SOLUTION OF THE UNITARIZABILITY PROBLEM FOR GENERAL LINEAR GROUP (NON-ARCHIMEDEAN CASE)

by

Marko Tadić

Planck-Institut
Mathematik
~~ied-Claren-Str. 26
00 Bonn 3

Department of Mathematics
University of Zagreb
41001 Zagreb
Yugoslavia

MPI/SFB 85-3

Introduction

Let G be a connected reductive group over a local field F. The set of all equivalence classes of (algebraicaly) irreducible admissible representations of G is denoted by $\tilde{G}$. The set of all equivalence classes of topologicaly irreducible unitary representations (on Hilbert spaces) is denoted by $\hat{G}$, and called a unitary dual of G. The unitary dual $\hat{G}$ is in a natural bijection with the subset of all unitarizable classes in $\tilde{G}$. In this way we shall identify $\hat{G}$ with the subset of all unitarizable classes in $\tilde{G}$. Thus, a description of the unitary dual can be done in two steps. The first step is to parametrize $\tilde{G}$, and the second one is to identify all unitarizable classes in $\tilde{G}$. The first step is called the problem of non-unitary dual, and the second one is called the unitarizability problem.

In this paper we give a solution of the unitarizability problem for the groups $GL(n)$ over a local non-archimedean field F. More precisely, Zelevinsky parameters and Langlands parameters of all unitarizable classes in $GL(n,F)^{\sim}$ are determined. Moreover, an explicit formula connecting Zelevinsky and Langlands parameters of $GL(n,F)^{\wedge}$ is proved. We prove also the Bernstein conjecture on complementary series from [1].

This paper finishes a study of unitary dual of $GL(n,F)$, started in [18]. The results of this paper were conjectured in [18], and proved for some n's. Note that the problem of non-unitary dual is not solved for reductive groups over non-archimedean fields, not even for $GL(n)$. This problem is solved for real reductive groups.

I.M. Gelfand and M.I. Graev solved in 1963 the unitarizability problem for $SL(2,F)$ ([5]). The similar ideas led to a solution of the unitarizability problem for closely related rank one groups. Before [18], those were the only cases of reductive groups over non-archimedean fields for which the unitarizability problem was solved (known to this author). There exist also two papers on the problem of unitarizability in the non-archimedean case. In [11], G.I. Olshansky constructed some complementary series for GL(n) over division algebras. J.N. Bernstein obtained in [1] some important general facts about $GL(n,F)^\wedge$. Our work on unitarizability for $GL(n,F)$ is founded on a nice theory of non-unitary dual of $GL(n,F)$. This theory was created by A.V. Zelevinsky. It is a continuation of research of J.N. Bernstein and A.V. Zelevinsky, and also of I.M. Gelfand and D.A. Kazhdan.

We shall now describe more detaily the results of this paper. Let F be a local non-archimedean field. Let Irr be the set of all equivalence classes of irreducible smooth admissible representations of $GL(n,F)$, with any n. The subset of all unitarizable classes in Irr is denoted by Irr^u. Set $\nu(g) = |\det g|_F$, $g \in GL(n,F)$. Let D^u be the set of all square integrable classes in Irr^u. For a smooth representation σ of $GL(n,F)$ and τ of $GL(m,F)$, let $\sigma \times \tau$ be the representation of $GL(m + n,F)$ induced by $\sigma \otimes \tau$, in a standard way.

Let n be a positive integer, and let $\delta \in D^u$. The representation

$$\nu^{\frac{n-1}{2}}\delta \times \nu^{\frac{n-1}{2}-1}\delta \times \dots \times \nu^{-\frac{n-1}{2}}\delta$$

has a unique irreducible quotient which is denoted by $u(\delta,n)$. In the classification of unitary dual of $GL(n,F)$ the role of the representations $u(\delta,n)$ is crutial. It follows from the following:

THEOREM A: Let B_t be the set of all

$$u(\delta,n),\ \nu^{\alpha}u(\delta,n) \times \nu^{-\alpha}u(\delta,n)$$

where n is a positive integer, $\delta \in D^u$ and $0 < \alpha < 1/2$.

i) If $\pi_1,\ldots,\pi_r \in B_t$, then $\pi_1 \times \ldots \times \pi_r \in \mathrm{Irr}^u$.

ii) If $\sigma \in \mathrm{Irr}^u$, then there exist $\pi_1,\ldots,\pi_s \in B_t$ so that $\sigma = \pi_1 \times \ldots \times \pi_s$. The elements $\pi_1,\ldots,\pi_s$ are unique up to a permutation.

This theorem describes the Langlands parameters of Irr^u. We also describe Zelevinsky parameters of Irr^u (see Theorems 3.1. and 3.3.).

Note that for $GL(n)$ over archimedean fields, irreducible square integrable representations exist only for

$$GL(1,\mathbb{C}) \cong \mathbb{C}^\times, GL(1,\mathbb{R}) = \mathbb{R}^\times \quad \text{and} \quad GL(2,\mathbb{R}).$$

If δ is an irreducible square integrable representation of $GL(1,\mathbb{R})$ or $GL(1,\mathbb{C})$, i.e. a unitary character of $\mathbb{R}^\times$ or $\mathbb{C}^\times$, then

$$u(\delta,n): g \to \delta(\det g)$$

is an one-dimensional unitary representation of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. If δ is an irreducible square integrable representation of $GL(2,\mathbb{R})$ than $u(\delta,n)$ were studied by B. Speh ([15]). In the non-archimedean case we have much more square integrable representations. Therefore we have much more representations $u(\delta,n)$.

At this point it could be interesting to compare I.M. Gelfand and M.A. Neumark list of representations of $SL(n,\mathbb{C})$ constructed in 1950 ([6]) with Theorem A, and to mention representations constructed by E.M. Stein ([17]).

In this paper we prove the Bernstein conjecture on complementary series from [1]. For $\pi \in \mathrm{Irr}$ let π^+ be the Hermitian contragradient of π. Rigid representations are defined in the third section.

THEOREM B (Bernstein conjecture on complementary series):

i) Suppose that $\nu^{\alpha}\sigma \times \nu^{-\alpha}\sigma^+$ is irreducible and unitarizable for all $\alpha \in (-1/2,1/2)$. Then σ is a unitarizable rigid representation.

ii) Suppose that σ is a rigid representation such that $\nu^{\alpha}\sigma \times \nu^{-\alpha}\sigma^+$ is an irreducible unitarizable representation for some $\alpha \in (0,1/2)$. Then there exist $\sigma_1,\sigma_2 \in \mathrm{Irr}^u$ so that

$$\sigma = \sigma_1 \times \bar{\nu}^{1/2}\sigma_2.$$

Now we shall describe connection between classifications of Zelevinsky and Langlands. Let C be the set of all cuspidal representations in Irr. The set of all segments in C is denoted by $S(C)$, and the set of all finite multisets in $S(C)$ is denoted by $M(S(C))$ (see section 1). Then Zelevinsky classification is a mapping

$$a \to Z(a)$$

which is a bijection of $M(S(C))$ onto Irr. We have also the bijection

$$a \to L(a), M(S(C)) \to \mathrm{Irr},$$

which is described in section 1. This is another parametrization of Irr, and it is equivalent to the standard Langlands classification. The classification $a \to L(a)$ was introduced, in this form, by F. Rodier in [12].

We consider a mapping

$$t: Z(a) \to L(a), \ a \in M(S(C))$$

introduced by F. Rodier in [12] which is a bijection on Irr. This mapping is identical to the restriction to Irr of the involution on the algebra of representations of all $GL(n,F)$ introduced by A.V. Zelevinsky in [24]. The mapping t contains complete information about connection between Zelevinsky and Langlands classifications. That means that the mapping t determines for $a \in M(S(C))$ an element $b \in M(S(C))$ so that

$$Z(a) = L(b).$$

In [25], A.V. Zelevinsky formulated a conjecture, in terms of involution on orbits of algebraic groups, which enables one to check if $Z(a) = L(b)$ or $Z(a) \neq L(b)$.

In [19], it is proved that

$$Z(a) \in Irr^u \Longleftrightarrow L(a) \in Irr^u, \ a \in M(S(C)).$$

It means that the unitarizable problem has the same solution in both classification. The above equivalence is equivalent to

$$t(Irr^u) = Irr^u$$

(this was a conjecture of J.N. Bernstein in [1]). Since in [19] we proved $t(Irr^u) = Irr^u$, here we obtain an explicit formula for the mapping

$$t: Irr^u \to Irr^u.$$

This implies an explicit formula expressing $b \in M(S(C))$ by $a \in M(S(C))$ such that

$$Z(a) = L(b),$$

when $Z(a)$ is unitarizable.

Set $C^u = C \cap \mathrm{Irr}^u$. For a positive integer n and $\rho \in C^u$, the representation

$$\left(\nu^{-\frac{n-1}{2}}\rho\right) \times \left(\nu^{-\frac{n-1}{2}+1}\rho\right) \times \ldots \times \left(\nu^{\frac{n-1}{2}}\rho\right)$$

has a unique irreducible quotient which is denoted by $\delta(\rho,n)$. The mapping

$$(\rho,n) \to \delta(\rho,n)$$

is a parametrization of D^u. This is a result of J.N. Bernstein

Theorem C: The mapping

$$t: \mathrm{Irr}^u \to \mathrm{Irr}^u$$

is an involutive homomorphism of the multiplicative semigroup Irr^u. The semigroup Irr^u is a free abelian semigroup over all

(*) $\qquad u(\delta,n), \ \nu^{\alpha}u(\delta,n) \times \nu^{-\alpha}u(\delta,n),$

where n is a positive integer, $\delta \in D^u$, $0 < \alpha < 1/2$. The homomorphism t is described on the basis (*) by

$$t(u(\delta(\rho,m),n)) = u(\delta(\rho,n),m),$$

$$t(\nu^{\alpha}u(\delta(\rho,m),n) \times \nu^{-\alpha}u(\delta(\rho,m),n)) =$$

$$\nu^{\alpha}u(\delta(\rho,n),m) \times \nu^{-\alpha}u(\delta(\rho,n),m).$$

For another formulation of Theorem C one can consult Theorem 3.3.

Together with Theorem 3.8. of [18], the above theorem proves Zelevinsky conjecture on the involution t for representations $u(\delta,n)$ (see Remark 3.3).

In the third section of this paper we are using a result of J.N. Bernstein for which this author does not know a reference for a written proof. We shall describe the role of this result in our paper. First we describe the result.

Let R be the free abelian group over basis Irr. The induction functor induces a structure of commutative associative ring on R. A.V. Zelevinsky showed that R is a polynomial ring over $Z(\Delta), \Delta \in S(C)$. Thus, the mapping

$$Z(\Delta) \to L(\Delta), \Delta \in S(C)$$

has a unique extension to a ring homomorphism on R. This homomorphism is denoted by

$$^t: \pi \to \pi^t, \quad ^t: R \to R$$

A.V. Zelevinsky proved that π is an involutive authomorphism of R and he conjectured that

$$(\mathrm{Irr})^t \subseteq \mathrm{Irr}.$$

Bernstein proof of this conjecture was announced in [24] (see also [1],[25] and [12]). This result will be denoted by (B). J.N. Bernstein used (B) in [1] to formulate the conjecture

$$(\mathrm{Irr}^u)^t \subseteq \mathrm{Irr}^u.$$

A.V. Zelevinksy used (B) to formulate in [25] a conjecture about ^{t}.

Now we shall describe some equivalent formulations of (B). The mapping $t: \mathrm{Irr} \to \mathrm{Irr}$ which we have introduced before can be uniquely extended to an additive homomorphism

$$t: R \to R.$$

Let us denote by (M) the statement that t is a multiplicative mapping (i.e. a ring homomorphism). F. Rodier showed in [12] that (B) implies $t = {}^t$. From this, one obtains directly equivalences

$$(B) \Longleftrightarrow {}^t = t \Longleftrightarrow (M).$$

A.V. Zelevinsky introduced in § 9 of [24] the involution ^{t} which he needed in § 10 of [24] to compare his classfication of irreducible representations of $GL(n,F)$ and n-dimensional semi-simple representations of Weil-Deligne group of F. As classification Z did not suit directly for that purpose, and he was not considering classification L, using ^{t} and (B), he obtained ${}^t: Irr \to Irr$ which indirectly define classification L

$$a \to Z(a) \to Z(a)^t = L(a)$$

and connection between classifications Z and L.

Therefore, it seems more natural to introduce the connection between two classifications directly

$$t: Irr \to Irr,\ Z(a) \to L(a),$$

instead of indirect definition of Zelevinsky in terms of algebra R which needs also (B).

When we introduce $t: Irr \to Irr$ as the connection between two classification, then (B) is equivalent to multiplicativity of t (more precise, multiplicativity of t lifted to R), i.e. (B) is just multiplicativity of the connection between classifications. This is a very usefull result, but even without this multiplicativity, connection t is important as we can see in the fourth section of this paper.

The preceding discussion suggested us to introduce in this paper t directly as connection between two classifications, as in [12]. Note that in this way one can formulate Zelevinsky conjecture describing t without assuming (B) (and also prove it for representations $u(\delta(\rho,n),m)$ if char $F = 0$, without assuming (B)) and to formulate the Bernstein conjecture

$$t(\mathrm{Irr}^u) \subseteq \mathrm{Irr}^u$$

(and prove it without using (B), in the characteristic zero case).

In the sections one and two of this paper we are using neither (B) nor the results depending on (B). We used (B) in the third section where Theorems A.,B. and C. are proved. In fact, (B) is used to prove the unitarizability of the representations $u(\delta,n)$. Here we use the result of [19].

As there is no reference for the proof of (B), we added Apendix in which Theorems A.,B. and C. are proved without assuming (B), when char $F = 0$. Therefore we are not using the result of [19] in this section. Using the idea of B. Speh in [15], we prove the unitarizability of $u(\delta,n)$ by global methods. Note that we show that $t:\mathrm{Irr}^u \to \mathrm{Irr}^u$ is multiplicative and involutive. We also obtain a new proof of the Bernstein conjecture that $t(\mathrm{Irr}^u) \subseteq \mathrm{Irr}^u$, which does not use (B). In [19] is another proof. For the restriction of characteristic in Apendix, one can apply similar observation to that of [10] there is the same restriction of characteristic.

From the point of view of real reductive groupes, it is interesting to have a proof of Theorem A. in terms of Langlands classification only. In the characteristic zero case, it is possible to prove Theorems A.

and B. dealing only with classification L, without assuming (B), as it is outlined in [22]. In order to do that, we need to obtain the description of composition factors of generalized principal series, without using (B). Using theory of intertwining operators developed in [14], we can reduce description to generalized rank one case, in the same manner as it was done for real reductive groups by B. Speh and D. Vogan (§ 3 of [16]). The Zelevinsky results in § 9 of [24] imply directly description for generalized rank one case.

The content of this paper is following. In the first section we introduce the notation used in this paper and recall of same basic results related to this notation. The exposition and results do not depend on (B). In the second section we prove without using (B), some technical statements necessary for proving the main results. In the third section, the main results are proved assuming (B) and only the local methods are used . In Apendix we prove the main results without assuming (B) when char F = 0. In order to do that, at the beginning of Apendix we prove some necessary results on the classification L without assuming (B). We are not using the result of [19] here as we do in the third section.

The notation which we are using in this paper is the same as that of [18] (or [24]), with two exceptions. In this paper we are dealing with t instead of ^{t} (that was discussed before). This new point of view demanded introducing of Langlands classification, and symmetric notation for classifications of Langlands and Zelevinsky. Therefore we accepted the notation [12] where that was realized.

Now we introduce some basic notation. The field of real numbers is denoted by $\mathbb{R}$, the subring of integers is denoted by $\mathbb{Z}$, the subset of non-negative integers is denoted by $\mathbb{Z}_+$ and the subset of positive integers is denoted by $\mathbb{N}$.

The following paper of this author considers the unitary dual of GL(n) over archimedean fields (i.e. over the real and the complex numbers).

Let us point out that $\hat{G}$ is in a natural way a topological space. In the case of reductive groups over a local field this topology has been found only for a few rank one groups. We have in preparation a paper dealing with the topology of $GL(n,F)^{\wedge}$ in terms of the parametrization of Theorem A., and describing this topology for, at least, $n \leq 17$. Roughly speaking the construction of unitarizable irreducible representation in [18] is based on identification of some limits in $GL(n,F)^{\wedge}$.

Some of the results of the present paper were announced in [22].

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1. Preliminaries

In this section we introduce notation and facts related to the non-unitary dual of $GL(n)$ over a non-archimedean field. For more details and also proofs, one may consult [24] and [12].

Let F be a locally compact non-archimedean field. Set $G_n = GL(n,F)$. The category of all smooth representatons of G_n of finite length is denoted by $\text{Alg } G_n$.

Let $\tau_i \in \text{Alg } G_{n_i}$, $i = 1,2$. Let P be the standard parabolic subgroup of $G_{n_1+n_2}$ whose Levi factor is naturally isomorphic to $G_{n_1} \times G_{n_2}$. Denote by $\tau_1 \times \tau_2$ the representation of $G_{n_1+n_2}$ induced from P by $\tau_1 \otimes \tau_2$ (induction is normalized).

Let R_n be the Grothendieck group of the category $\text{Alg} G_n$. There is a natural projection

$$\text{Alg } G_n \to R_n .$$

Set $R = \bigoplus_{n \geq 0} R_n$. The mapping

$$(\tau,\sigma) \to \tau \times \sigma,$$

$$\text{Alg } G_n \times \text{Alg } G_m \to \text{Alg } G_{n+m}$$

induces a bilinear mapping

$$R_n \times R_m \to R_{n+m}$$

which we shall denote by the same symbol. This bilinear mapping induces on R the structure of a graded commutative associative ring. Therefore, we have the notion of a homogenious elements in R, and the notion of a degree of elements of R.

Let $\tilde{G}_n$ be the set of all equivalence classes of irreducible smooth representations of G_n. By $\hat{G}_n$ we denote the subset of all unitarizable classes in $\tilde{G}_n$. We identify $\tilde{G}_n$ with the subset of R_n in a natural way. Then $\tilde{G}_n$ is a basis of a free abelian group R_n. Let $C(G_n)$ be the subset of all classes of cuspidal representations in $\tilde{G}_n$, i.e. the subset of all representations in $\tilde{G}_n$ whose matrix coefficients are compactly supported modulo the center of G_n. Set

$$\mathrm{Irr} = \bigcup_{n=0}^{\infty} \tilde{G}_n ,$$

$$\mathrm{Irr}^u = \bigcup_{n=0}^{\infty} \hat{G}_n ,$$

$$C = \bigcup_{n=1}^{\infty} C(G_n),$$

$$C^u = C \cap \mathrm{Irr}^u$$

Now Irr is a basis of free abelian group R.

If X is a set, then the additive semigroup of all functions from X into the non-negative integers, with finite support, is denoted by $M(X)$. Elements of $M(X)$ are called finite multisets in X. If $\{x_1,\ldots,x_n\}$ is the support of $f \in M(X)$, then we shall write f also as

$$f = (\underbrace{x_1,\ldots,x_1}_{f(x_1)\text{-times}}, \underbrace{x_2,\ldots,x_2}_{f(x_2)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{f(x_n)\text{-times}})$$

We shall identify the set of all subsets of X with the subset of $M(X)$, in a natural way. Also, we identify X with the subset of $M(X)$. The number

$$\sum_{x \in X} f(x)$$

is called the cardinal number of f and denoted by $\mathrm{card}\, f$.

Let $| \ |_F$ be the natural absolute value on F. The representation $g \to |\det g|_F$ of G_n is denoted by ν. For $\rho \in C$ and a positive integer n set

$$\Delta[n]^{(\rho)} = \{\nu^{-(n-1)/2}\rho, \nu^{1-(n-1)/2}\rho, \ldots, \nu^{(n-1)/2}\rho\} .$$

Then $\Delta[n]^{(\rho)}$ is called a segment in C. The set of all segments in C is denoted by $S(C)$.

Let $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$. As $\Delta_i \subseteq C$, we may consider $\Delta_i \in M(C)$. Define supp a by

$$\text{supp } a = \Delta_1 + \ldots + \Delta_n \in M(C).$$

Two segments $\Delta[n_i]^{(\rho_i)}$, $i = 1,2$, are called linked if

$$\Delta[n_1]^{(\rho_1)} \cup \Delta[n_2]^{(\rho_2)}$$

is again a segment, and

$$\Delta[n_1]^{(\rho_1)} \cup \Delta[n_2]^{(\rho_2)} \notin \{\Delta[n_1]^{(\rho_1)}, \Delta[n_2]^{(\rho_2)}\}$$

Let $\Delta[n_i]^{(\rho_i)}$, $i = 1,2$, be linked segments. Then there exists $\alpha \in \mathbb{R}$ so that $\rho_2 = \nu^{\alpha}\rho_1$. If $\alpha > 0$ then we say that $\Delta[n_1]^{(\rho_1)}$ precedes $\Delta[n_2]^{(\rho_2)}$ and write

$$\Delta[n_1]^{(\rho_1)} \to \Delta[n_2]^{(\rho_2)}$$

Let $\Delta = \{\rho, \nu\rho, \ldots, \nu^{n-1}\rho\} \in S(C)$ and $\alpha \in \mathbb{R}$. Set

$$\nu^{\alpha}\Delta = \{\nu^{\alpha}\rho, \nu^{\alpha+1}\rho, \ldots, \nu^{\alpha+n-1}\rho\}.$$

Let $a = (\Delta_1, \ldots, \Delta_n) \in M(S(C))$ and $\alpha \in \mathbb{R}$. Put

$$\nu^{\alpha}a = (\nu^{\alpha}\Delta_1, \ldots, \nu^{\alpha}\Delta_n).$$

For a representation $\pi \in \mathrm{Alg}\, G_n$, $\tilde{\pi}$ denotes the contragradient of π, and $\bar{\pi}$ the conjugated representation of π. The representation $\tilde{\bar{\pi}}$ is called the Hermitian contragradient of π and denoted by π^+.

For $\Delta = \{\rho, \nu\rho, \ldots, \nu^{n-1}\rho\} \in S(C)$ set:

$$\tilde{\Delta} = \{\tilde{\rho}, (\nu\rho)^\sim, \ldots, (\nu^{n-1}\rho)^\sim\} \in S(C),$$

$$\bar{\Delta} = \{\bar{\rho}, (\overline{\nu\rho}), \ldots, (\overline{\nu^{n-1}\rho})\} \in S(C),$$

$$\Delta^+ = (\bar{\Delta})^\sim = \{\rho^+, (\nu\rho)^+, \ldots, (\nu^{n-1}\rho)^+\} \in S(C).$$

If $a = (\Delta_1, \ldots, \Delta_n)$, then we set

$$\tilde{a} = (\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$$

$$\bar{a} = (\overline{\Delta_1}, \ldots, \overline{\Delta_n})$$

$$a^+ = (\Delta_1^+, \ldots, \Delta_n^+).$$

Note that

$$(\Delta[n]^{(\rho)})^\sim = \Delta[n]^{(\tilde{\rho})}$$

$$\overline{(\Delta[n]^{(\rho)})} = \Delta[n]^{(\bar{\rho})},$$

$$(\Delta[n]^{(\rho)})^+ = \Delta[n]^{(\rho^+)})$$

$$\nu^\alpha \Delta[n]^{(\rho)} = \Delta[n]^{(\nu^\alpha\rho)}.$$

For $\sigma_i \in \mathrm{Alg}\, G_i$, $i = 1,2$, we have

$$(\sigma_1 \times \sigma_2)^\sim \cong \tilde{\sigma}_1 \times \tilde{\sigma}_2,$$
$$(\overline{\sigma_1 \times \sigma_2}) \cong \overline{\sigma_1} \times \overline{\sigma_2},$$
$$(\sigma_1 \times \sigma_2)^+ \cong \sigma_1^+ \times \sigma_2^+$$
$$\nu^\alpha (\sigma_1 \times \sigma_2) \cong (\nu^\alpha\sigma_1) \times (\nu^\alpha\sigma_2).$$

Let $\Delta=\{\rho,\nu\rho,\ldots,\nu^{m-1}\rho\}\in S(C)$. Then the representation

$$\rho \times \nu\rho \times\ldots\times\nu^{n-1}\rho$$

possesses the unique irreducible sub-representation which is denoted by $Z(\Delta)$, and the unique irreducible quotient which is denoted by $L(\Delta)$.

Let $a = (\Delta_1,\ldots,\Delta_n)\in M(S(C))$. Choose such permutation σ of $\{1,\ldots,n\}$, so that holds:

$$\Delta_{\sigma(i)} \to \Delta_{\sigma(j)} \Rightarrow \sigma(i)>\sigma(j) \ , \qquad 1\leq i,j\leq n$$

The representation

$$Z(\Delta_{\sigma(1)})\times Z(\Delta_{\sigma(2)})\times\ldots\times Z(\Delta_{\sigma(n)})$$

does not depend on the choice of σ (up to an isomorphism), but only on a. (Proposition 6.4. of [24]). We denote this representation by

$$\zeta(a).$$

Now $\zeta(a)$ has a unique irreducible subrepresentation which is denoted by $Z(a)$.

The representation

$$L(\Delta_{\sigma(1)})\times L(\Delta_{\sigma(2)})\times\ldots\times L(\Delta_{\sigma(n)})$$

does not depend on σ as above. This can be proved in the same way like Proposition 6.4 of [24], using Theorem 9.7., (a) of [24]. We denote this representation by $\lambda(a)$. The representation $\lambda(a)$ has a unique irreducible quotient, which is denoted by $L(a)$.

By Theorem 6.1 of [24], the mapping

$$\begin{array}{c} a \to Z(a) \\ M(S(C)) \to \mathrm{Irr} \end{array}$$

is a bijection. This mapping parametrizes Irr. This is called Zelevinsky classification (see [24]).

The mapping

$$\begin{array}{c} a \to L(a) \\ M(S(C)) \to \mathrm{Irr} \end{array}$$

is a bijection. This is another parametrization of Irr and it is a version of Langlands classification of Irr. As presented here, this classification was presented by F. Rodier in [12] (see also [9]).

Let D^u denote the set of all classes of square-integrable representations in Irr^u. Set

$$D = \{\nu^\alpha\pi \,;\, \pi \in D^u,\ \alpha \in \mathbb{R}\}.$$

Elements of D are called essentially square-integrable representations. For $\delta = \nu^\alpha\pi \in D, \pi \in D^u, \alpha \in \mathbb{R}$ we define δ^u and $e(\delta)$ by

$$\delta^u = \pi \quad \text{and} \quad e(\delta) = \alpha.$$

By Theorem 9.3. of [24],

$$\Delta \to L(\Delta),\ S(C) \to D$$

is a bijection. Denote this bijection by φ. This bijection lifts to the bijection of $M(S(C))$ and $M(D)$ which is again denoted by φ. Now

$$\begin{array}{c} d \to L(\varphi^{-1}(d)) \\ M(D) \to \mathrm{Irr} \end{array}$$

is a bijection which will be again denote by L. This is a parametrization of Irr and can be described directly, without goint to $M(S(C))$, as follows. Let $d = (\delta_1,\ldots,\delta_n) \in M(D)$. Suppose that the ordering of δ_i satisfies

$$i < j \Rightarrow e(\delta_i) \geq e(\delta j).$$

Then $\lambda(d) = \delta_1 \times \ldots \times \delta_n$ possess a unique irreducible quotient, and it is equal to $L(d)$. This classification $d \to L(d)$, $M(D) \to \mathrm{Irr}$ is directly related to the Langlands classification of [9] in a simple manner.

For $d = (\delta_1,\ldots,\delta_n)$, $M(D)$, $\alpha \in \mathbb{R}$, set, as before:

$$\tilde{d} = (\tilde{\delta}_1,\ldots,\tilde{\delta}_n),$$

$$\bar{d} = (\overline{\delta_1},\ldots,\overline{\delta_n})$$

$$d^+ = \bar{\tilde{d}} = (\delta_1^+,\ldots,\delta_n^+).$$

$$\nu^\alpha d = (\nu^\alpha\delta_1,\ldots,\nu^\alpha\delta_n).$$

Let $\delta \in D, \delta = \nu^{e(\delta)}\,\delta^u$ and $\alpha \in \mathbb{R}$. Now

$$\tilde{\delta} = \nu^{-e(\delta)}(\delta^u)^\sim \quad \text{i.e.} \quad e(\tilde{\delta}) = -e(\delta) \quad \text{and} \quad (\tilde{\delta})^u = (\delta^u)^\sim ;$$

$$\bar{\delta} = \nu^{e(\delta)}(\overline{\delta^u}) \quad \text{i.e.} \quad e(\bar{\delta}) = e(\delta) \quad \text{and} \quad (\bar{\delta})^u = (\overline{\delta^u});$$

$$\delta^+ = \nu^{-e(\delta)}\delta^u \quad \text{i.e.} \quad e(\delta^+) = -e(\delta) \text{ and } (\delta^+)^u = \delta^u ,$$

$$\nu^\alpha\delta = \nu^{\alpha+e(\delta)}\delta^u \quad \text{i.e.} \quad e(\nu^\alpha\delta) = \alpha+e(\delta) \text{ and } (\nu^\alpha\delta)^u = \delta^u,$$

for $\alpha \in \mathbb{R}$.

Now we shall recall some very well known facts about classifications

1.1. Proposition: For $\alpha \in \mathbb{R}$, $a \in M(S(C))$ and $d \in M(D)$ we have:

(i) $\nu^{\alpha}\zeta(a) \cong \zeta(\nu^{\alpha}a), \overline{\zeta(a)} \cong \zeta(\bar{a})$;

$\nu^{\alpha}\lambda(a) \cong \lambda(\nu^{\alpha}a), \overline{\lambda(a)} \cong \lambda(\bar{a})$;

$\nu^{\alpha}\zeta(d) \cong \zeta(\nu^{\alpha}d), \overline{\zeta(d)} \cong \zeta(\bar{d})$.

(ii) $\nu^{\alpha}Z(a) = Z(\nu^{\alpha}a), \ \overline{Z(a)} = Z(\bar{a})$;

$\nu^{\alpha}L(a) = L(\nu^{\alpha}a), \ \overline{L(a)} = L(\bar{a})$;

$\nu^{\alpha}L(d) = L(\nu^{\alpha}d), \ \overline{L(d)} = L(\bar{d})$.

(iii) $Z(a)^{\sim} = Z(\tilde{a}), \ Z(a)^{+} = Z(a^{+})$;

$L(a)^{\sim} = L(\tilde{a}), \ L(a)^{+} = L(a^{+})$;

$L(d)^{\sim} = L(\tilde{d}), \ L(d)^{+} = L(d^{+})$.

Proof: For (i) one constructs desired isomorphisms directly. Clearly (i) implies (ii). By Theorem 7.10. of [24], $Z(a)^{\sim} = Z(\tilde{a})$, and now (ii) implies $Z(a)^{+} = Z(a^{+})$.

The relation $L(d)^{\sim} = L(\tilde{d})$ is another expression of the relation (3.3.13) of [9]. Now (ii) implies $L(d)^{+} = L(d^{+})$. We obtain $L(a)^{\sim} = L(\tilde{a})$ from the previous case $L(d)^{\sim} = L(\tilde{d})$ and Proposition 9.5. of [24] which states that $L(\Delta)^{\sim} = L(\Delta^{\sim})$.

Let $\pi \in \mathrm{Irr}$. Take $a, b \in M(S(C))$ such that

$$\pi = Z(a), \quad \pi = L(b).$$

Then supp a = supp b (Proposition 1.10 of [24]). Define

$$\text{supp } \pi = \text{supp } a = \text{supp } b.$$

Consider supp π as an element of $M(S(C))$ in a natural way. Then the set of all representations in Irr whose support is equal to supp π is just the set of all composition factors of

$$\zeta(\text{supp } \pi) = \lambda(\text{supp } \pi).$$

Suppose that $\pi_1, \pi_2 \in \text{Irr}$ and σ is a composition factor of $\pi_1 \times \pi_2$, then

$$\text{supp } \sigma = \text{supp } \pi_1 + \text{supp } \pi_2.$$

We introduce, like in [12], an additive homorphism t of R defined by

$$t(Z(a)) = L(a), \; a \in M(S(C)).$$

There exists a unique mapping

$$t: M(S(C)) \to M(S(C))$$

such that

$$t(Z(a)) = Z(t(a)), \; a \in M(S(C)),$$

i.e.

$$L(a) = Z(t(a)).$$

This implies

$$t(L(a)) = L(t(a)), \; a \in M(S(C)).$$

Formally, we have

$$t^{-1}(L(a)) = Z(a),$$

$$t^{-1}(L(a)) = L(t^{-1}(a))$$

$$t^{-1}(Z(a)) = Z(t^{-1}(a)).$$

The homorphism t contains all informations about connection of Zelevinsky and Langlands classification.

We have

$$\text{supp } t(\pi) = \text{supp } \pi \ , \quad \pi \in \text{Irr}.$$

A.V. Zelevinsky proved:

1.2. Proposition: The ring R is a $\mathbb{Z}$-polynomial algebra over $\{Z(\Delta); \Delta \in S(C)\}$.

We formulate the preceding proposition in another way which suits better to Langlands classification(and also to the case of archimedean fields).

1.3. Proposition: The ring R is a $\mathbb{Z}$-polynomial algebra over D i.e. over $\{L(\Delta); \Delta \in S(C)\}$.

This proposition is a consequence of Langlands classification (see, in particular, Lemma A. 4. f. of [2]) and Jacquet result stating that induced representation of G_m by square-integrabe one, is irreducible ([7]). This proof applies to the case of $GL(n)$ over any local field.

Formally, we obtain this proposition from the preceding one in the following way. By Proposition 1.2.

$$Z(\Delta) \to L(\Delta), \ \Delta \in S(C),$$

extends uniquely to a morphism of the ring R

$$^t: R \to R.$$

A.V. Zelevinsky showed that ^{t} is an involutive automorphism of R. This implies directly Proposition 1.3.

A.V. Zelevinsky conjectured $(\mathrm{Irr})^t \subseteq \mathrm{Irr}$. A proof of this was announced by J.N. Bernstein (see [24],[25],[12] and [1]), but unfortunately, there is no written proof of this result known to this author.

2. Some lemmas

For $n,d \in \mathbb{N}$ and $\rho \in C$ set

$$a(n,d)^{(\rho)} = (\nu^{-\frac{n-1}{2}}\Delta[d]^{(\rho)}, \nu^{1-\frac{n-1}{2}}\Delta[d]^{(\rho)}, \dots, \nu^{\frac{n-1}{2}}\Delta[d]^{(\rho)}).$$

This is a multisegment in C.

Denote by (U^m) the following statement

(U^m): if $n,d \in \mathbb{N}$ and $\rho \in C^u$ such that

$$(nd)\ \deg\rho \leq m$$

then $Z(a(n,d)^{(\rho)})$ is unitarizable.

Note that (U^m) is not the same statement as the statement (U_m) in 3.6. of [18].

Recall that Irr^u is a multiplicatively closed subset of R. The following lemma is contained implicitely in the fourth section of [18]. For $\sigma \in \mathrm{Irr}$ and $\alpha \in \mathbb{R}$ set

$$\pi(\sigma,\alpha) = \nu^{\alpha}\sigma \times \nu^{-\alpha}\sigma^{+} .$$

2.1. Lemma: Let $m \geq 1$. Suppose that (U^{m-1}) holds. Let

$$X_{m-1} = \{Z(a(n,d)^{(\rho)}),\ \pi(Z(a(n,d)^{(\rho)}),\alpha);$$

$$n,d \in \mathbb{N}, \rho \in C^u,\ (nd)\ \deg\rho \leq m-1,\ 0 < \alpha < 1/2\} .$$

Then:

(i) If $\sigma_1,\dots,\sigma_k \in X_{m-1}$, then $\sigma_1 \times \dots \times \sigma_k \in \mathrm{Irr}^u$. In particular, if

$\deg \sigma_1 + \dots + \deg\sigma_k = m$, then $\sigma_1 \times \dots \times \sigma_k \in \hat{G}_m$.

(ii) Set

$$I(\hat{G}_m) = \hat{G}_m \setminus \{Z(a(n,d)^{(\rho)});\ n,d \in \mathbb{N}, \rho \in C^u, (nd)\deg\rho = m\} .$$

If $\pi \in I(\hat{G}_m)$, then there exist $\sigma_1,\ldots,\sigma_i \in X_{m-1}$ such that

$$\pi = \sigma_1 \times \ldots \times \sigma_i .$$

Representations $\sigma_1,\ldots,\sigma_i$ are determined uniquely up to a permutation.

Proof: By (U^{m-1}) and Proposition 2.9. of [18], $X_{m-1} \subseteq \mathrm{Irr}^u$. Now (i) is a consquence of the fact that Irr^u is multiplicatively closed.

The uniqueness of a presentation of π in (ii) is a direct consequence of Proposition 3.18. of [18] (it can be obtained also without use of that proposition, but then argument should be longer).

We shall prove existence of a presentation of π in (ii) by induction. For $m = 1$ there is nothing to prove.

Let $m \geq 2$. Take $\pi \in I(\hat{G}_m)$. We can decompose

$$\pi = \tau_1 \times \ldots \times \tau_k$$

such that $\tau_i \in \mathrm{Irr}^u$, and such that there exists $\rho_i \in C^u$, $0 \leq \alpha_1 \leq 1/2$ so that

$$\mathrm{supp}\ \tau_i \in M(\{\nu^n(\nu^{\alpha_i}\rho_i), \nu^n(\nu^{-\alpha_i}\rho_i);\ n \in \mathbb{Z}\})$$

$i = 1,\ldots,k$ (4.1 of [18]). If the presentation $\pi = \tau_1 \times \ldots \times \tau_k$ is non-trivial, then the inductive assumption and (U^{m-1}) implies existence of the presentation.

Thus, we may suppose that

$$\text{supp } \pi \in M(\{\nu^n(\nu^{\alpha}\rho), \nu^n(\nu^{-\alpha}\rho);\ n \in \mathbb{Z}\}).$$

Let $\pi = Z(a)$, $a \in M(S(C))$.

We proceed now in the same way as in the proof of Lemma 4.11. of [18].

We shall consider first the case $\alpha_i \in \{0,1/2\}$. Since the highest shifted derivative π' of π is irreducible and unitarizable, we obtain by inductive assumption, considering the support of π, that

$$\pi' = Z(a(n_1,d_1)^{(\rho)} + \ldots + a(n_k,d_k)^{(\rho)}).$$

This implies that

$$\pi = Z(a(n_1,d_1+1)^{(\rho)} + \ldots + a(n_k,d_k+1)^{(\rho)} + \varphi)$$

$$= Z(a(n_1,d_1^*)^{(\rho)} + \ldots + a(n_k,d_k^*)^{(\rho)} + \varphi),\ d_i^* = d + 1$$

where $\varphi \in M(\{\nu^n(\nu^{\alpha}\rho);n \in \mathbb{Z}\})$. The assumption $\pi \in I(\hat{G}_m)$ implies

$$(n_i d_i^*) \deg\rho < m.$$

Thus $Z(a(n_i,d_i^*)^{(\rho)}) \in X_{m-1}$. Since π is unitary we have $\pi = \pi^+$, and thus

$$\varphi = \{\nu^{p_1+\alpha}\rho, \nu^{-(p_1+\alpha)}\rho, \ldots, \nu^{p_j+\alpha}\rho, \nu^{-(p_j+\alpha)}\rho\}$$

where $p_i \in \mathbb{Z}_+$. Now

$$\pi \times Z(\Delta[2p_1 + 2\alpha - 1]^{(\rho)}) \times \ldots \times Z(\Delta[2p_j + 2\alpha - 1]^{(\rho)}) =$$

$$= Z(a(n_1,d_1^*)^{(\rho)}) \times \ldots \times Z(a(n_k,d_k^*)^{(\rho)}) \times$$

$$\times Z(\Delta[2p_1 + 2\alpha + 1]^{(\rho)}) \times \ldots \times Z(\Delta[2p_j + 2\alpha + 1]^{(\rho)}),$$

where $\Delta[s]^{(\rho)} = \emptyset$ if $s \leq 0$. Proposition 3.18. of [18] implies that π is a subproduct of the right hand side and this implies existance of the presentation.

Suppose that $0 < \alpha < 1/2$. Using Lemma 4.10 of [18], we obtain a presentation in the similar way as in the preceding case. In fact, the proof of Lemma 4.11. contains a proof of this case.

2.2. <u>Lemma:</u> Let $m \geq 1$. Suppose that (U^{m-1}) holds. Let $n, d \in \mathbb{N}$, $\rho \in C^u$ such that

$$\text{(nd)} \quad \deg \rho = n \quad \text{and} \quad n \leq d.$$

Then $Z(a(n,d)^{(\rho)})$ is unitarizable.

<u>Proof:</u> If $n = 1$, then $Z(a(1,d)^{(\rho)}) = Z(\Delta[d]^{(\rho)})$ and this is unitarizable by [18] (see also [1]). Suppose $n \geq 2$. By (U^{m-1})

$$Z(a(n-1,d)^{(\rho)})$$

is unitarizable. Proposition 2.9. of [18] implies that all composition factors of

$$\nu^{1/2} Z(a(n-1,d)^{(\rho)}) \times \nu^{-1/2} Z(a(n-1,d)^{(\rho)})$$

are unitarizable. Thus

$$Z(\nu^{1/2} a(n-1,d)^{(\rho)} + \nu^{-1/2} a(n-1,d)^{(\rho)}) =$$

$$= Z(a(n,d)^{(\rho)} + a(n-2,d)^{(\rho)})$$

is unitarizable. Now Lemma 3.11 of [18] implies that

$$Z(a(n,d)^{(\rho)}) \times Z(a(n-2,d)^{(\rho)})$$

is irreducible, since

$$Z(a(n,d-1)^{(\rho)}) \times Z(a(n-2,d-1)^{(\rho)})$$

is irreducible. By (ii) of Theorem 2.5. in [18] $Z(a(n,d)^{(\rho)})$ is unitarizable.

□

2.3. <u>Lemma:</u> Let $n,d \in \mathbb{N}$ and $n < d$. Then

$$t(Z(a(n,d)^{(\rho)})) \neq Z(a(n,d)^{(\rho)}) \ , \quad \rho \in C,$$

i.e.

$$L(a(n,d)^{(\rho)}) \neq Z(a(n,d)^{(\rho)}).$$

<u>Proof:</u> Let $\rho \in C(G_m)$, i.e. $\deg \rho = m$. Suppose that

$$L(a(n,d)^{(\rho)}) = Z(a(n,d)^{(\rho)}).$$

Thus $\lambda(a(n,d)^{(\rho)})$ and $\zeta(a(n,d)^{(\rho)})$ have a common composition factor, by the definition of $L(a(n,d)^{(\rho)})$ and $Z(a(n,d)^{(\rho)})$.

Let P_0 (resp. P_1) be the unique standard parabolic subgroup of $GL(mdn,F)$ whose Levi factor M_0 (resp. M_1) is naturally isomorphic to $GL(m,F)^{dn}$ (resp. $GL(md,F)^n$). Let N_i be the unipotent radical of $P_i, i = 0,1$.

In the rest of the proof we shall use freely notation of § 1 of [24].

First of all, note that there is no standard parabolic subgroup of $GL(mnd,F)$ associated to P_0 different from P_0. Since $Z(a(n,d)^{(\rho)})$ is a composition factor of an induced representation from P_0 by a cuspidal irreducible representation of M_0, by § 6 of [3]

$$r_{(m,m,\ldots m),(nmd)}(Z(a(n,d)^{(\rho)}))\neq 0 \quad .$$

Now $Z(a(n,d)^{(\rho)}) = L(a(n,d)^{(\rho)})$ implies that representations

$$r_{(m,\ldots,m),(nmd)}(\zeta(a(n,d)^{(\rho)}))$$

and

$$r_{(m,\ldots mm),(nmd)}(\lambda(a(n,d)^{(\rho)}))$$

have a common non-trival irreducible composition factor.

Let $a(n,d)^{(\rho)} = (\Delta_1,\ldots,\Delta_n)$ where $\Delta_1 \to \Delta_2 \to \to \Delta_n$. Choose $\sigma \in C(G_m)$ so that

$$\begin{aligned}
\Delta_1 &= [\sigma,\nu^{d-1}\sigma], \\
\Delta_2 &= [\nu\sigma,\nu^{d}\sigma] , \\
&\ldots, \\
\Delta_m &= [\nu^{n-1}\sigma,\nu^{n-1+d-1}\sigma] .
\end{aligned}$$

Now

$$(*) \quad (\sigma\otimes\nu\sigma\otimes\ldots\otimes\nu^{d-1}\sigma)\otimes(\nu\sigma\otimes\ldots\otimes\nu^{d}\sigma)\otimes\ldots\otimes(\nu^{n-1}\sigma\otimes\ldots\otimes\nu^{n-1+d-1}\sigma)$$

is a composition factor of $r_{(m,\ldots,m),(nmd)}(\zeta(a(n,d)^{(\rho)}))$

and each other irreducible compostion factor is obtained from (*) after a permutation of factors of (*) with a permutation which preserves the order of elements of each bracket (see (3) of Proof of Proposition 6.9 of [24]).

In the similar way

$$(**)\quad (\nu^{d-1}\sigma\otimes\nu^{d-2}\sigma\otimes\ldots\otimes\sigma)\otimes(\nu^{d}\sigma\otimes\ldots\otimes\nu\sigma)\otimes\ldots\otimes(\nu^{n-1+d-1}\sigma\otimes\ldots\otimes\nu^{n-1}\sigma)$$

is a composition factor of $r_{(m,\ldots,),(nmd)}(\lambda(a(n,d)^{(\rho)}))$ and each other irreducible composition factor is obtained from (**) after a permutation of (**) with a permutation which preserves the order of elements of each bracket.

Let τ be an irreducible factor of $r_{(m,\ldots,m),(nmd)}(\zeta(a(n,d)^{(\rho)}))$. Let $\tau=\nu^{\alpha_1}\sigma\otimes\ldots\otimes\nu^{\alpha_{mnd}}\sigma$. Then there exist $1\leqq p_1<\ldots<p_d\leqq mnd$ so that

$$(***)\qquad \alpha_{p_1}=i-1,\quad i=1,\ldots,d.$$

Let ω be an irreducible composition factor of $r_{(m,\ldots,m),(nmd)}(\lambda(a(n,d)^{(\rho)}))$. Let $\omega=\nu^{\beta_1}\sigma\otimes\ldots\otimes\nu^{\beta_{mnd}}\sigma$. Now simple combinatorial observation implies that if $\beta_{g_1}<\beta_{g_2}<\ldots<\beta_{g_r}$, for some $1\leqq g_1<g_2<\ldots<g_r\leqq mnd$, then $r\leqq n$.

Therefore $r_{(m,\ldots,m),(nmd)}(\zeta(a(n,d)^{(\rho)}))$ and $r_{(m,\ldots,m),(nmd)}(\lambda(a(n,d)^{(\rho)}))$ can not have a common non-trival composition factor.

We obtained a contradiction. This proves the lemma.

3. Main results

The following theorem completly solves the unitarizability problem for GL(n) over non-archimedean field, and also presents explicit connection between Zelevinsky and Langlands classifications in the unitary case. The Bernstein Conjecture 8.10 of [1] was stating that $t(\mathrm{Irr}^u) \subseteq \mathrm{Irr}^u$. The following theorem describes completly $t:\mathrm{Irr}^u \to \mathrm{Irr}^u$.

3.1 Theorem: Let

$$B = \{Z(a(n,d)^{(\rho)}),\ \pi(Z(a(n,d)^{(\rho)}),\alpha)\ ;$$

$$n,d \in \mathbb{N},\ \rho \in C^u,\ 0 < \alpha < 1/2\} .$$

Fix $m \in \mathbb{N}$. Then

(i) If $\sigma_1,\dots,\sigma_k \in B$ such that

$$\deg \sigma_1 + \dots + \deg \sigma_k = m,$$

then $\sigma_1 \times \dots \times \sigma_k \in \hat{G}_m$.

(ii) If $\pi \in \hat{G}_m$, then there exists $\tau_1,\dots,\tau_j \in B$ so that

$$\pi = \tau_1 \times \dots \times \tau_j .$$

Such $\tau_1,\dots,\tau_j$ are unique up to a permutation and

$$\deg \tau_1 + \dots + \deg \tau_j = m.$$

(iii) The following formula holds

$$t(Z(a(n,d)^{(\rho)})) = Z(a(d,n)^{(\rho)})$$

$$t(\pi(Z(a(n,d)^{(\rho)}),\alpha)) = \pi(Z(a(d,n)^{(\rho)}),\alpha)$$

for elements of B.

3.2. Remark: Note that by (i), $B \subseteq \mathrm{Irr}^u$. The statement (iii), together with (i) and (ii) describes explicitly $t(Z(a))$ in terms of Zelevinsky classification, when $Z(a)$ is unitarizable. The same description is valid for Langlands classification.

Proof: We shall prove (i), (ii) and (iii) by induction on m (in (iii), $m = (nd) \deg \rho$). Define X_{m-1} as in Lemma 2.1.

Suppose that $m = 1$. Then (i), (ii) and (iii) holds. Here the only possibly $Z(a(n,d)^{(\rho)})$ is for $n = d = 1$ and ρ a unitary character of G_1, t is here identity.

Suppose that (i), (ii) and (iii) holds for $k \leq m-1$. Then (U^{m-1}) holds ((U^{m-1}) is defined in the beginning of the second section). Now Lemma 2.2. implies that $Z(a(n,d)^{(\rho)})$ is unitarizable, for $n \leq d$ and $(nd) \deg \rho = m$. By [19], $t(Z(a(n,d)^{(\rho)}))$ is unitarizable, i.e.

$$t(Z(a(n,d)^{(\rho)})) \in \hat{G}_m \quad .$$

From the inductive assumption one sees that

$$t(Z(a(n,d)^{(\rho)})) \notin I(\hat{G}_m) \quad ,$$

where $I(\hat{G}_m)$ is defined in (ii) of Lemma 2.1. (by the inductive assumption and Lemma 2.1. we know how t acts on $I(\hat{G}_m)$). One can obtain that also from Proposition 3.18 of [18] ($t(Z(a(n,d)^{(\rho)}))$ is a prime element of R since it is an image of the prime element under the automorphism of R, and elements in $I(G_m)$ are composite by Lemma 2.1).

The above discussion and Lemma 2.1 implies

$$t(Z(a(n,d)^{(\rho)})) \in \{Z(a(n_1,d_1)^{(\rho_1)});\ n_1,d_1 \in \mathbb{N}$$

$$\rho_1 \in C^u,\ (n_1 d_1) \deg \rho_1 = m\} .$$

Thus $t(Z(a(n,d)^{(\rho)})) = Z(a(n_1,d_1)^{(\rho_1)})$ for some n_1,d_1,ρ_1 as above. The fact

$$\operatorname{supp} a(n,d)^{(\rho)} = a(n_1,d_1)^{(\rho_1)}$$

implies

$$\rho = \rho_1$$

$$\{n,d\} = \{n_1,d_1\}$$

(support of $a(n,d)^{(\rho)}$ is computed in the proof of Theorem 3.8 of [18]). Therefore,

$$t(Z(a(n,d)^{(\rho)})) \in \{Z(a(n,d)^{(\rho)}),Z(a(d,n)^{(\rho)})\} .$$

If $n = d$ then

$$t(Z(a(n,n)^{(\rho)})) = Z(a(n,n)^{(\rho)}).$$

If $n < d$, then Lemma 2.3. implies

$$t(Z(a(n,d)^{(\rho)})) = Z(a(d,n)^{(\rho)}).$$

Thus $Z(a(d,n)^{(\rho)})$ is unitarizable. This means that $X_m \subseteq \mathrm{Irr}^u$. Since t is an involution, we have that

$$t(Z(a(d,n)^{(\rho)})) = Z(a(n,d)^{(\rho)}).$$

Thus, (iii) holds. Clearly (i) holds because if $\sigma_1,\dots,\sigma_k \in B$ and $\deg \sigma_1+\dots+\sigma_k = m$, then $\sigma_1,\dots,\sigma_k \in X_m$. Lemma 2.1. implies (ii).

The above theorem can be expressed in the following form:

3.3. THEOREM: Let

$$B = \{ a(n,d)^{(\rho)}, (\nu^{\alpha}a(n,d)^{(\rho)} + \nu^{-\alpha}a(n,d)^{(\rho)});$$

$$n,d \in \mathbb{N},\ \rho \in C^u, 0 < \alpha < 1/2\}.$$

Let $X(B)$ be the additive subsemigroup of $M(S(C))$ generated by
Then

$$a \to Z(a)$$

and

$$a \to L(a)$$

are bijections from $X(B)$ onto Irr^u.

The mapping

$$t: B \to B,$$

$$t: a(n,d)^{(\rho)} \to a(d,n)^{(\rho)};$$

$$(\nu^{\alpha}a(n,d)^{(\rho)} + \nu^{-\alpha}a(n,d)^{(\rho)}) \to (\nu^{\alpha}a(d,n)^{(\rho)} + \nu^{-\alpha}a(d,n)^{(\rho)})$$

extends uniquely to a morphism of semigroups $t: X(B) \to X(B)$.
Now

$$Z(a) = L(t(a))$$

and

$$t(Z(a)) = Z(t(a)).$$

3.4. <u>Remark</u>: It could be interesting to point out that since we have proved that

$$t(Z(a(n,d)^{(\rho)})) = Z(a(d,n)^{(\rho)}) ,$$

Theorem 3.8. of [18] implies that Conjecture 4.5. in [25] of Zelevinsky holds for representations,

$$Z(a(n,d)^{(\rho)})$$

(in fact, one should say; for $a(n,d)$).

This is a part of proof of the Zelevinsky conjecture in the unitary case. For the whole proof, one need to consider representations

$$Z(a(n_1,d_1)^{(\rho)}) \times \ldots \times Z(a(n_k,d_k)^{(\rho)})$$

(or $a(n_1,d_1)+ \ldots + a(n_k,d_k)$). One could expect that this case can be obtained by similar calculation like in the proof of Theorem 3.8. of [18].

Now we shall give one more consequence of Theorem 3.1. The following theorem describes the unitary dual of $GL(n)$ in terms of Langlands classification.

For $\delta \in D^u$ and $n \in \mathbb{N}$ set

$$u(\delta,n) = L(\nu^{-\frac{n-1}{2}}\delta , \nu^{1-\frac{n-1}{2}}\delta , \ldots , \nu^{\frac{n-1}{2}}\delta) .$$

We can characterize $u(\delta,n)$ as the unique irreducible quotient of

$$\nu^{\frac{n-1}{2}}\delta \times \nu^{\frac{n-1}{2}-1}\delta \times \ldots \times \nu^{-\frac{n-1}{2}}\delta$$

Clearly $L(a(n,d)^{(\rho)}) = u(L(\Delta[d]^{(\rho)}),n)$ and this implies $\{L(a(n,d)^{(\rho)});n,d \in \mathbb{N}, \rho \in C^u\} = \{u(\delta,n); \delta \in D^u, n \in \mathbb{N}\}$.

3.5. THEOREM: Let

$$B_t = \{u(\delta,n), \nu^{\alpha}u(\delta,n)\times \nu^{-\alpha}u(\delta,n);\ \delta\in D^u; n\in \mathbb{N},\ 0<\alpha<1/2\}.$$

(i) If $\pi_1,\dots,\pi_r \in B_t$, then $\pi_1\times\dots\times\pi_r \in \mathrm{Irr}^u$.

(ii) Let $\sigma\in\mathrm{Irr}^u$. Then there exist $\pi_1,\dots,\pi_r\in B_t$ so that $\sigma = \pi_1\times\dots\times\pi_r$. Multiset $(\pi_1,\dots,\pi_r)$ is uniquely determined by σ.

In [18] we introduced the notion of a rigid representation. Recall that for a representation $\pi\in\mathrm{Irr}$ we say that it is rigid if there exist $\rho_i\in C^u$ and $\alpha_i\in(1/2)\mathbb{Z}$ such that π is a composition factor of $(\nu^{\alpha_1}\rho_1)\times(\nu^{\alpha_2}\rho_2)\times\dots\times(\nu^{\alpha_k}\rho_k)$.

3.6. THEOREM: Let $\sigma\in\mathrm{Irr}$.

i) Suppose that $\pi(\sigma,\alpha)=\nu^{\alpha}\sigma\times\nu^{-\alpha}\sigma^+$ is irreducible and unitarizable for all $\alpha\in(-1/2,1/2)$. Then σ is a unitarizable rigid representation.

ii) Suppose that σ is a rigid representation such that $\pi(\sigma,\alpha)$ is an irreducible and unitarizable representation for some $\alpha\in(0,1/2)$. Then there exist unitarizable representations σ_1 and σ_2 so that

$$\sigma = \sigma_1\times\nu^{-1/2}\sigma_2.$$

Proof: The theorem is a consequence of Theorem 4.11. of [18] and Theorem 2.1.

3.7. Remark: (i) This theorem proves also Bernstein Conjecture 8.6. in [1] on complementary series and it proves conjectures (CI) and (CH) of [18].

(ii) The set B_t defined in Theorem 3.5. is equal to the set B defined in Theorem 3.1.

Appendix

In thisappendix we shall prove all the statements of the third section without using the Bernstein unpublished result that $(\mathrm{Irr})^t \subseteq \mathrm{Irr}$, if characteristic of F is zero. This means that we shall not use the result of [19].

We consider here the additive homomorphism

$$t: R \to R$$

defined by $t(Z(a)) = L(a)$, $a \in M(S(C))$. This is all we assume in this section about t (we do not assume that t is involutive, and also that t is multiplicative).

We shall first prove one result about classification L (without use of the Bernstein unpublished result).

A.1. Proposition: Let $a = (\Delta_1, \ldots, \Delta_n), b = (\Gamma_1, \ldots, \Gamma_m) \in M(S(C))$. Suppose that Δ_i and Γ_j are not linked for all $1 \leq i \leq n$, $1 \leq j \leq m$. Then:

(i) $\zeta(a) \times \zeta(b) \cong \zeta(a+b)$, $\lambda(a) \times \lambda(b) \cong \lambda(a+b)$.

(ii) $Z(a) \times Z(b) = Z(a+b)$.

Proof: The definition of ζ and λ implies (i). Proposition 8.5. of [24] gives $Z(a) \times Z(b) = Z(a+b)$.

A.2. Remark: One can obtain that $L(a) \times L(b) = L(a+b)$ when a, b are as in the above proposition, using multiplicity one of the Langlands representation in $\lambda(a+b)$. For our purposes, the following irreduciblity result will be sufficient.

A.3. Corollary: Let $r_i = (\rho_1^i, \ldots, \rho_{n_i}^i) \in M(C)$, $i = 1,2$. Suppose that

(*) $e(\rho_p^1) - e(\rho_q^2) \notin \{-1, 1\}$

for $1 \leq p \leq n_1$, $1 \leq q \leq n_2$. Let $a, b \in M(S(C))$ so that

$$\text{supp } a = r_1 \quad \text{and} \quad \text{supp } b = r_2.$$

Then $L(a) \times L(b)$ and $Z(a) \times Z(b)$ are irreducible and

$$L(a) \times L(b) = L(a+b) ,$$
$$Z(a) \times Z(b) = Z(a+b).$$

Proof: The assumption (*) implies that a and b are as in Proposition A.1., and (ii) of Proposition A.1. implies our statement for Zelevinsky classification.

Choose a^*, b^* so that

$$L(a) = Z(a^*) \quad \text{and} \quad L(b) = Z(b^*).$$

Now $\text{supp } a = \text{supp } a^*$ and $\text{supp } b = \text{supp } b^*$. The first part of our proof implies that

$$L(a) \times L(b) = Z(a^*) \times Z(b^*)$$

is irreducible.

Now. $L(a) \times L(b)$ is an irreducible quotient of $\lambda(a) \times \lambda(b)$. By (i) of Proposition A.1. we have

$$\lambda(a) \times \lambda(b) \cong \lambda(a+b)$$

since a and b satisfies the assumption of Proposition A.1. The representation $\lambda(a+b)$ has a unique irreducible quotient which is $L(a+b)$. Thus $L(a) \times L(b) = L(a+b)$.

A.4. Proposition: (i) Suppose that $a, b \in M(S(C)$. Then $L(a+b)$ is a composition factor of $L(a) \times L(b)$. In particular, if

(ii) If $c,d \in M(D)$, then $L(c+d)$ is a composition factor of $L(c) \times L(d)$.

Proof: First of all, note that it is enough to prove the proposition for $M(S(C))$.

Let $a,b \in M(S(C))$. Set

$$\text{supp } a = (\rho_1,\ldots,\rho_u), \text{ supp } b = (\sigma_1,\ldots,\sigma_v).$$

Let ε be the minimum of all

$$|1 + (e(\rho_i) - e(\sigma_j))| \text{ with } 1 + (e(\rho_i) - e(\sigma_j)) \neq 0,$$

$$|1 - (e(\rho_i) - e(\sigma_j))| \text{ with } 1 - (e(\rho_i) - e(\sigma_j)) \neq 0,$$

when $1 \leq i \leq u$, $1 \leq j \leq v$. Then $\varepsilon > 0$. Let $0 < \alpha < \varepsilon$. By the choice of α, a and $(\nu^\alpha b)$ satisfy the assumption of Corollary A.3. Thus $L(a) \times L(\nu^\alpha b)$ is irreducible and equals to $L(a + \nu^\alpha b)$. Proposition A.1. implies $\lambda(a) \times \lambda(\nu^\alpha b) \cong \lambda(a + \nu^\alpha b)$.

Suppose that a consists of n segments and b of m segments. We can denote that segments in the following way

$$a = (\Delta_{i(1)},\ldots,\Delta_{i(n)}),\ i(1) < \ldots < i(n),$$

$$b = (\Delta_{j(1)},\ldots,\Delta_{j(m)}),\ j(1) < \ldots < j(m)$$

with $\{i(1),\ldots,i(n)\} \cup \{j(1),\ldots,j(m)\} = \{1,2,\ldots,n+m\}$,

such that

$$\Delta_u \to \Delta_v \Rightarrow v < u .$$

For $0 \leq \alpha < \varepsilon$ set $\Delta^\alpha_{i(k)} = \Delta_{i(k)}$, $\Delta^\alpha_{j(k)} = \nu^\alpha \Delta_{j(k)}$.

Now $\quad a+\nu^{\alpha}b = (\Delta_1^{\alpha},\ldots,\Delta_{n+m}^{\alpha}).$

By construction

$$\lambda(a+\nu^{\alpha}b) \cong L(\Delta_1^{\alpha})\times\ldots\times L(\Delta_{n+m}^{\alpha})$$

for $0 \leq \alpha < \varepsilon$. The representation $\lambda(a+\nu^{\alpha}b)$ possess the unique irreducible quotient, which is equal to $L(a+\nu^{\alpha}b) = L(a)\times \nu^{\alpha}L(b)$ for $0<\alpha<\varepsilon$, and $L(a+b)$ for $\alpha = 0$.

Suppose that $L(a+b)$ is a representation of a group G_p. Let $H(G_p)$ be the Hecke algebra of G_p. For an admissible smooth representation π of G_p, ch_{π} will denote the character of π. With a fixed $f \in H(G_p)$

(*) $\quad \alpha \to ch_{L(a)\times\nu^{\alpha}L(b)}(f)$

is a continuous function. One can see this from the formula for the character of an induced representation in [23] (see also Lemma 2.1. of [20]).

Let (α_n) be a sequence of real numbers converging to 0 such that $0<\alpha_n<\varepsilon$ for all n. Suppose that we have proved that there exist a quotient π of $\lambda(a+b)$ and a subsequence $(\alpha_{n(k)})$ of (α_n) such that

(**) $\quad \lim_k ch_{L(a+\nu^{\alpha_{n(k)}}b)}(f) = ch_{\pi}(f)$

for all $f \in H(G_p)$.

Now $L(a+b)$ is the unique irreducible quotient of $\lambda(a+b)$, so it is a (unique irreducible) quotient of π. The relations (*) and (**) implies

$$\lim_k ch_{L(a+\nu^{\alpha_{n(k)}}b)}(f) =$$

$$= \lim_k ch_{L(a)\times \nu^{\alpha_n(k)} L(b)}(f) =$$

$$= ch_{L(a) \times L(b)}(f) = ch_{\pi}(f)$$

for all $f \in H(G_p)$. Since $L(a+b)$ is a subquotient of π, the last equality implies that $L(a+b)$ is the quotient of $L(a) \times L(b)$.

Thus, for a proof of the proposition we need to contruct π as above.

Roughly speaking, such π is contructed as follows. One can realize all representations

$$L(\Delta_1^{\alpha}) \times \ldots \times L(\Delta_{n+m}^{\alpha})$$

on the same vector space (by restriction to the standard maximal compact subgroup). In this way one obtains a continuous family of representations on the same vector space (for a precise formulation of "continuous family" see Lemma 3.5. of [21]). Now using the compactness of the Grassmanian manifold of a finite dimensional vector space, and the diagonal procedure (several times), one constructs a subsequence $(\alpha_{n(k)})$ and π as above.

For a formal proof ,to avoid the whole construction,we pass to the contragradient representations. Now $L(a+\nu^{\alpha_n}b)^{\sim} = L(a)^{\sim} \times \nu^{-\alpha_n} L(b)^{\sim}$ is a subrepresentation of $L(\Delta_1^{\alpha})^{\sim} \times \ldots \times L(\Delta_{n+m}^{\alpha})^{\sim}$, and we can as in the proof of Lemma 3.6 of [21], construct a subsequence $(\alpha_{n(k)})$ of (α_n) and a subrepresentation π_0 of $L(\Delta_1)^{\sim} \times \ldots \times L(\Delta_{n+m})^{\sim}$ such that

$$\lim_k ch_{L(a+\nu^{\alpha_n(k)}b)^{\sim}}(f) = ch_{\pi_0}(f)$$

for all $f \in H(G_p)$. Lemma 3.6. of [21] deals with induced representations by cuspidals, but the fact that inducing is by cuspidals ,is not used in the part of the proof of the lemma that we need (this fact is used at the end of the proof to reduce the lemma to the case of subrepresentations). Now $\tilde{\pi}_0$ is in a natural way a quotient of

$$L(\Delta_1) \times \ldots \times L(\Delta_{n+m}).$$

Using the fact that

$$ch_{\tilde{\sigma}}(f) = ch_{\sigma}(\tilde{f})$$

where $\tilde{f}$ is defined by $\tilde{f}(g) = f(g^{-1})$, one obtains that

$$\lim_k ch_{L(a + \nu^{\alpha_n(k)} b)}(f) = ch_{\tilde{\pi}_0}(f)$$

for all $f \in H(G_p)$. Thus, we can take $\pi = \tilde{\pi}_0$. This finishes the proof.

For another possible proof of the preceding proposition see (iii) of A.12.

A.5. Remark: Since the multiplicity of the Langlands representation in $\lambda(a+b)$ is one, then $L(c+d)$ is a composition factor of $L(c) \times L(d)$ whose multiplicity is one.

A.6. Corollary: (i) Let $c,d \in M(D)$. If $L(c) \times L(d)$ is irreducible, then

$$L(c) \times L(d) = L(c+d).$$

(ii) Let $a,b \in M(S(C))$. If $L(a) \times L(b)$ is irreducible then

$$L(a) \times L(b) = L(a+b).$$

Corollary 8.2., a) of [1] implies:

A.7. Corollary: (i) Let $c,d \in M(D)$. Suppose that $L(c)$ and $L(d)$ are unitarizable. Then $L(c+d)$ is unitarizable and

$$L(c+d) = L(c) \times L(d).$$

(ii) Let $a,b \in M(S(C))$. If $L(a)$ and $L(b)$ are unitarizable, then

$$L(a+b) = L(a) \times L(b).$$

Statement (ii) of the above Corollary remains true if one consider classification Z instead of L.

In the rest of this section we suppose that the <u>characteristic of the field F is zero.</u>

Let $\delta \in GL(m,F)^{\sim}$ be a square-integrable representation and $n \in \mathbb{N}$. Then the induced representation

$$\left(\nu^{\frac{n-1}{2}}\delta\right) \times \left(\nu^{\frac{n-1}{2}-1}\delta\right) \times \ldots \times \left(\nu^{-\frac{n-1}{2}}\delta\right)$$

has a unique irreducible quotient. This quotient was denoted by $u(\delta,n)$.

A.8. THEOREM: Suppose that char $F = 0$. Let $\delta \in \mathrm{Irr}$ be a square-integrable representation and let $n \in \mathbb{N}$. Then

$$u(\delta,n)$$

is a unitarizable representation.

Proof : The first part of the proof uses a result of [13] or [2], and the second part uses a result of [8].

Let $\delta \in \widetilde{G}_m = GL(m,F)^{\sim}$ be a square integrable representation and $n \in \mathbb{N}$.

There exist a divison algebra H central over F with dimension m^2 over F. We choose, like in § 5 of [13], a number field k, a place w of k, and a group G defined by a division algebra D over k such that: F is isomorphic to the completion k_w of k at w, the group $G(k_w)$ of k_w-rational points of G is isomorphic to the multiplicative group of H, G satisfies assumptions of § 5 of [13]. Let S_0 be the set of all places v such that $G(k_v)$ is ramified. Clearly $w \in S_0$. Let $\mathbb{A}$ be the Adele ring of k.

Since δ is an irreducible square-integrable representation of $GL(m,F) \cong GL(m,k_w)$, the proof of Proposition 5.15. of [13] implies that there exist an irreducible cuspidal automorphic representation σ of $GL(m,\mathbb{A})$ such that, in the factorisation

$$\sigma = \bigotimes_v \sigma_v$$

which corresponds to the factorisation of $GL(m,\mathbb{A})$ into the restricted product of all $GL(m,k_v)$ (see [4]), we have

$$\sigma_v \cong \delta.$$

Let Z^m be the center of the algebraic group $GL(m)$. Then Z^m is isomorphic to $GL(1)$. Now $Z^m(\mathbb{A})$ is naturally isomorphic to the restricted product of $Z^m(k_v)$. Let Z^m_+ be the group of all $z = (z_v) \in Z^m(\mathbb{A})$ such that $z_v = 1$ for all finite places, and z_v is a positive real number independent of v infinite.

Let η be the central character of the cuspidal automorphic representation σ. Suppose that η is trivial on Z_+^m.

Let P be the standard parabolic subgroup of GL(nm) whose Levi factor M is naturally isomorphic to $GL(m)^n$. We identify elements of $M(\mathbb{A})$ with n-triples $(g_1,\ldots,g_m)$, $g_i \in GL(m,\mathbb{A})$. Let π be the representation

$$(g_1,\ldots,g_n) \to \sigma(g_1)|\det g_1|^{\frac{n-1}{2}} \otimes \sigma(g_2)|\det g_2|^{\frac{n-1}{2}-1} \otimes \ldots \otimes \sigma(g_n)|\det g_n|^{-\frac{n-1}{2}}$$

Let $\pi = \otimes \pi_v$ be the decomposition of π into the restrict product of representations of $M(k_v)$. The induced representation from $P(\mathbb{A})$ to $GL(n\,m,\mathbb{A})$ (resp. $P(k_v)$ to $GL(nm,k_v)$) by π (resp. π_v) is denoted by $\mathrm{Ind}(\pi)$ (resp. $\mathrm{Ind}(\pi_v)$).

Since the center Z^{mn} of $GL(n,m)$ is isomorphic to $GL(1)$, we may consider η like a character of $Z^{mn}(\mathbb{A})$. Set $\omega = \eta^m$. Let $L^2(\omega, GL(mn,\mathbb{A}))$ be the space of (classes of) functions on $GL(mn,\mathbb{A})$ such that

$$f(\gamma z g) = \omega(z) f(g)$$

for all $\gamma \in GL(mn,k)$, $z \in Z^{mn}(\mathbb{A})$, $g \in GL(mn,\mathbb{A})$; and $|f|^2$ is integrable function on

$$GL(mn,k)\ Z^{mn}(\mathbb{A}) \backslash GL(mn,\mathbb{A})$$

with respect to a non-trivial right-invariant measure. Action of $GL(mn,\mathbb{A})$ o $L^2(\omega, GL(mn,\mathbb{A}))$ by right shifts defines a unitary representation of $GL(mn,\mathbb{A})$.

In § 2 of [8] it is proved that there exists an intertwining operator

$$E:\mathrm{Ind}(\pi) \to L^2(\omega, GL(mn,\mathbb{A}))$$

whose image is an irreducible representation. Let τ be the image of E. Decompose τ into restricted tensor product $\tau = \otimes\tau_v$.

Since $\mathrm{Ind}(\pi) \cong \underset{v}{\otimes} \mathrm{Ind}(\pi_v)$ we have the epimorphism

$$E:\underset{v}{\otimes} \mathrm{Ind}(\pi_v) \to \underset{v}{\otimes}\tau_v.$$

Now $\otimes \mathrm{Ind}(\pi_v)$ is, like a representation of $GL(mn,k_w)$, isomorphic to a direct sum of copies of $\mathrm{Ind}(\pi_w)$ (we need to fix a basis in each $\mathrm{Ind}(\pi_v)$, $v \neq w$, and use the fact that the local Hecke algebras are idempotented algebras). Since $\otimes\tau_v$ is also a direct sum of copies of τ_w, by the same reasons, we obtain directly that there exist a surjective intertwining operator

$$e:\mathrm{Ind}(\pi_w) \to \tau_w.$$

Now

$$\mathrm{Ind}(\pi_w) \cong (\nu^{\frac{n-1}{2}}\delta)\times \ldots\times(\nu^{\frac{n-1}{2}}\delta).$$

Thus

$$\tau_w \cong u(\delta,n).$$

Since τ is a subrepresentation of $L^2(\omega, GL(mn,\mathbb{A}))$, τ_w is unitarizable and therefore $u(\delta,n)$ is unitarizable.

It remains to consider the general case (without assumption $\eta|Z_+^m = 1$). This case reduces to the case of $\eta|Z_+^m = 1$ by twisting δ with a suitable character.

The following theorem is a direct consequence of the preceding theorem.

A. 9. Theorem: Representations

$$L(a(n,d)^{(\rho)}) \ , \ n,d \in \mathbb{N}, \ \rho \in C^u$$

are unitarizable.

A. 10. Theorem: Let char $F = 0$. Set

$$B = \{Z(a(n,d)^{(\rho)}), \pi(Z(a(n,d)^{(\rho)}),\alpha);$$

$$n,d \in \mathbb{N}, \rho \in C^u, \ 0 < \alpha < 1/2\}.$$

Fix $m \in \mathbb{N}$. Then

(i) If $\sigma_1, \ldots, \sigma_k \in B$ such that

$$\deg \sigma_1 + \ldots + \deg \sigma_k = m,$$

then $\sigma_1 \times \ldots \times \sigma_k \in \hat{G}_m$

(ii) If $\pi \in \hat{G}_m$, then there exist $\tau_1, \ldots, \tau_j \in B$ so that

$$\pi = \tau_j \times \ldots \times \tau_j.$$

(iii) If $n,d \in \mathbb{N}, \rho \in C^u$ so that $nd(\deg \rho) \leq m$, then $Z(a(n,d)^{(\rho)}) = L(a(d,n)^{(\rho)})$.

(iv) Let $n_i, d_i, m_j, e_j \in \mathbb{N}$, $\rho_i, \sigma_j \in C^u$, $0 < \alpha_j < 1/2$ for $1 \leq i \leq p, 1 \leq j \leq q$ where $p,q \in \mathbb{Z}_+$. Suppose that

$$\sum_{i=1}^{p} (n_i d_i) \deg \rho_i + 2 \sum_{j=1}^{p} (m_j e_j) \deg \sigma_j = m.$$

Then

$$L\left(\sum_{i=1}^{p} a(n_i,d_i)^{(\rho_i)} + \sum_{j=1}^{q}[\nu^{\alpha_j} a(m_j,e_j)^{(\sigma_j)} + \nu^{-\alpha_j} a(m_j,e_j)^{(\sigma_j)}]\right)$$

$$= \left[\prod_{i=1}^{p} L(a(n_i,d_i)^{(\rho_i)})\right] \times \left[\prod_{j=1}^{q} \pi(L(a(m_j,e_j)^{(\sigma_j)}),\alpha_j)\right] =$$

$$= \left[\prod_{i=1}^{p} Z(a(d_i,n_i)^{(\rho_i)})\right] \times \left[\prod_{j=1}^{q} \pi(Z(a(e_j,m_j)^{(\sigma_j)}),\alpha_j)\right] =$$

$$= Z\left(\sum_{i=1}^{p} a(d_i,n_i)^{(\rho_i)} + \sum_{j=1}^{q}[\nu^{\alpha_j} a(e_j,m_j)^{(\sigma_j)} + \nu^{-\alpha_j} a(e_j,m_j)^{(\sigma_j)}]\right)$$

<u>Proof:</u> We shall prove (i),(ii),(iii) and (iv) by induction on m. The proof is similar with the proof of Theorem 3.1.

For $m = 1$ there is nothing to prove. Let $m \geq 2$. Suppose that the theorem holds for $k \leq m-1$. Let X_{m-1} be defined as in Lemma 2.1. By our inductive assumption (U^{m-1}) holds ((U^m) is defined at the second section). Thus we can apply Lemma 2.1. Each element of $I(\hat{G}_m)$ is some product of elements of X_{m-1} ($I(\hat{G}_m)$ is defined in (ii) of Lemma 2.1.). By definition

$$I(\hat{G}_m) \subseteq \hat{G}_m$$

and

$$\hat{G}_m \setminus I(\hat{G}_m) \subseteq \{Z(a(n,d)^{(\rho)}),\ n,d \in \mathbb{N}, \rho \in C^u \text{ and } (nd)\deg\rho = m\}.$$

Let $\tau \in I(\hat{G}_m)$. Then

$$\tau = \prod_{i=1}^{p} Z(a(n_i,f_i)^{(\rho_i)}) \times \prod_{j=1}^{q} \pi(Z(a(m_j,e_j)^{(\sigma_j)}),\alpha_j)$$

for some $n_i, d_i \in \mathbb{N}, \rho_i, \sigma_j \in C^u, 0 < \alpha_j < 1/2, p,q \in \mathbb{Z}_+$, by Lemma 2.1. By inductive assumption, we have

$$Z(a(n_i,d_i)^{(\rho_i)}) = L(a(d_i,n_i)^{(\rho_i)}).$$

Also

$$\pi(L(a(e_j,m_j)^{(\sigma_j)}),\alpha_j) =$$

$$= \nu^{\alpha_j} L(a(e_j,m_j)^{(\sigma_j)}) \times \nu^{-\alpha_j} L(a(e_j, m_j)^{(\sigma_j)})$$

$$= \pi(Z(a(m_j,e_j)^{(\rho_j)}),\alpha_j).$$

Thus $\pi(L(a(e_j,m_j)^{(\sigma_j)},\alpha_j)$ is unitarizable. Using Corollary A.7. we obtain

$$\tau = \prod_{i=1}^{p} L(a(d_i,n_i)^{(\rho_i)}) \times \prod_{j=1}^{q} \pi(L(a(e_j,m_j)^{(\sigma_j)},\alpha_j).$$

This implies that (iv) holds for representations in $I(\hat{G}_m)$.

Let now $n,d \in \mathbf{N}, \rho \in C^u$ so that

$$(nd) \deg \rho = m.$$

Now $L(a(n,d)^{(\rho)})$ is unitarizable, by Theorem A.9 . By the preceding considerations

$$L(a(n,d)^{(\rho)}) \notin I(\hat{G}_m).$$

Thus

$$L(a(n,d)^{(\rho)}) \in \{Z(a(u,v)^{(\sigma)}); u,v \in \mathbf{N}, \sigma \in C^u \text{ and } (uv) \deg \sigma = m\}.$$

Therefore, $L(a(n,d)^{(\rho)}) = Z(a(u,v)^{(\sigma)})$ for some u,v and σ as above. The fact

$$\operatorname{supp} a(n,d)^{(\rho)} = \operatorname{supp} a(u,v)^{(\sigma)}$$

implies

$$L(a(n,d)^{(\rho)}) \in \{Z(a(n,d)^{(\rho)}), Z(a(d,n)^{(\rho)})\} .$$

By Lemma 2.3.

$$L(a(n,d)^{(\rho)}) = Z(a(d,n)^{(\rho)}) .$$

Thus $Z(a(d,n)^{(\rho)})$ is unitarizable. This implies (i),(ii), (iii) and the rest of (iv).

□

Let R^u be the additive subgroup of R generated by Irr^u. Then Irr^u is a $\mathbb{Z}$-basis of R^u, and R^u is a subring of r.

The following theorem is a direct consequence of the preceding one.

A.11. Theorem: (i) Let $a \in M(S(C))$. The representation $Z(a)$ is unitarizable if and only if

$$t(Z(a)) = L(a)$$

is unitarizable.

(ii) The mapping

$$t: Irr^u \to Irr^u$$

$$Z(a) \to L(a),\ Z(a) \in Irr^u,$$

is an involutive automorphism of the multiplicative semigroup Irr^u.

(iii) The homomorphism in (ii) satisfies

$$t(Z(a(n,d)^{(\rho)})) = Z(a(d,n)^{(\rho)}),$$

$$t(\pi(Z(a(n,d)^{(\rho)}),\alpha)) = \pi(Z(a(d,n)^{(\rho)}),\alpha),$$

$$n,d \in \mathbb{N}, \rho \in C^u, \ 0 < \alpha < 1/2.$$

(iv) The mapping $t|R_u$ is an involutive ring automorphism.

A.12. <u>Remark:</u> (i) Lemma 2.3. can not be omitted in our proof of Theorem 3.1., while we can prove (i) and (ii) of Theorem A.10., and also (i),(ii),(iv) of Theorem A.11. without using Lemma 2.3.
(ii) The statement (i) of the preceding theorem is a new proof of Conjecture 8.10. of [1] stated by J.N. Bernstein (in the zero characteristic case).
(iii) Now we shall give an outline of another possible proof of Proposition A.4. If $d \in M(D)$ then we have in R

$$\lambda(d) = \sum_{x \in M(D)} m_x^d L(x), \ m_x^d \in \mathbb{Z}_+, m_d^d = 1,$$

$$L(d) = \sum_{x \in M(D)} m(d,x)\lambda(x), \ m(d,x) \in \mathbb{Z}, m(d,d) = 1.$$

Take $d_1, d_2 \in M(D)$. In the ring R we have

$$L(d_1) \times L(d_2) = (\Sigma\, m(d_1,x_1)\lambda(x_1)) \times (\Sigma m(d_2,x_2)\lambda(x_2)) =$$

$$= \lambda(d_1+d_2) + \sum_{\substack{x_1 \neq d_1, \\ \text{or } x_2 \neq d_2}} m(d_1,x_1)m(d_2,x_2)\lambda(x_1+x_2) =$$

$$= L(d_1+d_2) + m(d_1,d_2)m(d_2,d_2) \sum_{y \neq d_1+d_2} m_y^{d_1+d_2} L(y$$

$$+ \sum_{\substack{x_1 \neq d_1, \\ \text{or } x_2 \neq d_2}} (m(d_1,x_1)m(d_2,x_2) \ \sum_y m_y^{x_1+x_2} \ L(y)).$$

For a proof of Proposition A.4 it is enough to show that if

$$m(d_1x_1)m(d_2,x_2)m_y^{x_1+x_2} \neq 0$$

where $x_1 \neq d_1$ or $x_2 \neq d_2$, then $y \neq d_1 + d_2$. This can be obtained using relation which exist between a and b when $m_b^a \neq 0$ (i.a. when L(b) is a composition factor of $\lambda(a)$. For this relation one can consult A.4. f. of [2].

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