f \leq \{0 \dots n\}} \tau(\bar{f})(\bar{f})_\sigma \otimes (f)_\sigma \tag{10}$$

Hence by (10) and (2.9) (3) one gets

$$\psi(s^{-1}\sigma) = (\sigma_\sharp \otimes \sigma_\sharp)\psi_I(e^m) \tag{11}$$

where $s^{-1}\sigma = \sigma_\sharp(e^m)$. This completes the proof of (4). We proceed similarly for the proof of (5). We have by (1.7) and (3.1) (6)

$$E_I(e^m) = \sum_{i=1}^{m} (-1)^{i-1} T\psi_I(e^{i-1})E_I(e)\psi_I(e^{m-i}) \tag{12}$$

$$= \sum_{i=1}^{m}(-1)^{i-1}(\sum_{A}\bar{\tau}_{a,b}(a,b)\otimes(\phi,a))\cdot(e\otimes e)\cdot(\sum_{B}\tau_{c,d}(\phi,c)\otimes(c,d)) \tag{13}$$

Here A is the set of all partitions (a, b) of $\{1 \dots (i-1)\}$ and B is the set of all partitions (c, d) of $\{(i+1)\dots m\}$. Moreover

$$\bar{\tau}_{a,b} = (-1)^{\sharp a\cdot\sharp b}\tau_{a,b} \tag{14}$$

Using the multiplication law in (1.1) we see that the summands in (13) have the form:

$$\begin{aligned}
&((a,b)\otimes(\phi,a))\cdot(e\otimes e)\cdot((\phi,c)\otimes(c,d))\\
&\quad = (-1)^{\sharp a}((a,b\cup\{i\})\otimes(\phi,a\cup\{i\}))\cdot((\phi,c)\otimes(c,d))\\
&\quad = (-1)^{\sharp a+(1+\sharp a)\sharp c}(a,b\cup\{i\}\cup c)\otimes(c,a\cup\{i\}\cup d)
\end{aligned} \tag{15}$$

Hence by (12) ... (15) we get

$$E_I(e^m) = \sum_{i=1}^{m}\sum_{A}\sum_{B}\tau\cdot(a,b\cup\{i\}\cup c)\otimes(c,a\cup\{i\}\cup d) \tag{16}$$

where τ is the sign

$$\begin{aligned}
&\tau = (-1)^t\tau_{a,b}\tau_{c,d} \quad \text{with}\\
&t = i-1+\sharp a+\sharp c+\sharp a\sharp c+\sharp a\sharp b
\end{aligned} \tag{17}$$

We have by (3)

$$\begin{aligned}
&\sigma_\sharp(a,b\cup\{i\}\cup c) = (f\cup\bar{g})_\sigma\\
&\sigma_\sharp(c,a\cup\{i\}\cup d) = (\bar{f}\cup g)_\sigma
\end{aligned} \tag{18}$$

with $g = d\cup\{i,n\}$ and $f = b\cup\{0,1\}$. As in (7) one gets $\tau(\bar{f}) = \tau_{a,b}$ and $\tau(\bar{g}) = \tau_{c,d}$. Since $\sharp a+\sharp b = i-1, \sharp c+\sharp d = m-i$ and $|f| = \sharp b, |g| = \sharp d$ we see that τ in (17) coincides with τ in (2.9) (5). This completes the proof of (5) above since we get by (16), (18) and (2.9) (4) the following equation

$$E(s^{-1}\sigma) = E(\sigma_\sharp e^m) = (\sigma_\sharp\otimes\sigma_\sharp)E_I(e^m) \tag{19}$$

q.e.d.

§ 4 The cobar construction as a cocommutative differential Hopf algebra

In this section we assume that the ring $\mathbb{Q}$ of rational numbers is a subring of the coefficient ring; that is $\mathbb{Q} \subseteq R$. Let V be a positive graded free module with $V_0 = 0$. The tensor algebra $T(V)$ is a graded Lie algebra with the Lie bracket

$$[x, y] = x \cdot y - (-1)^{|x||y|} y \cdot x \tag{4.1}$$

for $x, y \in T(V)$. Let $L(V)$ be the Lie subalgebra of $T(V)$ generated by $V \subset T(V)$. The inclusion $L(V) \subset T(V)$ induces the isomorphism $UL(V) = T(V)$ where $UL(V)$ is the universal enveloping algebra of $L(V)$. Let S_k denote the symmetric group on k letters. Let $\sigma \in S_k$ and let $\underline{x} = (x_1, \dots, x_k)$ be a list of (homogeneous) elements in $L(V)$. Define the graded sign $[\sigma : \underline{x}]$ of σ relative to $\underline{x}$ to be the sign (i.e., $+1$ or -1) that σ has when viewed as permuting k-blocks of sizes $|x_1|, \dots, |x_k|$. Put

$$C(\underline{x}) = C(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} [\sigma : \underline{x}] x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(k)} \tag{4.2}$$

which defines an element in $T(V)$. Let $U_0(V) = R$ and $U_1(V) = L(V)$ and

$$U_n(V) = span_R\{C(x_1, \dots, x_n) | x_i \in L(V)\} \subseteq T(V) \tag{1}$$

Then the Poincaré-Birkhoff-Witt theorem implies that the inclusions (1) induce an isomorphism

$$\bigoplus_{n \geq 0} U_n(V) = T(V) \tag{2}$$

which is natural in V; compare 0.4.3 [R]. The isomorphism yields the canonical projections

$$\pi_n = \pi_n^V : T(V) \to U_n(V) \tag{3}$$

which are natural in V. Clearly $\pi_0 = \varepsilon$ is the augmentation of $T(V)$. Moreover we have the map

$$C : U_1(V) \otimes U_n(V) \to U_{n+1}(V) \tag{4}$$

which carries $x \otimes C(x_1, \dots, x_n)$ to $C(x, x_1, \dots, x_n)$ with $x, x_1, \dots, x_n \in L(V)$. Again C is natural in V. Let

$$\mu_j : T(V) \otimes T(V) \xrightarrow{\pi_1 \otimes \pi_j} U_1(V) \otimes U_j(V) \xrightarrow{C} U_{j+1}(V) \tag{5}$$

be the composite given by (3) and (4). The formula at the bottom of page 60 [R] yields an explicit description for π_n in (3) and μ_j in (5).

(4.3) Definition. Let $I(V)$ be the augmentation ideal in the tensor algebra $T(V)$; that is, $I(V) = \oplus\{V^{\otimes n}, n \geq 1\}$. Anick defines on page 429 of [A] the following maps between graded modules which are natural in V.

$$I(V)^{\otimes 2} \xrightarrow{\phi} I(V)^{\otimes 2} \oplus I(V)^{\otimes 3} \xrightarrow{\lambda} I(V)^{\otimes 2} \xrightarrow{\mu} I(V)$$

We also write $\phi = \phi^V, \lambda = \lambda^V$ and $\mu = \mu^V$. For $a, b, c \in I(V)$ put

$$\mu(a \otimes b) = \sum_{j \geq 1} \frac{1}{1+j} \mu_j(a \otimes b) \tag{1}$$

$$\lambda(a \otimes b) = \sum_{i,j \geq 1} \frac{-i}{i+j} \mathcal{T}(\pi_i(a) \otimes \pi_j(b)) \tag{2}$$

$$\lambda(a \otimes b \otimes c) = \sum_{j,k \geq 1} \frac{1}{1+j+k} (1+\mathcal{T})(\mu_j(a \otimes b) \otimes \pi_k(c)) \tag{3}$$

Here $\mathcal{T}$ is the interchange map and $1 + \mathcal{T}$ is the sum of the identity and the interchange map. Moreover the coordinates of $\phi = (\phi_1, \phi_2)$ are

$$\begin{cases} \phi_1 = 1 - \mathcal{T} \\ \phi_2 = \bar{\Delta} \otimes 1 - 1 \otimes \bar{\Delta} \end{cases} \tag{4}$$

Here $\bar{\Delta}$ with $\bar{\Delta}(a) = \Delta(a) - 1 \otimes a - a \otimes 1$ for $a \in I(V) \subset T(V)$ is the reduced diagonal given by the algebra homomorphism

$$\Delta : T(V) \to T(V) \otimes T(V) \tag{5}$$

with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in V$.

(4.4) Definition. Let (A, ψ, G_1, G_2) be a homotopy Hopf algebra as in (1.8) for which A is a free differential algebra generated by a submodule $V \subset A$ with $V_0 = 0$. We use the inclusion $V \subset A$ to obtain the identification $T(V) = A$ and $I(V) = A_+$ where A_+ is the augmentation ideal. Let $n \geq 1$. We call

$$(A, \psi, G_1, G_2, V)$$

n-good if the following conditions (1), (2) and (3) hold. Let $V_{<n}$ and $V_{\leq n}$ be the submodules of V consisting of elements $x \in V$ with $|x| < n$ and $|x| \leq n$ respectively.

$$\psi = \mathcal{T}\psi \quad \text{and} \quad (\psi \otimes 1)\psi = (1 \otimes \psi)\psi \quad \text{on} \quad V_{\leq n} \tag{1}$$

$$d(V_{\leq n+1}) \subset L(V_{\leq n}) \subset T(V) = A \tag{2}$$

$$V_{\leq n} \subset \ker(\bar{\psi}) \tag{3}$$

Here $\bar{\psi} : A_+ \to A_+ \otimes A_+$ is the map defined by $\bar{\psi}(a) = \psi(a) - 1 \otimes a - a \otimes 1$ for $a \in A_+$.

(4.5) Anick theorem. *Let $n \geq 1$ and let $X = (A, \psi, G_1, G_2, V)$ be n-good. Then $\alpha(X) = (A, \chi, J_1, J_2, W)$ defined by the following conditions (1) ... (8) is $(n+1)$ -good. Moreover there is a derivation homotopy $F : \psi \simeq \chi$ with $F(x) = 0$ for $|x| < n, x \in A$.*

Let $q = 1 - \varepsilon : A \to A_+$. We also allow q to denote any of the homomorphisms $q^{\otimes 2} : A^{\otimes 2} \to A_+^{\otimes 2}, q^{\otimes 3} : A^{\otimes 3} \to A_+^{\otimes 3}$ or $q^{\otimes 2} \oplus q^{\otimes 3}$. Let $v \in V$. We define χ together with F by the following properties (1) ... (5). Via the identification $I(V) = A_+$ we can use the maps $\mu = \mu^V, \lambda = \lambda^V$ and $\phi = \phi^V$ in (4.3).

(1)
$$\chi(v) = \psi(v) \quad \text{for} \quad |v| \leq n$$

(2)
$$\chi(v) = \psi(v) - \lambda\phi\bar{\psi}(v) \quad \text{for} \quad |v| = n+1$$

(3)
$$F(v) = \lambda\phi\lambda q(G_1(v), G_2(v)) \quad \text{for} \quad |v| = n, n+1$$

(4)
$$F(v) = 0 \quad \text{for} \quad |v| < n \quad \text{and} \quad |v| > n+1$$

(5)
$$\chi(v) = \psi(v) - F(dv) \quad \text{for} \quad |v| > n+1$$

Here (4) and (5) are used to obtain F and χ inductively for $|v| > n+1$. Moreover we define J_1, J_2 and W by:

(6)
$$J_1 = -F + G_1 + \mathcal{T}F$$

$$J_2 = -(\chi \otimes 1 - 1 \otimes \chi)F - (F \otimes 1 - 1 \otimes F)\psi + G_2$$

(8)
$$W_i = \begin{cases} (1 - \mu\bar{\chi})V_i & \text{for} \quad i = n+1, n+2 \\ V_i & \text{otherwise} \end{cases}$$

The Anick-theorem above is essentially a reformulation of proposition 5.5 in [A] which we combine with corollary 3.12 in [A].

The construction of $\alpha(X)$ in (4.5) is natural in the following sense.

(4.6) Lemma. *Let $X = (A, \psi, G_1, G_2, V)$ and $X' = (A', \psi', G_1', G_2', V')$ be n-good and let $f : X \to X'$ be a map in* **hHA** *with $f(V) \subset V'$. Then $f : A \to A'$ is also a map $f : \alpha(X) \to \alpha(X')$ in* $\mathbf{hHA}_0$ *with $f(W) \subset W'$. Moreover $(f \otimes f)F = F'f$ for the homotopies $F : \psi \simeq \chi$ and $F' : \psi' \simeq \chi'$.*

The lemma follows from the naturality properties of μ, λ, ϕ in (4.3) by inspection of (4.5) (1) ... (8).

As an application of the Anick-Theorem (4.5) and theorem (2.10) we get the following cocommutative diagonal for the cobar construction.

(4.7) Theorem. *Assume $\mathbb{Q} \subseteq R$ and let X be a 1-reduced simplicial set. Then there is a diagonal ϕ on the cobar construction $\Omega C_* X$ such that $(\Omega C_* X, \phi)$ is a cocommutative differential Hopf algebra. Moreover there is a derivation homotopy $F : \psi \simeq \phi$ where ψ is the diagonal in (2.10). Both ϕ and F are natural for $X \in \mathbf{sSet}_1$ so that one gets a commutative diagram of functors*

$$\begin{tikzcd} \mathbf{sSet}_1 \arrow[rr, "\Omega C_*"] \arrow[dr, "(\Omega C_*)_\phi"'] & & \mathbf{DA}_0 \\ & \mathbf{CDA}_0 \arrow[ur, "\pi"'] & \end{tikzcd}$$

where $(\Omega C_*)_\phi(X) = (\Omega C_* X, \phi)$; see (1.9).

Proof of (4.7). One readily checks that

$$\Omega^{(1)} = (\Omega C_* X, \psi, G_1 = E, G_2 = 0, V = s^{-1}\tilde{C}_* X)$$

is 1-good in the sense of (4.4). Hence by applying the Anick theorem inductively we obtain objects $\Omega^{(n)}$ with $\Omega^{(n)} = \alpha(\Omega^{(n-1)})$ for $n \geq 2$ such that

$$\Omega^{(n)} = (\Omega C_* X, \psi^{(n)}, G_1^{(n)}, G_2^{(n)}, V^{(n)})$$

is n-good and such that $F^{(n)} : \psi^{(n)} \simeq \psi^{(n+1)}$ is a derivation homotopy. Since $\psi^{(n)}(x) = \psi^{(n+1)}(x)$ for $|x| \leq n, x \in \Omega C_* X$, and since $F^{(n)}(x) = 0$ for $|x| < n$ we see that ϕ, F in (4.7) are well defined by

$$\begin{aligned} \phi(x) &= \psi^{(n)}(x) \quad \text{for} \quad |x| = n \\ F(x) &= \sum_{n \geq 1} F^{(n)}(x) \end{aligned}$$

One readily checks that ϕ is cocommutative and coassociative; compare (4.4) (1). The naturality of ϕ and F follows from (4.6). q.e.d.

We start in the induction of theorem (4.7) with explicit data $\Omega^{(1)}$ given by (2.9). Therefore the diagonal ϕ in (4.7) is computable by applying inductively the Anick-construction (4.5). This yields a formula for ϕ in terms of generators of the cobar construction. Moreover we obtain by the construction in the proof of (4.7) the following result (4.8).

Let $\mathbf{DL}_0$ be the category of differential Lie algebras $L = (L, d)$ over $\mathbb{Q}$ with $H_0 L = 0$; see [Q]. For a graded rational vector space V let $L(V)$ be the free Lie algebra generated by V. The tensor algebra $T(V)$ is a Lie algebra with the bracket $[x, y] = xy - (-1)^{|x||y|}yx$. Then $L(V)$ is the sub Lie algebra of $T(V)$ generated by V and $T(V) = UL(V)$ is the universal enveloping algebra of $L(V)$. Clearly a degree 0-map $\varphi : V \to V'$ induces a Lie algebra map $L(\varphi) : L(V) \to L(V')$.

(4.8) Theorem. *Let X be a 1-reduced simplicial set. Then there is a differential $d = d_{\mathcal{L}}$ on the free Lie algebra $L(s^{-1}\tilde{C}_* X)$ such $\mathcal{L}(X) = (L(s^{-1}\tilde{C}_* X), d_{\mathcal{L}})$ defines a functor*

$$\mathcal{L} : \mathbf{sSet}_1 \to \mathbf{DL}_0$$

Here $\mathcal{L}$ carries a map $f : X \to Y$ to the induced map $\mathcal{L}(f) = L(s^{-1}\tilde{C}_* f)$ which is a differential map. Moreover there is an isomorphism of cocommutative differential Hopf algebras

$$\beta : (U\mathcal{L}(X), \Delta) \cong (\Omega C_* X, \phi)$$

Here $U\mathcal{L}(X) = (UL(s^{-1}\tilde{C}_* X), d)$ is the universal enveloping for which the diagonal Δ is primitive on $L(s^{-1}\tilde{C}_* X)$ and ϕ is the diagonal on the cobar construction in (4.7). The isomorphism β is natural in $X \in \mathbf{sSet}_1$ and β induces the identity on $s^{-1}\tilde{C}_* X$ if one divides out all decomposable elements.

Proof of (4.8). Using $\Omega^{(n)}$ in the proof of (4.7) we define

(1) $$V(X) \subset \Omega C_* X$$

by $V(X)_n = (V^{(n)})_n$. Then we see that $X \mapsto V(X)$ is a functor from $\mathbf{sSet}_1$ to the category of graded modules such that (1) is natural in X. Moreover by definition of ϕ we have the isomorphism of Hopf algebras

(2) $$(UL(V(X)), \Delta) = (T(V(X)), \Delta) = (\Omega C_* X, \phi)$$

which is induced by the inclusion (1). We now obtain a natural isomorphism

(3) $$\bar{\beta} : V(X) \cong s^{-1}\tilde{C}_* X$$

by dividing out in (2) all decomposable elements. Hence (2) and (3) yield the isomorphism β in (4.8). We define the differential $d_{\mathcal{L}}$ by $d_{\mathcal{L}} = \beta^{-1} d_\Omega \beta$. q.e.d.

The functor $\mathcal{L}$ in (4.8) induces an equivalence of rational homotopy categories

(4.9) $$Ho(\mathcal{L}) : Ho_{\mathbb{Q}}\mathbf{sSet}_1 \xrightarrow{\sim} Ho_{\mathbb{Q}}\mathbf{DL}_0$$

Here $Ho_{\mathbb{Q}}$ denotes the localization with respect to rational homology isomorphisms. In fact, by (4.11) below and the main result of Majewski [M] we know that $Ho(\mathcal{L})$ is naturally isomorphic to Quillen's functor $N\mathcal{P}\hat{Q}G$ which is the composite of four complicated functors; see [Q]. Hence theorem (4.8) replaces Quillen's functor by the functor $\mathcal{L}$ which is given by a differential $d_{\mathcal{L}}$ on $L(s^{-1}\tilde{C}_* X)$. A formula for $d_{\mathcal{L}}$ is obtained by an inductive process; see § 5 below.

As pointed out by Majewski [M] the complete simplicial Lie algebra

(1) $$\mathcal{L}'(X) = P\hat{Q}G(X), X \in \underline{\underline{sSet}}_1,$$

(given by Quillen's functor) can be described as follows. Let $\hat{L}V$ be the free complete Lie algebra over R generated by the free (filtered) graded module V. Then $\mathcal{L}'(X)$ is the simplicial free complete Lie algebra given by

(2)
$$\begin{aligned}(\mathcal{L}'X)_{n-1} &= \hat{L}(X_n - s_0X_{n-1})\\ &= \hat{L}(X_n)//\hat{L}(s_0X_{n-1})\end{aligned}$$

with $d_j, j \neq 0$ and s_j defined as in X and with $d_0(x)$ given by the Hausdorff formula

(3)
$$d_0(x) = \log(\exp(d_0x) \cdot \exp(-d_1x))$$

Quillen' s functor $NP\hat{Q}G$ is now obtained by the normalization (Moore chain complex) $N\mathcal{L}'(X)$ of $\mathcal{L}'(X)$. Using results of [M] we obtain a map

(4)
$$\mathcal{L}(X) \xrightarrow{\sim} N\mathcal{L}'(X) = NP\hat{Q}G(X)$$

in $\underline{\underline{DL}}_0$ which is a homology isomorphism and natural in the homotopy category $Ho(\underline{\underline{DL}}_0)$. Here the differential Lie algebra $\mathcal{L}(X)$ is considerably smaller than the differential Lie algebra $N\mathcal{L}'(X)$. For example we have in degree 1 the equation $\mathcal{L}(X)_1 = s^{-1}C_2X$ while $(N\mathcal{L}'(X))_1 = \hat{L}(s^{-1}C_2X)$. Since $\mathcal{L}'$ is defined by the Hausdorff formula (3) we consider the differential $d_{\mathcal{L}}$ of $\mathcal{L}(X)$ via the homology isomorphism (4) as a "simplicial form" of the classical <u>Hausdorff formula</u>:

(5)
$$\log(\exp(a) \cdot \exp(b)) = a + b + \frac{1}{2}[a, b] + \dots$$

Here ... denotes an intricate sum of Lie monomials of length ≥ 3; see [R]. For the differential $d_{\mathcal{L}}$ in (4.8) we have the canonical decomposition

(4.10)
$$d_{\mathcal{L}} = d_{\mathcal{L}}^{(1)} + d_{\mathcal{L}}^{(2)} + \dots + d_{\mathcal{L}}^{(t)} + \dots$$

where $d_{\mathcal{L}}^{(t)}(s^{-1}\sigma), \sigma \in X_n$, is a sum of Lie monomials of length t; that is

$$d_{\mathcal{L}}^{(t)}(s^{-1}\sigma) \in L(s^{-1}\tilde{C}_*X) \cap (s^{-1}\tilde{C}_*X)^{\otimes t}$$

For degree reasons we have $d_{\mathcal{L}}^{(t)}(s^{-1}\sigma) = 0$ for $t \geq n-1$. Now let R be a subring of $\mathbb{Q}$ and let $x \in span_{\mathbb{Q}}(E)$ be an element in the $\mathbb{Q}$ -vector space with fixed basis E. Then we say that x <u>is defined over</u> R if $x \in span_R(E) \subset span_{\mathbb{Q}}(E)$. Following Anik [A] let $\rho(R)$ be the least non invertible prime in the ring R.

(4.11) Theorem. *Let X be a 1-reduced simplicial set. Then the differential $d_{\mathcal{L}}$ of $\mathcal{L}(X)$ in (4.8) satisfies for $\sigma \in X_n$ the following formulas*

$$d_{\mathcal{L}}^{(1)}(s^{-1}\sigma) = -\sum_{i=0}^{n}(-1)^i s^{-1}(d_i\sigma); \quad \textit{see (2.5)}$$
$$d_{\mathcal{L}}^{(2)}(s^{-1}\sigma) = \frac{1}{2}\sum_{i=2}^{n-2}(-1)^i[s^{-1}\sigma(0\dots i), s^{-1}\sigma(i\dots n)]$$

Moreover $d_{\mathcal{L}}^{(t)}(s^{-1}\sigma)$ for $t \geq 1$ is defined over any subring R with $t < \rho(R), R \subset \mathbb{Q}$.

We prove theorem (4.11) in (5.15) below.

(4.12) Corollary. *Let X be an r-reduced simplicial set (i.e. $X_i = *$ for $i \leq r$) and let $\dim|X| \leq n$. Then $\mathcal{L}(X)$ is defined over $R \subset \mathbb{Q}$ with $n-2 < r \cdot \rho(R)$.*

The corollary says that the top attaching map $S^{n-1} \to X^{n-1}$ of X is "r-mild"; compare theorem 4.8 [A]. The corollary is the basis of Halperin's result [Ha] on $H_*(\Omega X, R)$ who however assumes $n-1 < r \cdot \rho(R)$.

Theorem (4.8) is also related to the following result of Anick; see 7.5 in [A].

(4.13) Theorem. *(Anick's cubical Hopf algebra): Let $\mathbb{Q} \subseteq R$ and let Y be a path connected topological monoid. Then the cubical chain complex $CU'_*(Y)$ is a differential algebra which admits a diagonal ϕ such that $(CU'_*(Y), \phi)$ is a cocommutative differential Hopf algebra. Moreover there is a derivation homotopy $F : \psi \simeq \phi$ where ψ is the diagonal in (3.2). Both ϕ and F are natural for $Y \in \mathbf{Mon}_0$ so that one has a commutative diagram of functors*

$$\begin{tikzcd}
\mathbf{Mon}_0 \arrow[rr, "CU'_*"] \arrow[dr, "(CU'_*)_\phi"'] & & \mathbf{DA}_0 \\
& \mathbf{CDA}_0 \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where $(CU'_*)_\phi(Y) = (CU'_*(Y), \phi)$.

The functor $\mathcal{E}$ of Anick is given by the primitive elements of $(CU'_*(Y), \phi)$; see (7.5) [A]. In theorem (3.5) we obtained the embedding of the cobar construction $\Omega C_* X$ into $CU'_*(\underline{\Omega} X)$ where $\underline{\Omega} X$ is the geometric cobar construction. This embedding was compatible with the structure as a homotopy Hopf algebra. We now show that it is as well an embedding of cocommutative differential Hopf algebras given by the cocommutative diagonal ϕ in (4.13) and (4.7) respectively.

(4.14) Theorem. *Let $\mathbb{Q} \subseteq R$ and let $X \in \mathbf{sSet}_1$. Then the inclusion of differential algebras in (3.5)*

$$i : \Omega(C_* X) = C_*(\underline{\Omega} X) \subset CU'_*(\underline{\Omega} X)$$

is compatible with the diagonal ϕ on both sides; that is, i is an inclusion of differential Hopf algebras in $\mathbf{CDA}_0$. Moreover the homotopy F in (4.7) is via the inclusion i the restriction of the homotopy F in (4.13).

Theorem (4.14) in connection with Majewski's result (4.9) (4) solves a problem of Anick; see the remark following 7.7 in [A]. In fact by (4.14) and (4.9) (4) we see immediately that Anick's functor defined by the primitives of the Hopf algebra $(CU'_*(\underline{\Omega} X), \phi)$ coincides up to natural equivalence in $Ho(\underline{\underline{DL}}_0)$ with Quillen's functor $NP\hat{Q}G$.

Proof of (4.14). The algebra $I(\infty)$ in (3.1) is termed S in [A]. Let S^1 be the submodule of elements $x \in I(\infty)$ with $|x| \geq 1$ and let TS^1 be the tensoralgebra generated by S^1. Anick describes in 7.2 [A] a homotopy Hopf algebra $(TS^1, D, \psi, G_1 = E, G_2 = 0)$ and in 7.3 [A] he applies (4.5) to obtain a cocommutative Hopf algebra (TS^1, D, ϕ) together with a derivation homotopy $F : \psi \simeq \phi$. Now let $\Delta(\infty)$ be the simplicial set in $\mathbf{sSet}_1$ for which the realization $|\Delta(\infty)|$ is the quotient $\Delta^\infty/\Delta^{\infty,1}$. Here $\Delta^{\infty,1}$ is the 1-skeleton of the infinite dimensional simplex Δ^∞ which is the direct limit of $\Delta^1 \subset \Delta^2 \subset \Delta^3 \subset \ldots$ where Δ^n is the standard simplex. Hence Δ^n correspond to an element $\sigma_n \in \Delta(\infty)_n$ which is non degenerate for $n \geq 2$. This yields the map of differential modules

$$\sigma_\sharp : I(\infty) = \bigoplus_{n \geq 0} I(n) \to \Omega C_*\Delta(\infty)$$

which is $(\sigma_n)_\sharp$ on $I(n-1), n \geq 1$; compare (2) in the proof of (3.5). Moreover $\sigma_\sharp$ induces canonically a map in $\mathbf{hHA}_0$

$$\sigma_* : (TS^1, D, \psi, E) \to (\Omega C_*\Delta(\infty), \psi, E)$$

We now apply the Anick theorem on both sides to obtain the diagonal ϕ and the homotopy $F : \psi \simeq \phi$ which both commute with σ_*. This implies the proposition of (4.14) since $(\Omega C_*\Delta(\infty), \phi)$ is the universal example for the Hopf algebra $(\Omega C_* X, \phi)$ in (4.14) with $X \in \mathbf{sSet}_1$. On the other hand ϕ on TS^1 is used by Anick to define ϕ on $CU'_*(Y)$ for $Y \in \mathbf{Mon}_0$; see 7.4 [A]. q.e.d.

§ 5 On the computation of the Lie differential $d_{\mathcal{L}}$

In this section we describe an inductive construction of elements $\beta(2), \beta(3), \ldots$ which determine the differential $d_{\mathcal{L}}$ by the formula in (5.9) below. This leads to a proof of theorem (4.11); see (5.13).

The graded set N is defined as in (2.8) with $|a| = \sharp(a) - 2$ for $a \in N$. Let

(5.1) $$\Omega = R\langle N\rangle / (N_0 \sim 1)$$

be the free algebra $R\langle N\rangle$ divided by the ideal generated by $a - 1$ with $a \in N_0$. We clearly have

(1) $$\Omega = R\langle N_{\geq 1}\rangle = T(U)$$

where $U = span_R(N_{\geq 1})$. For $a \in N$ let $N[a] = \{b \in N, b \subset a\}$. If $|a| = n - 1$ we have a canonical order preserving bijection

(2) $$j_a : \underline{n} = \{0 \ldots n\} \approx a$$

This defines the bijection $j_a : N[\underline{n}] \approx N[a]$ which carries b to $j_a(b)$. Moreover let $\Omega[a]$ be the subalgebra of Ω generated by $N[a]$. Then j_a also induces the isomorphism of algebras

(3) $$j_a : \Omega[\underline{n}] \approx \Omega[a]$$

We use the following convention. Given an element $w(\underline{n}) \in \Omega[\underline{n}]$ we obtain for any $a \in N$ with $|a| = n - 1$ the corresponding element

(5.2) $$w(a) = j_a(w(\underline{n})) \in \Omega[a] \subset \Omega$$

Hence if elements $d(\underline{n}), G(\underline{n}), \Lambda(\underline{n}), \ldots$ are defined in $\Omega[\underline{n}]$ then also the elements $d(a), G(a), \Lambda(a), \ldots \in \Omega$ are defined. We therefore can define a differential d_Ω on Ω by putting

$$d_\Omega(\underline{n}) = -\sum_{i=0}^{n}(-1)^i\partial_i(\underline{n}) + \sum_{i=2}^{n-2}\{0\ldots i\}\cdot\{i\ldots n\}$$

We also write

$$\partial(\underline{n}) = \sum_{i=0}^{n}(-1)^i\partial_i(\underline{n})$$
$$\partial^+(\underline{n}) = \sum_{i=2}^{n-2}\{0\ldots i\}\cdot\{1\ldots n\}$$

so that $d_\Omega(\underline{n}) = -\partial(\underline{n}) + \partial^+(\underline{n})$. This yields by (5.2) the formula for $d_\Omega(a)$ for all $a \in N_{\geq 1}$ and one can check that (Ω, d_Ω) is a well defined differential algebra. In fact, let $\Delta(\infty)$ be the simplicial set for which the realization is $\Delta^\infty/\Delta^{\infty,1}$ as in the proof of (4.11). Then $\Omega = \Omega(C_*\Delta(\infty))$ is the cobar construction defined in (2.5). Therefore we have by (2.9) the diagonal

$$\psi_\Omega : \Omega \to \Omega \otimes \Omega$$

and the derivation homotopy $E : \mathcal{T}\psi_\Omega \simeq \psi_\Omega$ with

$$(1) \qquad \psi_\Omega(\underline{n}) = \sum_{f\leq\underline{n}} \tau(\bar{f})\bar{f}\otimes f$$

$$(2) \qquad E(\underline{n}) = \sum_{i=1}^{n-1}\sum_{\substack{f\leq\{0\ldots i\}\\ g\leq\{i\ldots n\}}} \tau\cdot(f\cup\bar{g})\otimes(\bar{f}\cup g)$$

For the definition of τ see (2.9) (5). Using the convention (5.2) also $\psi_\Omega(a)$ and $E(a)$ are defined for $a \in N$. We are going to define inductively elements, $n \geq 2$,

$$(5.4) \qquad \begin{cases} \beta(\underline{n}) \in \Omega[\underline{n}], & \beta(\underline{2}) = \underline{2} \\ H(\underline{n-1}) \in \Omega[\underline{n-1}]\otimes\Omega[\underline{n-1}], & H(\underline{1}) = 0 \\ \beta'(\underline{n+1}) \in \Omega[\underline{n+1}], & \beta'(\underline{3}) = \underline{3} \\ H'(\underline{n}) \in \Omega[\underline{n}]\otimes\Omega[\underline{n}], & H'(\underline{2}) = 0 \end{cases}$$

If $Q(n) = \{\beta(\underline{n}), H(\underline{n-1})\}$ and $Q'(n) = \{\beta'(\underline{n+1}), H'(\underline{n}\}$ then we shall see that $Q(2), Q(3), \ldots, Q(n)$ and $Q'(n)$ determine $Q(n+1)$ and $Q'(n+1)$. Given the elements (5.4) we have by the convention (5.2) the corresponding elements $\beta(a), H(b), \beta'(c), H'(a)$ with $a, b, c \in N$ and $|a| = n-1, |b| = n-2, |c| = n$. The elements $\beta(a)$ define an isomorphism

$$\beta : \Omega \xrightarrow{\cong} \Omega \tag{1}$$

which carries a to $\beta(a)$. We define the differential $d_{\mathcal{L}}$ on Ω by

$$d_{\mathcal{L}}(a) = \beta^{-1} d_{\Omega} \beta(a) \tag{2}$$

so that β is an isomorphism of differential algebras. The choice of elements in (5.4) has the property that $d_{\mathcal{L}}(a)$ is a Lie element, that is

$$d_{\mathcal{L}}(a) \in L(U) \subset T(U) = \Omega \tag{3}$$

Here $\Delta : \Omega \to \Omega \otimes \Omega$ is the diagonal which is primitive on $L(U)$, that is $\Delta(a) = 1 \otimes a + a \otimes 1$ for $a \in N_{\geq 1}$. Then the elements $H(a)$ given by (5.4) determine a derivation homotopy for the diagram

$$\begin{tikzcd} \Omega \otimes \Omega \arrow[r, "\beta\otimes\beta", "\cong"'] & \Omega \otimes \Omega \\ \Omega \arrow[u, "\Delta"] \arrow[r, "\beta", "\cong"'] \arrow[ur, phantom, "\nwarrow H"] & \Omega \arrow[u, "\psi_\Omega"'] \end{tikzcd} \tag{4}$$

That is, $H : \Omega \to \Omega \otimes \Omega$ is the derivation homotopy

$$H : \psi_{\Omega}\beta \simeq (\beta \otimes \beta)\Delta \tag{5}$$

which is defined on generators $a \in N$ by the elements $H(a)$.

Let $\Omega(n)$ be the subalgebra of Ω generated by elements $a \in N$ with $|a| \leq n$. Hence $\underline{n+1} \in \Omega(n)$. We use the data $Q(2), Q(3), \ldots, Q(n)$ and $Q'(n)$ in (5.4) to define the following diagram which we denote by $D(n)$,

$$D(n) = \left\{ \begin{tikzcd} \Omega(n-1) \otimes \Omega(n-1) \arrow[r, "\beta\otimes\beta", "\cong"'] & \Omega(n-1) \otimes \Omega(n-1) \\ \Omega(n-1) \arrow[u, "\Delta"] \arrow[r, "\beta", "\cong"'] \arrow[ur, phantom, "\nwarrow H'_n"] & \Omega(n-1) \arrow[u, "\psi_\Omega"'] \\ \Omega(n) \arrow[u, phantom, "\cap"] \arrow[r, "\cong", "\beta'_{n+1}"'] & \Omega(n) \arrow[u, phantom, "\cap"] \end{tikzcd} \right. \tag{6}$$

Here β is the restriction of β in (1) and β'_{n+1} is an extension of $\beta|\Omega(n-1)$ defined by

$$(7) \qquad \beta'_{n+1}(a) = \begin{cases} \beta(a) & \text{for} \quad |a| \le n-1, \\ \beta'(a) & \text{for} \quad |a| = n, a \in N \end{cases}$$

Moreover the derivation homotopy H'_n is an extension of $H|\Omega(n-2)$ defined by

$$(8) \qquad H'_n(a) = \begin{cases} H(a) & \text{for} \quad |a| \le n-2, \\ H'(a) & \text{for} \quad |a| = n-1, a \in N. \end{cases}$$

The isomorphism β'_{n+1} defines a differential $d'_{\mathcal{L}}$ for $\Omega(n)$ by $d'_{\mathcal{L}}(a) = d_{\mathcal{L}}(a)$ for $|a| \le n-1$ and

$$(9) \qquad d'_{\mathcal{L}}(\underline{n+1}) = \beta^{-1} d_{\Omega} \beta'(\underline{n+1})$$

Again this is a Lie element in $L(U) \cap \Omega(n-1)$. Given a diagram $D(n)$ in (6) with these properties we shall use the Anick theorem (4.5) to construct a diagram $D(n+1)$; see the proof of (5.11) below. Then $D(n+1)$ determines $Q(n+1)$ and $Q'(n+1)$ so that the inductive step is obtained this way. We start the induction with diagram $D(2)$ with $H'_2 = 0$ and $\beta'_3 = $ identity.

(5.5) Notation. Using diagram $D(n)$ in (5.4) (6) we introduce the following types of elements. Let

$$\begin{aligned} \beta^{-1} &: \Omega(n-1) \cong \Omega(n-1), \\ (\beta'_{n+1})^{-1} &: \Omega(n) \cong \Omega(n) \end{aligned}$$

be the inverses of the isomorphisms β and β'_{n+1} respectively. Then the elements $\nu(\underline{n}), \nu'(\underline{n+1}), \bigtriangledown(\underline{n})$ and $\bigtriangledown'(\underline{n+1})$ defined by

$$\begin{aligned} \beta(\underline{n}) &= \underline{n} + \nu(\underline{n}), \\ \beta'(\underline{n+1}) &= \underline{n+1} + \nu'(\underline{n+1}) \\ \beta^{-1}(\underline{n}) &= \underline{n} + \bigtriangledown(\underline{n}) \\ (\beta'_{n+1})^{-1}(\underline{n+1}) &= \underline{n+1} + \bigtriangledown'(\underline{n+1}) \end{aligned}$$

are decomposable elements $\in \Omega_+ \cdot \Omega_+$. One readily checks the equations

$$(1) \qquad \begin{cases} \bigtriangledown(\underline{n}) = -\beta^{-1}\nu(\underline{n}) \\ \bigtriangledown'(\underline{n+1}) = -\beta^{-1}\nu'(\underline{n+1}) \end{cases}$$

Moreover we define for $j = 0, 1, \ldots, n+1$ the elements $v_j(\underline{n+1}), u_j(\underline{n+1}) \in \Omega_1 = U_1 = \operatorname{span}_R(N_1)$ by

$$(2) \qquad \bigtriangledown'(\underline{n+1}) = \xi + \sum_{j=0}^{n+1} (\partial_j(\underline{n+1}) \cdot v_j(\underline{n+1}) + u_j(\underline{n+1}) \cdot \partial_j(\underline{n+1})).$$

Here $\xi \in \Omega$ is an element which has no summands involving $a \in N$ with $|a| = n-1$. Let $q, \Delta, \lambda = \lambda^U, \phi = \phi^U, \mu = \mu^U$ be defined for $\Omega = T(U)$ as in (4.3). Then we set

$$\Lambda(\underline{n}) = \lambda\phi\lambda q(G_1(\underline{n}), G_2(\underline{n})) \tag{3}$$

where

$$\begin{aligned} G_1(\underline{n}) &= (\beta\otimes\beta)^{-1}((1-\mathcal{T})H'(\underline{n}) + E\,\beta(\underline{n})) \\ G_2(\underline{n}) &= (1\otimes\Delta - \Delta\otimes 1)(\beta\otimes\beta)^{-1}H'(\underline{n}) \\ &\quad + (1\otimes K - K\otimes 1)(\beta\otimes\beta)^{-1}\psi_\Omega\beta(\underline{n}) \end{aligned}$$

with $K = (\beta\otimes\beta)^{-1}H'_n$. Moreover we set

$$\Lambda'(\underline{n+1}) = \lambda\phi\lambda q(G'_1(\underline{n+1}), G'_2(\underline{n+1})) \tag{4}$$

where

$$\begin{aligned} G'_1(\underline{n+1}) &= (\beta'_{n+1}\otimes\beta'_{n+1})^{-1}E\,\beta'(\underline{n+1}) \\ G'_2(\underline{n+1}) &= (1\otimes(\beta\otimes\beta)^{-1}H'_n - (\beta\otimes\beta)^{-1}H'_n\otimes 1)(\beta'_{n+1}\otimes\beta'_{n+1})^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}) \end{aligned}$$

Recall that here $\bar{\psi}_\Omega(a) = \psi_\Omega(a) - 1\otimes a - a\otimes 1$ is the reduced diagonal defined by ψ_Ω and that E is the homotopy in (5.3)(2). Using the elements (1) ... (4) we are ready to describe the explicit formulas which yield the induction.

(5.6) Formulas for the induction.

(1)
$$\begin{aligned} \beta(\underline{n+1}) &= \beta'(\underline{n+1}) - \beta\mu(\beta\otimes\beta)^{-1}(A) \quad \text{with} \\ A = A(\underline{n+1}) &= \bar{\psi}_\Omega\beta'(\underline{n+1}) - H'_n\beta^{-1}d_\Omega\beta'(\underline{n+1}) \end{aligned}$$

$$H(\underline{n}) = H'(\underline{n}) + (\beta\otimes\beta)\Lambda(\underline{n}) \tag{2}$$

(3)
$$\begin{aligned} \beta'(\underline{n+2}) &= \underline{n+2} - \beta'_{n+1}\mu(X+Y+Z_1+Z_2) \quad \text{with} \\ X = X(\underline{n+2}) &= (\beta'_{n+1}\otimes\beta'_{n+1})^{-1}\bar{\psi}_\Omega(\underline{n+2}) \\ Y = Y(\underline{n+2}) &= (\beta\otimes\beta)^{-1}H'_n\left(\sum_{i=0}^{n+2}(-1)^i\,\bigtriangledown'\,(\partial_i(\underline{n+2})) - \beta^{-1}\sum_{i=2}^{n}(-1)^i\{0\dots i\}\cdot\{1\dots n+2\}\right) \\ Z_1 = Z_1(\underline{n+2}) &= \sum_{i=0}^{n+2}(-1)^i\sum_{j=0}^{n+1}Z_{ji} \\ Z_{ji} &= (\Lambda\,\partial_j\partial_i(\underline{n+2}))\cdot\Delta\,v_j(\partial_i(\underline{n+2})) - \Delta\,u_j(\partial_i(\underline{n+2}))\cdot(\Lambda\,\partial_j\partial_i(\underline{n+2})) \\ Z_2 = Z_2(\underline{n+2}) &= \Delta\{0,1,2\}\cdot\Lambda(\{2\dots n+2\}) - (-1)^n\Lambda(\{0\dots n\})\cdot\Delta\{n,n+1,n+2\} \end{aligned}$$

(4)

$$H'(\underline{n+1}) = (\beta'_{n+1}\otimes\beta'_{n+1})\Lambda'(\underline{n+1}) - R - \frac{1}{4}(\beta\otimes\beta)(S)$$

$$R = R(\underline{n+1}) = H'_n\mu(\beta\otimes\beta)^{-1}\bar\psi_\Omega\beta'(\underline{n+1})$$

$$\begin{aligned} S = S(\underline{n+1}) &= (\Lambda\underline{n})\cdot\Delta\{0,n,n+1\} + (-1)^n\Delta\{0,n,n+1\}\cdot(\Lambda\,\underline{n}) \\ &\quad - (-1)^n(\Lambda\{1\dots n+1\})\cdot\Delta\{0,1,n+1\} - \Delta\{0,1,n+1\}\cdot(\Lambda\{1\dots n+1\}) \\ &\quad + \sum_{i=1}^{n}(-1)^i\Big(\Delta\{i-1,i,i+1\}\cdot(\Lambda\partial_i\underline{n+1}) + (-1)^n(\Lambda\partial_i\underline{n+1})\cdot\Delta\{i-1,i,i+1\}\Big)\end{aligned}$$

In (3) and (4) we have $\Delta(x) = x\otimes 1 + 1\otimes x$ since the corresponding elements x in (3) and (4) are of degree 1. We point out that all equations (1) ... (4) are well defined in terms of the data $Q(2), Q(3), \dots, Q(N), Q'(n)$ in (8.9) so that the induction is well defined. The formulas above imply the following results (5.7) and (5.8) which simplify the expressions for the Lie differential $d_{\mathcal{L}}$.

(5.7) Proposition. $d_\Omega\beta(\underline{n+1}) = d_\Omega\beta'(\underline{n+1})$ *and hence*

$$d_{\mathcal{L}}(\underline{n+1}) = d'_{\mathcal{L}}(\underline{n+1}) = \beta^{-1}d_\Omega\beta'(\underline{n+1})$$

Proof. By (5.6) (1) we have:

$$d_\Omega\beta(\underline{n+1}) = d_\Omega\beta'(\underline{n+1}) - d_\Omega\beta\mu(\beta\otimes\beta)^{-1}(A) \tag{1}$$

$$\begin{aligned} d_\Omega\beta\mu(\beta\otimes\beta)^{-1} &= \beta d_{\mathcal{L}}\mu(\beta\otimes\beta)^{-1} \\ &= \beta\mu d_{\mathcal{L}}(\beta\otimes\beta)^{-1} \\ &= \beta\mu(\beta\otimes\beta)^{-1}d_\Omega \end{aligned} \tag{2}$$

since μ is a $\mathrm{Der}(L)$ -homomorphism by 3.7 (a) [A] with $d_{\mathcal{L}}\in\mathrm{Der}(L)$.

$$d_\Omega A = d_\Omega\bar\psi_\Omega\beta'(\underline{n+1}) - d_\Omega H'_n\beta^{-1}d_\Omega\beta'(\underline{n+1}) \quad \text{where} \tag{3}$$

$$\begin{aligned} d_\Omega H'_n &= \psi_\Omega\beta - (\beta\otimes\beta)\Delta - H'_n d_{\mathcal{L}} \quad \text{on} \quad \Omega(n-1) \\ &= \bar\psi_\Omega\beta - (\beta\otimes\beta)\bar\Delta - H'_n d_{\mathcal{L}} \end{aligned}$$

$$\begin{aligned} d_\Omega A &= d_\Omega\bar\psi_\Omega\beta'(\underline{n+1}) - \bar\psi_\Omega d_\Omega\beta'(\underline{n+1}) \\ &\quad + (\beta\otimes\beta)\bar\Delta\beta^{-1}d_\Omega\beta'(\underline{n+1}) + H'_n d_{\mathcal{L}}\beta^{-1}d_\Omega\beta'(\underline{n+1}) \end{aligned} \tag{4}$$

Here the last term vanishes since $d_{\mathcal{L}}\beta^{-1} = \beta^{-1}d_\Omega$. Moreover the first two terms cancel so that

(5) $$d_\Omega A = (\beta\otimes\beta)\bar\Delta\beta^{-1}d_\Omega\beta'(\underline{n+1}) = (\beta\otimes\beta)\bar\Delta d'_{\mathcal{L}}(\underline{n+1}) = 0$$

Here $\bar\Delta d'_{\mathcal{L}}(\underline{n+1}) = 0$ since $d'_{\mathcal{L}}(\underline{n+1})$ is a Lie element. Hence the lemma is a consequence of (1) and (2). q.e.d.

Recall that

$$\pi_1 = 1-\mu\,\bar\Delta : T(U)\to L(U)$$

is the canonical retraction from the tensor algebra $\Omega = T(U)$ to the Lie algebra.

(5.8) Theorem. *The Lie differential is given by*

$$\begin{aligned} d_{\mathcal{L}}(\underline{n+2}) = &-\sum_{i=0}^{n+2}(-1)^i\partial_i\underline{n+2} \\ &+(1-\mu\bar\Delta)\left(-\sum_{i=0}^{n+2}(-1)^i\bigtriangledown'(\partial_i\underline{n+2})+\beta^{-1}\sum_{i=2}^{n}(-1)^i\{0\dots i\}\cdot\{1\dots n+2\}\right)\end{aligned}$$

The proof of (5.8) is given in (5.12) below. We point out that $d_{\mathcal{L}}(\underline{n+2})$ in (5.8) is determined by the data $\beta(\underline{2}),\beta(\underline{3}),\dots,\beta(\underline{n})$ and $\beta'(\underline{n+1})$. Equivalently to (5.8) one has the following formulas

(5.9) Corollary.

$$\begin{aligned} d_{\mathcal{L}}(\underline{n+2}) &= (1-\mu\bar\Delta)((\beta'_{n+1})^{-1}d_\Omega\underline{n+2}) \\ d_{\mathcal{L}}(\underline{n+2}) &= (1-\mu\bar\Delta)(\beta^{-1}d_\Omega\underline{n+2})\end{aligned}$$

Proof. The first formula follows from (5.8) since by (5.3) and (5.5)

$$\begin{aligned}(\beta'_{n+1})^{-1}d_\Omega(\underline{n+2}) &= -(\beta'_{n+1})^{-1}\partial\,\underline{n+2}+\beta^{-1}(\partial^+\underline{n+2}) \\ &= -\partial\,\underline{n+2}-\bigtriangledown'(\partial\,\underline{n+2})+\beta^{-1}(\partial^+\,\underline{n+2})\end{aligned}$$

where $(1-\mu\bar\Delta)\partial\,\underline{n+2} = \partial\,\underline{n+2}$. The second formula is obtained as follows. We have by (5.5) (1)

$$(1-\mu\bar\Delta)\bigtriangledown'(\underline{n+1}) = -(1-\mu\bar\Delta)\beta^{-1}\nu'(\underline{n+1})$$

where by (5.6)(1)

$$\nu'(\underline{n+1}) = \nu(\underline{n+1})+\beta\mu(\beta\otimes\beta)^{-1}(A)$$

Since $(1-\mu\bar\Delta)\mu = 0$ we thus get by (5.5) (1)

$$\begin{aligned}(1-\mu\bar\Delta)\bigtriangledown'(\underline{n+1}) &= -(1-\mu\bar\Delta)\beta^{-1}\nu(\underline{n+1}) \\ &= (1-\mu\bar\Delta)\bigtriangledown(\underline{n+1})\end{aligned}$$

Hence we can replace $\bigtriangledown'$ in (5.8) by $\bigtriangledown$ and therefore one obtains the second formula in a similar way as the first formula above. q.e.d.

The formulas in (5.6) yield a well defined sequence of elements as in (5.4) and by the proof of the theorem below all the properties in (5.4) are satisfied. We now use $(\Omega, d_{\mathcal{L}})$ as a universal example. For this we define for each 1-reduced simplicial set X and $\sigma \in X_n$ the homomorphism of algebras

$$\sigma_* : \Omega[\underline{n}] \to UL(s^{-1}\tilde{C}_*X) = T(s^{-1}\tilde{C}_*X)$$

which carries $a \in N[\underline{n}]$ to $\sigma_*(a) = s^{-1}\sigma(a)$; see (2.2). Then we obtain a differential $d_{\mathcal{L}}$ on $L(s^{-1}\tilde{C}_*)$ by setting

$$d_{\mathcal{L}}(s^{-1}\sigma) = \sigma_* d_{\mathcal{L}}(\underline{n})$$

This defines a differential Lie algebra

(5.10) $$\mathcal{L}(X) = (L(s^{-1}\tilde{C}_*X), d_{\mathcal{L}})$$

We obtain an isomorphism

(1) $$\beta : U\mathcal{L}(X) \cong \Omega(C_*X)$$

by setting $\beta(s^{-1}\sigma) = \sigma_*\beta(\underline{n})$ and we obtain a derivation homotopy

(2) $$H : \psi\beta \simeq (\beta \otimes \beta)\Delta$$

by setting $H(s^{-1}\sigma) = (\sigma_* \otimes \sigma_*)H(\underline{n})$.

(5.11) Theorem. *The definition in (5.10) yields a well defined functor*

$$\mathcal{L} : \underline{\underline{sSet}}_1 \to \underline{\underline{DL}}_0$$

with the properties in (4.8).

In fact, this functor coincides with the one in (4.8) up to natural isomorphism.

Proof. Assume we have a diagram $D(n), n \geq 2$, with the properties described in (5.4). Using $D(n)$ we define a homotopy Hopf algebra

(1) $$X = (A, \psi, G_1, G_2, V)$$

which is $(n-1)$ -good as follows. Let $V = U_{\leq n+1}$ and let $A = T(V)$. Then an isomorphism

(2) $$\gamma : A = T(U_{\leq n+1}) \cong T(U_{\leq n+1}) = \Omega(n+1)$$

is defined by

$$\gamma(a) = \begin{cases} \beta(a) & \text{for} \quad |a| \leq n-1 \\ \beta'_{n+1}(a) & \text{for} \quad |a| = n \\ a & \text{for} \quad |a| = n+1, a \in N. \end{cases}$$

We define a differential d on A by the formula $d(a) = \gamma^{-1}d_\Omega\gamma(a)$ where d_Ω is the differential of $\Omega = T(U)$. Hence we have

$$
(3) \qquad d(a) = \begin{cases} d_{\mathcal{L}}(a) & \text{for} \quad |a| \leq n-1, \quad \text{see (5.4) (2)} \\ d'_{\mathcal{L}}(a) & \text{for} \quad |a| = n; \quad \text{see (5.4) (9)} \\ (\beta'_{n+1})^{-1}d_\Omega(a) & \text{for} \quad |a| = n+1, a \in N \end{cases}
$$

We now have the following commutative diagram

$$
(3a) \qquad \begin{tikzcd} A \otimes A \arrow[r, "\gamma\otimes\gamma"] & \Omega(n+1_\otimes\Omega(n+1) \\ A \arrow[u, "\psi'"] \arrow[r, "\gamma"'] & \Omega(n+1) \arrow[u, "\psi_\Omega"] \end{tikzcd}
$$

where $\psi' = (\gamma\otimes\gamma)^{-1}\psi_\Omega\gamma$. Since γ extends $\beta : \Omega(n-1) \cong \Omega(n-1)$ in $D(n)$ we see that $(\beta\otimes\beta)^{-1}H'_n$ is a derivation homotopy

$$
\psi'|\Omega(n-1) = (\beta\otimes\beta)^{-1}\psi_\Omega\beta \simeq \Delta|\Omega(n-1)
$$

The homotopy extension of ψ' via $(\beta\otimes\beta)^{-1}H'_n$ thus yields a diagonal

$$
(4) \qquad \psi : A \to A \otimes A
$$

and a homotopy $K : \psi' \simeq \psi$ with the properties:

$$
(5) \qquad K(a) = \begin{cases} (\beta\otimes\beta)^{-1}H'_n(a) & \text{for} \quad |a| \leq n-1 \\ 0 & \text{for} \quad |a| = n, n+1 \end{cases}
$$

$$
(6) \qquad \psi(a) = \begin{cases} \Delta(a) & \text{for} \quad |a| \leq n-1 \\ \psi'(a) - K(da) & \text{for} \quad |a| = n, n+1 \end{cases}
$$

We now obtain the homotopies G_1, G_2 in (1) as follows.

$$
\mathcal{T}\psi \xleftarrow{\mathcal{T}K} \mathcal{T}\psi' \xrightarrow{\gamma\otimes\gamma)^{-1}E\gamma} \psi' \xrightarrow{K} \psi
$$

$$
(7) \qquad G_1 = K + (\gamma\otimes\gamma)^{-1}E\gamma - \mathcal{T}K
$$

Here we use E in (5.3) (2). Moreover G_2 is given by

$$
\begin{tikzcd} (\psi\otimes 1)\psi & (\psi\otimes 1)\psi' \arrow[l, "(\psi\otimes 1)K"'] & (\psi'\otimes 1)\psi' \arrow[l, "(K\otimes 1)\psi'"'] \arrow[d, equal] \\ (1\otimes\psi)\psi & (1\otimes\psi)\psi' \arrow[l, "(1\otimes\psi)K"'] & (1\otimes\psi')\psi' \arrow[l, "(1\otimes K)\psi'"'] \end{tikzcd}
$$

Here we use the fact that ψ in (5.3) (1) is coassociative so that

$$G_2 = (1 \otimes \psi - \psi \otimes 1)K + (1 \otimes K - K \otimes 1)\psi' \tag{8}$$

This completes the definition of X in (1). The properties of $D(n)$ in (5.4) show that X is $(n-1)$ -good in the sense of (4.4). For this we need the assumption that $d'_{\mathcal{L}}(\underline{n+1})$ in (5.4) (9) is a Lie element. We now apply Anick's theorem (4.5) and we obtain

$$\alpha(X) = (A, \chi, J_1, J_2, W) \quad \text{and} \quad F : \psi \simeq \chi, \tag{9}$$

which is n-good. We point out that for this we have to <u>replace</u> n in (4.5) (1) ... (8) <u>by</u> $n-1$. We only use (A, χ, W, F) for the description of diagram $D(n+1)$. We define the isomorphism

$$\gamma' : \Omega(n+1) \cong A \tag{10}$$

by the map which carries V to W, that is

$$\gamma'(a) = \begin{cases} a & \text{for} \quad |a| \leq n-1 \\ (1 - \mu\bar{\chi})(a) & \text{for} \quad |a| = n, n+1 \end{cases}$$

Then diagram $D(n+1)$ is defined by

$$\beta'_{n+2} = \gamma\gamma' : \Omega(n+1) \cong \Omega(n+1), \tag{11}$$

$$H'_{n+1} = (\gamma \otimes \gamma)(K + F)\gamma', \quad \text{restricted to} \quad \Omega(n) \tag{12}$$

Accordingly we obtain the elements

$$\begin{cases} \beta(\underline{n+1}) = \gamma\gamma'(\underline{n+1}) \\ H(\underline{n}) = (\gamma \otimes \gamma)(K + F)\gamma'(\underline{n}) \quad \text{with} \quad \gamma'(\underline{n}) = \underline{n} \\ \beta'(\underline{n+2}) = \gamma\gamma'(\underline{n+2}) \\ H'(\underline{n+1}) = (\gamma \otimes \gamma)(K + F)\gamma'(\underline{n+1}) \end{cases} \tag{13}$$

It remains to show that the elements in (13) coincide with the elements described in (5.6). For this we need the definition of χ and F in (4.5) where we replace n by $n-1$. Recall that $|\underline{n+1}| = n$.

$$\begin{aligned} \beta(\underline{n+1}) &= \beta'_{n+1}(\underline{n+1} - \mu\bar{\chi}(\underline{n+1})); \quad \text{see (2) and (10)} \\ &= \beta'(\underline{n+1}) - \beta\mu(\bar{\psi}(\underline{n+1}) - \lambda\phi\bar{\psi}(n+1)) \end{aligned} \tag{14}$$

Here we have $\beta'_{n+1}\mu = \beta\mu$ in degree n since image (μ) is decomposable and we use (4.5) (2). By (3.8) [A] we know that $\bar{\Delta}\mu + \lambda\phi = 1$ where $\mu\bar{\Delta}\mu = \mu$ by 3.7 [A] since

μ has only coordinates in $U_{j+1}(V), j \geq 1$; see (4.2) (5). Hence we see that $\mu\lambda\phi = 0$ and we get by (6)

$$\beta(\underline{n+1}) = \beta'(\underline{n+1}) - \beta\mu(\bar{\psi}'(\underline{n+1}) - K(d(\underline{n+1})))$$

$$\begin{aligned}\bar{\psi}'(\underline{n+1}) &= (\gamma\otimes\gamma)^{-1}\bar{\psi}_\Omega\gamma(\underline{n+1}) \\ &= (\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}); \quad \text{see (2)},\end{aligned}$$

$$\begin{aligned}K(d(\underline{n+1})) &= K(d'_{\mathcal{L}}(\underline{n+1})); \quad \text{see (3)}, \\ &= (\beta\otimes\beta)^{-1}H'_n(\beta^{-1}d_\Omega\beta'(\underline{n+1})); \quad \text{see (5) and (5.4) (9)}.\end{aligned}$$

This completes the proof of (5.6) (1). Next we get the following formulas in degree $n-1$.

$$\begin{aligned}H(\underline{n}) &= (\gamma\otimes\gamma)(K(\underline{n}) + F(\underline{n})) \\ &= (\beta\otimes\beta)((\beta\otimes\beta)^{-1}H'(\underline{n}) + F(\underline{n})); \quad \text{see (5) and (5.8) (8)} \\ &= H'(\underline{n}) + (\beta\otimes\beta)F(\underline{n})\end{aligned} \tag{15}$$

By (4.5) (3) we have

$$F(\underline{n}) = \lambda\phi\lambda q(G_1(\underline{n}), G_2(\underline{n})) \tag{15a}$$

$$\begin{aligned}G_1(\underline{n}) &= (1-\mathcal{T})K(\underline{n}) + (\gamma\otimes\gamma)^{-1}E\,\gamma(\underline{n}); \quad \text{see (7)} \\ &= (1-\mathcal{T})(\beta\otimes\beta)^{-1}H'(\underline{n}) + (\beta\otimes\beta)^{-1}E\,\beta(\underline{n}); \quad \text{see (5) and (2)} \\ &= (\beta\otimes\beta)^{-1}((1-\mathcal{T})H'(\underline{n}) + E\,\beta(\underline{n}))\end{aligned}$$

$$\begin{aligned}G_2(\underline{n}) &= (1\otimes\psi - \psi\otimes 1)K(\underline{n}) + (1\otimes K - K\otimes 1)\psi'(\underline{n}), \quad \text{see (8)} \\ &= (1\otimes\Delta - \Delta\otimes 1)(\beta\otimes\beta)^{-1}H'(\underline{n}); \quad \text{see (5), (6)} \\ &+ (1\otimes K - K\otimes 1)(\beta\otimes\beta)^{-1}\psi_\Omega\beta(\underline{n}); \quad \text{see (3a), (2)}\end{aligned}$$

Here K can be replaced by $(\beta\otimes\beta)^{-1}H'_n$ since we are in degree $n-1$; see (5). By (15) we thus obtain a proof of (5.6) (2). Here we have $\Lambda(\underline{n}) = F(\underline{n})$ by (5.5) (3).

Next we obtain in degree $n+1$ the following equations.

$$\begin{aligned}\beta'(\underline{n+2}) &= \gamma\gamma'(\underline{n+2}) \\ &= \gamma(1-\mu\bar{\chi})(\underline{n+2}); \quad \text{see (10)}, \\ &= \gamma(\underline{n+2}) - \gamma\mu\bar{\chi}(\underline{n+2}) \\ &= \underline{n+2} - \beta'_{n+1}\mu\bar{\chi}(\underline{n+2}); \quad \text{see (2)}.\end{aligned} \tag{16}$$

By (4.5) (5) we have

$$\begin{aligned} \bar{\chi}(\underline{n+2}) &= \bar{\psi}(\underline{n+2}) - F(d(\underline{n+2})) \\ &= \bar{\psi}'(\underline{n+2}) - K(d(\underline{n+2})) - F(d(\underline{n+2})); \quad \text{see (6)} \\ X = \bar{\psi}'(\underline{n+2}) &= (\beta'_{n+1} \otimes \beta'_{n+1})^{-1} \bar{\psi}_\Omega(\underline{n+2}); \quad \text{see (2), (3a)} \\ d(\underline{n+2}) &= (\beta'_{n+1})^{-1} d_\Omega(\underline{n+2}); \quad \text{see (3)} \end{aligned}$$

Hence we get by (16)

$$\beta'(\underline{n+2}) = \underline{n+2} - \beta'_{n+1}\mu(X + Y + Z_1 + Z_2) \tag{17}$$

with X above and Y and Z_1, Z_2 computed as follows:

$$\begin{aligned} Y &= -K(\beta'_{n+1})^{-1} d_\Omega(\underline{n+2}) \\ &= K(\beta'_{n+1})^{-1}\Big(\sum_{i=0}^{n+2}(-1)^i \partial_i(\underline{n+2}) - \sum_{i=2}^{n}(-1)^i\{0\dots i\}\{i\dots n+2\}\Big) \end{aligned}$$

Here we have by definition of $\bigtriangledown'(\underline{n+1})$ in (5.5) (1)

$$(\beta'_{n+1})^{-1}(\partial_i(\underline{n+2})) = \partial_i(\underline{n+2}) + \bigtriangledown'(\partial_i(\underline{n+2})) \tag{18}$$

and we have $K\partial_i(\underline{n+2}) = 0$ by (5). Thus we get the formula for Y in (5.6) (3) since K coincides with $(\beta \otimes \beta)^{-1} H'_n$ and β'_{n+1} coincides with β on $\Omega(n-1)$. Next we compute by (18)

$$\begin{aligned} &-F(d(\underline{n+2})) = -F((\beta'_{n+1})^{-1} d_\Omega(\underline{n+2})) \\ &= F\left(\sum_{i=0}^{n+2}(-1)^i(\partial_i(\underline{n+2}) + \bigtriangledown'(\partial_i(\underline{n+2}))) - \beta^{-1}\sum_{i=2}^{n}(-1)^i\{0\dots i\}\{i\dots n+2\}\right) \end{aligned} \tag{19}$$

By (4.5) (4) we know that $F(a) = 0$ for $a \in N$ with $|a| < n-1$ and $|a| > n$. Here we have by (4.5) (3) for $v = \partial_i(\underline{n+2})$

$$\mu F(v) = \mu\lambda\phi\lambda q(G_1 v, G_2 v) = 0$$

since $\mu\lambda\phi = 0$; see the argument in (14). Hence we have

$$-\mu F(d(\underline{n+2})) = \mu(Z_1 + Z_2) \tag{20}$$

where

$$\begin{aligned} Z_1 &= F\Big(\sum_{i=0}^{n+2}(-1)^i \bigtriangledown' (\partial_i(\underline{n+2}))\Big) \\ Z_2 &= -F\beta^{-1}\sum_{i\in\{2,n\}}(-1)^i\{0\dots i\}\{1\dots n+2\} \end{aligned}$$

We know for ξ in (5.5) (2) that $F(\xi) = 0$ hence we get by (5.5) (2)

$$Z_1 = \sum_{i=0}^{n+2} (-1)^i \sum_{j=0}^{n+1} F\Big(\partial_j\partial_i(\underline{n+2}) \cdot v_j(\partial_i(\underline{n+2})) + u_j(\partial_i(\underline{n+2})) \cdot \partial_j\partial_i(\underline{n+1})\Big)$$

Now $F : \psi \simeq \chi$ is the derivation homotopy in (9) where ψ and χ coindide with Δ in degree 1. Hence we obtain the formula for Z_1 in (5.6) (3) since $F(\partial_j\partial_i(\underline{n+2})) = \Lambda(\partial_j\partial_i(\underline{n+2}))$ by (15a). Moreover since β is the identity in degree 1 we get

$$Z_2 = -F\Big(\{0,1,2\} \cdot \beta^{-1}\{2\ldots n+2\} + (-1)^n \beta^{-1}\{0\ldots n\} \cdot \{n, n+1, n+2\}\Big)$$

Here we have $\beta^{-1}\underline{n} = \underline{n} + \eta$ where $\eta = \bigtriangledown(\underline{n})$ is decomposable so that $F(\eta) = 0$. Hence we get $F\beta^{-1}\underline{n} = F(\underline{n}) = \Lambda(\underline{n})$ as computed in (15a). This proves the formula for Z_2 in (5.6) (3) and hence the proof of (5.6) (3) is complete.

Finally we compute by (10) and (5)

$$(21) \qquad \begin{aligned} H'(\underline{n+1}) &= (\gamma \otimes \gamma)(K + F)\gamma'(\underline{n+1}) \\ &= (\gamma \otimes \gamma)(K + F)(\underline{n+1} - \mu\bar{\chi}(\underline{n+1})) \\ &= (\gamma \otimes \gamma)(F(\underline{n+1}) - (K + F)\mu\bar{\chi}(\underline{n+1})) \end{aligned}$$

where we can replace $K\mu$ by $(\beta \otimes \beta)^{-1} H'_n \mu$ (see (5)) and where by (4.5) (2)

$$\mu\bar{\chi}(\underline{n+1}) = \mu(\bar{\psi}(\underline{n+1}) - \lambda\phi\bar{\psi}(\underline{n+1}))$$

with $\mu\lambda\phi = 0$ (see (14)) so that

$$\begin{aligned} \mu\bar{\chi}(\underline{n+1}) &= \mu\bar{\psi}(\underline{n+1}) \\ &= \mu(\bar{\psi}'(\underline{n+1}) - F(d\,\underline{n+1})), \quad \text{see (6)} \\ &= \mu((\gamma \otimes \gamma)^{-1}\bar{\psi}_\Omega\gamma(\underline{n+1}) - F(d'_{\mathcal{L}}\underline{n+1})), \quad \text{see (3) and (3a)} \\ &= \mu((\beta \otimes \beta)^{-1}\bar{\psi}_\Omega\beta'(n+1) - F(\beta^{-1}d_\Omega\beta'(\underline{n+1}))), \quad \text{see (2) and (5.4) (9).} \end{aligned}$$

Here we have $\beta'(\underline{n+1}) = \underline{n+1} + \xi$ where ξ is decomposable so that:

$$\begin{aligned} \mu F\beta^{-1}d_\Omega\beta'(n+1) &= \mu F\beta^{-1}d_\Omega(\underline{n+1} + \xi) \\ &= \mu F\beta^{-1}(-\sum_{i=0}^{n-1}(-1)^i\partial_i\underline{n+1}), \quad \text{since} \quad F(a) = 0 \quad \text{for} \quad |a| < n-1, \\ &= \mu F(-\sum_{i=0}^{n-1}(-1)^i\partial_i\underline{n+1}) = 0 \end{aligned}$$

since $\beta^{-1}(\underline{n}) = \underline{n} + \eta$ with $F\eta = 0$ and since $\mu F(\underline{n}) = 0$ by (15a) and $\mu\lambda\phi = 0$. Hence we have proved

$$\mu\bar{\chi}(\underline{n+1}) = \mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1})$$

so that by (21):

$$\begin{aligned} H'(\underline{n+1}) &= (\gamma\otimes\gamma)F(\underline{n+1}) \\ &\quad - (\gamma\otimes\gamma)(\beta\otimes\beta)^{-1}H'_n\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(n+1) \\ &\quad - (\gamma\otimes\gamma)F\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}) \end{aligned}$$

and hence by (2)

$$(22)\qquad \begin{aligned} H'(\underline{n+1}) &= (\beta'_{n+1}\otimes\beta'_{n+1})F(\underline{n+1}) - R - \frac{1}{4}(\beta\otimes\beta)(S) \\ R &= H'_n\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(n+1) \\ S &= 4\cdot F\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}) \end{aligned}$$

Taking into account that $F(a) = 0$ for $|a| < n-1$ we obtain the following formulas

$$\begin{aligned} \frac{1}{4}S &= F\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}) \\ &= F\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega(\underline{n+1}) \\ &= F\mu\bar{\psi}_\Omega(\underline{n+1}) = F\mu(S') \quad \text{with} \\ S' &= \sum_{f\in M}\tau(\bar{f})\bar{f}\otimes f \end{aligned}$$

where M is the set of all f, $f \le \{0\dots n+1\}$, $f \neq \{0, n+1\}$ and $f \neq \underline{n+1}$ for which $|f| = n-1$ or for which $\bar{f}$ contains a factor of degree $n-1$. Hence M consists of the following subsets of $\underline{n+1}$,

$$\{0,1,n+1\}, \{0,n,n+1\}, \partial_i(\underline{n+1}) \quad \text{with} \quad i = 1,\dots,n$$

Hence we have

$$\begin{aligned} S' = S'(\underline{n+1}) &= +\{0\dots n\}\otimes\{0,n,n+1\} \\ &\quad - (-1)^n\{1\dots n+1\}\otimes\{0,1,n+1\} \\ &\quad - \sum_{i=1}^{n}(-1)^i\{i-1,i,i+1\}\otimes\partial_i(\underline{n+1}) \end{aligned}$$

By definition of μ in (4.3) we thus get

$$\mu(S') = \frac{1}{2}C(S') = \frac{1}{4}m(1+\mathcal{T})(S') \in \Omega$$

Here m denotes the multiplication $m : \Omega\otimes\Omega \to \Omega$. Thus we get

$$(23)\qquad S = 4F\mu(S') = F(1+\mathcal{T})(S')$$

By (15a) this coincides with the formula for S in (5.6) (4) since ψ and χ coincide with Δ in degree 1. Finally we consider the first summand in (22). By (4.5) (3) we have

$$(24) \qquad F(\underline{n+1}) = \lambda\phi\lambda q(G_1\underline{n+1}, G_2\underline{n+1})$$

Here (7) yields

$$\begin{aligned} G_1(\underline{n+1}) &= (1-\mathcal{T})K(\underline{n+1}) + (\gamma\otimes\gamma)^{-1}E\,\gamma(\underline{n+1}) \\ &= (\beta'_{n+1}\otimes\beta'_{n+1})^{-1}E\,\beta'(\underline{n+1}), \quad \text{see (5) and (2).} \end{aligned}$$

For this we have to check that $E\beta'(\underline{n+1}) \in \Omega(n)\otimes\Omega(n)$; but this is a consequence of the fact that $E(\underline{n+1}) \in \Omega(n)\otimes\Omega(n)$. Hence $G_1(\underline{n+1})$ coincides with the element $G'_1(\underline{n+1})$ in (5.5) (4). On the other hand we get by (8) the formula

$$(25) \qquad \begin{aligned} G_2(\underline{n+1}) &= (1\otimes\psi - \psi\otimes 1)K(\underline{n+1}) + (1\otimes K - K\otimes 1)\psi'(\underline{n+1}) \\ &= (1\otimes K - K\otimes 1)(\gamma\otimes\gamma)^{-1}\psi_\Omega\gamma(\underline{n+1}); \quad \text{see (5), (3a).} \end{aligned}$$

Here γ can be replaced by β'_{n+1} and K can be replaced by $(\beta\otimes\beta)^{-1}H'_n$. To see this we write $\psi_\Omega(x) = 1\otimes x + x\otimes 1 + \bar{\psi}_\Omega(x)$ so that

$$\begin{aligned} (\gamma\otimes\gamma)^{-1}\psi_\Omega\gamma(\underline{n+1}) &= (\gamma\otimes\gamma)^{-1}(1\otimes\gamma(\underline{n+1}) + \gamma(\underline{n+1})\otimes 1 + \bar{\psi}_\Omega\gamma(\underline{n+1})) \\ &= 1\otimes\underline{n+1} + \underline{n+1}\otimes 1 + (\gamma\otimes\gamma)^{-1}\bar{\psi}_\Omega\beta'(\underline{n+1}); \quad \text{see (2).} \end{aligned}$$

Here we have $(\gamma\otimes\gamma)^{-1}\bar{\psi}_\Omega = (\beta'_{n+1}\otimes\beta'_{n+1})^{-1}\bar{\psi}_\Omega$ in degree n. Moreover we have $K(\underline{n+1}) = 0$ by (5) so that we get

$$G_2(\underline{n+1}) = G'_2(n+1)$$

where the right hand side is the element in (5.4) (4). Hence $F(\underline{n+1}) = \Lambda'(\underline{n+1})$ coincides by (24) with the element in (5.4) (4) and therefore the proof of (5.6) (4) is complete. q.e.d.

(5.12) Proof of (5.8). We have by (5.6) (3)

$$(1) \qquad \begin{aligned} d_{\mathcal{L}}(\underline{n+2}) &= \beta^{-1}d_\Omega\beta'(\underline{n+2}) \\ &= \beta^{-1}d_\Omega(\underline{n+2}) - \beta^{-1}d_\Omega\beta'_{n+1}\mu(X+Y+Z_1+Z_2) \end{aligned}$$

where $d_\Omega\beta'_{n+1} = \beta d_{\mathcal{L}}$ by (5.7) and $d_{\mathcal{L}}\mu = \mu d_{\mathcal{L}}$ since μ is a $\mathrm{Der}(L)$ -homomorphism; see 3.7 (a) [A]. Hence we get

$$(2) \qquad d_{\mathcal{L}}(\underline{n+2}) = \beta^{-1}(-\partial+\partial^+) - \mu d_{\mathcal{L}}(X+Y+Z_1+Z_2)$$

where we put $\partial = \partial(\underline{n+2}), \partial^+ = \partial^+(\underline{n+2})$ as in (5.3). We have

$$\beta^{-1}(\partial) = \partial + \bigtriangledown(\partial)$$

where $\bigtriangledown$ is the linear map defined by $\bigtriangledown$ in (5.5). Using (5.6) (A) and (5.5) we get

$$\begin{aligned}
\bigtriangledown(\partial) &= -\beta^{-1}\nu(\partial)\\
&= -\beta^{-1}(\nu'(\partial) - \beta\mu(\beta\otimes\beta)^{-1}(A(\partial))\\
&= \bigtriangledown'(\partial) + \mu(\beta\otimes\beta)^{-1}(A(\partial))\\
A(\partial) &= \bar{\psi}_\Omega\beta'(\partial) - H'_n d_{\mathcal{L}}(\partial)\\
&= \bar{\psi}_\Omega(\partial + \nu'(\partial)) - H'_n d_{\mathcal{L}}(\partial)
\end{aligned}$$

where $\nu'(\partial) = -\beta \bigtriangledown' (\partial)$. Hence we get for the summand $-\beta^{-1}(\partial)$ of (2) the equation

$$\begin{aligned}
&-\beta^{-1}(\partial) = -\partial - \bigtriangledown'(\partial)\\
&-\mu(\beta\otimes\beta)^{-1}(\bar{\psi}_\Omega\partial - \bar{\psi}_\Omega\beta\bigtriangledown'(\partial) - H'_n d_{\mathcal{L}}(\partial))
\end{aligned}\tag{3}$$

On the other hand the summand $-\mu\, d_{\mathcal{L}}X$ of (2) has the form

$$\begin{aligned}
-\mu\, d_{\mathcal{L}}X &= -\mu\, d_{\mathcal{L}}(\beta'_{n+1}\otimes\beta'_{n+1})^{-1}\bar{\psi}_\Omega(\underline{n+2})\\
&= -\mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega d_\Omega\underline{n+2}\\
&= \mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega(\partial - \partial^+)
\end{aligned}\tag{4}$$

Here the first summand of (4) and the corresponding summand of (3) cancel. Next we get

$$-\mu\, d_{\mathcal{L}}Y = \mu\, d_{\mathcal{L}}(\beta\otimes\beta)^{-1}H'_n(-\bigtriangledown'(\partial) + \beta^{-1}(\partial^+))$$

with

$$\begin{aligned}
d_{\mathcal{L}}(\beta\otimes\beta)^{-1}H'_n &= (\beta\otimes\beta)^{-1}d_\Omega H'_n\\
&= (\beta\otimes\beta)^{-1}(\bar{\psi}_\Omega\beta - (\beta\otimes\beta)\bar{\Delta} - H'_n d_{\mathcal{L}}).
\end{aligned}$$

See (5.4) (6). Hence we get

$$\begin{aligned}
-\mu\, d_{\mathcal{L}}Y &= \mu(\beta\otimes\beta)^{-1}\bar{\psi}_\Omega(-\beta\bigtriangledown'(\partial) + \partial^+)\\
&\quad - \mu\bar{\Delta}(-\bigtriangledown'(\partial) + \beta^{-1}(\partial^+)) && (5)\\
&\quad - \mu(\beta\otimes\beta)^{-1}H'_n d_{\mathcal{L}}(-\bigtriangledown'(\partial) + \beta^{-1}\partial^+) && (6)
\end{aligned}$$

Here we have

$$\begin{aligned}
-\bigtriangledown'(\partial) + \beta^{-1}(\partial^+) &= \beta^{-1}\nu'(\partial) + \beta^{-1}(\partial^+)\\
&= \beta^{-1}(\beta'_{n+1}(\partial) - \partial + \partial^+)\\
&= \beta^{-1}(\beta'_{n+1}(\partial) + d_\Omega\underline{n+2}) = \xi_1
\end{aligned}$$

with

$$\begin{aligned}
d_{\mathcal{L}}\xi_1 &= \beta^{-1}d_\Omega\beta'_{n+1}(\partial)\\
&= \beta^{-1}\beta\, d_{\mathcal{L}}(\partial) = d_{\mathcal{L}}(\partial)
\end{aligned}$$

Hence we get

$$-\mu(\beta\otimes\beta)^{-1}H'_n d_{\mathcal{L}}(\partial) = (6) \tag{7}$$

It remains to check that

$$\mu\, d_{\mathcal{L}}(Z_1+Z_2) = 0. \tag{8}$$

Then putting all equations together the proof of (5.8) is complete. For the proof of (8) we use (20) in the proof of (5.11) so that we get

$$\mu\, d_{\mathcal{L}}(Z_1+Z_2) = \mu\, d_{\mathcal{L}}F(d(\underline{n+2})) \tag{9}$$

Recall that $F: A_+ \to A_+\otimes A_+$ is a homotopy $F:\bar{\psi}\simeq\bar{\chi}$. Moreover in degree $\leq n$ the differential of A coincides with $d_{\mathcal{L}}$, see (5.11) (3) and (5.7). Hence we have in (9)

$$\begin{aligned}\mu\, d_{\mathcal{L}}(Z_1+Z_2) &= \mu(dF)(d\,\underline{n+2}) \\ &= \mu(\bar{\psi}-\bar{\chi}-Fd)(d\,\underline{n+2}) \\ &= \mu(\bar{\psi}-\bar{\chi})d\,\underline{n+2}\end{aligned} \tag{10}$$

Here we have $\bar{\psi}=\bar{\chi}$ in degree $\leq n-1$ by (4.5) (1) (where we have to replace n by $n-1$). This implies by (5.11) (3)

$$\begin{aligned}(\bar{\psi}-\bar{\chi})d\,\underline{n+2} &= (\bar{\psi}-\bar{\chi})(\beta'_{n+1})^{-1}d_\Omega\underline{n+2} \\ &= (\bar{\psi}-\bar{\chi})(\beta'_{n+1})^{-1}(-\sum_{i=0}^{n+2}(-1)^i\partial_i\underline{n+2}) \\ &= (\bar{\psi}-\bar{\chi})\left(-\sum_{i=0}^{n+2}(-1)^i\partial_i\underline{n+2}\right)\end{aligned} \tag{11}$$

Here the last equation uses the fact that $\nabla'(\underline{n+1})$ in (5.5) is decomposable. Finally we get by (4.5) (2) (where we replace n by $n-1$) the equation

$$\mu(\bar{\psi}-\bar{\chi})(\underline{n+1}) = \mu\lambda\phi\bar{\psi}(\underline{n+1}) = 0 \tag{12}$$

where $\mu\lambda\phi = 0$; compare (5.11) (14). q.e.d.

Let $m:\Omega\otimes\Omega\to\Omega$ be the multiplication of the tensor algebra $\Omega = T(U)$. The following elements are needed in (5.8).

(5.13) Lemma. $(1-\mu\bar{\Delta})\bigtriangledown' \underline{n+1} = \frac{1}{4}(1-\mu\bar{\Delta})m(1+\mathcal{T})A'(\underline{n+1})$ *where*

$$\begin{aligned} A'(\underline{n+1}) &= \mu(\beta\otimes\beta)^{-1}A\{0\dots n\}\otimes\{0,n,n+1\} \\ &- (-1)^n\mu(\beta\otimes\beta)^{-1}A\{1\dots n+1\}\otimes\{0,1,n+1\} \\ &- \sum_{i=1}^{n}(-1)^i\{i-1,i,i+1\}\otimes\mu(\beta\otimes\beta)^{-1}A(\partial_i\underline{n+1}) \end{aligned}$$

Here A is defined as in (5.6) (1).

Proof. For $\pi_1 = 1-\mu\bar{\Delta}$ we have by (5.5) and (5.6) (3)

$$\begin{aligned} \pi_1\bigtriangledown' \underline{n+1} &= -\pi_1\beta^{-1}\nu'(\underline{n+1}) \\ &= +\pi_1\beta^{-1}\beta'_n\mu(X+Y+Z_1+Z_2)(\underline{n+1}) \end{aligned}$$

Here we have $\beta'_n=\beta$ in degree $<n-1$ so that $\beta^{-1}\beta'_n\mu(Y+Z_1+Z_2)=\mu(Y+Z_1+Z_2)$ since Y,Z_1,Z_2 are decomposable. Using this type of argument we get the following equations since $\pi_1\mu=0$.

$$\begin{aligned} \pi_1\bigtriangledown' \underline{n+1} &= \pi_1\beta^{-1}\beta'_n\mu(X(\underline{n+1})) \\ &= \pi_1\beta^{-1}\beta'_n\mu(\beta'_n\otimes\beta'_n)\bar{\psi}_\Omega\underline{n+1} \\ &= \pi_1\beta^{-1}\beta'_n\mu(\beta'_n\otimes\beta'_n)S'(\underline{n+1}) \\ &= \pi_1\beta^{-1}\beta'_n\mu S'(\underline{n+1}) \\ &= \pi_1\beta^{-1}\beta'_n(\frac{1}{4}m(1+\mathcal{T}))S'(\underline{n+1}), \quad \text{see (4.3) (1),} \\ &= \frac{1}{4}\pi_1 m(1+\mathcal{T})(\beta^{-1}\beta'_n\otimes\beta^{-1}\beta'_n)S'(\underline{n+1}) \end{aligned}$$

Here we use $S'(\underline{n+1})$ in (5.11) (23) so that $S'(\underline{n+1}) = \underline{n}\otimes\{0,n,n+1\}+\dots$. Here we have

$$\begin{gathered} (\beta^{-1}\beta'_n\otimes\beta^{-1}\beta'_n)(\underline{n}\otimes\{0,n,n+1\}) = \\ \beta^{-1}\beta'_n(\underline{n})\otimes\{0,n,n+1\} = \\ (\underline{n}+\mu(\beta\otimes\beta)^{-1}A(\underline{n}))\otimes\{0,n,n+1\} \end{gathered}$$

In the last equation we use (5.6) (1). This yields the equation in the lemma since

$$\frac{1}{4}\pi_1 m(1+\mathcal{T})(\underline{n}\otimes\{0,n,n+1\}) = \pi_1\mu(\underline{n}\otimes\{0,n,n+1\}) = 0$$

q.e.d.

(5.14) Corollary. $(1-\mu\bar{\Delta})\bigtriangledown'(\underline{n+1})\in\Omega^3_+ = \otimes_{i\geq 3}U^{\otimes i}$.

The corollary is a consequence of the lemma since $A(\underline{n})$ is decomposable; i.e. $A(\underline{n})\in\Omega^2_+$. For this we need the fact that π_1 carries a word $w\in N^*$ to a linear combination of words obtained by permuting the letters of w, see 3.2.3 [R].

(5.15) Proof of (4.11). By (5.8) we already know $d^{(1)}_{\mathcal{L}}$. Next we see by (5.8) and (5.14) that $d^{(2)}_{\mathcal{L}}(\underline{n+2})$ is the quadratic part of

$$(1-\mu\bar{\Delta})(\sum_{i=2}^{n}(-1)^i\beta^{-1}\{0\ldots i\}\cdot\beta^{-1}\{1\ldots n+2\})$$

where $\beta^{-1}\underline{n}=\underline{n}+\bigtriangledown(\underline{n})$ where $\bigtriangledown(\underline{n})$ is decomposable. Hence we get

$$d_{\mathcal{L}}^{(2)}(\underline{n+2})=(1-\mu\bar{\Delta})(\sum_{i=2}^{n}(-1)^i\{0\ldots i\}\cdot\{2\ldots n+2\})$$

and this yields the formula for $d_{\mathcal{L}}^{(2)}$ in (4.11) since

$$(1-\mu\bar{\Delta})\{0\ldots i\}\cdot\{i\ldots n+2\}=\frac{1}{2}[\{0\ldots i\},\{i\ldots n+2\}].$$

Finally we refer to the properties of μ and λ in (4.3) for the proof of the fact that $d_{\mathcal{L}}^{(t)}$ is defined over R with $t<\rho(R)$; for this compare the corresponding argument of Anick in [A]. q.e.d.

(5.16) Example. Below we consider only subsets of $\{0,1,\ldots,9\}$. Therefore we can simplify the notation by setting $135=\{1,3,5\}$ or $\underline{2}=012$, $\underline{3}=0123$. Moreover the product in Ω is denoted by $012\cdot 234=\{0,1,2\}\cdot\{2,3,4\}$. With this notation we get by (5.2) (1), (2):

$$\begin{aligned}(1)\qquad \bar{\psi}_\Omega(\underline{2})&=0 \quad\text{and}\\ \bar{\psi}_\Omega(\underline{3})&=-123\otimes 013+012\otimes 023\quad\text{and}\\ \bar{\psi}_\Omega(\underline{4})&=1234\otimes 014+0123\otimes 034\\ &\quad+012\otimes 0234+234\otimes 0124-123\otimes 0134\\ &\quad-(012\cdot 234)\otimes 024\end{aligned}$$

$$\begin{aligned}(2)\qquad E(\underline{2})&=012\otimes 012\quad\text{and}\\ E(\underline{3})&=-0123\otimes(013+123)\\ &\quad+(023+012)\otimes 0123\end{aligned}$$

We now consider the first step in the induction (5.6) when we put $n=2$. We get by (5.5) (3), (4) the formulas

$$(3)\qquad\begin{aligned}\Lambda(\underline{2})&=\lambda\phi\lambda q(E(\underline{2}))=\frac{1}{4}(1-\mathcal{T})(E(\underline{2}))\\ &=\frac{1}{4}[012,012]\quad\text{and}\\ \Lambda'(\underline{3})&=\lambda\phi\lambda q(E(\underline{3}))=\frac{1}{4}(1-\mathcal{T})(E(\underline{3}))\\ &=-\frac{1}{4}[0123,013+123]+\frac{1}{4}[023+012,0123]\end{aligned}$$

Here we see by the definition of λ,ϕ that we can put $\lambda=-\frac{1}{2}\mathcal{T}$ and $\phi=(1-\mathcal{T})$. Moreover we get by definition of μ

$$
(4) \qquad \begin{aligned} \mu\,\bar\psi_\Omega(\underline{3}) &= \frac{1}{4}m(1+\mathcal{T})\bar\psi_\Omega\underline{3} \\ &= \frac{1}{4}(-123\cdot 013 + 013\cdot 123 + 012\cdot 023 - 023\cdot 012) \end{aligned}
$$

Now we get by (5.6) and (5.4) the formulas

$$
(5) \qquad \begin{aligned} \beta(\underline{3}) &= \underline{3} - \mu\,\bar\psi_\Omega(\underline{3}), \\ H(\underline{2}) &= \Lambda(\underline{2}), \\ \beta'(\underline{4}) &= \underline{4} - \mu(\bar\psi_\Omega(\underline{4}) + \Delta(012)\cdot\Lambda(234) - \Lambda(012)\cdot\Delta(234)), \\ H'(\underline{3}) &= \Lambda'(\underline{3}) - S(\underline{3}) \end{aligned}
$$

Compare the definition of S in (5.6) (4). We are now reday to compute $d_{\mathcal{L}}\underline{5}$. Using (5) we have

$$
(6) \qquad \beta^{-1}(\underline{3}) = \underline{3} + \mu\,\bar\psi_\Omega(\underline{3})
$$

Hence we get by (5.8) the following formula with $\pi_1 = 1 - \mu\,\bar\Delta$.

$$
\begin{aligned} d_{\mathcal{L}}\underline{5} = &- \sum_{i=0}^{5}(-1)^i(\partial_i\underline{5} + \pi_1 \bigtriangledown' (\partial_i\underline{5})) \\ &+ \pi_1((012)\cdot\beta^{-1}(2345) - \beta^{-1}(0123)\cdot(345)) \end{aligned}
$$

Hence we obtain $d_{\mathcal{L}}(\underline{5}) = d_{\mathcal{L}}^{(1)}(\underline{5}) + d_{\mathcal{L}}^{(2)}(\underline{5}) + d_{\mathcal{L}}^{(3)}(\underline{5})$ with

$$
\begin{aligned} d_{\mathcal{L}}^{(1)}(\underline{5}) &= -\sum_{i=0}^{5}(-1)^i\partial_i\underline{5} \\ d_{\mathcal{L}}^{(2)}(\underline{5}) &= \pi_1(012\cdot 2345 - 0123\cdot 345) \\ &= \frac{1}{2}[012, 2345] - \frac{1}{2}[0123, 345] \\ d_{\mathcal{L}}^{(3)}(\underline{5}) &= -\sum_{i=0}^{5}(-1)^i\pi_1 \bigtriangledown' (\partial_i\underline{5}) \\ &\quad + \pi_1((012)\cdot\mu\bar\psi_\Omega(2345) - \mu\bar\psi_\Omega(0123)\cdot(345)) \end{aligned}
$$

Here we have by (5.13)):

$$
\begin{aligned} \pi_1 \bigtriangledown' (\underline{4}) &= \frac{1}{4}\pi_1 m(1+\mathcal{T})A'(\underline{4}) \quad \text{where} \\ A'(\underline{4}) &= \mu\,\bar\psi_\Omega(0123)\otimes 034 + \mu\,\bar\psi_\Omega(1234)\otimes 014 \\ &\quad + 012\otimes\mu\,\bar\psi_\Omega(0234) - 123\otimes\mu\,\bar\psi_\Omega(0134) \\ &\quad + 234\otimes\mu\bar\psi_\Omega(0124) \end{aligned}
$$

Now $d_{\mathcal{L}}^{(3)}(5)$ is completely computed by use of (4).

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