$\mathcal{U}_q(sl(n))$-invariant quantization of symmetric coadjoint orbits via reflection equation algebra

J. Donin

Max-Planck-Istitute für Mathematik, Vivatsgasse 7, 53111 Bonn,
Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel;
e-mail: donin@mpim-bonn.mpg.de, donin@macs.biu.ac.il

A. Mudrov

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel;
e-mail: mudrova@macs.biu.ac.il

Abstract

We study relations between the two-parameter $\mathcal{U}_q(sl(n))$-invariant deformation quantization on $sl^*(n)$ and the reflection equation algebra. The latter is described by a quantum permutation on $\mathrm{End}(\mathbb{C}^n)$ given explicitly. The reflection equation algebra is used for constructing the one-parameter quantization on coadjoint orbits, including symmetric and certain bisymmetric and nilpotent ones. Our approach is based on embedding the quantized function algebras on orbits into the algebra of functions on the quantum group $SL_q(n)$ via reflection equation algebra characters.

1 Introduction

This paper is devoted to a particular case of the following problem. Let M be a manifold with a Poisson structure invariant under an action of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of the group G and $\mathcal{U}_h(\mathfrak{g})$ the corresponding quantized universal enveloping algebra, [3]. The problem is to construct a quantization $\mathcal{A}_h$ of the function algebra $\mathcal{A}$ on M invariant under the action of $\mathcal{U}_h(\mathfrak{g})$. In this paper, we consider the case $\mathfrak{g} = sl(n)$ or $gl(n)$ and M is either $\mathfrak{g}^*$ or an orbit in $\mathfrak{g}^*$. We quantize the algebra of polynomial functions on M and present it as a quotient of the corresponding tensor algebra by very natural relations, which are similar to those in the classical case. We prove the flatness of the deformations obtained. In this way, we quantize all quantum symmetric orbits and some bisymmetric ones, including the twistor space. Note that the analogous relations can be written for all semisimple orbits but there arises the question whether the quotient algebras by those relations are flat.

Let us recall some facts about the quantization on $\mathfrak{g}^*$ and its orbits. It was shown in [4] that in the case $\mathfrak{g} = sl(n)$ there exists a two-parameter deformation of the polynomial algebra $\mathrm{S}(\mathfrak{g})$ on $\mathfrak{g}^*$ which can be viewed as a $\mathcal{U}_h(\mathfrak{g})$-invariant quantization of the Lie-Poisson bracket on $\mathfrak{g}^*$. Recall that in the classical case a natural one-parameter $\mathcal{U}(\mathfrak{g})$-invariant quantization of $\mathrm{S}(\mathfrak{g})$ is given by the family $\mathrm{S}_t(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})[t]/J_t$, where $\mathrm{T}(\mathfrak{g})$ is the tensor algebra of the vector space $\mathfrak{g}$ and the ideal J_t is generated by the elements $x \otimes y - \tau(x \otimes y) - t[x, y]$, $x, y \in \mathfrak{g}$. Here, τ is the flip operator on $\mathfrak{g}^{\otimes 2}$. So, $\mathrm{S}_t(\mathfrak{g})$ is a quadratic-linear algebra. By the Poincaré-Birkhoff-Witt theorem, $\mathrm{S}_t(\mathfrak{g})$ is a free module over $\mathbb{C}[t]$. As was proven in [4], [5], that picture can be extended to the quantum case for $\mathfrak{g} = sl(n)$. Namely, there exist deformations τ_h and $[\,\cdot,\cdot\,]_h$ of both maps τ and $[\,\cdot,\cdot\,]$ such that the ideal $J_{h,t}$ generated by the elements $x \otimes y - \tau_h(x \otimes y) - t[x, y]_h$, $x, y \in \mathfrak{g}$, gives the two-parameter $\mathcal{U}_h(\mathfrak{g})$-invariant quantization $\mathrm{S}_{h,t}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})[[h]][t]/J_{h,t}$ of the Lie-Poisson bracket on $\mathfrak{g}^*$. It was also shown that such a family does not exist for $\mathfrak{g} \neq sl(n)$.

It is possible to prove that $\mathrm{S}_{h,t}(g)$ can be restricted to any semisimple orbit to give a two-parameter $\mathcal{U}_h(\mathfrak{g})$-invariant quantization on it. On the other hand, one can obtain a one-parameter invariant quantization on a semisimple orbit M as the subalgebra of quantized functions on G invariant under the action of the quantized stabilizer, [6]. Note that these statements do not give any explicit form of the quantized algebras. In this paper, we explicitly describe $\mathrm{S}_{h,t}(\mathfrak{g})$ and the quantum symmetric orbits in $\mathfrak{g}^*$. We also describe quantizations of certain nilpotent and bisymmetric orbits. We conjecture that all bisymmetric orbits can be quantized within our approach.

The paper is organized as follows. Sections 2 and 3 contain some basic material about the quantum group $\mathcal{U}_q(\mathfrak{g})$, $\mathfrak{g} = sl(n)$, and its defining representation in the vector space $V = \mathbb{C}^n$. In this paper, we work with the quantum group $\mathcal{U}_q(\mathfrak{g})$ in the sense of Lusztig, [17], instead of $\mathcal{U}_h(\mathfrak{g})$. The latter may be considered as the completion of $\mathcal{U}_q(\mathfrak{g})$ at the point $q = 1$. Correspondingly, instead of $\mathrm{S}_{h,t}(\mathfrak{g})$ we deal with the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$ as a free module over $\mathbb{C}(q)$. In Section 4, we study two different algebra structures on $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ invariant with respect to the actions of either $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$ or $\mathcal{U}_q(\mathfrak{g})$. As a result, we come to two inequivalent embeddings of $\mathrm{T}(V \otimes V^*)$ into $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$. That, in its turn, leads to two different quantizations of the polynomial algebra on $V \otimes V^*$ studied in Sections 5 and 6. We prove that they are the reflection equation (RE) algebra (see [14]) and the algebra of functions on the quantum group known as the FRT algebra (after Faddeev, Reshetikhin, and Takhtajan), [8]. It turns out that both algebras can be described uniformly by quantum permutations on $(V \otimes V^*)^{\otimes 2}$ but within different categories. In Section 7 we evaluate the lowest weight vectors of the irreducible $\mathcal{U}_q(\mathfrak{g})$-modules in $\mathcal{M}^{\otimes 2}$, where $\mathcal{M}$ is the algebra of $n \times n$ complex matrices. Using this information, we derive the involutive permutation τ_{RE} giving the commutation relations in the reflection equation algebra. That is done in Section 8. In Section 9, we reduce this permutation to the submodule $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{\otimes 2}$, $\mathfrak{g} = sl(n)$.

In Section 7, we formulate a relation between $\mathrm{S}_{q,t}(\mathfrak{g})$ and the RE algebra, following [5]. That relation involves the reduction of τ_{RE} from $\mathcal{M}$ to $\mathfrak{g}$, and a quantum deformation of the Lie commutator $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ determined by τ_{RE} up to a factor. Specifying values of the quantum trace in the RE algebra, we obtain one-parameter subfamilies of $\mathrm{S}_{q,t}(\mathfrak{g})$ as quotients of the RE algebra.

Section 11 is devoted to the quantization on coadjoint orbits in $\mathfrak{g}^*$. We work, actually, with the adjoint orbits in the matrix space using the isomorphism between adjoint and coadjoint modules and employing the relation between the RE algebra and the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$.

The RE algebra appeared as an abstraction of algebraic constructions, [1], [14], arising from the theory of integrable models, [2]. It is also related to the braid group for a solid handlebody, [13]. In this paper, we consider the RE algebra as the $\mathcal{U}_q(\mathfrak{g})$-invariant quantization of the polynomial algebra on the matrix space. It is generated by elements arranged in the matrix L and subject to a set of quadratic relations called the reflection equation.

The conjugation transformation of L with T, the generating matrix of the FRT algebra whose entries commute with the entries of L, is again an RE matrix, [15]. It follows from this observation that any character of the RE algebra specifies a homomorphism to the FRT algebra. This is done in the same way as in the classical geometry, where maximal ideals of the function algebra on M correspond to the points of M. Each point defines an embedding of the function algebra on the orbit into that on the group G. We use this idea for constructing quantized manifolds as quotient spaces of the quantized group. Characters of the RE algebra are exactly solutions to the *matrix* reflection equation, [16]. We find such solutions in the form of projectors of rank $k < n$. Thus, we come to presenting the one-parameter quantizations on symmetric orbits as quotients of the RE algebra. There are also solutions other than projectors, corresponding to different paths in the parameter space of the two-parameter quantization $\mathrm{S}_{h,t}(\mathfrak{g})$.

We prove that there is an epimorphism from the RE algebra defined on $n \times n$ matrices onto the RE algebra defined on $k \times k$ matrices for $k < n$. That enables us to build new solutions to the matrix RE by embedding a given solution into a bigger matrix as the left upper block and extending it to the whole matrix with zeros. In particular, starting from a semisimple non-degenerate RE matrix we gain an additional, zero eigenvalue. We use this method for constructing examples of quantized bisymmetric orbits. The complete classification of solutions to the matrix RE is unknown. In particular, it is interesting to find nilpotent matrices providing quantization of nilpotent orbits. We managed to build such solutions among the matrices with the zero square.

2 The quantum universal enveloping algebra $\mathcal{U}_q(sl(n))$

By $\mathfrak{g}$ we mean the complex Lie algebra $sl(n)$, and $\mathcal{U}_q(\mathfrak{g})$ is the corresponding quantum group. The latter is understood in the sense of Lusztig [17], i.e., a free module over the field of rational functions in q. In the classical limit $q \to 1$, $\mathcal{U}_q(\mathfrak{g})$ turns into the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The quantum group $\mathcal{U}_q(\mathfrak{g})$ is generated by the elements H_i, $X_i^{\pm}$, $i = 1, \ldots, n-1$, satisfying the commutation relations

$$[H_i, X_i^{\pm}] = \pm 2X_i^{\pm}, \quad [H_i, X_{i\pm1}^{\pm}] = \mp X_{i\pm1}^{\pm},$$

$$[H_i, X_j^{\pm}] = 0, \quad |i-j| > 1,$$

$$[X_i^{+}, X_j^{-}] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}.$$

Besides, the Serre relations hold:

$$(X_i)^2 X_{i\pm1} - 2_q X_i X_{i\pm1} X_i + X_{i\pm1}(X_i)^2 = 0, \quad \text{where} \quad X_k = X_k^{\pm}.$$

Here $2_q \equiv q + q^{-1}$. In general, the quantum integer numbers are defined as $n_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}}$. The coproduct Δ, counit ε, and antipode γ are

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes X_i^{\pm}.$$

$$\varepsilon(H_i) = \varepsilon(X_i^{\pm}) = 0, \quad \gamma(H_i) = -H_i, \quad \gamma(X_i^{\pm}) = -q^{\mp 1} X_i^{\pm}.$$

Remark 2.1. *The conjugation of the Lie algebra $\mathfrak{g}$ by the skew-diagonal matrix $J = \sum e_\alpha^{n+1-\alpha}$ extends to an automorphism of $\mathcal{U}_q(\mathfrak{g})$. The subset of fixed points with respect to this automorphism is a quantum subgroup. It is the stabilizer of J in the representation on matrices. We use this fact in the sequel.*

In the tensor square of $\mathcal{U}_q(\mathfrak{g})$, there is an element $\mathcal{R}$ called the universal R-matrix. It satisfies the conditions

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{1}$$

$$\mathcal{R}\Delta(x) = \Delta'(x)\mathcal{R}, \quad x \in \mathcal{U}_q(\mathfrak{g}). \tag{2}$$

The subscripts distinguish the legs of tensor objects and the prime stands for the opposite coproduct. We adopt the Sweedler symbolic notations $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for the coproduct and denote the antipode by the bar: $\gamma(x) = \bar{x}$, $x \in \mathcal{U}_q(\mathfrak{g})$. The exact expression for $\mathcal{R}$ can be found in [20, 12, 11]. We do not use it in the sequel, rather the image of $\mathcal{R}$ in the fundamental representation on $\mathbb{C}^n$.

3 Fundamental representation of $\mathcal{U}_q(sl(n))$

By $\mathcal{M}$ we denote the algebra of $n \times n$ complex matrices with the multiplication $e_j^i e_l^k = e_l^i \delta_j^k$ defined on the matrix units e_j^i. Here δ_j^k stand for the Kronekker symbol. The elements of $\mathcal{M}$

are considered as right endomorphisms of $V = \mathbb{C}^n$ with the action $e_i e^k_l = e_l \delta^k_i$ on the basis elements $e_i \in V$. Formulas

$$\rho(H_i) = e^i_i - e^{i+1}_{i+1}, \quad \rho(X^+_i) = e^i_{i+1}, \quad \rho(X^-_i) = e^{i+1}_i, \quad i = 1, \dots, n-1.$$

define a homomorphism ρ of the algebras $\mathcal{U}_q(\mathfrak{g})$ into $\mathcal{M}$. On the Chevalley generators, it is given by the same formulas as in the classical limit $q \to 1$.

The homomorphism ρ defines the right *adjoint* and left *coadjoint* actions of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{M}$:

$$A \triangleleft \mathrm{ad}_\rho(x) = \rho(\overline{x_{(1)}}) A \rho(x_{(2)}), \tag{3}$$

$$\mathrm{ad}^*_\rho(x) \triangleright A = \rho(x_{(2)}) A \rho(\overline{x_{(1)}}) \tag{4}$$

for $x \in \mathcal{U}_q(\mathfrak{g})$ and $A \in \mathcal{M}$. These actions are conjugate via the trace pairing on $\mathcal{M}$:

$$\mathrm{Tr}\Big(\big(A \triangleleft \mathrm{ad}_\rho(x)\big)\ B \Big) = \mathrm{Tr}\Big(A\ \big(\mathrm{ad}^*_\rho(x) \triangleright B\big) \Big).$$

The left coadjoint module $\mathcal{M}$ equipped with action (4) will be denoted $\mathcal{M}^*$.

The element

$$R = q^{-\frac{1}{n}} \big(q \sum_i e^i_i \otimes e^i_i + \sum_{i \neq j} e^i_i \otimes e^j_j + \omega \sum_{i<k} e^k_i \otimes e^i_k\big) \in \mathcal{M} \otimes \mathcal{M}, \tag{5}$$

where $\omega \equiv q - q^{-1}$, satisfies the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

It differs from the R-matrix used in [8] by the overall factor $q^{-\frac{1}{n}}$. In this form, this is the image of the universal R-matrix $\mathcal{R}$ under the homomorphism $\rho^{\otimes 2}$.

Consider the braid matrix $S = q^{\frac{1}{n}} \sigma_{V \otimes V} R$, where $\sigma_{V \otimes V}$ is the ordinary (classical) permutation on $V^{\otimes 2}$. The matrix S satisfies the Hecke condition $S^2 = \omega S + 1$. The projectors

$$P^+_q = \frac{1}{2_q}(q - S), \qquad P^-_q = \frac{1}{2_q}(q^{-1} + S), \tag{6}$$

determine a $\mathcal{U}_q(\mathfrak{g})$-invariant decomposition of the tensor square $V^{\otimes 2}$ into the sum of two submodules. They are deformations of the subspaces of symmetric and antisymmetric 2-tensors.

Let V^* be the space of linear functionals on V. We choose a basis $\{e_i\}$ in V and denote $\{f^i\}$ its dual with respect to the canonical pairing between V and V^*.

Proposition 3.1. *The homomorphism $\rho\colon \mathcal{U}_q(\mathfrak{g}) \to \mathcal{M}$ induces the right and left actions of $\mathcal{U}_q(\mathfrak{g})$ on the tensor algebra $\mathrm{T}(V \oplus V^*)$. They are given on the basis elements $e_i \in V$ and $f^i \in V^*$ by*

$$e_i \triangleleft x = \sum_{\alpha=1}^n e_\alpha \rho(x)^\alpha_i, \qquad f^i \triangleleft x = \sum_{\alpha=1}^n f^\alpha \rho(\bar{x})^i_\alpha, \tag{7}$$

$$x \triangleright e_i = \sum_{\alpha=1}^n e_\alpha \rho(\bar{x})^\alpha_i, \qquad x \triangleright f^i = \sum_{\alpha=1}^n f^\alpha \rho(x)^i_\alpha, \tag{8}$$

for $x \in \mathcal{U}_q(\mathfrak{g})$. Formula $\langle e_i, f^j\rangle = \delta_i^j$ defines a $\mathcal{U}_q(\mathfrak{g})$-invariant right pairing between V and V^*. The right module $V^* \otimes V$ with the action induced by (7) is isomorphic to the adjoint module $\mathcal{M}$ via the correspondence $f^i \otimes e_j \to e^i_j$. The left module $V \otimes V^*$ with the action induced by (8) is isomorphic to the coadjoint module $\mathcal{M}^*$ via the correspondence $e_j \otimes f^i \to e^i_j$.

Proof. Direct verification. □

One can introduce the following $\mathcal{U}_q(\mathfrak{g})$-invariant involutive permutations among the elements of V and V^*.

Proposition 3.2. *Formulas*

$$\tau_{V\otimes V}(e_i \otimes e_j) = q e_j \otimes e_i,\ i > j, \tag{9}$$

$$\tau_{V^*\otimes V^*}(f^i \otimes f^j) = q f^j \otimes f^i,\ i < j, \tag{10}$$

$$\tau_{V\otimes V^*}(e_j \otimes f^i) = q^{-\frac{1}{n}}\Bigl((1-\delta^i_j) f^i \otimes e_j + \delta^i_j (q f^i \otimes e_i + \omega \sum_{k>i} f^k \otimes e_k)\Bigr). \tag{11}$$

define a $\mathcal{U}_q(\mathfrak{g})$-invariant involutive permutation on $(V \oplus V^*)^{\otimes 2}$.

Proof. Operations (9) and (10) are determined by projectors (6). Operation (11) is readily obtained from the action of the universal R-matrix. To compute it, one needs to know only the image of $\mathcal{R}$ in $\mathcal{M}^{\otimes 2}$, which is given by (5). □

4 Algebra $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ in braided categories

In this section, we consider the tensor algebras $\mathrm{T}(V)$ and $\mathrm{T}(V^*)$ from the two points of view: as objects from the category of either $\mathcal{U}_q(\mathfrak{g})$- or $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-representations. Correspondingly, the algebra structure on $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ may be introduced in two ways, as the tensor product of algebras in the two categories. The particular choice of the category determines embedding of $\mathrm{T}(V \otimes V^*)$, the tensor algebra of the space $V \otimes V^*$, into $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$. That leads to two different quantizations of the polynomial algebra on $V \otimes V^*$.

Let us remind the construction of the tensor product of algebras in the category of a quantum group representation. Let $\mathcal{H}$ be a quantum group with an R-matrix $\mathcal{R}$. Recall that a (left) $\mathcal{H}$-module algebra $\mathcal{A}$ for a quantum group $\mathcal{H}$ is an associative algebra with unity in the category of $\mathcal{H}$-representations:

$$x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), \quad x \triangleright 1 = \varepsilon(x) 1, \quad x \in \mathcal{U}_q(\mathfrak{g}), \quad a, b \in \mathcal{A}. \tag{12}$$

Given two module algebras $\mathcal{A}$ and $\mathcal{B}$, one can introduce their (braided) tensor product $\mathcal{A}\check{\otimes}\mathcal{B}$, which is again an $\mathcal{H}$-module algebra.

Proposition 4.1. *Let $\mathcal{R}_1 \otimes \mathcal{R}_2$ denote the symbolic decomposition of the universal R-matrix into the two tensor factors. The formula*

$$(a_1 \check{\otimes} b_1)(a_2 \check{\otimes} b_2) = a_1(\mathcal{R}_2 \triangleright a_2) \check{\otimes} (\mathcal{R}_1 \triangleright b_1) b_2, \quad a_i \in \mathcal{A}, \quad b_i \in \mathcal{B} \tag{13}$$

defines an associative multiplication on $\mathcal{A}\check{\otimes}\mathcal{B}$ turning it into an $\mathcal{H}$-module algebra. Embeddings $\mathcal{A} \to \mathcal{A}\check{\otimes}1$ and $\mathcal{B} \to 1\check{\otimes}\mathcal{B}$ are homomorphisms of algebras.

Proof. Associativity of multiplication (13) follows from (1). Compatibility with the action of $\mathcal{H}$ is a consequence of (2). The last statement of the proposition is immediate due to the equality $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1 \otimes 1$ following from the definition of the universal R-matrix. $\square$

One sees that multiplication (13) is characterized by the permutation relation

$$ba = (\mathcal{R}_2 \triangleright a)(\mathcal{R}_1 \triangleright b) \tag{14}$$

between the elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Proposition 4.2. *Let V and U be $\mathcal{H}$-modules. The subalgebra in $\mathrm{T}(V)\check{\otimes}\mathrm{T}(U)$ generated by the submodule $V\check{\otimes}U$ is isomorphic to the tensor algebra $\mathrm{T}(V \otimes U)$. It is an algebra in the category of $\mathcal{H}$-modules.*

Proof. Embedding $V \otimes U \to \mathrm{T}(V)\check{\otimes}\mathrm{T}(U)$ is extended to a homomorphism of the free algebra $\mathrm{T}(V \otimes U)$. It is an monomorphism due to invertibility of permutation (14). The subalgebra generated by $V \otimes U$ is $\mathcal{H}$-invariant, so it is a module algebra. $\square$

We will apply this proposition when $U = V^*$.

Let us apply the construction above to the algebras $\mathcal{A} = \mathrm{T}(V)$, $\mathcal{B} = \mathrm{T}(V^*)$ considered as $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-module algebras in the following way. The representation of the quantum group $\mathcal{U}_q(\mathfrak{g})$ in V generates the actions of $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$ on V and V^*:

$$(x \otimes y) \triangleright v \equiv \varepsilon(y) x \triangleright v, \qquad (x \otimes y) \triangleright \phi \equiv \varepsilon(x) y \triangleright \phi, \tag{15}$$

where $x \otimes y \in \mathcal{U}_q^{\otimes 2}(\mathfrak{g})$, $v \in V$, and $\phi \in V^*$. These actions are extended to the actions on $\mathrm{T}(V)$, $\mathrm{T}(V^*)$, and on the tensor product $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$.

Note that the multiplications in the algebras $\mathrm{T}(V)$ and $\mathrm{T}(V^*)$ considered as $\mathcal{U}_q(\mathfrak{g})$- or $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-module algebras coincide with the usual classical multiplication.

Proposition 4.3. *The algebra $\mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ in the category of $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-modules has the classical multiplication:*

$$(a_1 \check{\otimes} b_1)(a_2 \check{\otimes} b_2) = a_1 a_2 \check{\otimes} b_1 b_2, \quad a_i \in \mathrm{T}(V), \quad b_i \in \mathrm{T}(V^*). \tag{16}$$

Proof. The universal R-matrix of $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$ is the product of two $\mathcal{R}$-s supported in the two different copies of $\mathcal{U}_q(\mathfrak{g})$. By construction (15), each copy acts trivially either on $\mathrm{T}(V)$ or on $\mathrm{T}(V^*)$, so the universal R-matrix turns into the identity operation in (13). $\square$

Note that being considered in the category of $\mathcal{U}_q(\mathfrak{g})$-modules, the algebra $\mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ has a non-classical multiplication characterized by the permutation relation (14), where $\mathcal{R}$ is the R-matrix for $\mathcal{U}_q(\mathfrak{g})$.

Due to Proposition 4.2, the tensor algebra $\mathrm{T}(V \otimes V^*)$ has two different realizations in the categories of either $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$- or $\mathcal{U}_q(\mathfrak{g})$-representations. One can introduce additional relations in $\mathrm{T}(V \otimes V^*)$ resulting in two different quantizations of the polynomial algebra on $V \otimes V^*$; they are the FRT and RE algebras. We consider them in Sections 5 and 6 and present their defining commutation relations in terms of involutive permutations τ_{FRT} and τ_{RE}.

Taking into account Proposition 4.3, we reserve the symbol $\check{\otimes}$ only for the tensor product in the category of $\mathcal{U}_q(\mathfrak{g})$-modules where the permutation (14) between V and V^* is non-classical. So, by $\mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ ($\mathrm{T}(V) \otimes \mathrm{T}(V^*)$) we will denote the tensor product considered as $\mathcal{U}_q(\mathfrak{g})$- ($\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-) module.

5 Algebra $\mathcal{A}_{FRT}(\mathcal{M})$

Let $\mathcal{A}_{FRT}(\mathcal{M})$ be the associative unital algebra over $\mathbb{C}(q)$ generated by the matrix elements T^i_j subject to relations

$$ST_1T_2 \quad = \quad T_1T_2S. \tag{17}$$

Here, S is the Hecke matrix introduced in Section 3. The algebra $\mathcal{A}_{FRT}(\mathcal{M})$ was introduced in [8] as a quantized polynomial algebra on the space of matrices. It is endowed with the $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-module structure coming from that on $V \otimes V^*$, since relations (17) are $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-invariant. They are given by a quantum permutation τ_{FRT} on the space $V \otimes V^*$, which we are going to present explicitly.

Consider $\mathrm{T}(V \otimes V^*)$ as a subalgebra of $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ in the tensor category of $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-modules, along the line of Proposition 4.2. Consider the permutation

$$\tau_{FRT} \quad = \quad \sigma_{V\otimes V^*} \circ \tau_{V\otimes V} \circ \tau_{V^*\otimes V^*} \circ \sigma^{-1}_{V\otimes V^*}, \tag{18}$$

which is the composition of elementary permutations (9), (10) and the classical flip $\sigma_{V\otimes V^*}$. This permutation is $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-invariant. Let $(V \otimes V^*)^{\otimes 2} = I^+_{FRT} \oplus I^-_{FRT}$ be the decomposition into symmetric and skew-symmetric submodules with respect to τ_{FRT}. It is easy to see that I^-_{FRT} equals $(P^+_q \otimes P^-_q + P^-_q \otimes P^+_q)(V^{\otimes 2} \otimes V^{*\otimes 2})$, where $P^\pm_q$ are projectors (6), and is a deformation of the exterior square $(V \otimes V^*)^{\wedge 2}$.

Theorem 5.1. *Let $\mathcal{J}_{FRT}$ be the ideal in $\mathrm{T}(V \otimes V^*)$ generated by I^-_{FRT}. The quotient algebra $\mathrm{T}(V \otimes V^*)/\mathcal{J}_{FRT}$ is isomorphic to $\mathcal{A}_{FRT}(\mathcal{M})$.*

Proof. Let us take a basis $\{e_i\} \subset V$ and its right dual $\{f^i\} \subset V^*$, i. e., $\langle e_i, f^j\rangle = \delta^j_i$. The elements $e_i \otimes f^j$ transform as entries of the matrix T, under the action of $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$, so we can identify them. The product of two matrix entries can be rewritten as $T^i_j T^m_n = e_j \otimes f^i e_n \otimes f^m = e_j e_n \otimes f^i f^m$, since the elements of V and V^* commute. This implies that the ideal determined by (17) turns into $\mathcal{J}_{FRT}$. □

6 Algebra $\mathcal{A}_{RE}(\mathcal{M})$

The reflection equation algebra $\mathcal{A}_{RE}(\mathcal{M})$ is generated by the entries of the matrix L subject to the quadratic relations, [1, 14],

$$L_2SL_2S \quad = \quad SL_2SL_2. \tag{19}$$

Here, S is the Hecke matrix introduced in Section 3. The generators L^i_j form the coadjoint module $\mathcal{M}^*$ via the correspondence $L^i_j \to e^i_j$. Action (4) is extended over $\mathcal{A}_{RE}(\mathcal{M})$ turning it into a $\mathcal{U}_q(\mathfrak{g})$-module algebra;

The element $Q = \mathcal{R}_{21}\mathcal{R}$ defines a morphism of $\mathcal{U}_q(\mathfrak{g})$-modules $\mathcal{A}_{RE}(\mathcal{M}) \to \mathcal{U}_q(\mathfrak{g})$ by the correspondence $L \to Q_2 \otimes \rho(Q_1)$, [5], where $\mathcal{U}_q(\mathfrak{g})$ is considered as a left $\mathcal{U}_q(\mathfrak{g})$-module via the adjoint action. This allows to prove that the algebra $\mathcal{A}_{RE}(\mathcal{M})$ is a flat deformation of the polynomial algebra on the matrix space $\mathcal{M}$, (see also [18]).

The algebra $\mathcal{A}_{RE}(\mathcal{M})$ can be described in the same way as $\mathcal{A}_{FRT}(\mathcal{M})$ by a quantum permutation τ_{RE} . Following Proposition 4.2, consider the tensor algebra $\mathrm{T}(V\otimes V^*)$ as the subalgebra $\mathrm{T}(V\check{\otimes}V^*)\subset \mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ in the category of $\mathcal{U}_q(\mathfrak{g})$-modules. Let

$$\tau_{RE} \quad = \quad \tau_{V\otimes V^*}\circ\tau_{V\otimes V}\circ\tau_{V^*\otimes V^*}\circ\tau^{-1}_{V\otimes V^*}, \tag{20}$$

which is the composition of elementary permutations (9–11). The permutation τ_{RE} is $\mathcal{U}_q(\mathfrak{g})$-invariant. Consider the decomposition $(V\otimes V^*)^{\otimes 2}=I^+_{RE}\oplus I^-_{RE}$ into the symmetric and skew-symmetric submodules with respect to τ_{RE}. It is easy to see that the submodule I^-_{RE} is equal to $(P^+_q\otimes P^-_q+P^-_q\otimes P^+_q)(V^{\otimes 2}\check{\otimes}V^{*\otimes 2})$, where $P^\pm_q$ are projectors (6), and is a deformation of the exterior square $(V\otimes V^*)^{\wedge 2}$.

Theorem 6.1. *Let $\mathcal{J}_{RE}$ be the ideal in $\mathrm{T}(V\otimes V^*)$ generated by I^-_{RE}. The quotient algebra $\mathrm{T}(V\check{\otimes}V^*)/\mathcal{J}_{RE}$ is isomorphic to $\mathcal{A}_{RE}(\mathcal{M})$.*

Proof. As in the proof of Theorem 5.1, let $\{f^i\}\subset V^*$ be the right dual to the basis $\{e_i\}\subset V$. Due to Proposition 3.1, the elements $e_i\check{\otimes}f^j$ form the coadjoint module $\mathcal{M}^*$ for $U_q(\mathfrak{g})$, so we can put $L^j_i=e_i\check{\otimes}f^j$. By Proposition 4.2, the algebra $\mathrm{T}(V\check{\otimes}V^*)$ is isomorphic to the free associative algebra generated by L^i_j. It remains to show that the quadratic submodule generating the ideal $\mathcal{J}_{RE}$ is isomorphic to the submodule specified by the reflection equation. They are conjugate by the $\tau_{V\otimes V^*}$ which is the restriction to $V\otimes V^*$ of permutation (14). The permutation $\tau_{V\otimes V^*}$ is given by the formula (11), and it acts by the rule matrix is exactly $q^{-\frac{1}{n}}S$, $e_ie_l\check{\otimes}f^kf^n=q^{-\frac{1}{n}}S^{km}_{lj}(e_i\check{\otimes}f^j)(e_m\check{\otimes}f^n)$. ☐

Remark 6.2. *In the classical case, the polynomial algebra on $V\otimes V^*$ can be built equivalently as a $\mathcal{U}(\mathfrak{g})$- or $\mathcal{U}^{\otimes 2}(\mathfrak{g})$-module. In the quantum case, dependingly of the point of view, we come to either FRT or RE algebras, which are different.*

7 Irreducible submodules of $\mathcal{M}^{\otimes 2}$

This section is technical, it contains information about submodules of $\mathcal{M}^{\otimes 2}$ and its dual in terms of highest and lowest weight vectors. They will be used in computing the quantum permutation τ_{RE} on $(V\otimes V^*)^{\otimes 2}\sim\mathcal{M}^{*\otimes 2}$. It is convenient to evaluate the conjugate operator τ^*_{RE} in the dual space first. For that reason we consider three $\mathcal{U}_q(\mathfrak{g})$-module structures on $\mathcal{M}^{\otimes 2}$:

$$(A\otimes B)\triangleleft x \quad = \quad \rho(\overline{x_{(2)}})A\rho(x_{(3)})\otimes\rho(\overline{x_{(1)}})B\rho(x_{(4)}), \tag{21}$$

$$(A\otimes B)\triangleleft x \quad = \quad \rho(\overline{x_{(1)}})A\rho(x_{(2)})\otimes\rho(\overline{x_{(3)}})B\rho(x_{(4)}), \tag{22}$$

$$x\triangleright(A\otimes B) \quad = \quad \rho(x_{(2)})A\rho(\overline{x_{(1)}})\otimes\rho(x_{(4)})B\rho(\overline{x_{(2)}}). \tag{23}$$

Actions (21) and (22) are nothing else than $\mathrm{ad}_{\rho\otimes\rho}$ and $\mathrm{ad}_\rho\otimes\mathrm{ad}_\rho$, correspondingly. The left action (23) is conjugate to (22) with respect to the ordinary trace pairing and it is just $\mathrm{ad}^*_\rho\otimes\mathrm{ad}^*_\rho$.

A basis of the lowest weight vectors of irreducible $\mathcal{U}_q(\mathfrak{g})$-submodules with respect to the action (23) may be calculated directly. It is:

$$
\begin{aligned}
\delta_1 &= \sum_{i=1}^{n} q^{-2i+1} e_i^i \otimes e_1^n, \\
\delta_2 &= \sum_{i=1}^{n} e_1^n \otimes q^{-2i+1} e_i^i, \\
\delta_3 &= \sum_{i=1}^{n} e_1^i \otimes e_i^n, \\
\delta_4 &= \sum_{i=1}^{n} a_i e_i^n \otimes e_1^i, \quad a_i = 1, q^{-3}, \dots, q^{-2n+3}, q^{-2n}, \\
\delta_5 &= \sum_{i,k=1}^{n} q^{-(2i-1)} q^{-(2k-1)} e_i^i \otimes e_k^k, \\
\delta_6 &= \sum_{i,k=1}^{n} a_i e_i^k \otimes e_k^i, \quad a_i = q^{-(2n-1)}, q^{-(2n-3)}, \dots, q^{-3}, q^{-1}, \\
\delta_7 &= q e_1^n \otimes e_2^{n-1} + q^{-1} e_2^{n-1} \otimes e_1^n - e_1^{n-1} \otimes e_2^n - e_2^n \otimes e_1^{n-1}, \\
\delta_8 &= e_1^n \otimes e_1^n, \\
\delta_9 &= e_1^n \otimes e_1^{n-1} - q^{-1} e_1^{n-1} \otimes e_1^n, \\
\delta_{10} &= e_1^n \otimes e_2^n - q^{-1} e_2^n \otimes e_1^n,
\end{aligned}
$$

The coefficients a_i entering δ_4 and δ_6 decrease by q^{-2} each step within the interval $i = 1, \dots, n$. Here, we assume $n > 2$. The case $n = 2$ is simple and left to the reader.

The vectors δ_5 and δ_6 are invariant elements under action (23). They are symmetric with respect to the permutation τ^*_{RE}, since that is so for the invariant elements in the classical situation. The vectors δ_7 and δ_8 belong to the symmetric part, too, while δ_9 and δ_{10} are antisymmetric with respect to τ^*_{RE}. Indeed, they generate submodules of multiplicity one in $\mathcal{M}^{\otimes 2}$ and are deformations of classical submodules belonging to symmetric and antisymmetric parts, respectively. The situations is more complicated in what concerns vectors $\delta_1, \dots, \delta_4$ because they generate the isotypic component of type $\mathfrak{g}$, and in the classical limit $q = 1$ the corresponding classical component intersects the symmetric and skew-symmetric parts of $\mathcal{M}^{\otimes 2}$ with multiplicity two. So the problem of evaluating the permutation τ_{RE} reduces to calculating its restriction to the $\mathfrak{g}$-isotypic component of $\mathcal{M}^{*\otimes 2}$.

First, we compute the dual conjugate operation to τ_{RE} on the right module $\mathcal{M}^{\otimes 2}$ with action (22). A basis of lowest weight vectors has the dual consisting of highest weight ones; by that reason we also consider the highest weight vectors α^i and β^i of type $\mathfrak{g}$ with respect to actions

(21) and (22), respectively:

$$\begin{aligned}
\alpha^1 &= 1\otimes e^1_n,\\
\alpha^2 &= \sum_{i=1}^n a_i e^1_n\otimes e^i_i, \quad a_i = q,1,\dots,1,q^{-1},\\
\alpha^3 &= \sum_{i=1}^n a_i e^1_i\otimes e^i_n, \quad a_i = q,1,\dots,1,q^{-1},\\
\alpha^4 &= \sum_{i=1}^n a_i e^i_n\otimes e^1_i, \quad a_i = 1,\dots,1,q^{-1},\\
\beta^1 &= 1\otimes e^1_n,\\
\beta^2 &= e^1_n\otimes 1,\\
\beta^3 &= \sum_{i=1}^n a_i e^1_i\otimes e^i_n, \quad a_i = q^{-(2n-1)},q^{-(2n-3)},\dots,q^{-3},q^{-1},\\
\beta^4 &= \sum_{i=1}^n a_i e^i_n\otimes e^1_i, \quad a_i = q,1,\dots,1,q^{-1}. \qquad (24)
\end{aligned}$$

Let us denote the isotypic $\mathfrak{g}$-type components with respect to actions (21), (22), and (23) by $\mathcal{M}^{\otimes 2}_\alpha$, $\mathcal{M}^{\otimes 2}_\beta$, and $\mathcal{M}^{*\otimes 2}_\delta$. They are generated by the vectors α^i, β^i, and δ_i, $i=1,\dots,4$, respectively.

8 The quantum permutation τ_{RE}

In this section, we calculate the permutation τ_{RE} in terms of the lowest weight vectors introduced in the previous section. The permutations τ_{FRT} and τ_{RE} define quadratic relations in the deformed algebras of functions on matrices. In both cases, relations (17) and (19) are formulated using the algebra structure on $\mathcal{M}^{\otimes 2}$, i. e., in the dual setting. Thus, the FRT and RE algebra permutations are introduced through the dual operations τ^*_{FRT} and τ^*_{RE} conjugate to τ_{FRT} and τ_{RE} by the trace pairing. The involutions τ^*_{FRT} and τ^*_{RE} are invariant with respect to right actions (21) and (22). It is natural to compute them first, using the algebra structure on $\mathcal{M}^{\otimes 2}$, and then evaluate τ_{RE} by duality.

The subspace of $\Omega\in\mathcal{M}^{\otimes 2}$ such that $S\Omega=\Omega S$ is the annulator of the submodule defining FRT relations (17). Therefore, the involution τ^*_{FRT} is determined by the conjugation transformation with the Hecke matrix S. Solutions to the equation $S\sigma(\Omega)=\sigma(\Omega)S$, where the map σ is defined as $\Omega\to(1\otimes\Omega_1)S(1\otimes\Omega_2)$, $\Omega=\Omega_1\otimes\Omega_2\in\mathcal{M}^{\otimes 2}$, form the submodule annihilating the RE relations. Hence, the involutions τ^*_{RE} and τ^*_{FRT} are σ-conjugate: $\tau^*_{RE}=\sigma^{-1}\tau^*_{FRT}\sigma$. Since the Hecke matrix S is invariant under the adjoint action $\mathrm{ad}_{\rho\otimes\rho}$, the map σ intertwines actions (22) and (21). It is an isomorphism of modules, being a deformation of the classical flip $\sigma_{\mathcal{M}\otimes\mathcal{M}}$.

As was noted in Section 7, only the restriction of τ^*_{RE} to the isotypic $\mathfrak{g}$-type component requires an additional consideration. Restricted to the $\mathfrak{g}$-component $\mathcal{M}^{\otimes 2}_\alpha$, the conjugation operation with the matrix S has eigenvalues $+1$ and $-q^{\pm 2}$. The eigenspace $\mathcal{M}^{\otimes 2}_{\alpha;+}$ corresponding to the eigenvalue $+1$ consists of symmetric tensors, in the sense of the permutation τ^*_{FRT}. Eigenvectors of the eigenvalues $-q^{\pm 2}$ may be called τ^*_{FRT}-antisymmetric. We denote this submodule $\mathcal{M}^{\otimes 2}_{\alpha;-}$. Thus, the conjugation with the matrix S determines decomposition

$$\mathcal{M}^{\otimes 2}_{\alpha} = \mathcal{M}^{\otimes 2}_{\alpha;+} \oplus \mathcal{M}^{\otimes 2}_{\alpha;-}, \tag{25}$$

where both $\mathcal{M}^{\otimes 2}_{\alpha;\pm}$ contain irreducible submodule $\mathfrak{g}$ with multiplicity two. Decomposition (25) induces decompositions

$$\mathcal{M}^{\otimes 2}_{\beta} = \mathcal{M}^{\otimes 2}_{\beta;+} \oplus \mathcal{M}^{\otimes 2}_{\beta;-}, \tag{26}$$

$$\mathcal{M}^{*\otimes 2}_{\delta} = \mathcal{M}^{*\otimes 2}_{\delta;+} \oplus \mathcal{M}^{*\otimes 2}_{\delta;-}. \tag{27}$$

into the symmetric and antisymmetric parts. The subspaces $\mathcal{M}^{\otimes 2}_{\beta;\pm}$ are the images of $\mathcal{M}^{\otimes 2}_{\alpha;\pm}$ via the inverse transformation σ^{-1}. They yield the involution τ^*_{RE}. The subspaces $\mathcal{M}^{*\otimes 2}_{\delta;\pm}$ are the annulators of $\mathcal{M}^{\otimes 2}_{\beta;\mp}$ with respect to the trace pairing between $\mathcal{M}^{\otimes 2}$ and $\mathcal{M}^{*\otimes 2}$. The permutation τ_{RE} is determined by the submodules $\mathcal{M}^{*\otimes 2}_{\delta;\pm}$.

Proposition 8.1. *The $\mathfrak{g}$-type submodules of symmetric and antisymmetric tensors are generated by the following highest and lowest weight vectors:*

$$\begin{array}{lllll}
\mathcal{M}^{\otimes 2}_{\alpha;+} : & \alpha^3+\alpha^4, & \alpha^1+\alpha^2+\omega\alpha^3, & \mathcal{M}^{\otimes 2}_{\alpha;-} : & q^{\pm 1}\alpha^1 - q^{\mp 1}\alpha^2 \mp \alpha^3 \pm \alpha^4,\\
\mathcal{M}^{\otimes 2}_{\beta;+} : & \beta^1+\beta^2-\omega\beta^3, & \beta^3+\beta^4+\omega\beta^1, & \mathcal{M}^{\otimes 2}_{\beta;-} : & \beta^1-\beta^2, \quad \beta^3-\beta^4,\\
\mathcal{M}^{*\otimes 2}_{\delta;+} : & \delta_3+\delta_4-\omega\delta_2, & \delta_1+\delta_2+\omega\delta_3, & \mathcal{M}^{*\otimes 2}_{\delta;-} : & \delta_1-\delta_2, \quad \delta_3-\delta_4.
\end{array}$$

Proof. The eigenvectors of the conjugation by S restricted to $\mathcal{M}^{\otimes 2}_{\alpha}$ are computed directly. Thus we obtain (25). The transformation σ acts on the highest weight vectors of the $\mathfrak{g}$-type components as

$$\sigma\colon \begin{pmatrix} \beta^1\\ \beta^2\\ \beta^3\\ \beta^4 \end{pmatrix} \to \begin{pmatrix} \alpha^3\\ \alpha^4+\omega\alpha^1\\ \alpha^1\\ \alpha^2 \end{pmatrix}.$$

Decomposition (26) is the σ-preimage of (25). Decomposition (27) is computed by pairing δ_i with β^k via the ordinary trace. $\square$

Corollary 8.2. *The submodule of τ_{RE}-symmetric tensors in $\mathcal{M}^{*\otimes 2}$ is generated by the lowest weight vectors*

$$\{\delta_1+\delta_2+\omega\delta_3, \delta_3+\delta_4-\omega\delta_2, \delta_5, \delta_6, \delta_7, \delta_8\}$$

The submodule of τ_{RE}-antisymmetric tensors in $\mathcal{M}^{*\otimes 2}$ is generated by the lowest weight vectors

$$\{\delta_1-\delta_2, \delta_3-\delta_4, \delta_9, \delta_{10}\}$$

Proof. Immediate from the last proposition. $\square$

The permutation τ_{RE} can be reduced to the submodule $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. We study this problem in the next section.

9 Reducing τ_{RE} to $\mathfrak{g}^{\otimes 2}$

The left coadjoint module $\mathcal{M}^*$ contains a one-dimensional submodule $\mathfrak{m}_0$ spanned by the invariant element $\sum_{i=1}^n q^{-2i+2} e_i^i$ and the $\mathfrak{g}$-type submodule of q-traceless matrices. The decomposition $\mathcal{M}^* = \mathfrak{m}_0 \oplus \mathfrak{g}$ leads to the decomposition

$$\mathcal{M}^* \otimes \mathcal{M}^* = \mathfrak{m}_0 \otimes \mathfrak{m}_0 \;\oplus\; \mathfrak{m}_0 \otimes \mathfrak{g} \;\oplus\; \mathfrak{g} \otimes \mathfrak{m}_0 \;\oplus\; \mathfrak{g} \otimes \mathfrak{g}.$$

The lowest weight vectors $\tilde{\delta}_3$ and $\tilde{\delta}_4$ belonging to $\mathfrak{g}\otimes\mathfrak{g}$ are obtained by subtracting proper linear combinations of δ_1 and δ_2 from δ_3 and δ_4:

$$\tilde{\delta}_3 = \delta_3 - \frac{\omega}{1-q^{-2n}}\delta_1 - \frac{\omega}{1-q^{-2n}}\delta_2, \quad \tilde{\delta}_4 = \delta_4 - \frac{\omega}{q^{2n}-1}\delta_1 - \frac{\omega}{1-q^{-2n}}\delta_2. \tag{28}$$

Now we can evaluate the reduction $\tilde{\tau}_{RE}$ of the permutation τ_{RE} to the submodule $\mathfrak{g}^{\otimes 2}$.

Proposition 9.1. *The composition*

$$\mathfrak{g}\otimes\mathfrak{g} \to \mathcal{M}^*\otimes\mathcal{M}^* \xrightarrow{\tau_{RE}} \mathcal{M}^*\otimes\mathcal{M}^* \to \mathfrak{g}\otimes\mathfrak{g}, \tag{29}$$

where the left arrow is embedding and the right one is the projection along $\mathfrak{m}_0\otimes\mathcal{M}^* + \mathcal{M}^*\otimes\mathfrak{m}_0$, defines an involutive $\mathcal{U}_q(\mathfrak{g})$-invariant permutation $\tilde{\tau}_{RE}$ on $\mathfrak{g}^{\otimes 2}$. The lowest weight vectors

$$\tilde{\delta}_+ \equiv \left(1-\frac{\omega q^{-n}}{n_q}\right)\tilde{\delta}_3 + \left(1+\frac{\omega q^{n}}{n_q}\right)\tilde{\delta}_4, \qquad \tilde{\delta}_- \equiv \tilde{\delta}_4 - \tilde{\delta}_3, \tag{30}$$

where $\tilde{\delta}_i$ are introduced by (28), generate the $\mathfrak{g}$-type submodules in $\mathfrak{g}^{\otimes 2}$ of symmetric and anti-symmetric tensors with respect to $\tilde{\tau}_{RE}$, correspondingly.

Proof. The first statement is immediate. It easy to check that the vector $\tilde{\delta}_+ \in \mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{\otimes 2}$ is τ_{RE}-symmetric. As to the vector $\tilde{\delta}_-$, it is the image of τ_{RE}-antisymmetric vector $\delta_4-\delta_3$ under the projection $\mathcal{M}^{*\otimes 2} \to \mathfrak{g}^{\otimes 2}$. $\square$

Corollary 9.2. *The submodules of symmetric and antisymmetric tensors are generated by the sets*

$$\{\delta_6 - \frac{\omega}{1-q^{-2n}}\delta_5,\ \delta_7,\ \tilde{\delta}_+\}, \quad \textit{and} \quad \{\delta_9,\ \delta_{10},\ \tilde{\delta}_-\},$$

of lowest weight vectors respectively.

Proof. The vector δ_5 is the square of the invariant element $\sum_{i=1}^n q^{-2i+2} e_i^i$ spanning $\mathfrak{m}_0$. The combination $\delta_6 - \frac{\omega}{1-q^{-2n}}\delta_5$ is the Casimir element of $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. The $\mathfrak{g}$-type vectors $\tilde{\delta}_\pm$ were considered in the above proposition. Concerning the other lowest weight vectors, they already lye in $\mathfrak{g}^{\otimes 2} \subset M^{*\otimes 2}$, so the proof is an immediate consequence of Corollary 8.2. $\square$

Remark 9.3. *Unlike in the classical case, the submodule $\mathfrak{g}^{\otimes 2}$ in $M^{*\otimes 2}$ is not preserved by the permutation τ_{RE}. By this reason, $\tilde{\tau}_{RE}$ is not a simple restriction of τ_{RE} to $\mathfrak{g}^{\otimes 2}$.*

As a left $\mathcal{U}_q(\mathfrak{g})$-module, the tensor square $\mathcal{M}^{*\otimes 2}$ is endowed with another permutation coming from the universal R-matrix representation: $A\otimes B \to \mathcal{R}_2\triangleright B\otimes \mathcal{R}_1\triangleright A, \quad A\otimes B\in \mathcal{M}^{*\otimes 2}$. It can be shown that its restriction to the $\mathfrak{g}$-component $\mathcal{M}^{*\otimes 2}_\delta \subset \mathcal{M}^{*\otimes 2}$ acts according to the rule

$$\begin{pmatrix} \delta_1\\ \delta_2\\ \tilde\delta_3\\ \tilde\delta_4 \end{pmatrix} \to \begin{pmatrix} \delta_2\\ \delta_1\\ \tilde\delta_4\\ q^{-2n}\tilde\delta_3 \end{pmatrix}.$$

It is easy to see, that this operator has eigensubspaces distinct from those of τ_{RE}. With this remark, we complete the study of the RE algebra. In the remainder of the paper, we investigate relations between $\mathcal{A}_{RE}(\mathcal{M})$ and the quantization on $\mathfrak{g}^*$ and its orbits.

10 Algebra $\mathcal{A}_{RE}$ and the quantization on $\mathfrak{g}^*$

It was shown in [5] that a two-parameter $\mathcal{U}_q(\mathfrak{g})$-invariant quantization $\mathrm{S}_{q,t}(\mathfrak{g})$ of the Lie-Poisson bracket on $\mathfrak{g}^*$ exists only in the $sl(n)$ case. It is realized as the quotient algebra of $\mathrm{T}(\mathfrak{g})[t](q)$ by the quadratic-linear relations

$$x\otimes y - \tilde\tau_{RE}(x\otimes y) \quad = \quad t[x,y]_q, \tag{31}$$

where $[\,\cdot,\cdot\,]_q\colon \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ is a deformed Lie bracket. This is a $\mathcal{U}_q(\mathfrak{g})$-invariant map $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ sending the submodule of $\tilde\tau_{RE}$-symmetric tensors to zero . The commutator $[\,\cdot,\cdot\,]_q$ is uniquely defined up to a factor, since the module of $\tilde\tau_{RE}$-antisymmetric tensors contains the submodule isomorphic to $\mathfrak{g}$ with multiplicity one. In the limit $q\to 1$, the commutator turns into the classical Lie bracket. In this way, one recovers the algebra $U(\mathfrak{g})[t]$ viewed as a quantization of the Lie-Poisson bracket. Another limit $t\to 0$ yields a one-parameter family $\mathrm{S}_{q,0}(\mathfrak{g})$, which can be interpreted as a $\tilde\tau_{RE}$-commutative algebra of polynomials on $\mathfrak{g}^*$.

The RE algebra is closely related to $\mathrm{S}_{q,t}(\mathfrak{g})$. To formulate that relation, let us introduce the quantum trace of the RE matrix L:

$$\mathrm{Tr}_q(L) \quad\equiv\quad \sum_{i=1}^n q^{-2i+2}L^i_i = \mathrm{Tr}(DL), \quad \text{where} \quad D\equiv \sum_{i=1}^n q^{-2i+2}e^i_i. \tag{32}$$

Remark that this definition makes sense for matrices with entries being elements of any associative algebra $\mathcal{A}$ over $\mathbb{C}(q)$. The quantum trace is the map $\mathrm{id}\otimes\mathrm{Tr}_q\colon \mathcal{A}\otimes\mathcal{M}\to\mathcal{A}\otimes\mathbb{C}(q)\sim\mathcal{A}$. For the generating matrix L of the RE algebra, the elements $\mathrm{Tr}_q(L^k)$, $k\in\mathbb{N}$, are invariant. It follows from the commutation relations (19) that they belong to the center of $\mathcal{A}_{RE}(\mathcal{M})$.

Theorem 10.1. *The quotient algebra of $\mathcal{A}_{RE}(\mathcal{M})$ by the ideal $(\mathrm{Tr}_q(L)-\lambda)$, where $\lambda\in\mathbb{C}$, is isomorphic to the one-parameter sub-family $t=\lambda\frac{q-q^{-1}}{Tr_q(Id)}$ in $S_{q,t}(\mathfrak{g})$.*

Proof. In its essential part, this theorem was proven in [5]. Here we specify the exact relation between the parameter t and the value of λ, using the explicit expression for the permutation τ_{RE} derived in Section 8, in terms of the lower weight vectors. By Proposition 8.1, the τ_{RE}-symmetric $\mathfrak{g}$-type submodules are generated by the vectors $\delta_1-\delta_2$ and $\delta_4-\delta_3$. The first one is irrelevant because it turns zero when projected to $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. The second vector is represented as $\delta_4 - \delta_3 = \tilde{\delta}_4 - \tilde{\delta}_3 - \omega\delta_1$, where $\tilde{\delta}_i$ belong to $\mathfrak{g}^{\otimes 2}$. Vector δ_1 is expressed through the matrix D introduced by (32) and the lowest weight vector e_1^n of the adjoint representation in $\mathfrak{g}$: $\delta_1 = q^{-1} D \otimes e_1^n$. Comparing this with (31), we find the value of the parameter t. $\square$

11 The quantization of coadjoint orbits

One can show that the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$ can be restricted to any semisimple orbit in $\mathfrak{g}^*$. It this connection, there arises the problem of explicit description of the quantized manifolds in terms of ideals in $\mathrm{S}_{q,t}(\mathfrak{g})$. We solve this problem (in a one-parameter setting) for certain classes of orbits including the symmetric ones, using the relation between $\mathcal{A}_{RE}$ and $\mathrm{S}_{q,t}(\mathfrak{g})$ (Theorem 10.1). Let $\mathcal{O}_A \subset \mathcal{M}$ be the orbit passing through the matrix A. Consider two commutative diagrams

$$
\begin{tikzcd}
G \times \mathcal{M} \arrow[r] & \mathcal{M} \\
G \times \{A\} \arrow[u] \arrow[r] & \mathcal{O}_A \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}(G) \otimes \mathrm{Fun}(\mathcal{M}) \arrow[d, "\mathrm{id} \otimes \chi^A"'] & \mathrm{Fun}(\mathcal{M}) \arrow[l] \arrow[d] \\
\mathrm{Fun}(G) \otimes \mathbb{C} & \mathrm{Fun}(\mathcal{O}_A), \arrow[l]
\end{tikzcd}
$$

where the horizontal arrows correspond to the action of the group G. The right square represents morphisms of the polynomial algebras induced by the maps of manifolds depicted on the left. The map χ^A is the character of the algebra $\mathrm{Fun}(\mathcal{M})$ corresponding to the point $A \in \mathcal{M}$. We are going to quantize the right square; that will give us realization of the quantized algebra $\mathrm{Fun}_q(\mathcal{O}_A)$, on the one hand, as a quotient of $\mathcal{A}_{RE}(\mathcal{M})$ and as a subalgebra in the quantized function algebra on G, on the other.

Let $\mathcal{F}_q(G)$ be the Hopf algebra of quantized polynomial functions on the group G. It is the quotient of the FRT algebra by the additional relation $\det_q(T) = 1$, where $\det_q$ is the quantum determinant, [8]. Since $\mathcal{U}_q(\mathfrak{g})$ acts on the RE algebra $\mathcal{A}_{RE}(\mathcal{M})$, this action generates the coaction of $\mathcal{F}_q(G)$ on $\mathcal{A}_{RE}(\mathcal{M})$. Thus, we come to

Theorem 11.1. *[15]. Let T^{-1} be the matrix with entries $\gamma(T^i_j)$, where T is the generating matrix of the algebra $\mathcal{F}_q(G)$ and γ the antipode. The conjugation transformation $L \to T^{-1}LT$ of an RE matrix L with the FRT matrix T whose entries commute with the entries of L is again an RE matrix.*

It follows that the correspondence $L \to T^{-1}LT$ extends to an homomorphism $\mathcal{A}_{RE}(\mathcal{M}) \to \mathcal{F}_q \otimes \mathcal{A}_{RE}(\mathcal{M})$. Theorem 11.1 yields the quantization of the upper arrow of the right square on the above diagram. To quantize the other maps, we replace χ^A by χ^A_q, a character of the reflection equation algebra. It is determined by the correspondence $L^i_j \to A^i_j \in \mathbb{C}(q)$ such that the numeric matrix A satisfies the reflection equation, [16],

$$A_2 S A_2 S \;=\; S A_2 S A_2 \tag{33}$$

supported in $\mathcal{M}^{\otimes 2}$. Any solution to this equation gives rise to the algebra $\mathrm{Fun}_q(\mathcal{O}_A)$ closing the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_q(G)\otimes\mathcal{A}_{RE}(\mathcal{M}) & \longleftarrow & \mathcal{A}_{RE}(\mathcal{M}) \\
{\scriptstyle \mathrm{id}\otimes\chi_q^A}\Big\downarrow & & \Big\downarrow \\
\mathcal{F}_q(G)\otimes\mathbb{C} & \longleftarrow & \mathrm{Fun}_q(\mathcal{O}_A),
\end{array}$$

so that the bottom arrow is embedding. Thus, we obtain

Theorem 11.2. *Let A be a solution of the numeric RE (33). Then the algebra $Fun_q(\mathcal{O}_A)$ in the diagram above is the quantization of the polynomial algebra on the orbit $\mathcal{O}_A$ passing through the matrix A.*

Proof. It is clear that at $q = 1$ the algebra $\mathrm{Fun}_q(\mathcal{O}_A)$ coincides with the polynomial algebra on the orbit. The flatness of $\mathrm{Fun}_q(\mathcal{O}_A)$ over q follows from the fact that it is simultaneously a quotient of the flat module $\mathcal{A}_{RE}(\mathcal{M})$ and a submodule of the flat module $\mathcal{F}_q(G)$. □

The matrix $L(A) = T^{-1}AT$ possesses the following properties:

Lemma 11.3. *For any polynomial function $\mathcal{P}$,*

$$\mathcal{P}\big(L(A)\big) = L\big(\mathcal{P}(A)\big). \tag{34}$$

The quantum trace is invariant under the conjugation:

$$\mathrm{Tr}_q(L(A)) = \mathrm{Tr}_q(A). \tag{35}$$

Proof. The first statement is evident. The second one is checked using the commutation relations in the algebra $\mathcal{F}_q(G)$. □

Note that solutions A of (33) yield quantizations which are quotients of one-parameter subfamilies in $\mathrm{S}_{q,t}(\mathfrak{g})$. Those subfamilies are defined by paths in the parameter space (q, t) crossing the axis $q = 1$ at the point $t = 0$. In particular, the solutions with $\mathrm{Tr}_q(A) = 0$ correspond to the path $t = 0$. The limit $q \to 1$ for t separated from 0 cannot be reached within our approach. In the last subsections, we consider quantizations of various types of orbits, along the line of Theorem 11.2.

11.1 Quantized symmetric orbits

To begin with, let us prove a statement relating reflection equation algebras in different dimensions. Let $\mathcal{M}(n)$ and $\mathcal{M}(k)$ be the matrix algebra of $n \times n$ and $k \times k$ matrices, $0 < k < n$. The homomorphism $\mathcal{M}(k) \to \mathcal{M}(n)$ of embedding as the left upper block induces a contravariant epimorphism of the function algebras. The same holds in the quantum situation:

Proposition 11.4. *The quotient algebra $\mathcal{A}_{RE}\big(\mathcal{M}(n)\big)$ by the relations $L^i_j = 0, \quad i > k$ or $j > k$, is isomorphic to $\mathcal{A}_{RE}\big(\mathcal{M}(k)\big)$.*

Proof. Let P^+ and P^- be the projectors from $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ to the first and the second addends, respectively. Denote R^{++} the projection of the R-matrix (5) to $\mathcal{M}(k) \otimes \mathcal{M}(k)$. Up to a nonzero scalar factor, this is the R-matrix in dimension k. The equalities

$$(P^+ \otimes 1)R = R^{++} + q^{-\frac{1}{n}} P^+ \otimes P^-, \qquad R(1 \otimes P^+) = R^{++} + q^{-\frac{1}{n}} P^- \otimes P^+ \tag{36}$$

follows directly from (5). The matrix L is equal to $L^+ = P^+ L P^+$ modulo the ideal specified in the hypothesis. In terms of the matrix R instead of the Hecke matrix S, the reflection equation is rewritten as

$$R_{21} L_1^+ R L_2^+ = L_2^+ R_{21} L_1^+ R$$

modulo the relations of concern. Using (36), the matrix R can be replaced by R^{++} thus leading to the RE in dimension k. $\square$

Proposition 11.4 suggests a method of building solutions to (33) by those in smaller dimensions extending them by zero matrix elements. For example, the projectors $P_k = \sum_{i=1}^k e_i^i$ of rank $k = 1, \ldots, n$ satisfy (33). Indeed, each can be obtained from the $k \times k$ unit matrix (which is apparently a solution to the RE) by extending it with zeros to the $n \times n$ matrix.

Let us introduce the quantum integer numbers

$$\hat{k}_q \equiv \sum_{i=1}^{k} q^{-2i+2} = \mathrm{Tr}_q(P_k). \tag{37}$$

Theorem 11.5. *The quotient algebra of $\mathcal{A}_{RE}(\mathcal{M})$ by the relations*

$$L^2 = L, \quad Tr_q(L) = \hat{k}_q \tag{38}$$

yields a $\mathcal{U}_q(\mathfrak{g})$-invariant quantization $Fun_q(\mathcal{O}_{P_k})$ of the symmetric orbit $\mathcal{O}_{P_k}$ passing through the projector P_k

Proof. Relations (38) follow from identities (34) and (35) applied to $A = P_k$. The projectors P_k are stable under the right adjoint action (3) of $\mathcal{U}_q(sl(k)) \otimes \mathcal{U}_q(sl(n-k)) \otimes \mathcal{U}_q(sl(1))$ as a quantum subgroup in $\mathcal{U}_q(sl(n))$. Therefore, the subalgebra in $\mathcal{F}_q(G)$ generated by the RE matrix $L(P_k)$ is invariant under the right action of $\mathcal{U}_q(sl(k)) \otimes \mathcal{U}_q(sl(n-k)) \otimes \mathcal{U}_q(sl(1))$. $\square$

Remark 11.6. *The quantum projective spaces may be obtained directly from the description of $\mathcal{A}_{RE}$ given in Section 6. Indeed, imposing the additional conditions $V^{\wedge 2} = 0$ and $V^{*\wedge 2} = 0$ one comes to the subalgebra in $Sym_q(V) \check{\otimes} Sym_q(V^*)$ generated by $e_i \otimes f^j$ with e_i and f^j subject to relations (9–11). The matrix elements $L_i^j = e_i f^j$ satisfy the equality $L_i^l L_l^j = e_i (f^l e_l) f^j = Tr_q(L) L_j^i$ following from the commutation relations. So, conditions (38) for the case $k = 1$ hold simultaneously in this algebra. Another description of the quantum projective space is found in [7].*

As in the classical situation, we can consider solution λP_k with arbitrary $\lambda \neq 0$. This will lead to the relation $L^2 = \lambda L$ and the corresponding rescaling of the quantum trace. The resulting conditions give the deformation quantization of the orbit passing through λP_k. It may be regarded as the quantization of the same manifold but along the path $t = \lambda\omega \frac{\hat{k}_q}{\hat{n}_q}$ in the two parameter space of the universal family $\mathrm{S}_{q,t}(\mathfrak{g})$ (Theorem 10.1).

11.2 The Cayley-Hamilton identity

The goal of this subsection is to exhibit correspondence with identities in $\mathcal{A}_{RE}(\mathcal{M})$ of the Cayley-Hamiltonian type, [19]. In the algebra $\mathcal{A}_{RE}(\mathcal{M})$, the k-th powers L^k of the generating matrix form the left coadjoint module $\mathcal{M}^*$. The quantum traces $\mathrm{Tr}_q(L^k)$, $k = 1, 2, \dots$ are $\mathcal{U}_q(sl(n))$-invariant and central, (see, e.g., [19]).

Theorem 11.7. *[19]. The generating matrix of the reflection equation algebra $\mathcal{A}_{RE}(\mathcal{M})$ obeys the relation*

$$\sum_{j=0}^{n} \sigma_q^j(L)(-L)^{n-j} = 0,$$

where $\sigma_q^j(L)$ are central elements expressed through the quantum traces of powers in L by the recursive formula

$$\hat{k}_q \sigma_q^k(L) \;=\; \sum_{j=0}^{k-1} \sigma_q^j(L)(-1)^{k-j+1}\mathrm{Tr}_q(L^{k-j}), \quad \sigma_q^0(L) = 1. \tag{39}$$

In particular, $\sigma_q^{n+1}(L) = 0$.

It follows that imposing the projector condition on the matrix L specifies the quantum trace modulo the finite set of values.

Corollary 11.8. *The condition $L^2 = L$ on the generating matrix of the $\mathcal{A}_{RE}(\mathcal{M})$ implies the discrete set $\{\hat{0}_q, \hat{1}_q, \dots, \hat{n}_q\}$ of values for the quantum trace.*

Proof. The statement follows from the formula (39) since, under the hypothesis made, one has

$$\sigma_q^{n+1}(L) = \frac{\mathrm{Tr}_q(L)\big(\mathrm{Tr}_q(L) - \hat{1}_q\big)\dots\big(\mathrm{Tr}_q(L) - \hat{n}_q\big)}{\widehat{(n+1)}_q!} = 0.$$

□

All the possible values of $\mathrm{Tr}_q(L)$ are realized, by Theorem 11.5.

The Cayley-Hamiltonian identity in the RE algebra was used for construction of the quantum sphere $\mathbb{S}_q^2$ in [10].

11.3 On the quantization of non-semisimple orbits

As was already mentioned, the two-parameter quantization $\mathrm{S}_{q,t}(sl(n))$ may be "restricted" to every semisimple orbits. There are no definite assertions of that kind concerning orbits passing through nilpotent elements. In this section, we prove

Theorem 11.9. *There exists a one-parameter $\mathcal{U}_q(\mathfrak{g})$-invariant quantization of nilpotent orbits in $\mathcal{M}$ satisfying the matrix equation $A^2 = 0$. It is a restriction of the one-parameter subfamily $t = 0$ in the universal two-parameter quantization $S_{q,t}(\mathfrak{g})$.*

Proof. Like in the case of symmetric orbits, we seek for a solution to the matrix reflection equation, which will realize the quantized algebra as a subalgebra in $\mathcal{F}_q(G)$. Specializing to the skew-diagonal matrices,

$$A = \sum_{i=1}^{n} \lambda_i e_i^{i'}, \quad i' \equiv n + 1 - i,$$

we find it in the form $\lambda_i \lambda_{i'} = \lambda^2, \quad i = 1, \dots, n$. In case of non-zero λ we come to nondegenerate matrices with two eigenvalues. These solutions were found in [16]. They lead to other realization of quantum symmetric orbits than by means of projectors. In case of $\lambda = 0$, we obtain a solution with arbitrary values of λ_i either above or below the principal diagonal. All these matrices being squared are zero. By re-enumerating the basis elements, they can be brought to the sum of 2×2 jordanian blocks yielding all such matrices. Let us fix the solution $A = \sum_{i=1}^{k} e_{i'}^{i}$, for $k \leq \frac{n}{2}$. The stabilizer is a subalgebra in the parabolic Lie algebra with the semisimple part $sl(k) \oplus sl(n-k) \oplus sl(k)$. It is selected by the identification between symmetrically allocated $sl(k)$ via the involutive conjugation by the skew-diagonal unit of dimension k. As was noted in Remark 2.1, this automorphism extends to the quantized algebras. $\square$

11.4 On the quantization of bisymmetric orbits

To quantize the symmetric orbits in g^*, we used the projectors, i. e., the semisimple elements with the eigenvalues 1 and 0. There are non-degenerate semisimple solutions to the matrix reflection equation with two eigenvalues. For example, one can take $\sum_{i=1}^{n} e_{n+1-1}^{i}$. Such matrices can be used for constructing solutions with three eigenvalues by the embedding method, along the line of Proposition 11.4. In this way, one gains the additional zero eigenvalue. One might have expected that this will provide a tool for quantizing all the semisimple orbits, which are classified as homogeneous spaces by the number of eigenvalues and their multiplicities. However, that is probably impossible, due to certain indications. Indeed, in [16], there were written out all non-degenerate solutions up to $n = 4$. They have at most two eigenvalues and lead to different quantizations for the symmetric orbits than by projectors. Among them, there are also q-traceless matrices. We conjecture that non-degenerate matrix solutions to the reflection equation with arbitrary value of the quantum trace do exist for each symmetric orbit. This implies the existence of the quantization via the RE algebra characters for every bisymmetric orbit, i. e., consisting of matrices with three eigenvalues. Let us present, following [16], the non-degenerate q-traceless RE matrices up to $n = 4$.

$$A^{1,1} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad A^{2,1} = \begin{pmatrix} -q^{-2} & & \hat{2}_q \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$A^{2,2} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad A^{3,1} = \begin{pmatrix} -q^{-2}\hat{2}_q & & & \hat{3}_q \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix}.$$

The matrix $A^{2,2}$ is interesting from the physical point of view because it yields the quantization of the twistor space. Using these RE matrices together with the projectors P_k, we can recover all the semisimple orbits for $n \leq 4$, excepting the maximal orbit in dimension $n = 4$. But the maximal orbits can be quantized by specifying values of the Casimir elements, i. e., $\mathrm{Tr}(L^i)$, $i = 1, \ldots, n-1$. So we conclude that, at least to dimension $n = 4$, all the semisimple orbits can be quantized explicitly as quotients of the reflection equation algebra.

References

[1] [AFS] A. Alekseev, L. Faddeev, M. Semenov-Tian-Shansky, *Hidden quantum group inside Kac-Moody algebra,* Proceedings of the Euler International Mathematical Institute on Quantum Groups, Lect. Notes Math. **1510** (Springer, Berlin, 1992) 148.

[2] [Cher] I. V. Cherednik, *Factorizing particles on a half line, and root systems,* Teoret. Mat. Fiz. **61** (1984), no. 1, 35–44.

[3] [D] V. G. Drinfeld, *Quantum Groups*, in Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A.V. Gleason, AMS, Providence (1987) 798.

[4] [Do] J. Donin, *Double quantization on the coadjoint representation of $sl(n)$,* Czech J. of Physics, **47**, n.11, 1997, 1115-1122.

[5] [Do1] J. Donin, *$U_h(\mathfrak{g})$-invariant quantization of coadjoint orbits and vector bundles over them,* Preprint 54 (2000), Max-Plank-Institute, J. of Geometry and Physics, **38** (2001) 54.

[6] [DoG] J. Donin, D. Gurevich, *Some Poisson structures associated to Drinfeld-Jimbo R-matrices and their quantization,* Israel J. of Math. **92** (1995) 23.

[7] [DGK] J. Donin, D. Gurevich, S. Khoroshkin, *Double quantization of $\mathbb{C}P^n$ type by generalized Verma modules,* Preprint math.QA/9803155, J. of Geom. and Phys., v.28, 1998, 384.

[8] [FRT] L. Faddeev, N. Reshetikhin, and L. Takhtajan, *Quantization of Lie groups and Lie algebras* Leningrad Math. J. **1** (1990) 193.

[9] [G] D. Gurevich, *Algebraic aspects of quantum Yang-Baxter equation,* Leningrad Math. J. **2** (1991) 802.

[10] [GS] D. Gurevich, P. A. Saponov *Quantum sphere via reflection equation algebra,* Preprint math.QA/9911141.

[11] [KT] S. M. Khoroshkin, V. N. Tolstoy *Universal R-matrix for quantized (super) algebras,* Comm. Math. Phys. **141** (1991) 559.

[12] [KR] A. N. Kirillov, N. Yu. Reshetikhin, *q-Weyl group an multiplicative formula for R-matrices,* Comm. Math. Phys. **130** (1990) 421.

[13] [K] P. P. Kulish, *Quantum groups, q-oscillators, and covariant algebras,* Theor. Math. Phys. **94** (1993) 193.

[14] [KSkl] P. P. Kulish, E. K. Sklyanin *Algebraic structure related to the reflection equation,* J. Phys. A **25** (1992) 5963.

[15] [KS] P. P. Kulish, R. Sasaki *Covariance properties of reflection equation algebras,* Prog. Theor. Phys. **89** #3 (1993) 741.

[16] [KSS] P. P. Kulish, R. Sasaki, and C. Schweibert, *Constant solutions of reflection equations and quantum groups,* J. Math. Phys **34** #1 (1993) 286.

[17] [Lu] G. Lusztig, "Introduction to quantum groups", Progress in Mathematics, 110. Birkh. Boston, Inc., Boston, MA, 1993.

[18] [Mj] S. Majid, "Foundations of quantum group theory", Cambridge University Press, 1995.

[19] [PS] P. N. Pyatov, P. A. Saponov *Characteristic relations for quantum matrices,* J. Phys. A **28** #3 (1995) 4415.

[20] [Ros] M. Rosso, *An analog of P.B.W. theorem and the universal* $U_h sl(N+1)$, Comm. Math. Phys. **124** (1991) 307.