TOPOLOGICAL ASPECTS OF KLEINIAN GROUPS IN HIGHER DIMENSIONS

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Abstract. This is a survey of higher-dimensional Kleinian groups, i.e., discrete isometry groups of the hyperbolic n-space $\mathbb{H}^n$ for $n \geq 4$. Our main emphasis is on the topological and geometric aspects of higher-dimensional Kleinian groups and their contrast with the discrete groups of isometry of $\mathbb{H}^3$.

To the memory of Sasha Reznikov

1. Introduction

The goal of this survey is to give an overview (mainly from the topological prospective) of the theory of Kleinian groups in higher dimensions. The survey grew out of a series of lectures which I gave in the University of Maryland in Fall of 1991. An early version of this paper appeared as the preprint [107]. In this survey I collected well-known facts as well as less-known and new results. Hopefully, this will make the survey interesting to both non-experts and experts. We also refer the reader to Tukia's short survey of higher-dimensional Kleinian groups, [201].

There is a vast variety of Kleinian groups in higher dimensions: It appears that there is no hope for a comprehensive structure theory similar to the theory of discrete groups of isometries of $\mathbb{H}^3$. I do not know a good guiding principle for taxonomy of the higher-dimensional Kleinian groups. In this paper the higher-dimensional Kleinian groups are organized according to the topological complexity of their limit sets. In this setting, one of the key questions that I will address is the interaction between the geometry and topology of the limit set and the algebraic and topological properties of the Kleinian group.

This paper is organized as follows. In Section 2 we consider the most basic concepts of the theory of Kleinian groups, e.g. domain of discontinuity, limit set, geometric finiteness, etc. In Section 3 we discuss various ways to construct Kleinian groups and list the tools of the theory of Kleinian groups in higher dimensions. In Section 4 we review the homological algebra used in the paper. In Section 5 we discuss various notions of equivalence between Kleinian groups: From the weakest (isomorphism) to the strongest (conjugacy).

In Section 6 we consider groups with zero-dimensional limit sets; such groups are relatively well-understood. Convex-cocompact groups with 1-dimensional limit sets are discussed in Section 7. Although the topology of the limit sets of such groups is well-understood, their group-theoretic structure is a mystery. We know very little about Kleinian groups with higher-dimensional limit sets, thus we restrict the discussion to Kleinian groups whose limit sets are topological spheres (Section 8). We

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then discuss Ahlfors finiteness theorem and its failure in higher dimensions (Section 9). We then consider the representation varieties of Kleinian groups (Section 10). Lastly we discuss algebraic and topological constrains on Kleinian groups in higher dimensions (Section 11).

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Contents

2. Basic definitions

Moebius transformations. For the lack of space, our discussion of the basics of Kleinian groups below is somewhat sketchy. For the detailed treatment we refer the reader to [130, 16, 35, 173, 113]. We let $\mathbb{B}^{n+1}$ denote the closed ball $\mathbb{H}^{n+1} \cup \mathbb{S}^n$; its boundary $\mathbb{S}^n$ is identified via the stereographic projection with $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Let $\mathbf{Mob}(\mathbb{S}^n)$ denote the group of all *Moebius* transformations of the n-sphere $\mathbb{S}^n$, i.e., compositions of inversions in $\mathbb{S}^n$. The group $\mathbf{Mob}(\mathbb{S}^n)$ admits an extension to the hyperbolic space $\mathbb{H}^{n+1}$, so that $\mathbf{Mob}(\mathbb{S}^n) = \mathrm{Isom}(\mathbb{H}^{n+1})$, the isometry group of $\mathbb{H}^{n+1}$.

For elements $\gamma \in \mathbf{Mob}(\mathbb{S}^n)$ define the *displacement function*

$$d_\gamma(x) := d(x, \gamma(x)), \quad x \in \mathbb{H}^{n+1}.$$

The elements γ of $\mathbf{Mob}(\mathbb{S}^n)$ are classified as:

1. *Hyperbolic:* The function d_γ is bounded away from zero. Its minimum is attained on a geodesic $A_\gamma \subset \mathbb{H}^{n+1}$ invariant under γ. The ideal end-points of A_γ are the fixed points of γ in $\mathbb{S}^n$.

2. *Parabolic*: The function d_γ is positive but has zero infimum on $\mathbb{H}^{n+1}$; such elements have precisely one fixed point in $\mathbb{S}^n$.

3. *Elliptic*: γ fixes a point in $\mathbb{H}^{n+1}$.

The group $\mathbf{Mob}(\mathbb{S}^n)$ is isomorphic to an index 2 subgroup in the Lorentz group $O(n+1,1)$, see e.g. [173]. In particular, $\mathbf{Mob}(\mathbb{S}^n)$ is a matrix group. Selberg's lemma (see [186]) implies that every finitely generated group of matrices contains a finite index subgroup which is torsion-free. This, of course, applies to the finitely generated subgroups of $\mathbf{Mob}(\mathbb{S}^n)$ as well.

A group Γ is said to *virtually* satisfy a property X if it contains a finite index subgroup $\Gamma' \subset \Gamma$, such that Γ' satisfies X. Therefore, every finitely generated group of matrices is *virtually torsion-free.*

Definition 2.1. A discrete subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is called a *Kleinian group.*

Dynamical notions. The *discontinuity set* $\Omega(\Gamma)$ of a group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$, is the largest open subset in $\mathbb{S}^n$ where Γ acts properly discontinuously. Its complement $\mathbb{S}^n \setminus \Omega(\Gamma)$ is the *limit set* $\Lambda(\Gamma)$ of the group Γ. Equivalently, the limit set of a Kleinian group can be described as the accumulation set in the sphere $\mathbb{S}^n$ of an orbit $\Gamma \cdot o$. Here o is an arbitrary point in $\mathbb{H}^{n+1}$. A Kleinian group is called *elementary* if its limit set is finite, i.e., is either empty, or consists of one or of two points.

We will use the notation $M^n(\Gamma)$ for the n-dimensional quotient $\Omega(\Gamma)/\Gamma$ and $\bar{M}^{n+1}(\Gamma)$ for the $n+1$-dimensional quotient $(\mathbb{H}^{n+1} \cup \Omega(\Gamma))/\Gamma$.

For a closed subset $\Lambda \subset \mathbb{S}^n$, let $Hull(\Lambda)$ denote its *convex hull* in $\mathbb{H}^n$, i.e., the smallest closed convex subset H of $\mathbb{H}^n$ such that

$$cl_{\mathbb{B}^{n+1}}(H) \cap \mathbb{S}^n = \Lambda.$$

Clearly, if Λ is a point, then $Hull(\Lambda)$ does not exist. Otherwise, $Hull(\Lambda)$ exists and is unique. We declare $Hull(\Lambda)$ to be empty in the case when Λ is a single point.

One way to visualize the convex hull $Hull(\Lambda)$ is to consider the *projective model* of the hyperbolic space, where the geodesic lines are straight line segments contained in the interior of $\mathbb{B}^{n+1}$. Therefore, the Euclidean notion of convexity coincides with the hyperbolic notion. Hence the convex hull in this model can be described as follows: $Hull(\Lambda)$ is the intersection of the Euclidean convex hull of Λ with the interior of $\mathbb{B}^{n+1}$.

Suppose that $\Lambda = \Lambda(\Gamma)$ is the limit set of a Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$. The quotient $Hull(\Lambda)/\Gamma$ is called the *convex core* of the orbifold $N = \mathbb{H}^{n+1}/\Gamma$. It is characterized by the property that it is the smallest closed convex subset in N, whose inclusion to N is a homotopy-equivalence. For $\epsilon > 0$ consider the open ϵ-neighborhood $Hull_\epsilon(\Lambda)$ of $Hull(\Lambda)$ in $\mathbb{H}^{n+1}$. Since $Hull_\epsilon(\Lambda)$ is Γ-invariant, we can form the quotient $Hull_\epsilon(\Lambda)/\Gamma$. Then $Hull_\epsilon(\Lambda)/\Gamma$ is the ϵ-neighborhood of the convex core.

Geometric finiteness. We now arrive to one of the key notions in the theory of Kleinian groups:

Definition 2.2. A Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is called *geometrically finite* if:

(1) Γ is finitely generated,

and

(2) $vol(Hull_\epsilon(\Lambda(\Gamma))/\Gamma) < \infty$.

In a number of important special cases, e.g. when Γ is torision-free, or $n = 2$, or when $\Lambda(\Gamma) = \mathbb{S}^n$, the assumption (1) follows from (2), see [35]. However, E. Hamilton [89] constructed an example of a Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ for which (2) holds but (1) fails. This group contains finite order elements of arbitrarily high order; by Selberg's lemma, such groups cannot be finitely-generated.

A Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is called a *lattice* if $\mathbb{H}^{n+1}/\Gamma$ has finite volume. Equivalently, $\Lambda(\Gamma) = \mathbb{S}^n$ and Γ is geometrically finite. A lattice is *cocompact* (or *uniform*) if $\mathbb{H}^{n+1}/\Gamma$ is compact.

One can characterize geometrically finite groups in terms of their limit sets. Before stating this theorem we need two more definitions.

Definition 2.3. A limit point $\xi \in \Lambda(\Gamma)$ is called a *conical limit point* if there exists a geodesic $\alpha \subset \mathbb{H}^n$ asymptotic to ξ, a point $o \in \mathbb{H}^n$, a number $r < \infty$, and a sequence $\gamma_i \in \Gamma$ so that

1. $\lim_i \gamma_i(o) = \xi$.
2. $d(\gamma_i(o), \alpha) \leq r$.

The reason for this name comes from the shape of the r-neighborhood of the vertical geodesic α in the upper half-space model of $\mathbb{H}^{n+1}$: It is a Euclidean cone with the axis α. Equivalently, one can describe the conical limit points of nonelementary groups by the following (see [35]):

$\xi \in \Lambda(\Gamma)$ is a conical limit point if and only if for every $\eta \in \Lambda(\Gamma) \setminus \{\xi\}$ there exists a point ψ and a sequence $\gamma_i \in \Gamma$ such that:

1. $\lim_i \gamma_i(\zeta) = \xi$ for every $\zeta \in \Lambda(\Gamma) \setminus \{\psi\}$.
2. $\lim_i \gamma_i^{-1}(\xi) \neq \lim_i \gamma_i^{-1}(\eta)$.

The set of conical limit points of a Kleinian group Γ is denoted $\Lambda_c(\Gamma)$.

Definition 2.4. A point $\xi \in \Lambda(\Gamma)$ is called a *bounded parabolic fixed point* if it is the fixed point of a parabolic subgroup $\Pi \subset \Gamma$ and $(\Lambda(\Gamma) - \{\xi\})/\Pi$ is compact.

The following is one of the many characterizations of geometrically finite groups:

Theorem 2.5. *(A. Beardon and B. Maskit* [12], *B. Bowditch* [35]*) A Kleinian group Γ is geometrically finite if and only if each limit point $\xi \in \Lambda(\Gamma)$ is either a* conical limit point *or a* bounded parabolic fixed point*.*

C. Bishop proved in [28] that one can drop the word *bounded* in the above theorem.

We refer the reader to [35] for other characterizations of geometric finiteness. For instance, if Γ has no parabolic elements then geometric finiteness is equivalent to the assumption that $\bar{M}^{n+1}(\Gamma)$, or, equivalently, $Hull(\Lambda(\Gamma))/\Gamma$, is compact. In this case the group Γ is called *convex-cocompact*. We will frequently use the fact that every convex-cocompact Kleinian group is *Gromov-hyperbolic* (see e.g. [39]).

A sufficient, but not necessary, condition for geometric finiteness of $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is existence of a convex fundamental polyhedron $\Phi \subset \mathbb{H}^{n+1}$, which has only finitely many faces.

If $n = 1$, then every finitely generated Kleinian group is geometrically finite. The proof is rather elementary, see e.g. [48]. For $n \geq 2$ this implication is no longer true. The first (implicit) examples were given by L. Bers, they are *singly-degenerate groups*:

Definition 2.6. A finitely generated nonelementary Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^2)$ is *singly degenerate* if its domain of discontinuity is simply-connected, i.e., homeomorphic to the 2-disk.

L. Bers [17] proved that singly degenerate Kleinian groups exist and are never geometrically finite. The first *explicit* examples of finitely generated geometrically infinite Kleinian subgroups Γ of $\mathbf{Mob}(\mathbb{S}^2)$ were given by T. Jorgensen [100]. In Jorgensen's examples Γ appears as a normal subgroup of a lattice $\hat{\Gamma} \subset \mathbf{Mob}(\mathbb{S}^2)$ with $\hat{\Gamma}/\Gamma \cong \mathbb{Z}$. Remarkably, all known examples of finitely-generated geometrically infinite Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^n)$ can be traced to the 2-dimensional examples (i.e., contained in $\mathbf{Mob}(\mathbb{S}^2)$), or quasiconformal deformations of such examples.

Problem 2.7. *Construct examples of finitely-generated geometrically infinite subgroups of $\mathbf{Mob}(\mathbb{S}^n)$, $n \geq 3$, which do not have the 2-dimensional origin as above.*

Assumption 2.8. *From now on we will assume that all Kleinian groups are finitely generated and torsion free, unless we state otherwise.*

Note that the second part of this assumption is not very restrictive because of Selberg's lemma.

Cusps. Γ-conjugacy classes $[\Pi]$ of maximal parabolic subgroups Π of a Kleinian group Γ are called *cusps* of Γ. More geometrically, cusps of Γ can be described using the *thick-thin decomposition* of the quotient manifold $M = \mathbb{H}^{n+1}/\Gamma$. Given a positive number $\epsilon > 0$ let $M_{(0,\epsilon]}$ denote the collection of points x in M such that there exists a homotopically nontrivial loop α based at x, so that the length of α is at most ϵ. Then $M_{(\epsilon,\infty)}$ is the complement of $M_{(0,\epsilon]}$ in M. According to Kazhdan-Margulis lemma [124], there exists a number $\mu = \mu_{n+1} > 0$ such that for every Kleinian group Γ, every component of $M_{(0,\mu]}$ has *virtually abelian* fundamental group. The submanifold $M_{(0,\mu]}$ is called the *thin part* of M and its complement the *thick part* of M.

Now, the cusps of Γ are in bijective correspondence with the noncompact components of $M_{(0,\mu]}$ (also called cusps):

For every cusp $[\Pi]$ in Γ, there exists a noncompact component $C \subset M_{(0,\mu]}$, so that $\Pi = \pi_1(C)$. Conversely, for each cusp $C \subset M$, there exists a maximal parabolic subgroup $\Pi \subset \Gamma$ such that $\Pi = \pi_1(C)$.

Taking the Γ-conjugacy class of Π reflects ambiguity in the choice of the base-point needed to identify $\pi_1(M)$ and Γ.

Moebius structures. In this paper we shall also discuss the subject closely related to the theory of Kleinian groups, namely *Moebius structures*. When M is a smooth manifold of dimension ≥ 3, a *Moebius* (or *flat conformal*) structure K on a M is the conformal class of a conformally-Euclidean Riemannian metric on M. Topologically, K is a maximal *Moebius atlas* on M, i.e., an atlas with Moebius transition maps. Thus, for each Kleinian group Γ and a Γ-invariant subset $\Omega \subset \Omega(\Gamma)$, the standard Moebius structure on $\Omega \subset \mathbb{S}^n$ projects to a Moebius structure K_Γ on the manifold Ω/Γ. The Moebius structures of this type are called *uniformizable*.

Complex-hyperbolic Kleinian groups. Instead of considering the isometry group of the hyperbolic n-space, one can consider other negatively curved symmetric spaces, for instance, the *complex–hyperbolic n-space* $\mathbb{CH}^n$ and its group of biholomorphic isometries $PU(n,1)$. From the analytical viewpoint, $\mathbb{CH}^n$ is the unit ball in $\mathbb{C}^n$ and $PU(n,1)$ is the group of biholomorphic automorphisms of this ball. The Bergman metric on $\mathbb{CH}^n$ is a Kähler metric of negative sectional curvature. The discrete subgroups of $PU(n,1)$ are *complex–hyperbolic Kleinian groups*. They share many properties with Kleinian groups. In fact, *nearly all positive results* stated in this survey for Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^n)$ $(n \geq 3)$ are also valid for the complex–hyperbolic Kleinian groups! (One has to replace *virtually abelian* with *virtually nilpotent* in the discussion of cusps.) There exists an isometric embedding $\mathbb{H}^n \to \mathbb{CH}^n$ which induces an embedding of the isometry groups. Therefore, complex–hyperbolic Kleinian groups $(n \geq 4)$ also inherit the pathologies of the higher–dimensional Kleinian groups.

We refer the reader to [76, 77, 182] for the discussion of complex–hyperbolic Kleinian groups.

3. Ways and means of Kleinian groups

3.1. **Ways: Sources of Kleinian groups.** The following is a list of ways to construct Kleinian groups.

(a) **Poincare fundamental polyhedron theorem** (see e.g. [173] for the very detailed discussion, as well as [146]). This source is, in principle, the most general. The Poincare fundamental polyhedron theorem asserts that given a polyhedron Φ in $\mathbb{H}^{n+1}$ and a collection of elements $\gamma_1, \gamma_2, ..., \gamma_k, ...$ of $\mathbf{Mob}(\mathbb{S}^n)$, pairing the faces of Φ, under certain condition on this data, the group Γ generated by $\gamma_1, \gamma_2, ..., \gamma_k, ...$ is Kleinian and Φ is a *fundamental domain* for the action of the group Γ on $\mathbb{H}^{n+1}$.

Every Kleinian group has a convex fundamental polyhedron (for example, the *Dirichlet fundamental domain*). However, in practice, Poincare fundamental polyhedron theorem is not always easy to use, especially if Φ has many faces and n is large. This theorem was used, for instance, to construct non-arithmetic lattices in

$\mathbf{Mob}(\mathbb{S}^n)$ (see [140, 141, 203]), as well as other interesting Kleinian groups, see e.g. [55, 131, 108, 94, 174].

(b) **Klein–Maskit Combination Theorems** (see e.g. [146] and [130]). The Combination Theorems provide conditions that guarantee that a group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ generated by two Kleinian subgroups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$, is again Kleinian and algebraically splits as an amalgam of Γ_1 and Γ_2. (These theorems generalize the classical "ping-pong" argument due to Schottky and Klein.) Combination Theorems also show that the quotient manifold $M^n(\Gamma)$ of the group Γ is obtained from $M^n(\Gamma_1)$, $M^n(\Gamma_2)$ via some "cut-and-paste" operation. Moreover, Combination Theorems generalize to graph of groups. There should be a generalization of Combination Theorems to *complexes of groups* (see e.g. [39] for the definition); however, to the best of my knowledge, nobody worked out the general result, see [114] for a special case.

(c) **Arithmetic groups and their subgroups** (see e.g. [204] and [139]). A subgroup $\Gamma \subset O(n,1)$ is called *arithmetic* if there exists an embedding

$$\iota : O(n,1) \hookrightarrow GL(N, \mathbb{R}),$$

such that the image $\iota(\Gamma)$ is *commensurable* with the intersection

$$\iota(O(n,1)) \cap GL(N, \mathbb{Z}).$$

Recall that two subgroups $\Gamma_1, \Gamma_2 \subset G$ are called *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

Below is a specific construction of arithmetic groups. Let f be a quadratic form of signature $(n,1)$ in $n+1$ variables with coefficients in a totally real algebraic number field $K \subset \mathbb{R}$ satisfying the following condition:

(*) For every nontrivial (i.e., different from the identity) embedding $\sigma : K \to \mathbb{R}$, the quadratic form f^σ is positive definite.

Without loss of generality one may assume that this quadratic form is diagonal. For instance, take

$$f(x) = -\sqrt{2}x_0^2 + x_1^2 + ... + x_n^2.$$

We now define discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ using the form f. Let A denote the ring of integers of K. We define the group $\Gamma := O(f, A)$ consisting of matrices with entries in A preserving the form f. Then Γ is a discrete subgroup of $O(f, \mathbb{R})$. Moreover, it is a lattice: its index 2 subgroup

$$\Gamma' = O'(f, A) := O(f, A) \cap O'(f, \mathbb{R})$$

acts on $\mathbb{H}^n$ so that $\mathbb{H}^n/\Gamma'$ has finite volume. Such groups Γ (and subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ commensurable to them) are called *arithmetic subgroups of the simplest type* in $O(n,1)$, see [204].

Remark 3.1. If $\Gamma \subset O(n,1)$ is an arithmetic lattice so that either Γ is non-cocompact or n is even, then it follows from the classification of rational structures on $O(n,1)$ that Γ is commensurable to an arithmetic lattice of the simplest type. For odd n there is another family of arithmetic lattices given as the groups of units of appropriate skew-Hermitian forms over quaternionic algebras. Yet other families of arithmetic lattices exist for $n = 3$ and $n = 7$. See e.g. [204].

We refer the reader to [139] for the detailed treatment of geometry and topology of arithmetic subgroups of $\mathbf{Mob}(\mathbb{S}^2)$.

(d) **Small deformations of a given Kleinian group**. We discuss this construction in details in Section 10.1. The idea is to take a Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ and to "perturb it a little bit", by modifying the generators slightly (within $\mathbf{Mob}(\mathbb{S}^n)$) and preserving the relators. The result is a new group Γ' which may or may not be Kleinian and even if it is, Γ' is not necessarily isomorphic to Γ. However if Γ is convex-cocompact, Γ' is again a convex-cocompact group isomorphic to Γ, see Theorem 10.14.

(e) **Limits of sequences of Kleinian groups**, see Section 10.3. Take a sequence Γ_i of Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^n)$ and assume that it has a limit Γ: It turns out that there are two ways to make sense of this procedure (*algebraic* and *geometric* limit). In any case, Γ is again a Kleinian group. Even if the (algebraic) limit does not exist as a subgroup of $\mathbf{Mob}(\mathbb{S}^n)$, there is a way to make sense of the limiting group as a group of isometries of a metric tree. This logic turns out to be useful for proving compactness theorems for sequences of Kleinian groups.

(f) **Differential-geometric constructions of hyperbolic metrics.** The only (but spectacular) example where it has been used is Perelman's work on Ricci flow and proof of Thurston's geometrization conjecture. However applicability of this tool at the moment appears to be limited to 3-manifolds.

A beautiful example of application of (b) and (c) is the construction of M. Gromov and I. Piatetski-Shapiro [84] of *non-arithmetic* lattices in $\mathbf{Mob}(\mathbb{S}^n)$. Starting with two arithmetic groups Γ_j $(j = 1, 2)$ they first "cut these groups in half", take "one half" $\Delta_j \subset \Gamma_j$ of each, and then combine Δ_j's via Maskit Combination. The construction of Kleinian groups in [85] (see also Section 8.2) is an application of (b), (c) and (d). Thurston's hyperbolic Dehn surgery theorem is an example of (e). One of the most sophisticated constructions of Kleinian groups is given by Thurston's hyperbolization theorem (see e.g. [165], [166], [113]); still, it is essentially a combination (a very complicated one!) of (b), (d) and (e).

Remark 3.2. There is potentially the sixth source of Kleinian groups in higher dimensions: monodromy of linear ordinary differential equations. However, to the best of my knowledge, the only example of its application relevant to Kleinian groups, is the construction of lattices in $PU(n, 1)$ (i.e., the isometry group of the complex-hyperbolic n–space) by Deligne and Mostow, see [57].

3.2. **Means: Tools of the theory of Kleinian groups in higher dimensions.** Several key tools of the "classical" theory of Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$ (mainly, the Beltrami equation and pleated hypersurfaces) are missing in higher dimensions. Below is the list of main tools that are currently available.

(a) **Dynamics, more specifically, the convergence property:** Every sequence of Moebius transformations $\gamma_i \in \mathbf{Mob}(\mathbb{S}^n)$ either contains a convergent subsequence or contains a subsequence which converges to a constant map away from a point in $\mathbb{S}^n$. See e.g. [113].

(b) **Kazhdan-Margulis lemma** and its corollaries.

It turns out that the lion share of the general results about higher-dimensional Kleinian groups is a combination of (a) and (b), together with some hyperbolic geometry.

(c) **Group actions on trees and Rips theory**. This is a very potent tool for proving compactness results for families of representations of Kleinian groups, see for instance Theorem 10.18.

(d) **Barycentric maps**. These maps were originally introduced by A. Douady and C. Earle [60] as a tool of the Teichmüller theory of Riemann surfaces. In the hands of of G. Besson, S. Gallot and G. Courtois these maps became a powerful analytic tool of the theory of Kleinian groups in higher dimensions, see e.g. [19, 20], as well as Theorems 9.21 and 10.24 in this survey. In contrast, equivariant harmonic maps which proved so useful in the study of, say, Kähler groups, seem at the moment to be only of a very limited use in the theory of Kleinian groups in higher dimensions.

(e) **Ergodic theory of the actions of Γ on its limit set and Patterson–Sullivan measures.** See for instance [191], [163] and the survey of P. Tukia [201].

(f) **Conformal geometric analysis.** This is a branch of (conformal) differential geometry concerned with the analysis of the conformally-flat Riemannian metrics on $M^n(\Gamma) = \Omega(\Gamma)/\Gamma$. This tool tends to work rather well in the case when $M^n(\Gamma)$ is compact. The most interesting examples of this technique are due to R. Schoen and S-T. Yau [181], S. Nayatani [162], A. Chang, J. Qing, J. and P. Yang, [49], and H. Izeki [96, 97, 98].

(g) **Infinite-dimensional representation theory** of the group $\mathbf{Mob}(\mathbb{S}^n)$. The only (but rather striking) example of its application is Y. Shalom's work [187].

(h) **Topological rigidity theorems of Farrell and Jones**: See Section 8.1.

4. A bit of homological algebra

(Co)homology of groups. Throughout this section we let R be a commutative ring with a unit. The *group ring* $R\Gamma$ of a group Γ consist of finite linear combinations of the form

$$\sum_{\gamma\in\Gamma} r_\gamma \gamma,$$

with $r_\gamma \in R$ equal to zero for all but finitely many $\gamma \in \Gamma$. The examples that the reader should have in mind are $R = \mathbb{Z}$ and $R = \mathbb{R}$. Let V be a (left) $R\Gamma$-module. Basic examples include $V = R$ (with the trivial $R\Gamma$-module structure) and $V = R\Gamma$. If R is a field, then V is nothing but a vector space over R equipped with a linear action of the group Γ. The very useful (for the theory of Kleinian groups) example is the following:

Let $G = \mathbf{Mob}(\mathbb{S}^n)$, $\mathfrak{g}$ be the Lie algebra of G. Then G acts on $\mathfrak{g}$ via the *adjoint representation* $Ad = Ad_G$. Therefore $\mathfrak{g}$ becomes an $\mathbb{R}G$-module. For every abstract group Γ and a representation $\rho : \Gamma \to G$ we obtain the $\mathbb{R}\Gamma$-module

$$V = \mathfrak{g}_{Ad(\rho)}$$

where the action of Γ is given by the composition $Ad \circ \rho$. We will abbreviate this module to $Ad(\rho)$. From the theory of Kleinian groups viewpoint, the most important example of this module is when Γ is a Kleinian subgroup of G and ρ is the identity embedding.

Given an $R\Gamma$-module V, one defines the *homology* and *cohomology* groups

$$H_*(\Gamma, V), \quad H^*(\Gamma, V)$$

of Γ with coefficients in V; we refer the reader to [26, 42] for the precise definitions and discussion. Similarly, given a collection Π of conjugacy classes of subgroups of Γ, one defines relative (co)homology $H^*(\Gamma, \Pi; V)$ and $H_*(\Gamma, \Pi; V)$. Whenever discussing (co)homology with $R = \mathbb{Z}$ and $\mathbb{Z}$ as the trivial $\mathbb{Z}\Gamma$-module, we will use the notation $H^*(\Gamma), H_*(\Gamma)$.

An example to keep in mind is the following. Suppose that M is a manifold, or, more generally, a cell complex, which is an Eilenberg-MacLane space $K(\Gamma, 1)$. Then for the trivial Γ-module R we have

$$H^*(\Gamma, R) \cong H^*(M, R), \quad H_*(\Gamma, R) \cong H_*(M, R).$$

The (co)homology of groups behave in a manner similar to the more familiar (co)homology of cell complexes. For instance, if Γ admits an n-dimensional $K(\Gamma, 1)$, then $H^i(\Gamma, V) = H_i(\Gamma, V) = 0$ for all $i > n$ and all $R\Gamma$-modules.

(Co)homological dimension. For a group Γ let $cd_R(\Gamma)$ and $hd_R(\Gamma)$ denote the *cohomological* and *homological* dimensions of Γ (over R):

$$cd_R(\Gamma) = \sup\{n : \exists \text{ an } R\Gamma\text{–module } V \text{ so that } H^n(\Gamma, V) \neq 0\},$$

$$hd_R(\Gamma) = \sup\{n : \exists \text{ an } R\Gamma\text{–module } V \text{ so that } H_n(\Gamma, V) \neq 0\}.$$

Using the relative (co)homology one defines the relative (co)homological dimension of Γ with respect to a collection Π of the conjugacy classes of its subgroups, $cd_R(\Gamma, \Pi)$ and $hd_R(\Gamma, \Pi)$. We will use this definition in the case when Γ is a Kleinian group and Π is the set of its cusps. We will omit the subscript $\mathbb{Z}$ whenever $R = \mathbb{Z}$.

If Γ admits a compact $K(\Gamma, 1)$ then

$$hd_R(\Gamma) = cd_R(\Gamma), \quad \forall \text{ rings } R,$$

see for instance [26]. In general,

$$hd_R(\Gamma) \leq cd_R(\Gamma) \leq hd_R(\Gamma) + 1.$$

Example 4.1. Let Γ be a free group of finite rank $k > 0$. Then $hd_R(\Gamma) = cd_R(\Gamma)$ for all rings R. Indeed, Γ admits a finite $K(\Gamma, 1)$ which is the bouquet B of k circles. Since B is 1-dimensional,

$$hd_R(\Gamma) = cd_R(\Gamma) \leq 1.$$

On the other hand, by taking the trivial $R\Gamma$-module $V = R$ we obtain

$$H_1(B, R) = R^k,$$

the direct sum of k copies of R, and hence is nontrivial.

It turns out that the converse to this example is also true, which is an application of the famous theorem of J. Stallings on ends of groups:

Theorem 4.2. *(J. Stallings,* [188]*) If Γ is a finitely generated group with $cd(\Gamma) = 1$, then Γ is free.*

This result was generalized by M. Dunwoody:

Theorem 4.3. *(M.Dunwoody,* [61]*) Let R be an arbitrary commutative ring with a unit.*

1. If Γ is a finitely generated torsion-free group with $cd_R(\Gamma) = 1$, then Γ is free.

2. If Γ is finitely-presented and $cd_R(\Gamma) = 1$ then Γ is a free product of finite and cyclic groups with amalgamation over finite subgroups. In particular, Γ is virtually free.

Poincare duality groups. Poincare duality groups are homological generalizations of the fundamental groups of closed aspherical manifolds. We give the definition under the assumption that Γ admits a compact $K(\Gamma, 1)$.

Definition 4.4. The group Γ is an (oriented) n-dimensional Poincare duality group over R (a $PD(n)$-group for short) if

$$H^i(\Gamma, R\Gamma) \cong R, \text{ for } i = n \text{ and } H^i(\Gamma, R\Gamma) = 0, \text{ for } i \neq n.$$

The basic examples are the fundamental groups of closed oriented aspherical n-manifolds.

This definition generalizes to (possibly non-oriented) $PD(n)$-groups, where we have to twist the module $V = R\Gamma$ by an appropriate orientation character $\chi : R\Gamma \to R$. The basic examples are the fundamental groups of closed aspherical n-manifolds M. The character χ in this case corresponds to the orientation character $\pi_1(M) \to R$.

The following is one of the major problems in higher-dimensional topology:

Problem 4.5. *(C.T.C. Wall, see a very detailed discussion in* [125]*.) Suppose that Γ is an n-dimensional Poincare duality group over $\mathbb{Z}$. Then there exits a closed n-dimensional manifold M which is $K(\Gamma, 1)$.*

This problem is open for all $n \geq 3$. The case $n = 1$ is an easy corollary of the Stallings-Dunwoody theorem. In the case $n = 2$, the positive solution is due to Eckmann, Linnel and Muller, see [63], [62]. This result was extended to the case of fields R by B. Bowditch [36] and for general rings R by M. Kapovich and B. Kleiner [118] and B. Kleiner [127].

Cannon's conjecture below is a special case (after Perelman's work) of Wall's problem:

Conjecture 4.6. *(J. Cannon) Suppose that Γ is a Gromov-hyperbolic group whose ideal boundary is homeomorphic to $\mathbb{S}^2$. Then Γ admits a cocompact properly discontinuous isometric action on $\mathbb{H}^3$.*

5. Notions of equivalence for Kleinian groups

In this section we discuss various equivalence relations for Kleinian subgroups Γ_1, Γ_2 of $\mathbf{Mob}(\mathbb{S}^n)$. We start with the weakest one and end with the strongest.

(0) **Algebraic**: Γ_1 is isomorphic to Γ_2 as an abstract group.

(1) **Dynamical**: there exists a homeomorphism $f : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ such that $f\Gamma_1 f^{-1} = \Gamma_2$; i.e., the groups Γ_1 and Γ_2 have the same topological dynamics on their limit sets. Thus, Γ_1 is geometrically finite iff Γ_2 is, since geometric finiteness can be stated in terms of topological dynamics of a group on its limit set (Theorem 2.5).

(2) **Topological conjugation**: there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $f\Gamma_1 f^{-1} = \Gamma_2$. (One can relax this by assuming that f is defined only on the domain of discontinuity.)

(3) **Quasiconformal conjugation**: in (2) one can find a quasiconformal homeomorphism. In the case $n = 1$ one should replace *quasiconformal* with *quasisymmetric*.

(4) **Topological isotopy**: in (2) there exists a continuous family of homeomorphisms $h_t : \mathbb{S}^n \to \mathbb{S}^n$ such that: $h_0 = id$, $\forall t$, $h_t \Gamma_1 h_t^{-1} \subset \mathbf{Mob}(\mathbb{S}^n)$ and $h_1 \Gamma_1 h_1^{-1} = \Gamma_2$.

(5) **Quasiconformal isotopy**: in (4) all homeomorphisms are quasiconformal (quasisymmetric).

(6) **Moebius conjugation:** there is $f \in \mathbf{Mob}(\mathbb{S}^n)$ such that $f\Gamma_1 f^{-1} = \Gamma_2$.

We refer the reader to [90, 95] for the definitions of quasisymmetric and quasiconformal homeomorphisms.

Below is a collection of facts about the relation between different notions of equivalence of Kleinian groups.

Suppose that both groups Γ_j are *geometrically finite* and $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism which preserves the *type* of elements, i.e., for $\gamma \in \Gamma_1$, $\varphi(\gamma)$ is hyperbolic if and only if γ is hyperbolic. It is clear that the above assumptions are necessary for getting the equivalence (1). The following theorem shows that these assumptions are also sufficient.

Theorem 5.1. *(P. Tukia* [199]*). Under the above assumptions, the isomorphism φ can be realized by the equivalence (1), i.e., there exists a (quasisymmetric) homeomorphism f of the limit sets, so that $f\gamma f^{-1} = \varphi(\gamma)$ for all $\gamma \in \Gamma_1$. Moreover, if $f : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$ is a φ-equivariant quasiconformal (quasisymmetric) homeomorphism, then f admits a φ-equivariant quasiconformal (quasisymmetric) extension to the entire sphere.*

Question 5.2. (Quasiconformal vs. topological.) Suppose that two Kleinian groups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$ are topologically conjugate by a homeomorphism f (defined either on $\Omega(\Gamma_1)$, or on $\Lambda(\Gamma_1)$, or on the entire $\mathbb{S}^n$), which induces a type-preserving isomorphism $\varphi : \Gamma_1 \to \Gamma_2$. Does it imply that φ is induced by a quasiconformal (quasisymmetric) homeomorphism with the same domain as f?

Note that, for every n, the above question actually consists of 3 subquestions, depending on the domain of f. Here is what is currently known about these questions:

1. If $n = 1$ then all three questions have the affirmative answer and the proof is rather elementary. It also follows for instance from Theorem 5.1.

2. If $n = 2$ then the answer to all three questions is again positive, but the proof is highly nontrivial. The easiest case is when the homeomorphism f is defined on $\Omega(\Gamma_1)$. Then we get the induced homeomorphism $\bar{f}$ of the quotient surfaces $S_1 \to S_2$, where

$S_i = \Omega(\Gamma_i)/\Gamma_i$. The existence of a diffeomorphism $S_1 \to S_2$ homotopic to $\bar{f}$ follows from the uniqueness of the smooth structure on surfaces. If S_1 is compact, then this diffeomorphism lifts to an equivariant quasiconformal homeomorphism $\Omega(\Gamma_1) \to \Omega(\Gamma_2)$. Two noncompact surfaces can be diffeomorphic but not quasiconformally homeomorphic: For instance, open disk is not quasiconformally equivalent to the complex plane. However, since φ is type-preserving, Ahlfors Finiteness Theorem [4] in conjunction with a lemma of Bers and Maskit (see e.g. [113, Corollary 4.85]), implies the existence of a quasiconformal homeomorphism $S_1 \to S_2$.

If f is defined on the limit set and Γ_1, Γ_2 are geometrically finite, then the positive answer is a special case of Tukia's theorem 5.1. However, if Γ_i's are not geometrically finite, the proof becomes very difficult and is a corollary of the solution of the Ending Lamination Conjecture in the work of J. Brock, R. Canary and Y. Minsky in [156], [155], [41], and M. Rees [176].

Combination of the Ahlfors' Finiteness Theorem with the Ending Lamination Conjecture also gives the positive answer in the case when f is defined on $\mathbb{S}^2$.

3. If f is defined on $\Omega(\Gamma_1)$ then the answer is positive provided that $n \neq 4$ and $M^n(\Gamma_1)$ is compact. This is a consequence of the theorem of D. Sullivan [190], who proved uniqueness of the quasiconformal structure on compact n-manifolds ($n \neq 4$): Apply Sullivan's theorem to the manifolds $M^n(\Gamma_i)$, $i = 1, 2$ and lift the quasiconformal homeomorphism to the domain of discontinuity.

Remark 5.3. An alternative proof of Sullivan's theorem and its generalization was given by J. Luukkainen in [138]. See also [202].

If $n = 4$, f is defined on $\Omega(\Gamma_1)$, and $M^4(\Gamma_1)$ is compact, then the situation is unclear but one probably should expect the negative answer since the uniqueness of quasiconformal structures in dimension 4 was disproved by S. Donaldson and D. Sullivan, [59].

Question 5.4. Is there a pair of Kleinian groups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$ so that the manifolds $M^n(\Gamma_1), M^n(\Gamma_2)$ are homeomorphic but not diffeomorphic?

Note that in view of the examples in [65], the positive answer to the above question would not be too surprising.

If f is defined on $\Omega(\Gamma_1)$ and we do not assume compactness of $M^n(\Gamma_1)$, then the answer to Question 5.2 is negative in a variety of ways.

(a) For instance, take *singly degenerate groups* $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^2)$, which are both isomorphic to the fundamental group of a closed oriented surface S, contain no parabolic elements and have distinct ending laminations. Then $\Omega(\Gamma_i) \subset \mathbb{S}^2$ are open disks D_i for both i. There exists an equivariant homeomorphism $h : D_1 \to D_2$, which induces an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$. However, since the ending laminations are different, there is no equivariant homeomorphism $\Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$.

Now extend both groups to the 3-sphere, so that $\Gamma_i \subset \mathbf{Mob}(\mathbb{S}^3)$, $i = 1, 2$. Then the 3-dimensional domains of discontinuity B_i of both groups are diffeomorphic to the open 3-ball, $i = 1, 2$; the quotient manifolds are

$$M^3(\Gamma_i) = B_i/\Gamma_i \cong S \times \mathbb{R}, \quad i = 1, 2.$$

Therefore there exists an equivariant diffeomorphism $f : B_1 \to B_2$. We claim that this map cannot be quasiconformal. Indeed, otherwise it would extend to an equivariant homeomorphism of the limit sets (which are planar subsets of $\mathbb{R}^3$). This is a contradiction.

(b) One can construct geometrically finite examples as well. The reason is that even though all (orientation-preserving) parabolic elements of $\mathbf{Mob}(\mathbb{S}^2)$ are quasiconformally conjugate, the analogous assertion is false for the parabolic elements of $\mathbf{Mob}(\mathbb{S}^3)$. Suppose that τ is the translation in $\mathbb{R}^3$ by a nonzero vector v. Let R_{θ_i}, $i = 1, 2$, denote the rotations around v by the angles $\theta_1, \theta_2 \in [0, \pi]$. Then the skew motions

$$\gamma_i = R_{\theta_i} \circ \tau_i, \quad i = 1, 2$$

are parabolic elements of $\mathbf{Mob}(\mathbb{S}^3)$. One can show that

Proposition 5.5. *γ_1 and γ_2 are quasiconformally conjugate in $\mathbb{S}^3$ if and only if $\theta_1 = \theta_2$.*

The proof is based on a calculation of the extremal length of a certain family of curves in $\mathbb{R}^3$ and we will not present it here.

Therefore the cyclic groups $\Gamma_i = \langle \gamma_i \rangle$ are geometrically finite, the isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ sending γ_1 ot γ_2 is type-preserving. The quotient manifolds $M^3(\Gamma_i)$ are both diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$. Therefore there exists a φ-equivariant diffeomorphism $f : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$ which, of course, extends to a homeomorphism $\mathbb{S}^3 \to \mathbb{S}^3$. However this homeomorphism cannot be made quasiconformal.

These examples do not resolve the following:

Question 5.6. Suppose that $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$, $n \geq 3$, are Kleinian groups and $f : \Lambda(\Gamma_1) \to \Lambda(\Gamma_2)$ is a homeomorphism which induces an isomorphism $\Gamma_1 \to \Gamma_2$. Does it follow that f is quasisymmetric?

If $n \leq 2$, then (in the list of equivalences between Kleinian groups) we have the implication

$$(3) \Rightarrow (5).$$

Indeed, consider a quasiconformal homeomorphism f conjugating Kleinian groups Γ_1 and Γ_2 and let μ denote the Beltrami differential of f. Then for $t \in [0, 1]$ the solutions of the Beltrami equation

$$\frac{\partial f_t}{\partial \bar{z}} = t\mu \frac{\partial f}{\partial z}$$

also conjugate Γ_1 to Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$, see e.g. [18]. This gives the required quasiconformal isotopy. Since (2) is equivalent to (3) for $n \leq 2$, it follows that for $n \leq 2$ we have

$$(2) \iff (3) \iff (4) \iff (5)$$

This argument however fails completely in higher dimensions, since the Beltrami equation in $\mathbb{R}^n$ for $n \geq 3$ is overdetermined.

Question 5.7. In the list of equivalences between Kleinian groups:

(a) Does (2) $\Rightarrow$ (4) ?

(b) Does (3) $\Rightarrow$ (5) ?

One can show (using quasiconformal stability, see Section 10.2, cf. [143, Theorem 7.2]) that for convex-cocompact groups parts (a) and (b) of the above question are equivalent. In Theorem 10.13 we give examples of convex-cocompact Kleinian groups in $\mathbf{Mob}(\mathbb{S}^n)$, $n \geq 5$, for which the answer to Question 5.7 is negative. The situation in dimensions 3 and 4 at the moment is unclear, but we expect in these dimensions the answer to be negative as well.

The implications (i)$\Rightarrow$(6) for i$\leq$ 5 are, of course, extremely rare. The most celebrated example is provided by the Mostow rigidity theorem:

Theorem 5.8. *Suppose that* $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$ *are lattices and* $n \geq 2$. *Then (0)$\Rightarrow$(6) for these groups.*

See [161] for G. Mostow's original proof or [116] for a more elementary argument along the same lines which uses only the analytical properties of quasiconformal mappings. A completely different argument due to M. Gromov can be found in [16]. Yet another proof is an application of the barycentric maps [19]. Note that, presently, there are no proofs using equivariant harmonic maps.

Mostow's ergodic arguments were greatly generalized by D. Sullivan in [191]:

Theorem 5.9. *(D. Sullivan,* [191]*) Suppose that* $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$ *are Kleinian groups whose limit set is the entire* $\mathbb{S}^n$ *and so that the action of* Γ_1 *on* $\mathbb{S}^n$ *is* recurrent. *Then (3)$\Rightarrow$(6) for these groups.*

The action of $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ on $\mathbb{S}^n$ is called *recurrent* if for every measurable subset $E \subset \mathbb{S}^n$ of positive Lebesgue measure, the measure of the intersection $\gamma(E) \cap E$ is positive for some $\gamma \in \Gamma \setminus \{1\}$.

6. Groups with zero-dimensional limit sets

In what follows, we let dim denote the covering dimension of topological spaces, see for instance [92].

Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is a non-elementary Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ and $\dim(\Lambda(\Gamma)) = 0$; hence $\Lambda(\Gamma)$ is totally disconnected (its only connected components are points). Recall that a *discontinuum* is a nonempty perfect totally disconnected compact topological space, [6]. Hence $\Lambda(\Gamma)$ is a discontinuum. It follows (see e.g. [6]) that $\Lambda(\Gamma)$ is homeomorphic to the standard Cantor set $K \subset [0, 1]$.

Below is a couple of examples of Kleinian groups whose limit sets are totally disconnected.

Example 6.1. (A **Schottky group**, see e.g. [130, 146].) Let $n, k \geq 1$. Suppose that we are given a collection of disjoint closed topological n-balls

$$B_1^+, \dots, B_k^+, B_1^-, \dots, B_k^- \subset \mathbb{S}^n$$

and Moebius transformations $\gamma_j \in \mathbf{Mob}(\mathbb{S}^n)$ so that $\gamma_j(\mathrm{int}(B_j^+)) = \mathrm{ext}(B_j^-)$. We assume that for each pair B_j^+, B_j^- there exists a diffeomorphism of $\mathbb{S}^n$ which carries these balls to the round balls.[1] Then

$$\Phi := \mathbb{S}^n - \bigcup_{j=1}^{k} (B_j^+ \cup \mathrm{int}(B_j^-))$$

is a fundamental domain for the group Γ generated by $\gamma_1, \ldots, \gamma_k$. The group Γ is called a *Schottky group*. It is isomorphic to a free group of rank k, and the limit set of Γ is a discontinuum provided that $k \geq 2$. Every nontrivial element of Γ is hyperbolic.

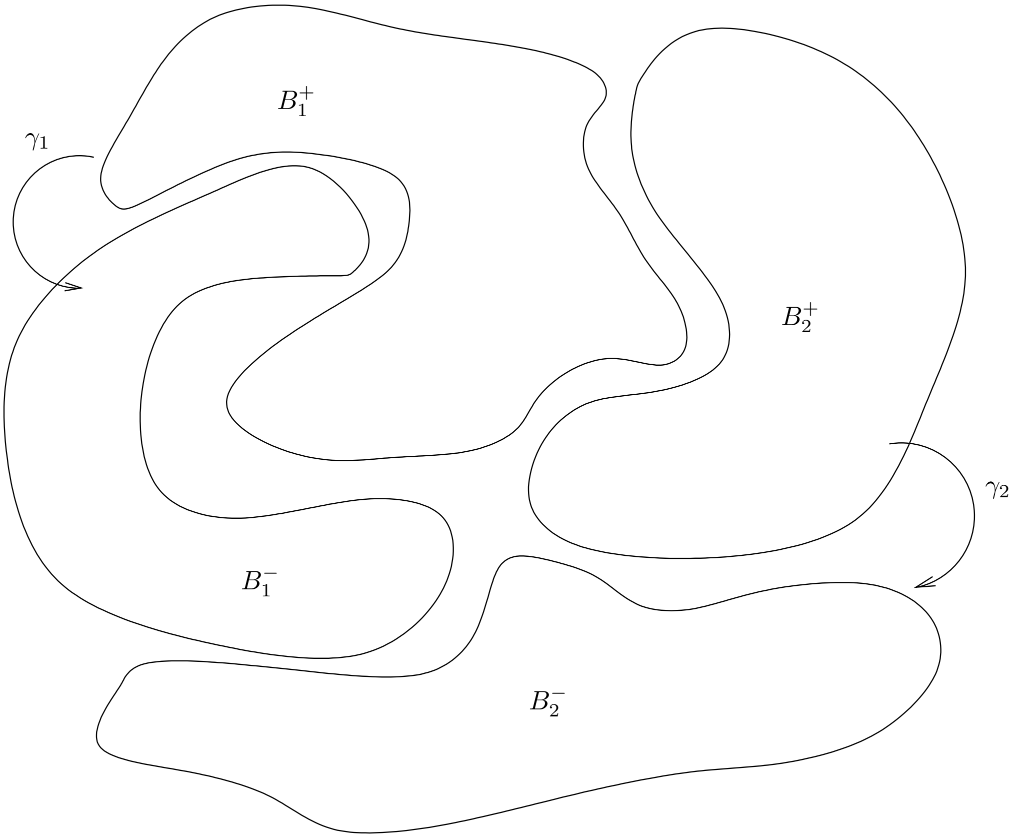

FIGURE 1. *A Schottky group.*

Before giving the next example we need a definition. Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is a nontrivial elementary subgroup. Then, after conjugating Γ if necessary, we can assume that either:

1. Γ fixes $0, \infty \in \overline{\mathbb{R}^n} = \mathbb{S}^n$ and therefore is generated by $\gamma(x) = Ax$, where A is the product of a scalar $c > 1$ by an orthogonal matrix.
2. Or Γ acts on $\mathbb{R}^n \subset \mathbb{S}^n$ by Euclidean isometries.

[1]By the smooth Schoenflies theorem, for $n \neq 4$ it suffices to assume that the balls B_j^+ have smooth boundary.

In the first case we take the fundamental domain Φ for the action of Γ on $\mathbb{S}^n$ to be an annulus bounded by two disjoint round spheres. In the second case we take a Dirichlet fundamental domain $\Phi \subset \mathbb{R}^n$ for Γ.

We refer to the fundamental domains Φ as *standard* fundamental domains. A fundamental domain for Γ is *topologically standard* if it is the image of a standard fundamental domain of Γ under a diffeomorphism of $\Omega(\Gamma)$ commuting with Γ. For instance, the fundamental domain for a rank 1 Schottky group is topologically standard. Therefore, the fundamental domain Φ for the Schottky group satisfies the property that it is the intersection of topologically standard fundamental domains

$$\Phi_j = \mathbb{S}^n - (B_j^+ \cup \mathrm{int}(B_j^-))$$

for the groups $\Gamma_j = \langle \gamma_j \rangle$.

Given a domain $\Phi \subset \mathbb{S}^n$, we let $\Phi^c \subset \mathbb{S}^n$ denote the closure of the complement of Φ. We are now ready for the second example which is a generalization of the first.

Example 6.2. (**Schottky-type groups, see e.g.** [130, 146]**.**) Start with a collection of elementary Kleinian groups $\Gamma_i \subset \mathbf{Mob}(\mathbb{S}^n)$, $i = 1, ..., k$. Let $\Phi_i \subset \mathbb{S}^n$ be topologically standard fundamental domains for these groups. Assume that

$$\Phi_i^c \cap \Phi_j^c = \emptyset$$

for all $i \neq j$. Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ be the group generated by $\Gamma_1, ..., \Gamma_k$. Then:

1. As an abstract group, Γ is isomorphic to the free product $\Gamma_1 * ... * \Gamma_k$.
2. $\Phi := \Phi_1 \cap ... \cap \Phi_k$ is a fundamental domain for the group Γ.
3. The limit set of Γ is totally disconnected.

The groups Γ obtained in this fashion are called *Schottky-type groups*.

A Schottky-type group is called *classical* if it admits a fundamental domain $\Phi := \Phi_1 \cap ... \cap \Phi_k$, so that each Φ_i is geometrically standard. It is not difficult to see that Schottky-type groups are geometrically finite. For instance, consider the case of a Schottky group Γ of rank k, for $n \geq 2$. We have the map

$$j : \mathbb{Z} = H_n(M^n(\Gamma)) \to H_n(\bar{M}^{n+1}(\Gamma))$$

induced by the inclusion of manifolds. Since the manifold $\bar{M} = \bar{M}^{n+1}(\Gamma)$ is $K(\Gamma, 1)$, it follows that

$$H^n(\bar{M}) = H^n(\Gamma) = H^n(B) = 0$$

where B is the bouquet of k circles. Therefore $j = 0$. Hence $\bar{M}$ is compact and hence Γ is convex-cocompact. A similar argument works for Schottky-type groups, provided that one uses cohomology relative to the cusps.

The quotient manifolds of the Schottky-type groups Γ have rather simple topology, as it follows from the explicit description of their fundamental domains. Namely, let $M_i = M^n(\Gamma_i)$, $i = 1, ..., k$. Then we get the smooth connected sum decomposition

$$M^n(\Gamma) = M_1 \# ... \# M_k.$$

By combining this with Theorem 5.1 we obtain

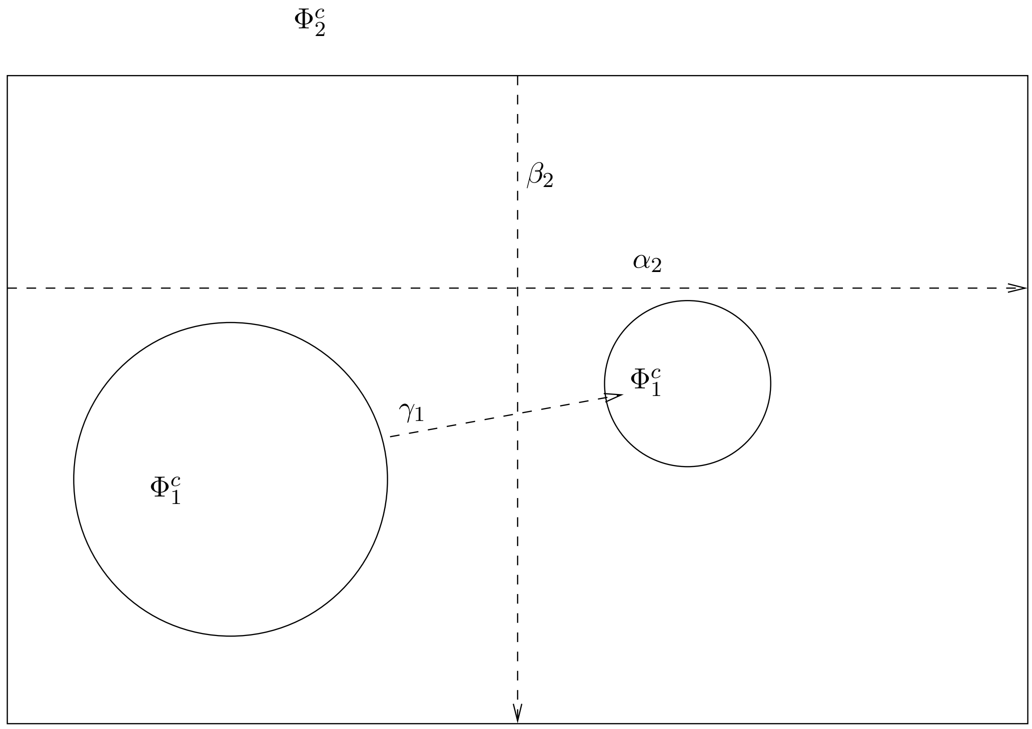

FIGURE 2. *A classical Schottky-type group. Is is isomorphic to $\mathbb{Z} * \mathbb{Z}^2$ with $\mathbb{Z} = \langle \gamma_1 \rangle$, $\mathbb{Z}^2 = \langle \alpha_2, \beta_2 \rangle$.*

Proposition 6.3. *1. Suppose that Γ, Γ' are Schottky groups of the same rank. Then there exists a quasiconformal homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which conjugates Γ to Γ': $f\Gamma f^{-1} = \Gamma'$.*

2. Suppose that Γ, Γ' are Schottky-type groups and $\varphi : \Gamma \to \Gamma'$ is a type-preserving isomorphism, so that for every free factor Γ_i in Γ, the restriction $\varphi : \Gamma_i \to \Gamma_i' \subset \Gamma'$ is induced by a quasiconformal homeomorphism of $\mathbb{S}^n$. Then there exists a quasiconformal homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which induces the isomorphism φ.

Question 6.4. Let $n \geq 3$. Is there a quasiconformal isotopy between Γ and Γ' in the above theorem (either part 1 or part 2)?

In the case when Γ and Γ' are both classical, the positive answer follows rather easily. In the non-classical case the above question is open even if $n = 3$ and Γ is a Schottky group.

According to B. Maskit [144], Schottky subgroups of $\mathbf{Mob}(\mathbb{S}^2)$ can be characterized as follows:

Theorem 6.5. *A Kleinian subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ is a Schottky group if and only if Γ is free, has nonempty domain of discontinuity in $\mathbb{S}^2$ and consists only of hyperbolic elements.*

This result was generalized by N. Gusevskii and N. Zindinova [88] as follows:

Theorem 6.6. *Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ be a Kleinian subgroup, which has nonempty domain of discontinuity in $\mathbb{S}^2$ and is isomorphic to a Schottky-type group Γ' via a type-preserving isomorphism $\Gamma \to \Gamma'$. Then Γ is a Schottky-type group.*

Both theorems are easy under the assumption that Γ is geometrically finite, the key point here is that (in dimension 2) one can prove geometric finiteness under the above mild assumptions.

If Γ is a Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$, then the above results are not longer true, moreover, Γ can be geometrically infinite. For instance, take a free finitely generated purely hyperbolic discrete subgroup of $PSL(2, \mathbb{C})$, whose limit set is the 2-sphere (the existence of such groups was first established by V. Chuckrow [51]). The Moebius extension of this group to $\mathbb{S}^3$ has nonempty domain of discontinuity, but is not geometrically finite.

Tameness of limit sets. Below we address the following:

Question 6.7. Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is a Kleinian group, whose limit set is a discontinuum. What can be said about the embedding $\Lambda(\Gamma) \subset \mathbb{S}^n$?

A discontinuum $D \subset \mathbb{S}^n$ is called *tame* if there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which carries D to the Cantor set $K \subset [0, 1]$ and is called *wild* otherwise. It is a classical (and easy) result that every discontinuum in $\mathbb{S}^2$ is tame, see e.g. [27].

The first example of a wild discontinuum was the *Antoine's necklace* $A \subset \mathbb{S}^3$: The fundamental group $\pi_1(\mathbb{S}^3 \setminus A)$ is nontrivial which explains why A is wild, see [27]. D. DeGryse and R. Osborne [56] constructed for every $n \geq 3$ examples of wild discontinua $D_n \subset \mathbb{S}^n$, such that

$$\pi_1(\mathbb{S}^n \setminus D_n) = \{1\}.$$

See also [73] for infinitely many inequivalent 3-dimensional examples of this type.

Theorem 6.8. *(R. Kulkarni,* [133]*.) Suppose that a Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ has totally disconnected limit set. Then Γ is isomorphic to a Schottky-type group.*

Theorem 6.9. *(N. Gusevskii,* [86]*.) Suppose that Γ is as in the above theorem. Then Γ admits a fundamental domain Φ of the same shape as in Example 6.2, only the fundamental domains Φ_i for Γ_i's are not required to be topologically standard.*

Corollary 6.10. *Every Kleinian group with totally disconnected limit set is geometrically finite.*

Proof. Repeat the arguments which we used to establish geometric finiteness of Schottky groups. ☐

The proof of Theorem 6.9 is based on the following

Theorem 6.11. *(M. Brin,* [40]*.) Let $\tilde{M}$ be a smooth oriented n-manifold of dimension > 2, so that $H^1(\tilde{M}) = 0$. Let $\Gamma \curvearrowright \tilde{M}$ is a smooth properly discontinuous free action.*

Then, for every smooth oriented compact hypersurface Σ in $\tilde{M}$ and an open neighborhood U of $\Gamma \cdot \Sigma$, there exists a smooth compact connected oriented hypersurface $\Sigma^* \subset U$ such that for every $\gamma \in \Gamma$ either $\gamma\Sigma^* \cap \Sigma^* = \emptyset$ or $\gamma\Sigma^* = \Sigma^*$.

This theorem allows one to split (inductively) the Kleinian group Γ as a free product in a "geometric fashion": Start with a compact hypersurface in $\Omega(\Gamma)$ which separates components of $\Lambda(\Gamma)$. Find Σ^* as in Brin's theorem which still separates. Then cut open the manifold $M^n(\Gamma)$ along the projection of Σ^*. This decomposition yields a free product decomposition $\Gamma = \Gamma' * \Gamma''$ so that Γ is a Klein combination of the groups Γ', Γ''. Continue inductively. Finite generation of Γ implies that the decomposition process will terminate and the terminal groups must be elementary. Note that if all Σ^* were spheres then this decomposition would be Schottky-type.

Problem 6.12. *Suppose that* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is such that* $\Lambda(\Gamma)$ *is a tame discontinuum. Does is follow that* Γ *is a Schottky-type group?*

If $n = 2$ then the affirmative answer to this question follows for instance from Maskit's theorem. If $n = 3$ and $\pi_1(\Omega(\Gamma)) = 1$, then the answer is again affirmative: The reason is that in this case $\pi_2(\Omega(\Gamma))$ is nontrivial and hence by the Sphere Theorem (see e.g. [91]) we can find a smooth hypersurface Σ^* as in Brin's theorem, so that Σ^* is diffeomorphic to $\mathbb{S}^2$. This argument however fails for $n \geq 4$, where the above problem is still open.

On the other hand, there are Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^3)$ with wild discontinua as limit sets. The first such example was given by M. Bestvina and D. Cooper:

Theorem 6.13. *(M. Bestvina, D. Cooper* [22]*.) There exists a (geometrically finite) Kleinian group* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ *which contains parabolic elements and has totally disconnected limit set with non-simply connected complement in* $\mathbb{S}^3$*.*

The proof that $\pi_1(\Omega(\Gamma)) \neq 1$ presented in [22] was incomplete; however the gap was filled several years later by S. Matsumoto:

Theorem 6.14. *(S. Matsumoto* [147, 148]*, see also* [87]*.) There are Kleinian groups* Γ *in* $\mathbf{Mob}(\mathbb{S}^3)$ *without parabolic elements whose limit sets are wild discontinua.*

7. Groups with one-dimensional limit sets

The simplest examples of 1-dimensional limit sets of Kleinian groups are topological circles. For instance, the limit set of a lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^2)$ is the round circle. Of course, even if the limit set of $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is a topological circle, its embedding in $\mathbb{S}^n$ can be complicated. We will discuss this issue later on. For now, we are only interested in the topology of the limit set itself.

Given convex-cocompact Kleinian groups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^n)$ with 1-dimensional limit sets, one can use Klein-Maskit Combination theorems in order to get convex-cocompact Kleinian groups $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ isomorphic to

$$\Gamma_1 *_\Delta \Gamma_2$$

where Δ is either trivial or infinite cyclic. The limit sets of the resulting groups are again 1-dimensional. For instance, if $\Lambda(\Gamma_i)$ is a topological circle for $i = 1, 2$ then the limit set of $\Gamma = \Gamma_1 * \Gamma_2$ will be disconnected: The connected components of $\Lambda(\Gamma)$ are topological circles and points. Similarly, if $\Delta = \mathbb{Z}$, then the limit set of $\Gamma = \Gamma_1 *_\Delta \Gamma_2$ will have *cut pairs*: The complement to the 2-point set $\Lambda(\Delta)$ in $\Lambda(\Gamma)$ is disconnected.

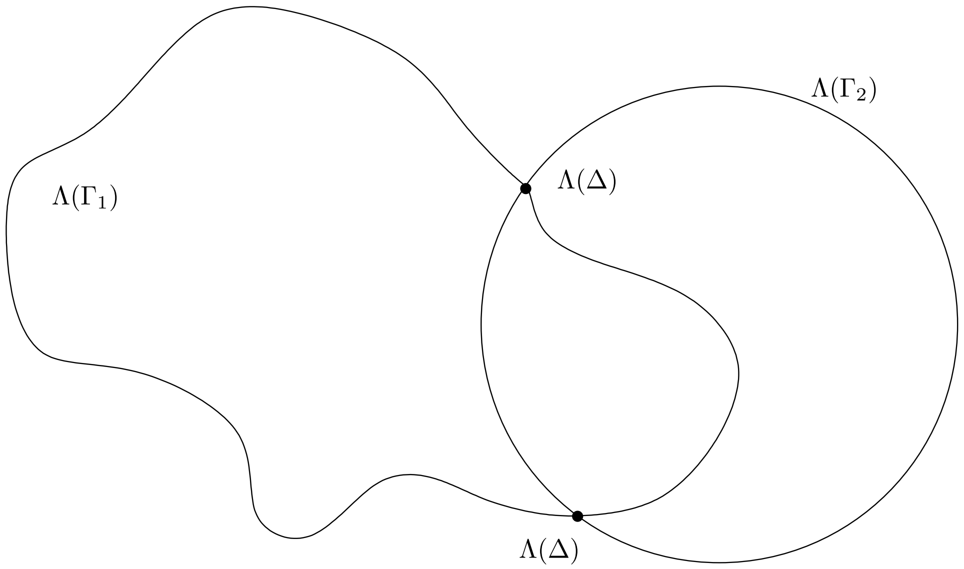

FIGURE 3. *Combination of two 1-quasifuchsian groups:* $\Gamma = \Gamma_1 *_{\mathbb{Z}} \Gamma_2$.

These constructions are, of course, not very illuminating. Therefore we are interested in examples of 1-dimensional limit sets which are connected and which do not contain cut-pairs. It turns out that there are only two such examples:

1. The Sierpinski carpet $\mathcal{S}$. Start with the unit square $S = I \times I$. Subdivide this square into 9 squares of the size $\frac{1}{3} \times \frac{1}{3}$ and then remove from S the open middle square $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. Repeat this for each of the remaining $\frac{1}{3} \times \frac{1}{3}$ subsquares in S and continue inductively. After removing a countable collection of open squares we are left with a compact subset $\mathcal{S} \subset \mathbb{R}^2$, called the *Sierpinski carpet*.

2. The Menger curve $\mathcal{M}$. Start with the unit cube $Q = I \times I \times I$. Each face F_i of Q contains a copy of the Sierpinski carpet $\mathcal{S}_i$. Let $p_i : Q \to F_i$ denote the orthogonal projection. Then

$$\mathcal{M} := \bigcap_i p_i^{-1}(\mathcal{S}_i)$$

is called the *Menger curve*.

Example 7.1. There exists a convex-cocompact subgroup $G \subset \mathbf{Mob}(\mathbb{S}^2)$ whose limit set is homeomorphic to the Sierpinski carpet $\mathcal{S}$.

To construct such example start with a compact hyperbolic manifold M^3 with nonempty totally-geodesic boundary. Thus we get an embedding of $\Gamma = \pi_1(M^3)$ into $\mathbf{Mob}(\mathbb{S}^2)$ as a convex-cocompact Kleinian subgroup. The limit set of Γ is homeomorphic to the Sierpinski carpet. To see this note that the convex hull $Hull(\Lambda(\Gamma))$ in $\mathbb{H}^3$ is obtained by removing from $\mathbb{H}^3$ a countable collection of disjoint open half-spaces $H_j \subset \mathbb{H}^3$. The ideal boundary of each H_j is the open round disk $D_j \subset \mathbb{S}^2$. Thus

$$\Lambda(\Gamma) = \mathbb{S}^2 \setminus \bigcup_j \operatorname{int}(D_j).$$

Clearly, $D_j \cap D_i = \emptyset$, unless $i = j$; since $\Lambda(\Gamma)$ has empty interior, we conclude that $\Lambda(\Gamma)$ is homeomorphic to $\mathcal{S}$, see [52]. Moreover, it is easy to see that this homeomorphism extends to the 2-sphere, since it sends the boundary circles of $\Lambda(\Gamma)$ to the boundary squares of $\mathcal{S}$.

The construction of Kleinian groups whose limit sets are homeomorphic to $\mathcal{M}$ is more complicated:

Example 7.2. (M. Bourdon, [33]; see also [114].) There exists a convex-cocompact subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ whose limit set is homeomorphic to the Menger curve $\mathcal{M}$.

The following theorem is proved in [117] in the more general context of Gromov-hyperbolic groups:

Theorem 7.3. *(M. Kapovich, B. Kleiner,* [117]*) Suppose that* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is a (torsion-free) nonelementary convex-cocompact subgroup such that: (a)* Γ *does not split as a free product, (b)* Γ *does not split as an amalgam over* $\mathbb{Z}$*, (c)* $\dim(\Lambda(\Gamma)) = 1$*. Then* $\Lambda(\Gamma)$ *is either homeomorphic to the Sierpinski carpet or to the Menger curve.*

Conjecture 7.4. *(M. Kapovich, B. Kleiner,* [117]*) If* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is a (torsion-free) convex-cocompact Kleinian group whose limit set is homeomorphic to the Sierpinski carpet, then* Γ *is isomorphic to a convex-cocompact subgroup in* $\mathbf{Mob}(\mathbb{S}^2)$*.*

It was proved in [117] that this conjecture would follow either from the positive solution of the 3-dimensional Wall's problem (Problem 4.5) or from Cannon's conjecture (Conjecture 4.6).

Topology of the limit sets of geometrically infinite Kleinian groups can be more complicated. A *dendroid* is a compact locally connected simply-connected 1-dimensional topological space.

Theorem 7.5. *(J. Cannon and W. Thurston* [46]*, see also* [1] *and* [53]*.) There exist singly-degenerate Kleinian groups whose limit sets are* dendroids.

Conjecturally, limit sets of all singly-degenerate Kleinian groups are dendroids and the following problem is open even for $n = 2$:

Problem 7.6. *Suppose that* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is a Kleinian group whose limit set is connected and 1-dimensional. Is it true that* $\Lambda(\Gamma)$ *is locally connected?*

See [154] and [152] for partial results in dimension 2.

8. Groups whose limit sets are topological spheres

Definition 8.1. A Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is called *i-fuchsian*[2] if $\Lambda(\Gamma)$ is a round i-dimensional sphere in $\mathbb{S}^n$.

[2]Our definition is somewhat different from the classical: fuchsian subgroups of $PSL(2, \mathbb{C})$ are usually required to preserves a round disk in $\mathbb{S}^2$.

To construct examples of i-fuchsian groups start with a lattice $\Gamma \subset \mathbf{Mob}(\mathbb{S}^i)$. The limit set of Γ is the round sphere $\mathbb{S}^i$. Define the *canonical embedding*

$$\iota : \mathbf{Mob}(\mathbb{S}^i) \hookrightarrow \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$$

induced by the embedding of the Lorentz groups

$$O(i+1,1) \hookrightarrow O(n+1,1)$$

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

where I is the identity matrix. Therefore we get the *canonical* embedding

$$\iota : \Gamma \hookrightarrow \mathbf{Mob}(\mathbb{S}^{\mathrm{n}}).$$

One can modify this construction as follows. Note that the stabilizer of $\mathbb{S}^i$ in $\mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ contains $\mathbf{Mob}(\mathbb{S}^i) \times SO(n-i)$. Choose a homomorphism $\phi : \Gamma \to SO(n-i)$. For instance, being residually finite, the group Γ will have many epimorphisms to finite groups, which then can be embedded in $SO(n-i)$ if $n-i$ is sufficiently large. Alternatively, in many cases the group Γ will have infinite abelianization Γ^{ab}. The abelian group Γ^{ab} admits many embeddings into $SO(n-i)$ provided that $n-i \geq 2$. Then the image of

$$\rho = \iota \times \phi : \Gamma \to \mathbf{Mob}(\mathbb{S}^i) \times SO(n-i) \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$$

is also an i-fuchsian group, since $\rho(\Gamma)$ preserves $\mathbb{S}^i$ and the action of $\rho(\Gamma)$ on $\mathbb{S}^i$ is the same as the action of Γ.

Definition 8.2. A Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ is called *i-quasifuchsian* if its limit set is a topological i-dimensional sphere.

We will refer to the number $n-i$ as the *codimension* of a (quasi)fuchsian group Γ.

Example 8.3. Suppose that Γ is an i-fuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ and $\Gamma' \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ is another group which is topologically conjugate to Γ (with a homeomorphism f defined on the entire n-sphere). Then Γ' is i-quasifuchsian. However, as we will see, there are i-quasifuchsian groups (for $n \geq 3$) which cannot be obtained in this fashion.

The (quasiconformal) homeomorphisms f as in the previous example exist in abundance if $i = 1, n = 2$, due to solvability of the Beltrami equation. If $i \geq 2$, the situation is very different and it is not so easy to construct nontrivial examples of i-quasifuchsian groups which are not fuchsian.

The following result was proved by M. Bestvina and G. Mess [25] in the context of Gromov-hyperbolic groups:

Theorem 8.4. *Each convex-cocompact i-quasifuchsian group is a* Poincare duality group *(over $\mathbb{Z}$) of dimension $i+1$.*

Question 8.5. Is it true that each convex-cocompact quasifuchsian group is isomorphic to the fundamental group of a closed aspherical manifold?

This is, of course, a special case of Wall's problem.

Question 8.6. Is there a convex-cocompact group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ whose limit set is a homology sphere which is not homeomorphic to a sphere?

8.1. **Topological rigidity.** The deepest (to this date) results on topology of higher-dimensional Kleinian groups are the topological rigidity theorems of T. Farrell and L. Jones and their corollaries:

Theorem 8.7. *(T. Farrell and L. Jones,* [66]*) Suppose that* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is a convex-cocompact Kleinian group,* $n \geq 4$*,* N *is a compact aspherical manifold (possibly with nonempty boundary* ∂N*) and* $f : (\bar{M}^{n+1}(\Gamma), M^n(\Gamma)) \to (N, \partial N)$ *is a homotopy-equivalence which is a homeomorphism on the boundary. Then* f *is homotopic to a homeomorphism (rel.* $M^n(\Gamma)$*).*

Theorem 8.8. *(T. Farrell and L. Jones,* [67, Theorem 0.1]*) Suppose that* X *is a nonpositively curved closed Riemannian manifold,* Y *is a closed aspherical manifold of dimension* ≥ 5 *and* $f : X \to Y$ *is a homotopy-equivalence. Then* f *is homotopic to a homeomorphism.*

Theorem 8.9. *(T. Farrell and L. Jones,* [68, Proposition 0.10]*) For each Kleinian group* Γ *the Whitehead group* $Wh(\Gamma)$ *is trivial.*

By combining Theorem 8.9 with the s-cobordism theorem (see e.g. [169, 179, 129]), one gets:

Corollary 8.10. *Suppose that* W^{n+1} *is a topological (resp. PL, smooth) h-cobordism so that* $n \geq 5$ *and* $\pi_1(W^{n+1})$ *is isomorphic to a Kleinian group. Then* W *is trivial in the topological (resp. PL, smooth) category.*

8.2. **Quasifuchsian groups of codimension 1.** The situation in the case of $n = 2$ is completely understood due to the following:

Theorem 8.11. *(B. Maskit* [145]*, see also* [142]*.) Let* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ *be a Kleinian group whose domain of discontinuity* $\Omega(\Gamma)$ *consists of precisely two components. Then:*

1. Γ *is 1-quasifuchsian and geometrically finite.*
2. Γ *is quasiconformally conjugate to a 1-fuchsian group.*
3. $\bar{M}^3(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ *is homeomorphic to an interval bundle over a surface* S*, which is 2-fold covered by* $\Omega(\Gamma)/\Gamma$*.*

Our goal is to compare the higher-dimensional situation with this theorem. Suppose that Γ is a quasifuchsian group of codimension 1 in $\mathbf{Mob}(\mathbb{S}^n)$. Then $\Omega(\Gamma)$ consists of two components, Ω_1, Ω_2. After replacing Γ by an appropriate index 2 subgroup, we can assume that each Ω_i is Γ-invariant; hence $M^n(\Gamma) = M_1 \cup M_2$, where $M_i := \Omega_i/\Gamma$. Then $H_*(\Omega_i) \cong H_*(point)$, $i = 1, 2$. Therefore, if Ω_i is simply-connected, then Ω_i is contractible.

Theorem 8.12. *Suppose that both* M_i *are compact and both* Ω_i *are simply-connected. Then* $\bar{M}^{n+1}(\Gamma)$ *is diffeomorphic to* $M_1 \times [0, 1]$ *provided that* $n \geq 5$*.*

Proof. Note that, for homological reasons, $W = \bar{M}^{n+1}(\Gamma)$ is compact, hence Γ is convex-cocompact. Since both $\Omega_1, \Omega_2, \mathbb{H}^{n+1}$ are contractible, the inclusions

$$M_i \hookrightarrow W, \quad i = 1, 2$$

are homotopy-equivalences. Therefore W defines a smooth h-cobordism between the aspherical manifolds M_1 and M_2. According to Corollary 8.10, this h-cobordism is smoothly trivial. $\square$

Suppose that $n = 3$ and both Ω_1, Ω_2 are contractible. Then

$$\pi_1(M_1) \cong \pi_1(M_2) \cong \Gamma,$$

the manifolds M_1 and M_2 are irreducible and have infinite fundamental group. If Γ were to contain a subgroup Π isomorphic to $\mathbb{Z}^2$, the subgroup Π would be parabolic. This would contradict compactness of W. Therefore, according to Perelman's solution of Thurston's hyperbolization conjecture, there exists a closed hyperbolic 3-manifold M_0 which is homeomorphic to M_1 and M_2. The fundamental group of $\Gamma_0 = M_0$ acts as a 2-fuchsian group on $\mathbb{S}^3$. Therefore, according to our discussion in Section 5, the group Γ is quasiconformally conjugate to Γ_0. It is not difficult to see that passage to the index 2 subgroup which we used above does no harm and we obtain:

Proposition 8.13. *Suppose that Γ is a codimension 1 quasifuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$, so that both components of $\Omega(\Gamma)$ are simply-connected and $M^3(\Gamma)$ is compact. Then Γ is quasiconformally conjugate to a 2-fuchsian group $\Gamma_0 \subset \mathbf{Mob}(\mathbb{S}^3)$.*

The situation is radically different for $n \geq 4$:

Theorem 8.14. *For each $n \geq 4$ there are codimension 1 quasifuchsian subgroups Γ in $\mathbf{Mob}(\mathbb{S}^n)$ which are not isomorphic to fuchsian groups.*

Sketch of the proof: Fix $n \geq 4$. M. Gromov and W. Thurston in [85] construct examples of negatively curved compact conformally-flat n-manifolds M^n, so that M^n is not homotopy-equivalent to any closed hyperbolic n-manifold N^n. (See also [115] for a review of the Gromov–Thurston examples and for a construction of a convex projective structure on M^n.)

By choosing parameters in the construction of [85] more carefully, one can construct an example of a *uniformizable* flat conformal manifold M^n with the same properties. Moreover, $M^n = \Omega_1/\Gamma$, $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is convex-cocompact, and $\Omega(\Gamma) = \Omega_1 \cup \Omega_2$ is the union of two simply-connected components. Then $\Lambda(\Gamma)$ is homeomorphic to $\mathbb{S}^{n-1}$, since the limit set of Γ is homeomorphic to the ideal boundary of the universal cover of M^n. If Γ were isomorphic to an $n-1$-fuchsian group Γ', then Γ' would be isomorphic to the fundamental group of a closed hyperbolic n-manifold N^n, which is a contradiction. $\square$

The above examples have another interesting property. Let Ω^{n+1} denote the domain of discontinuity of the group Γ (regarded as a subgroup of $\mathbf{Mob}(\mathbb{S}^{n+1})$). Note that Ω^{n+1} is connected and $\pi_1(\Omega^{n+1}) \cong \mathbb{Z}$. Since both Ω_1, Ω_2 are contractible, it follows that

$$\pi_i(\Omega^{n+1}) = 0, \quad i \geq 2.$$

Set $M^{n+1} = \Omega^{n+1}/\Gamma$. We have the short exact sequence

$$1 \to \mathbb{Z} = \pi_1(\Omega^{n+1}) \to \pi_1(M^{n+1}) \to \Gamma \to 1.$$

The embedding $M_1 \to M^{n+1}$ determines a splitting of this sequence. Hence the manifolds M^{n+1} and $\mathbb{S}^1 \times M^n$ are homotopy-equivalent. Given the existence of a metric of negative curvature on M^n, we obtain a metric of nonpositive curvature on $\mathbb{S}^1 \times M^n$. Therefore, by Theorem 8.8, the manifolds M^{n+1} and $\mathbb{S}^1 \times M^n$ are homeomorphic.

Let $k\mathbb{Z} \subset \mathbb{Z} \subset \pi_1(M^{n+1})$ be the index k subgroup in the center of $\pi_1(M^{n+1})$. Then we obtain the k-fold covering $X_k \to X_1 = M^{n+1}$, where

$$\pi_1(X_k) = k\mathbb{Z} \times \Gamma \subset \pi_1(X_1).$$

Since the manifolds X_k have isomorphic fundamental groups and $\pi_i(X_k) = 0$ for all $i \geq 2, k \in \mathbb{N}$, these manifolds are all homeomorphic to the smooth manifold X_1 by Theorem 8.8. By [126, Essay IV], there only finitely many smooth structures on the manifold X_1. Therefore we obtain an infinite family of diffeomorphic manifolds

$$M_j^{n+1} := X_{k_j}, j \in \mathbb{N}$$

and smooth covering maps $p_j : M_j^{n+1} \to M^{n+1}$.

The $(n+1)$-manifold $M^{n+1} = \Omega^{n+1}/\Gamma$ has the flat conformal structure K_1 uniformized by the group Γ. Let K_j denote the flat conformal structure on M^{n+1}, which is the lift of K_1 via p_j.[3] We thus obtain an infinite family of diffeomorphic flat conformal manifolds

$$(M_j^{n+1}, K_j), j = 1, 2, ...$$

Question 8.15. Suppose that M is a closed hyperbolic n-manifold. Is there a finite cover $f : M' \to M$ such that the pull-back map $f^* : H^3(M, \mathbb{Z}/2) \to H^3(M', \mathbb{Z}/2)$ is trivial? (Recall [126] that the group $H^3(M, \mathbb{Z}/2)$ classifies PL structures on M if $n \geq 5$.)

We regard the structures K_j as elements of $\mathfrak{M}(X)$, the moduli space of the flat conformal structures on a fixed smooth manifold X. The proof of the following claim is similar to [105], where it was proved in the context of 3-manifolds:

Claim 8.16. *For different i, j the structures K_i, K_j lie in different connected components of the moduli space $\mathfrak{M}(X)$. Thus $\mathfrak{M}(X)$ consists of infinitely many connected components.*

We note that K. Scannell in [180] constructed examples of hyperbolic 3-manifolds X for which $\mathfrak{M}(X)$ consists of infinitely many connected components.

To get the same phenomenon in dimension 4 consider one of the hyperbolic manifolds M^3 obtained by Dehn surgery on a 2-bridge knot, so that the natural embedding

$$\Gamma = \pi_1(M) \hookrightarrow \mathbf{Mob}(\mathbb{S}^3)$$

is locally rigid (see Example 10.12). Then the natural embedding of Γ to $\mathbf{Mob}(\mathbb{S}^4)$ is also locally rigid and hence the manifold $M^4 = M^4(\Gamma) \cong M \times \mathbb{S}^1$ has locally rigid flat conformal structure. Taking k-fold covers of this manifold we obtain infinitely many locally rigid flat conformal structures on M^4. By combining these results we obtain

Theorem 8.17. *For every $n \geq 3$ there exists a smooth compact n-manifold X^n such that $\mathfrak{M}(X^n)$ consists of infinitely many connected components.*

[3] The structures K_j are obtained via *grafting* of (M^{n+1}, K_1) along the hypersurface M^n.

We now return to our discussion of Kleinian groups, restricting to $n = 3$. Suppose that Γ is a convex-cocompact 2-quasifuchsian group, such that both components of $\Omega(\Gamma)$ are simply-connected. Then, by proposition 8.13, the limit set of Γ is *tame*, i.e., there is a homeomorphism of $\mathbb{S}^3$ which maps $\Lambda(\Gamma)$ to the round sphere.

Theorem 8.18. *(B. Apanasov and A. Tetenov* [9]*.) There exists a convex-cocompact 2-quasifuchsian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ whose limit set is a* wild *2-sphere, i.e., there is no homeomorphism of $\mathbb{S}^3$ which maps $\Lambda(\Gamma)$ to the round sphere. Moreover, one component of $\Omega(\Gamma)$ is simply-connected.*

8.3. **1-quasifuchsian subgroups of $\mathbf{Mob}(\mathbb{S}^3)$.** Given a Kleinian subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ whose limit set is a topological circle C, we would like to analyze the embedding $C \hookrightarrow \mathbb{S}^3$. It is clear that C could be an unknot in $\mathbb{S}^3$ (i.e., there exists a homeomorphism of $\mathbb{S}^3$ which maps C to a round circle), take for instance any 1-fuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$.

A topological circle C in $\mathbb{S}^3$ is called *tame* if it is isotopic to a polygonal knot in $\mathbb{S}^3$; if C is not tame, it is called *wild*.

Proposition 8.19. *1. If Γ is a 1-quasifuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$ then either $\Lambda(\Gamma)$ is an unknot or it is a wild knot K such that $\pi_1(\mathbb{S}^3 \setminus K)$ is infinitely generated.*
2. Each 1-quasifuchsian group is geometrically finite.

Proof. Since Γ is a 1-quasifuchsian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$, this group is nonelementary. The fundamental group of $M = M^3(\Gamma)$ is finitely generated (since Γ is) and we have the exact sequence:

$$1 \to \pi_1(\Omega(\Gamma)) \to \pi_1(M) \to \Gamma \to 1.$$

Suppose that $\pi_1(\Omega(\Gamma))$ is finitely generated. Then, according to Jaco-Hempel's Theorem [91], $\pi_1(\Omega(\Gamma)) \cong \mathbb{Z}$. This immediately excludes tame nontrivial knots (the result proved by R. Kulkarni, [132]). It remains to exclude wild knots with

$$\Delta := \pi_1(\Omega(\Gamma)) \cong \mathbb{Z}.$$

Without loss of generality (after passing to an index 2 subgroup in Γ), we can assume that Δ is contained in the center of $\pi_1(M)$. Note that M is a Seifert manifold (since its fundamental group has infinite center and M is irreducible). Hence M admits an $\mathbb{S}^1$-action. Lift this action to $\Omega(\Gamma)$ and then extend it to the entire 3-sphere (so that the fixed point set is the limit set). Raymond's classification [175] of topological $\mathbb{S}^1$ actions on $\mathbb{S}^3$ implies that this $\mathbb{S}^1$-action is topologically conjugate to the orthogonal action, hence $\Lambda(\Gamma)$ is an unknot. This proves (1).

To prove (2) note that the group Γ acts as a *convergence group* on $\mathbb{S}^1 = \Lambda(\Gamma)$ (see [200]). Hence, according to [200], there exists a homeomorphism $f : \Lambda(\Gamma) \to \mathbb{S}^1$ such that

$$f\Gamma f^{-1} = \Gamma' \subset PSL(2, \mathbb{R}).$$

Since finitely generated discrete subgroups of $PSL(2, \mathbb{R})$ are geometrically finite, it follows that Γ' is geometrically finite. Since geometric finiteness is an invariant of topological dynamics on the limit set, the group Γ is geometrically finite as well. $\square$

Theorem 8.20. *(M. Gromov, B. Lawson, W. Thurston,* [83], *N. Kuiper* [131], *and M. Kapovich* [104, 109].*) There are 1-quasifuchsian groups* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ *such that* $\Lambda(\Gamma)$ *are unknotted but* Γ *are not topologically conjugate to 1-fuchsian groups.*

In the examples constructed in this theorem, the manifolds $M^3(\Gamma)$ are nontrivial oriented circle bundles over orientable surfaces. On the other hand, for every 1-fuchsian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$, the manifold $M^3(\Gamma)$ is a 3-dimensional Seifert manifold with the zero Euler number.

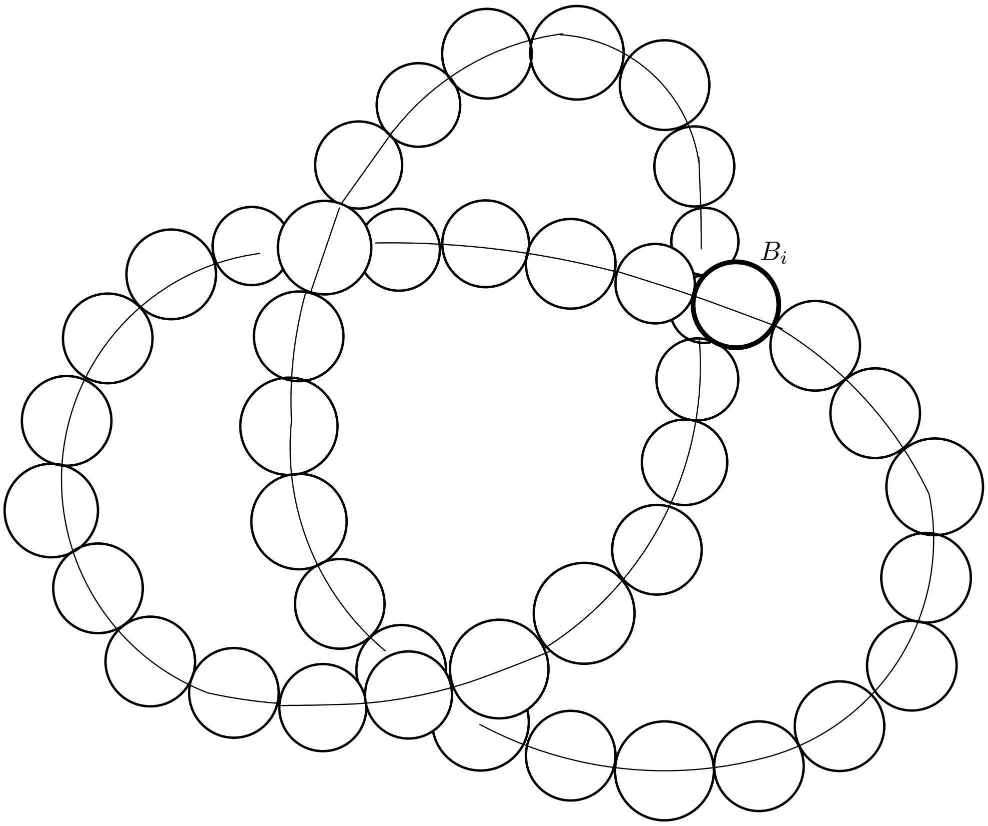

Figure 4

Theorem 8.21. *(B. Apanasov* [8], *B. Maskit* [146], *see also* [83].*) There are 1-quasifuchsian groups* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ *such that* $\Lambda(\Gamma)$ *are wild knots.*

Sketch of the proof: Start with a necklace of round balls

$$B_0, B_1, ..., B_{m-1} \subset \mathbb{S}^3,$$

so that B_i is tangent to B_j, if $j = i + 1 \in \mathbb{Z}/m\mathbb{Z}$ and is disjoint otherwise. Assume that this necklace is *knotted*, i.e., the polygonal knot obtained by connecting the consecutive points of tangency is a nontrivial knot $K \subset \mathbb{S}^3$. See Figure 4.

Let $\gamma_i \in \mathbf{Mob}(\mathbb{S}^3)$ denote the inversion in the sphere ∂B_i, $i = 0, 1, ..., m-1$. Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ be the group generated by these inversions. By Poincare's fundamental polyhedron theorem,

$$\Phi = \mathbb{S}^3 \setminus \bigcup_{i=0}^{m-1} B_i$$

is a fundamental domain for Γ. Tukia's theorem 5.1 implies that Γ is 1-quasifuchsian. By Seifert–Van-Kampen Theorem, $\pi_1(\mathbb{S}^3 \setminus K)$ embeds in $\pi_1(\Omega(\Gamma))$. Therefore $\pi_1(\Omega(\Gamma))$ is not isomorphic to $\mathbb{Z}$ which implies that the limit set of Γ is a wild knot. $\square$

By modifying the above construction, S. Hwang [93] proved

Theorem 8.22. *(S. Hwang.) Let L be a polygonal link in $\mathbb{S}^3$. Then there exists a (torsion-free) convex-cocompact Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ with a fundamental domain $\Phi \subset \mathbb{S}^3$ such that the complement $\mathbb{S}^3 \setminus \Phi$ is isotopic to a regular neighborhood of L.*

The above theorem is the key for proving

Theorem 8.23. *(S. Hwang [93].) Let M be a closed oriented 3-manifold. Then there exists a closed oriented 3-manifold M' such that the connected sum $M \# M'$ admits a Moebius structure.*

Very little is known about quasifuchsian groups in $\mathbf{Mob}(\mathbb{S}^n)$ whose limit sets have dimension between 2 and $n-2$. Perhaps the most interesting result here is obtained by I. Belegradek [13] who used the construction from [83] to get

Theorem 8.24. *There exist convex-cocompact 2-quasifuchsian subgroups $\Gamma_1, \Gamma_2 \subset \mathbf{Mob}(\mathbb{S}^4)$ so that:*

1) $\Lambda(\Gamma_1)$ is a wild 2-sphere in $\mathbb{S}^4$.

2) $\Lambda(\Gamma_2)$ is tame but the group Γ_2 is not topologically conjugate to a 2-fuchsian group: $M^4(\Gamma_2)$ is a nontrivial circle bundle over a hyperbolic 3-manifold.

Similar results probably hold for codimension 2 quasifuchsian subgroups in $\mathbf{Mob}(\mathbb{S}^n)$, $n \geq 5$.

9. Ahlfors finiteness theorem in higher dimensions: Quest for the holy grail

9.1. **The holy grail.** One of the most fundamental results of the theory of Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$ is the *Ahlfors' Finiteness Theorem* (the "Holy Grail"), which we state here together with its companions:

Theorem 9.1. *Suppose that $\Gamma \subset PSL(2, \mathbb{C})$ is a Kleinian group[4] which may have torsion. Then the following hold:*

1. (L. Ahlfors [4], L. Greenberg [79]) The group Γ is analytically finite, i.e., the quotient $O := \Omega(\Gamma)/\Gamma$ is a complex orbifold of finite conformal type[5]. In particular, O is homotopy-equivalent to a finite CW complex.

[4]Recall that all Kleinian groups are assumed to be finitely generated.

[5]I.e., as a Riemann surface it biholomorphic to a compact Riemann surface with a finite subset removed; as an orbifold it has only finitely many singular cone-points.

2. (D. Sullivan [192]*) Γ has only finitely many cusps.*

3. (M. Feighn and G. Mess [71]*) Γ has only finitely many Γ-conjugacy classes of finite order elements.*

4. (P. Scott [184, 185]*) Γ is finitely presentable and the orbifold $\mathbb{H}^3/\Gamma$ is finitely covered by a manifold $\mathbb{H}^3/\Gamma'$, which is homotopy-equivalent to a compact 3-manifold.*

5. (L. Ahlfors) The action of Γ on $\Lambda(\Gamma)$ is recurrent *with respect to the Lebesgue measure μ.*

Corollary 9.2. *If Γ is as above then:*

a. For each component Ω_0 of $\Omega(\Gamma)$, the limit set of the stabilizer of Ω_0 in Γ equals $\partial\Omega_0$ (follows directly from Part 1 of Theorem 9.1). In particular, no component of $\Omega(\Gamma)$ has trivial stabilizer.

b. Kleinian subgroups Γ of $\mathbf{Mob}(\mathbb{S}^2)$ are coherent*, i.e., each finitely generated subgroup of Γ is also finitely presented (follows from Part 4 of Theorem 9.1).*

c. (W. Thurston, see [157]*) If $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ is geometrically finite with $\Omega(\Gamma) \neq \emptyset$ then each finitely generated subgroup $\Delta \subset \Gamma$ is geometrically finite as well.*

We also now fully understand the topology of the manifold (orbifold) $\mathbb{H}^3/\Gamma$:

Theorem 9.3. *(Former tameness conjecture.) The quotient $\mathbb{H}^3/\Gamma$ is* tame*, i.e., it is homeomorphic to the interior of a manifold (orbifold) with boundary.*

The above theorem was proved for freely indecomposable groups Γ by F. Bonahon [30] and by I. Agol [2], D. Calegari and D. Gabai [44] in the general case.

The next theorem is a combination of a result by Thurston [196], who proved ergodicity for tame Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$, and the proof of the tameness conjecture:

Theorem 9.4. *If Γ is as above, then the action of Γ on $\Lambda(\Gamma)$ is ergodic with respect to the Lebesgue measure: each measurable Γ-invariant function on $\Lambda(\Gamma)$ is constant a.e..*

Note, that the conglomerate of assertions presented above contains statements of different nature: algebraic, topological, dynamical. For a while it was hoped that a theorem analogous to Theorem 9.1 can be proved for Kleinian groups in higher dimensions; an attempt to develop analytical technique to achieve this was made by Ahlfors in [3] (see also [164]).

Nearly all algebraic and topological assertions of Theorem 9.1 and the Corollary 9.2 have been disproved in the case of Kleinian groups acting in higher dimensions (M. Kapovich and L. Potyagailo, [122], [121], [111], [170], [171]):

Theorem 9.5. *There exist Kleinian subgroups $\Gamma_1, ..., \Gamma_5 \subset \mathbf{Mob}(\mathbb{S}^3)$ so that:*

1. The group Γ_1 is not finitely presentable.

2. For each i, the manifold $M(\Gamma_i) = \Omega(\Gamma_i)/\Gamma_i$ contains a component with infinitely generated fundamental group.

3. Γ_2 is free and has infinitely many cusps (of rank 1).

4. Γ_3 *is not torsion-free and has infinitely many conjugacy classes of finite order elements.*

5. Γ_4 *is a normal subgroup of a convex-cocompact group* $\widehat{\Gamma}_4 \subset \mathbf{Mob}(\mathbb{S}^3)$ *and satisfies (1), (2) and (4).*

6. (B. Bowditch, G. Mess [37]*) The group* Γ_5 *satisfies (1) and (2) and is contained in a cocompact lattice* $\widehat{\Gamma}_5 \subset \mathbf{Mob}(\mathbb{S}^3)$*.*

7. Groups Γ_i*,* $i = 1, ..., 4$ *are normal subgroups of geometrically finite groups* $\widehat{\Gamma}_i$ *so that* $\widehat{\Gamma}_i/\Gamma_i \cong \mathbb{Z}$*.*

Remark 9.6. By modifying Γ_3 one can also construct an example $\Gamma_6 \subset \mathbf{Mob}(\mathbb{S}^3)$ such that $\Omega(\Gamma_6)/\Gamma_6$ has infinitely many connected components.

At the time when the above examples were constructed, they were regarded as a "rare pathology". It appears however that such examples are rather common:

Conjecture 9.7. *(M. Kapovich, L. Potyagailo, E. Vinberg, see* [123]*.) Suppose that* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *is an arithmetic lattice, where* $n \geq 3$*. Then* Γ *is* noncoherent*, i.e., it contains a finitely generated subgroup* Δ *which is not finitely presentable.*

This conjecture was proved in [123] in a number of special cases, e.g. for all non-cocompact arithmetic lattices provided that $n \geq 5$.

All the examples Γ_i in the above theorem are based upon existence of hyperbolic 3-manifolds M^3 of finite volume which fiber over the circle: the groups Γ_i are obtained by manipulating with the normal surface subgroups in $\pi_1(M^3)$.

Problem 9.8. *Find examples similar to* Γ_i*'s without using hyperbolic 3-manifolds fibering over the circle.*

Problem 9.9. *Construct a finitely generated Kleinian subgroup* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ *such that Part (a) of Corollary 9.2 fails for* Γ*.*

Problem 9.10. *(G. Mess.) Construct a finitely-presented Kleinian subgroup of* $\mathbf{Mob}(\mathbb{S}^n)$ *(*$n \geq 3$*) which contains no parabolic elements and for which any of the assertions of Theorem 9.1 fail. (In Part (a) one would need to replace analytical finiteness with finiteness of the homotopy type.)*

Problem 9.11. *Construct a finitely generated Kleinian subgroup* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ *such that* $\Omega(\Gamma)$ *contains a contractible component* Ω_0 *so that:*

The stabilizer Γ_0 *of* Ω_0 *in* Γ *is finitely-generated, but the manifold* Ω_0/Γ_0 *is not tame.*

Note however that although algebra and topology fail, the assertions of dynamical nature (Theorem 9.1, part 5, Corollary 9.2, part (b) and Theorem 9.4) remain open in higher dimensions. Moreover, an attempt to construct a higher-dimensional counter-example to Theorem 9.1 (part 5) along the lines of the examples Γ_i, is doomed to failure:

Theorem 9.12. *(K. Matsuzaki, [149]) Let $\widehat{\Gamma}$ be a geometrically finite subgroup*[6] *in* $\mathbf{Mob}(\mathbb{S}^n)$*. Suppose that $\Gamma \subset \widehat{\Gamma}$ is a normal subgroup (which does not have to be finitely generated). Then the action of Γ on its limit set is recurrent.*

Ergodicity fails however for discrete subgroups of $PU(2,1)$ (it probably also fails for Kleinian groups in higher dimensions but an example would be difficult to construct):

Theorem 9.13. *There exists a finitely generated (but not finitely presentable!) discrete group Γ of isometries of complex-hyperbolic 2-plane $\mathbb{CH}^2$ so that the limit set of Γ is the 3-sphere and the action of Γ on $\mathbb{S}^3$ is not ergodic.*

Proof. There are examples (the first was constructed by R. Livne in his thesis [136], see also [57]) of cocompact torsion-free discrete subgroups $\widehat{\Gamma} \subset PU(2,1)$ such that the complex 2-manifold $M = \mathbb{CH}^2/\widehat{\Gamma}$ admits a nonconstant holomorphic map $f : M \to S$ to a Riemann surface S of genus ≥ 2. The fundamental group of the generic fiber of f maps onto a normal subgroup Γ in $\widehat{\Gamma}$, so that Γ is finitely generated but is not finitely presentable [112]. By lifting f to the universal covers we get a nonconstant holomorphic function

$$\tilde{f} : \mathbb{CH}^2 \to \mathbb{H}^2$$

which is Γ-invariant. Then the bounded harmonic function $Re(\tilde{f})$ is Γ-invariant and nonconstant. This harmonic function admits a measurable extension h to $\mathbb{S}^3$, the boundary of the complex ball $\mathbb{CH}^2$, so that h is Γ-invariant and not a.e. constant. $\square$

9.2. **Groups with small limit sets.** So far, our quest for the holy grail mostly resembled Monty Python's: We are not sure what to look for in higher dimensions. Nevertheless, there is a glimmer of hope.

Recall that the Hausdorff dimension $\dim_H$ of a subset $E \subset \mathbb{R}^n$ is defined as follows. For each $\alpha > 0$ consider the α-Hausdorff measure of E:

$$mes_\alpha(E) = \lim_{\rho \to 0} \inf\{\sum_i r_i^\alpha : \quad r_i \leq \rho, E \text{ is contained in the union of } r_i\text{-balls}\}.$$

The Hausdorff dimension of E is

$$\dim_H(E) = \inf_\alpha \{\alpha : mes_\alpha(E) = 0\}.$$

According to [92], for every bounded subset $E \subset \mathbb{R}^n$ one has the inequality

$$\dim(E) \leq \dim_H(E)$$

between topological and Hausdorff dimensions. In particular, if Γ is a Kleinian group with $\dim_H(\Lambda(\Gamma)) < 1$ then Γ is geometrically finite and is isomorphic to a Schottky-type group, see Theorem 6.9.

Conjecture 9.14. *If $\dim_H(\Lambda(\Gamma)) < 1$, then Γ is a Schottky-type group. Moreover, Γ is classical.*

[6]Actually, the proof also works for subgroups of any rank 1 Lie group.

The *critical exponent* of a Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is

$$\delta(\Gamma) := \inf\{s > 0 : \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} < \infty\},$$

where d is the hyperbolic metric in $\mathbb{H}^{n+1}$. The following theorem is the result of combined efforts of a large number of mathematicians, including P. Tukia, D. Sullivan and P. Nicholls, we refer to [163], [29] for the proofs:

Theorem 9.15. *For every Kleinian subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$,*

$$\delta(\Gamma) = \dim_H(\Lambda_c(\Gamma)).$$

Recall that $\Lambda_c(\Gamma)$ is the conical limit set of Γ.

The critical exponent relates to λ_0, the bottom of the spectrum of Laplacian on the hyperbolic manifold $\mathbb{H}^{n+1}/\Gamma$, by the following

Theorem 9.16. *(D. Sullivan [194])*

$$\lambda_0 = \left(\frac{n}{2}\right)^2, \ \textit{if} \quad 0 \le \delta(\Gamma) \le \frac{n}{2},$$
$$\lambda_0 = \delta(\Gamma)(n - \delta(\Gamma)), \ \textit{if} \quad \frac{n}{2} < \delta(\Gamma) \le n.$$

The expectation is that Kleinian groups in $\mathbf{Mob}(\mathbb{S}^n)$ with *small* limit sets behave analogously to the Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$.

Conjecture 9.17. *Suppose that Γ is a (finitely generated) subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ so that $\Lambda(\Gamma)$ has Hausdorff dimension < 2. Then Γ is geometrically finite.*

For $n = 2$, this conjecture is a theorem of C. Bishop and P. Jones [29]. A partial generalization of [29] was proved by A. Chang, J. Qing, J. and P. Yang, in [49]:

Theorem 9.18. *Suppose that Γ is a (finitely generated)* conformally finite[7] *subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ such that $\dim_H(\Lambda(\Gamma)) < n$. Then Γ is geometrically finite.*

The converse to the above theorem was proved earlier by P. Tukia in [198].

Theorem 9.19. *(Y. Shalom [187]) Suppose that Γ is a geometrically finite subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ such that $\dim_H(\Lambda(\Gamma)) < 2$ and $\Delta \subset \Gamma$ is a finitely generated normal subgroup. Then Δ has finite index in Γ. In particular, Δ is geometrically finite as well.*

Thus, attempts to construct geometrically infinite groups using normal subgrooups in geometrically finite Kleinian groups with *small* limit sets, are doomed to failure. On the other hand, the assumption that $\delta(\Gamma)$ is small should impose strong restrictions on the algebraic properties of the group Γ.

Conjecture 9.20. *Suppose that Γ is a Kleinian group in $\mathbf{Mob}(\mathbb{S}^n)$ which does not contain parabolic elements. Then:*

1. *$cd(\Gamma) - 1 \le \delta(\Gamma)$.*
2. *In the case of equality, Γ is an i-fuchsian convex-cocompact group, $i = \delta(\Gamma)$.*

[7] I.e., $M^n(\Gamma) = \Omega(\Gamma)/\Gamma$ is compact modulo cusps.

Recall that cd and hd stand for the cohomological and homological dimensions of a group. A partial confirmation of Part 1 of this conjecture is obtained in

Theorem 9.21. *(M. Kapovich [103]) Suppose that Γ is a Kleinian group in $\mathbf{Mob}(\mathbb{S}^n)$. Then for every ring R,*

$$hd_R(\Gamma, \Pi) - 1 \leq \delta(\Gamma)$$

where $\Pi \subset \Gamma$ is the set of rank ≥ 2 abelian subgroups of Γ and $hd_R(\Gamma, \Pi)$ is the relative homological dimension.

We refer the reader to the series of papers by H. Izeki [96, 97, 98] for the related results.

Corollary 9.22. *Suppose that Γ is finitely-presented and $\delta(\Gamma) < 1$. Then Γ is free.*

Proof. Since $\delta(\Gamma) < 1$, it follows that Γ contains no rank 2 abelian subgroups. Then we have the inequalities

$$cd(\Gamma) \leq 1 + hd(\Gamma) \leq \delta(\Gamma) + 2 < 3.$$

Combined with finite presentability of Γ, the inequality $cd(\Gamma) \leq 2$ implies that Γ admits a finite $K(\Gamma, 1)$; therefore

$$cd(\Gamma) = hd(\Gamma) \leq \delta(\Gamma) + 1 < 2.$$

Hence Γ is free by Stallings' theorem. □

An inequality similar to Conjecture 9.20 was proved by A. Reznikov: For a (finitely-generated) group Γ define

$$\alpha(\Gamma) := \inf\{p \in [1, \infty] : \ell_p H^1(\Gamma) \neq 0\}.$$

Here $\ell_p H^1$ is the 1-st ℓ_p-cohomology of the group Γ, see [34] for the precise definition. Then

Theorem 9.23. *(A. Reznikov, [177], see also [34] for the detailed proof in the case of isometries of $CAT(-1)$ spaces.) For every Kleinian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$,*

$$\alpha(\Gamma) \leq \max(\delta(\Gamma), 1).$$

Question 9.24. What can be said about Γ in the case of equality in Reznikov's theorem?

In the case of geometrically finite groups, Part 2 of Conjecture 9.20 holds:

Theorem 9.25. *(Chenbo Yue, [209], see also [31] and [20].) Suppose that Γ is convex-cocompact and $i = \delta(\Gamma) = cd(\Gamma) - 1$. Then Γ is i-fuchsian.*

Remark 9.26. Chenbo Yue states his theorem in the context of i-quasifuchsian groups, but his proof actually does not need this assumption.

Chenbo Yue's theorem is generalized in [103] to all geometrically finite groups.

Conjecture 9.27. *Suppose that Γ is a Kleinian group in $\mathbf{Mob}(\mathbb{S}^n)$ whose limit set is not totally disconnected and has Hausdorff dimension 1. Then Γ is 1-fuchsian.*

This conjecture is known to be true for $n = 2$, see [45].

Problem 9.28. *(The* gap *problem, L. Bowen, cf.* [189]*.) 1. Is there a number $d_n < n$ such that for every Schottky subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$, $n \geq 3$, we have:*

$$\delta(\Gamma) < d_n.$$

2. More generally, consider a sequence $\Gamma_j \subset \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ of convex-cocompact groups isomorphic to a fixed group Γ so that: $\Lambda(\Gamma_j) \neq \mathbb{S}^n$ for each j. Is it true that

$$\limsup_{j\to\infty} \delta(\Gamma_j) < n\ ?$$

By the work of Phillips and Sarnak [168], the answer to the above question is positive in the class of classical Schottky groups.

10. Representation varieties of Kleinian groups

For a finitely-generated Γ consider the *representation variety* of Γ:

$$R_n(\Gamma) := Hom(\Gamma, \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})).$$

If Γ has the presentation

$$\Gamma = \langle x_1, ..., x_m | r_1, ..., r_k, ...\rangle,$$

the representation variety is given by

$$\{(g_1, ..., g_m) \in (\mathbf{Mob}(\mathbb{S}^{\mathrm{n}}))^m : r_1(g_1, ..., g_m) = 1, ...r_k(g_1, ..., g_m) = 1, ...\}.$$

The group $\mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$ acts on $R_n(\Gamma)$ via conjugation:

$$\theta \cdot \rho(\gamma) = \theta\rho(\gamma)\theta^{-1}, \quad \theta \in \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})\,.$$

Given this action, one can form the quotient variety

$$X_n(\Gamma) := R_n(\Gamma)/\!/\,\mathbf{Mob}(\mathbb{S}^{\mathrm{n}}),$$

called the *character variety*. Roughly speaking, the elements of $X_n(\Gamma)$ are represented by conjugacy classes of representations $\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$. This is literally true for "most" representations, the ones for which $\rho(\Gamma)$ does not contain a normal parabolic subgroup, see [99]. In general, the representations ρ_1, ρ_2 project to the same point in $X_n(\Gamma)$ iff the closures of their $\mathbf{Mob}(\mathbb{S}^{\mathrm{n}})$-orbits have nonempty intersection. We let $[\rho]$ denote the projection of $\rho \in R_n(\Gamma)$ to $X_n(\Gamma)$.

A *trivial deformation* of a representation $\rho_0 \in R_n(\Gamma)$ is a curve $\rho_t \in R_n(\Gamma)$ which projects to a point in $X_n(\Gamma)$. A representation ρ_0 is called *locally rigid* if it admits no nontrivial deformations.

We will be mostly interested in representations $\rho \in R_n(\Gamma)$ which have discrete, nonelementary image, however much of our discussion is more general.

10.1. **Local theory.** We start by considering the *local* structure of $X_n(\Gamma)$. Given an abstract group Γ and a representation $\rho \in R_n(\Gamma)$, we have the adjoint action of $\rho(\Gamma)$ on the Lie algebra $\mathfrak{g}$ of $\mathbf{Mob}(\mathbb{S}^n)$ and the associated 1-st cohomology group

$$H^1(\Gamma, Ad(\rho)) = Z^1(\Gamma, Ad(\rho))/B^1(\Gamma, Ad(\rho)),$$

see Section 4. It was first observed by A. Weil [207] (in the general context of representations to Lie groups) that if $X_n(\Gamma)$ is smooth at $[\rho] \in X_n(\Gamma)$ then $H^1(\Gamma, Ad(\rho))$ is isomorphic to the tangent space to $X_n(\Gamma)$ at $[\rho]$. Moreover, Weil proved that if $H^1(\Gamma, Ad(\rho)) = 0$ then $[\rho]$ is an isolated point on $X_n(\Gamma)$, i.e., ρ is locally rigid.

Therefore, the elements of $H^1(\Gamma, Ad(\rho))$ can be regarded as *infinitesimal deformations* of the representation ρ. An infinitesimal deformation $[\xi] \in H^1(\Gamma, Ad(\rho))$ is called *integrable* if it is tangent to a smooth curve in $X_n(\Gamma)$. The obstructions to integrability are cohomological in nature, they are certain elements of $H^2(\Gamma, Ad(\rho))$, called *Massey products*. However, in practice, these cohomology classes are very difficult to compute. The first such obstruction is the *cup-product*:

$$\phi([\xi]) = [\xi] \cup [\xi] \in H^2(\Gamma, Ad(\rho)),$$

see for instance [78]. Here $\phi([\xi])$ is represented by the 2-cocycle

$$\tau(x, y) = [\xi(x), Ad \circ \rho(x)\xi(y)]$$

where $[,]$ is the Lie bracket on the Lie algebra $\mathfrak{g}$. If the first obstruction vanishes and Γ is the fundamental group of a surface, then $X_n(\Gamma)$ is smooth at ρ, see [78], where a much more general result is proved.

We will be mostly interested in the case where $\rho : \Gamma \hookrightarrow \mathbf{Mob}(\mathbb{S}^n)$ is a discrete embedding, whose image we will identify with Γ. Then, by abusing the terminology, we will talk of small deformations of ρ in $R_n(\Gamma)$ as *small deformations of* Γ itself.

10.1.1. *Small deformations of* 1*-quasifuchsian groups.* Recall that the group $\mathbf{Mob}(\mathbb{S}^n)$ has dimension $d = (n+2)(n+1)/2$. Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is 1-quasifuchsian; in this subsection we allow Γ to have nontrivial finite order elements. We assume however that Γ contains no elements fixing the circle $C = \Lambda(\Gamma)$ pointwise. Therefore we obtain the injective map

$$\Gamma \to \mathrm{Isom}(\mathbb{H}^2)$$

given by the restriction of the elements of Γ to the round circle C. To simplify the discussion we assume that Γ preserves the orientation on C. Then Γ embeds as a lattice in $PSL(2, \mathbb{R})$.

If Γ contains no parabolic elements then it has the presentation:

$$\langle a_1, b_1, ..., a_q, b_q, c_1, ..., c_k | [a_1, b_1] \cdot ...[a_q, b_q] \cdot c_1 \cdot ... \cdot c_k = 1, c_j^{r_j} = 1, j = 1, ..., k\rangle.$$

For a representation $\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^n)$ we let

$$e_j := d - \dim\{\xi \in \mathfrak{g} : Ad \circ \rho(c_j)(\xi) = \xi\};$$

in other words, e_j is the codimension of the centralizer of $\rho(c_j)$ in $\mathbf{Mob}(\mathbb{S}^n)$. Let s denote the dimension of centralizer of $\rho(\Gamma)$ in $\mathbf{Mob}(\mathbb{S}^n)$.

Theorem 10.1. *(A. Weil [207])*

$$h = \dim H^1(\Gamma, Ad(\rho)) = (2q-2)d + 2s + e_1 + ... + e_k. \tag{10.2}$$

Moreover, if $s = 0$ then $X_n(\Gamma)$ near $[\rho]$ is a smooth h-dimensional manifold.

For instance, if $n = 1$, $\Gamma \subset PSL(2,\mathbb{R}) \subset \mathbf{Mob}(\mathbb{S}^1)$; therefore $d = 3$, we get $e_i = 1$ for each $i = 1, ..., k$, $s = 0$. Hence

$$h = 6q - 6 + k,$$

which is the familiar formula for the dimension of the Teichmüller space of the orbifold $O = \mathbb{H}^2/\Gamma$. If $n = 2$, we, of course, obtain $h = 2(6q - 6 + k)$ which is the (real) dimension of the space of the quasifuchsian deformations of Γ in $PSL(2,\mathbb{C})$.

To better understand the difficulties which one encounters in the case of i-fuchsian groups for $i \geq 2$, we consider the *hyperbolic triangle groups* Γ. The reason for considering these groups is that they are rigid in $PSL(2,\mathbb{R})$ (similarly to rigidity of lattices in $\mathbf{Mob}(\mathbb{S}^n)$, $n \geq 2$).

The triangle groups are the 1-fuchsian groups with $q = 0$, $k = 3$; every such Γ has the presentation

$$\langle c_1, c_2, c_3 | c_1 \cdot c_2 \cdot c_3 = 1, c_j^{r_j} = 1, j = 1, 2, 3\rangle,$$

where $r_1^{-1} + r_2^{-1} + r_3^{-1} < 1$. Such group embeds discretely into $PSL(2,\mathbb{R})$ and we will denote the image of this embedding by $\Delta = \Delta(r_1, r_2, r_3)$

As a subgroup of $\mathbf{Mob}(\mathbb{S}^2)$, the group Δ is locally rigid (which follows from vanishing of H^1). Moreover, triangle groups are “strongly rigid” in $\mathbf{Mob}(\mathbb{S}^2)$, i.e., every discrete embedding of Γ into $\mathbf{Mob}(\mathbb{S}^2)$ is induced by conjugation of the identity embedding, see [80] for the complete description of $X_2(\Gamma)$.

The situation changes somewhat if we consider representations into $\mathbf{Mob}(\mathbb{S}^3)$. First, let $\rho_0 : \Delta \to \Gamma' \subset \mathbf{Mob}(\mathbb{S}^3)$ be the embedding obtained as the composition

$$\Delta \subset \mathbf{Mob}(\mathbb{S}^1) \hookrightarrow \mathbf{Mob}(\mathbb{S}^2) \hookrightarrow \mathbf{Mob}(\mathbb{S}^3).$$

of natural embeddings. For ρ_0 we get: $d = 10, s = 1, e_j = 6$ $(j = 1, 2, 3)$. Thus (10.2) tells us that ρ_0 is still locally rigid in $\mathbf{Mob}(\mathbb{S}^3)$. However, instead of ρ_0 we can take a *twisted* extension. Suppose that we can find numbers $m_j \in \mathbb{Z}, 1 < |m_j| < r_j - 1$ $(j = 1, 2, 3)$ such that:

$$m_1^{-1} + m_2^{-1} + m_3^{-1} = 0, \text{ and } \forall j,\ m_j \text{ divides } r_j.$$

(This is satisfied for instance by $m_1 = m_2 = 4, m_3 = -2$ and $r_j = 8$ for all j.)

Define a homomorphism $\theta : \Delta \to SO(2)$ by sending c_i to the rotation by $2\pi/m_i$. Then define $\rho : \Delta \to \mathbf{Mob}(\mathbb{S}^1) \times SO(2) \subset \mathbf{Mob}(\mathbb{S}^3)$ by twisting ρ_0 via θ:

$$\rho(\gamma) = \rho_0(\gamma) \times \theta(\gamma), \quad \gamma \in \Delta.$$

It is clear that $\rho(\Delta)$ is again a 1-fuchsian subgroup in $\mathbf{Mob}(\mathbb{S}^3)$. If $r_j > 3$ for each j, then $e_j = 8$, $s = 1$ and the formula (10.2) gives the dimension $h = 6$ for $H^1(\Delta, Ad(\rho))$. I do not know if any of these *infinitesimal* deformations is integrable. To decide this one has to analyze the quadratic form

$$\phi : H^1(\Delta, Ad(\rho)) \to H^2(\Delta, Ad(\rho)) = \mathbb{R},$$

given by the cup-product. According to [78], the quadratic cone $\{\phi = 0\}$ is analytically isomorphic to a neighborhood of $[\rho]$ in $X_3(\Delta)$; hence it suffices to find a nontrivial 1-cocycle ξ for which $\phi([\xi]) = 0$ to get nontrivial deformations of the representation ρ.

On the other hand, one can use (10.2) to show that every representation ρ of the group $\Delta = \Delta(2, 3, r_3)$ into $\mathbf{Mob}(\mathbb{S}^3)$ has zero cohomology $H^1(\Delta, Ad(\rho))$. Therefore $X_3(\Delta)$ is a zero-dimensional algebraic variety and, hence, is a finite set.

This situation is somewhat typical for representations of lattices in $\mathbf{Mob}(\mathbb{S}^n)$ ($n \geq 2$) into $\mathbf{Mob}(\mathbb{S}^{n+1})$: In a number of cases we can prove local rigidity by making cohomological computations; in some cases we can only conclude that H^1 is nonzero, without being able to make a definitive conclusion about existence of nontrivial deformations.

10.1.2. *Small deformations of i-quasifuchsian groups for $i \geq 2$.* In the case of $(n-1)$-quasifuchsian groups Γ ($n \geq 2$), the existence of nontrivial deformations of Γ in $\mathbf{Mob}(\mathbb{S}^n)$ is not at all clear. Suppose that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a cocompact lattice. We have the identity embedding

$$\iota : \Gamma \hookrightarrow \mathrm{Isom}(\mathbb{H}^n),$$

which is locally rigid by Mostow's theorem.

Remark 10.3. Actually, local rigidity of ι was known prior to the work of Mostow; it was first established by E. Calabi [43], whose proof was later generalized by A. Weil [205, 206]. These arguments were based on proving that $H^1(\Gamma, Ad(\iota)) = 0$.

Consider now the composition of ι with the natural embedding:

$$\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n) \hookrightarrow \mathrm{Isom}(\mathbb{H}^{n+1}).$$

Then

$$H^1(\Gamma, Ad(\iota)) \cong H^1(\Gamma, V_n),$$

where $V_n = \mathbb{R}^{n,1}$ and Γ acts on the Lorentz space $\mathbb{R}^{n,1}$ via the usual embedding $\Gamma \hookrightarrow O(n, 1)$.

It turns out that ρ may or may not be locally rigid, even if Γ is torsion-free: Rigidity depends on the lattice Γ. One has the following list ((a) through (e)) of constructions of deformations and infinitesimal deformations of $[\rho]$ in $X_n(\Gamma)$:

(a) *Bending*, see [99], [128]. Given a connected properly embedded totally-geodesic hypersurface $S \subset M = \mathbb{H}^n/\Gamma$, one associates with S a smooth curve through $[\rho]$ in $X_n(\Gamma)$, called the *bending deformation* of $[\rho]$. More generally, given a disjoint collections of such hypersurfaces $S_1, ..., S_k$, one obtains a k-dimensional smooth submanifold in $X_n(\Gamma)$ containing $[\rho]$. This construction is completely analogous to bending deformations of 1-fuchsian subgroups in $\mathbf{Mob}(\mathbb{S}^2)$ defined by Thurston in [196]. We let $[\xi_S]$ denote the element of $H^1(\Gamma, Ad(\rho))$ corresponding to a connected totally-geodesic hypersurface $S \subset M$.

There are numerous group Γ satisfying assumptions of the bending construction. Namely, start with an arithmetic group $O'(f, A)$ of the simplest type (see Section 2), where

$$f = a_0x_0^2 + a_1x_1^2 + ... + a_nx_n^2,$$

and $a_0 < 0, a_i > 0, i = 1, ..., n$. Identify $\mathbb{H}^n$ with a component of the hyperboloid $\{f(x) = -1\}$. Then the stabilizer of the hyperplane $P = \{x_n = 0\}$ in $O'(f, A)$ is an arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^{n-1})$. The intersection $H = P \cap \mathbb{H}^n$ is a hyperplane in $\mathbb{H}^n$. After passing to an appropriate finite-index subgroup Γ in $O'(f, A)$, one obtains a totally-geodesic embedding of the hypersurface $S = H/\Gamma'$ into H/Γ, where $\Gamma' = \Gamma \cap O'(f, A)$. We refer the reader to [153] for the details.

Problem 10.4 (I. Rivin)**.** *Construct examples of hyperbolic n-manifolds M of finite volume (n ≥ 4) such that M contains a* separating *properly embedded totally-geodesic hypersurface S ⊂ M. Note that the main objective of* [153] *was to construct* nonseparating *hypersurfaces.*

The idea of bending deformations of representations is quite simple and has nothing to do with the hyperbolic space. Below is a general description of bending as defined by D. Johnson and J. Millson in [99]. Suppose that we are given a graph of groups $\mathcal{G}$ with the vertex groups Γ_v and the edge groups Γ_e. Let $\Gamma = \pi_1(\mathcal{G})$ be the fundamental group of $\mathcal{G}$. For instance, the amalgam

$$\Gamma = \Gamma_{v_1} *_{\Gamma_e} \Gamma_{v_2} \tag{10.5}$$

is the fundamental group of a graph of groups which is a single edge e with two vertices v_1, v_2. Let $\rho_0 : \Gamma \to G$ be a representation of Γ to a Lie group G. A *bending deformation* of ρ_0 is a curve of representations $\rho_t : \Gamma \to G, t \in [-1, 1]$, such that for each vertex group Γ_v, we have

$$\rho_t|\Gamma_v = g_{v,t}(\rho_0|\Gamma_v)g_{v,t}^{-1},$$

for some curve $g_{v,t}$ of elements of G. Therefore, the restriction of ρ_t to each vertex group determines a trivial deformation of the representation of this group. The trick is that the deformation of the representation of the entire group Γ may be still nontrivial. For instance, in the case of the amalgam (10.5), let $g_t \in G, g_0 = 1$, be a curve of elements centralizing $\rho(\Gamma_e)$ but not $\rho(\Gamma_{v_1}), \rho(\Gamma_{v_2})$. Define the family of representations

$$\rho_t : \Gamma \to G, \quad \rho_t|\Gamma_{v_1} = \rho_0|\Gamma_{v_1}, \quad \rho_t|\Gamma_{v_2} = g_t(\rho_0|\Gamma_{v_1})g_t^{-1}.$$

This is a nontrivial deformation of the representation ρ_0. In the case of the bending of $\pi_1(M)$ in $\mathbf{Mob}(\mathbb{S}^n)$ as above, the point is that:

1. The centralizer of each $\pi_1(S_i)$ in $\mathbf{Mob}(\mathbb{S}^n)$ is 1-dimensional (the group of elliptic rotations around the limit set of $\pi_1(S_i)$).

2. The centralizer in $\mathbf{Mob}(\mathbb{S}^n)$ of the fundamental group of each component of $M \setminus (S_1 \cup ... \cup S_k)$ is zero-dimensional.

Therefore one obtains nontrivial bending deformations ρ_t of the identity embedding of $\rho : \Gamma \hookrightarrow \mathbf{Mob}(\mathbb{S}^n)$. The set of bending parameters $t = (t_1, ..., t_k)$ can be identified with $(\mathbb{S}^1)^k$, as the centralizer of each $\pi_1(S_i)$ in $\mathbf{Mob}(\mathbb{S}^n)$ is the circle $SO(2)$.

For each $n \geq 4$, D. Johnson and J. Millson [99] constructed a lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ and intersecting hypersurfaces $S_1, S_2 \subset \mathbb{H}^n/\Gamma$, so that

$$[\xi_{S_1}] \cup [\xi_{S_2}] \in H^2(\Gamma, Ad(\rho)) \neq 0.$$

In particular, $X_n(\Gamma)$ is not smooth at $[\rho]$. In contrast with this result, the character varieties $X_2(\Gamma)$ tend to be smooth:

Theorem 10.6. *(See* [113]*.) Let* $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ *be a discrete subgroup. Then the identity embedding* $\iota : \Gamma \to \mathbf{Mob}(\mathbb{S}^2)$ *determines a smooth point on* $X_2(\Gamma)$*.*

On the other hand, there are cocompact lattices $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ and (nondiscrete) representations $\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^2)$ for which $X_2(\Gamma)$ has nonquadratic singularity at $[\rho]$, see [120].

(b) *Generalized bending* associated with a collection of compact totally-geodesic submanifolds with boundary in M^n, see [10][8], [119], [150], [11].

The idea of the generalized bending is that instead of considering fundamental groups of graphs of groups, one looks at the more general *complexes of groups.* The only examples which had been worked out are 2-dimensional complexes of groups. Let $\mathcal{X}$ be such a complex with the vertex groups Γ_v. Let $\pi_1(\mathcal{X}) = \Gamma$ and $\rho_0 : \Gamma \to G$ be a representation to a Lie group. Then, as in the definition of bending, a *generalized bending* of ρ_0 is a curve of representations $\rho_t : \Gamma \to G, t \in [-1, 1]$, whose restrictions to each vertex subgroup Γ_v are trivial deformations of $\rho_0|\Gamma_v$.

(c) Suppose that a lattice $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a *reflection group*, i.e., it is generated by reflections in the faces of a convex acute polyhedron $\Phi \subset \mathbb{H}^n$ of finite volume (the fundamental domain of Γ). If f is the number of facets of Φ, then one can show that

$$\dim H^1(\Gamma, Ad(\rho)) = f - n - 1,$$

see [110]. The facets of Φ correspond to vectors spanning H^1. If $n \geq 4$, it is unclear which (if any) of these infinitesimal deformations are integrable. Of course, in *some* examples *some* of these infinitesimal deformations are integrable, since they appear as infinitesimal bending deformations. If $n = 3$, then $X_3(\Gamma)$ is smooth near $[\rho]$ and has dimension $f - 4$, see [110].

And that's all for $n \geq 3$.

Problem 10.7 (P. Storm)**.** *Let M be a compact hyperbolic $(n+1)$-dimensional manifold with nonempty totally-geodesic boundary, $n \geq 3$. Let $\Gamma := \pi_1(M) \subset \mathbf{Mob}(\mathbb{S}^n)$. Is it true that Γ is locally rigid in $\mathbf{Mob}(\mathbb{S}^n)$?*

Note that (by Mostow rigidity) local rigidity of Γ would follow if we knew that for each component S of ∂M, the fundamental group $\Gamma_S := \pi_1(S)$ is locally rigid in $\mathbf{Mob}(\mathbb{S}^n)$. At the moment, we do not have results in either direction of this problem:

1. It is unclear if any of the locally rigid hyperbolic 3-manifolds, or their disjoint union (see Example 10.12), bounds a compact hyperbolic 4-manifold.

2. Even if some Γ_S is not locally rigid, it is unclear if its deformations extend to deformations of Γ.

Remark 10.8. Consider the 120-cell $D^4 \subset \mathbb{H}^4$ which appears in [55]. Pick a facet $F \subset D^4$. Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ be the Kleinian group generated by reflections in all facets of D^4 except for F. Then Γ is the fundamental group of a right-angled 4-dimensional reflection orbifold $\mathcal{O}$ with boundary (the boundary $S = \partial\mathcal{O}$ corresponds to the facet F). The group $\Gamma_S := \pi_1(S)$ is the Coxeter group generated by reflections in the facets of the regular right-angled hyperbolic dodecahedron. In particular, $X_3(\Gamma)$

[8]Some of the theorems stated in this paper are probably incorrect since they do not take into account the restrictions on the angles between the totally-geodesic submanifolds.

is a smooth 8-dimensional manifold near $[\iota]$, where $\iota : \Gamma_S \hookrightarrow \mathbf{Mob}(\mathbb{S}^2) \hookrightarrow \mathbf{Mob}(\mathbb{S}^3)$ is the identity embedding.

One can show however that Γ is locally rigid in $\mathbf{Mob}(\mathbb{S}^3)$.

Assume now that $n = 3$.

(d) The first obstruction to integrability of infinitesimal deformations is always zero, see [120].

Question 10.9. Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ is a cocompact lattice. It it true that the character variety $X_3(\Gamma)$ is smooth at the point $[\rho]$?

(e) Finally, there are several constructions which work for specific examples of lattices $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$, e.g. *stumping deformations* [7], generalized in [195].

We recall the following

Conjecture 10.10. *Suppose that $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is a lattice. Then Γ contains a finite-index subgroup Γ' such that Γ' has infinite abelianization, i.e., $H^1(\Gamma', \mathbb{R}) \neq 0$.*

We refer the reader to [153, 135, 172, 137, 183] for various results towards this conjecture in the case of arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$. The methods used in these papers for proving virtual nonvanishing of the first cohomology group usually also apply to the cohomology groups $H^1(\Gamma, Ad(\rho))$, where $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is the natural embedding. On the other hand, the proofs of special cases of Conjecture 10.10 for hyperbolic 3-manifolds which use the methods of 3-dimensional topology (see e.g. [134]), usually provide no information about local rigidity of Γ in $\mathrm{Isom}(\mathbb{H}^4)$.

Conjecture 10.11. *Suppose that $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is a lattice. Then there exists a finite-index subgroup $\Gamma' \subset \Gamma$ so that $H^1(\Gamma', Ad(\rho)) \neq 0$.*

On the other hand, some uniform torsion-free lattices in $\mathbf{Mob}(\mathbb{S}^2)$ are locally rigid in $\mathbf{Mob}(\mathbb{S}^3)$:

Example 10.12. In [110] we constructed examples of (torsion-free) cocompact lattices Γ in $\mathbf{Mob}(\mathbb{S}^2)$ for which $H^1(\Gamma, Ad(\rho)) = 0$, where $\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^3)$ is the natural embedding. The quotient manifolds $\mathbb{H}^3/\Gamma$ in these examples are non-Haken. K. Scannell [180] constructed analogous examples with Haken quotients $\mathbb{H}^3/\Gamma$.

More specifically, it was proved in [110] that for every hyperbolic 2-bridge knot $K \subset \mathbb{S}^3$, there are infinitely many (hyperbolic) Dehn surgeries on K, so that for the resulting manifolds M_j, $j \in \mathbb{N}$, we have

$$H^1(\Gamma_j, Ad(\rho)) = 0, \text{ where } \Gamma_j = \pi_1(M_j).$$

10.1.3. *Failure of quasiconformal isotopy.*

Theorem 10.13. *There exists a pair of convex-cocompact Kleinian groups $\Delta_1, \Delta_2 \subset \mathbf{Mob}(\mathbb{S}^5)$ and a quasiconformal homeomorphism $f : \mathbb{S}^5 \to \mathbb{S}^5$ conjugating Δ_1 to Δ_2, which is not isotopic to the identity through homeomorphisms $h_t : \mathbb{S}^5 \to \mathbb{S}^5$ such that*

$$h_t \Delta_1 h_t^{-1} \subset \mathbf{Mob}(\mathbb{S}^5)\,.$$

Proof. We begin with a lattice $\Gamma = \pi_1(N)$, where $N = M_j$ is as in the discussion of Example 10.12 and $K \subset \mathbb{S}^3$ is the figure 8 knot. Consider the representation

$$\rho_1 : \Gamma \hookrightarrow \mathbf{Mob}(\mathbb{S}^2) \hookrightarrow \mathbf{Mob}(\mathbb{S}^5)$$

obtained by the composition of natural embeddings. Then

$$H^1(\Gamma, Ad(\rho_1)) = H^1(\Gamma, V_3 \oplus V_3 \oplus V_3 \oplus \mathbb{R}^3),$$

where $V_3 = \mathbb{R}^{3,1}$ and $\mathbb{R}^3$ is the trivial 3-dimensional $\mathbb{R}\Gamma$-module. Since $H^1(\Gamma, V_3) = 0$ by [110] and $H^1(\Gamma, \mathbb{R}^3) = 0$ since N is a rational homology sphere, we obtain

$$H^1(\Gamma, Ad(\rho_1)) = 0.$$

Therefore ρ_1 is locally rigid. If N is an integer homology 3-sphere, then nonvanishing of the Casson invariant of K implies that Γ admits a nontrivial homomorphism

$$\theta : \Gamma \to SO(3)$$

which lifts to $SU(2)$, see [5]. If M_j is not an integer homology sphere, then Γ has nontrivial abelianization and hence we also obtain a nontrivial homomorphism $\theta : \Gamma \to SO(3)$ with cyclic image. In any case, we twist the representation ρ_1 by θ:

$$\rho_2 = \rho_1 \times \theta : \Gamma \to \mathbf{Mob}(\mathbb{S}^2) \times SO(3) \subset \mathbf{Mob}(\mathbb{S}^5)\,.$$

It is clear that $[\rho_1], [\rho_2]$ are distinct points of $X_5(\Gamma)$. The images of ρ_1 and ρ_2 are 2-fuchsian, convex-cocompact groups $\Delta_1, \Delta_2 \subset \mathbf{Mob}(\mathbb{S}^5)$. We obtain the isomorphism

$$\rho := \rho_2 \circ \rho_1^{-1} : \Delta_1 \to \Delta_2$$

Clearly,

$$M^5(\Delta_1) = N \times \mathbb{S}^2,$$

while $M^5(\Delta_2)$ is the 2-sphere bundle over N associated with θ. It is easy to see that the latter bundle is (smoothly) trivial. Therefore we obtain a diffeomorphism

$$h : M^5(\Delta_1) \to M^5(\Delta_2)$$

which lifts to a ρ-equivariant diffeomorphism $f : \Omega(\Delta_1) \to \Omega(\Delta_2)$. The latter extends to a quasiconformal homeomorphism $f : \mathbb{S}^5 \to \mathbb{S}^5$ by Theorem 5.1. If there was a continuous family of homomorphisms ρ_t connecting ρ to the identity embedding $\Delta_1 \to \Delta_2$, then the representation ρ_1 would not be locally rigid in $X_5(\Gamma)$. Contradiction. $\square$

By embedding naturally the groups Δ_1, Δ_2 to $\mathbf{Mob}(\mathbb{S}^n)$ for $n \geq 6$ one obtains higher-dimensional examples.

10.2. **Stability theorem.** Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ be a geometrically finite Kleinian group. Consider the set of cusps in Γ:

$$[\Pi_1], ..., [\Pi_m],$$

where Π_i are maximal parabolic subgroups of Γ. We define the *(topologically) relative* representation variety

$$R_n^{top}(\Gamma) = \{\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^n) : \rho(\Pi_i) \text{ is topologically conjugate to } \Pi_i \text{ in } \mathbb{S}^n, \forall i\}$$

and the *(quasiconformally) relative* representation variety

$$R_n^{qc}(\Gamma) = \{\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^n) : \rho(\Pi_i) \text{ is quasiconformally conjugate to } \Pi_i \text{ in } \mathbb{S}^n, \forall i\}.$$

Let $Homeo(\mathbb{S}^n)$ and $QC(\mathbb{S}^n)$ be the groups of homeomorphisms and quasiconformal homeomorphisms of $\mathbb{S}^n$ with the topology of uniform convergence. Let $X_n^{top}(\Gamma), X_n^{qc}(\Gamma)$ be the projections of $R_n^{top}(\Gamma), R_n^{qc}(\Gamma)$ to $X_n(\Gamma)$. Let $\iota : \Gamma \to \mathbf{Mob}(\mathbb{S}^n)$ be the identity embedding. Then the *Stability Theorem* for geometrically finite groups states that every homomorphism ρ of Γ sufficiently close to ι is induced by a (quasiconformal) homeomorphism h_ρ close to the identity and depending continuously on ρ. More precisely:

Theorem 10.14. *(Stability theorem, see* [142], [64] [193], [75], [105]*.) There exist neighborhoods U^{top}, U^{qc} of ι in $R_n^{top}(\Gamma), R_n^{qc}(\Gamma)$ respectively, and continuous maps*

$$L^{top} : U^{top} \to Homeo(\mathbb{S}^n), \quad L^{qc} : U^{qc} \to QC(\mathbb{S}^n)$$

so that

$$L^{top}(\iota) = L^{qc}(\iota) = Id,$$

and for every $\rho \in U^{top}$, resp. $\rho \in U^{qc}$, the homeomorphism $L^{top}(\rho)$, resp. $L^{qc}(\rho)$ is ρ-equivariant.

This theorem was first proved by A. Marden in [142] in the case $n = 3$. Marden was using convex finitely-sided fundamental domains with simplicial links of vertices: Such polyhedra are generic among the Dirichlet fundamental domains, see [102]. Marden then argued that a small perturbation of such fundamental domain is again a fundamental domain (by Poincare's fundamental polyhedron theorem). Moreover, the *simplicial* assumption implies that the combinatorics of the fundamental domain does not change under a small perturbation. This allowed Marden to construct an equivariant quasiconformal homeomorphism close to the identity. This argument does not readily generalize to higher dimensions, mainly because finiteness of the number of faces is not equivalent to geometric finiteness. (Otherwise, the same argument goes through.)

D. Sullivan [193] considered the case of general n, but assumed that Γ is convex-cocompact. Then he proved the existence of a homeomorphism h_ρ defined on the limit set of Γ and the fact that it depends continuously on ρ. The fact that

$$h_\rho : \Lambda(\Gamma) \to \Lambda(\rho(\Gamma))$$

is necessarily quasi-symmetric then follows from Tukia's theorem 5.1. One then has to show existence of a ρ-equivariant diffeomorphism of the domains of discontinuity

$$f_\rho : \Omega(\Gamma) \to \Omega(\rho(\Gamma))$$

smoothly depending on ρ. This is achieved by appealing to Thurston's holonomy theorem (see [64, 75]) for the Moebius structures on the manifold $M^n(\Gamma)$, as it is done in [105], [97]. The homeomorphisms h_ρ and f_ρ yield a ρ-equivariant quasiconformal homeomorphism of the n-sphere by Theorem 5.1.

The proof in [64] is a good alternative to the above argument; it is also sufficiently flexible to hande the case of geometrically finite Kleinian groups with cusps. Namely, instead of working with the n-dimensional manifold $M^n(\Gamma)$ one works with the convex hyperbolic $(n+1)$-manifold

$$H(\Gamma) := Hull_\epsilon(\Lambda(\Gamma))/\Gamma.$$

An analogue of Thurston's holonomy theorem for manifolds with boundary applies in this case. Thus, for $\rho \in U^{top}$, there exists a hyperbolic structure $s(\rho)$ (with the holonomy ρ) on the thick part

$$H(\Gamma)_{[\mu,\infty)}$$

of the manifold $H(\Gamma)$. Moreover, convexity of the boundary for the new hyperbolic structures (away from the cusps) persists under small perturbations of the hyperbolic structure. Therefore, if Γ is convex-cocompact, $\Gamma' := \rho(\Gamma)$ is again convex-cocompact and $\rho : \Gamma \to \Gamma'$ is an isomorphism. If Γ is merely geometrically finite, because ρ belongs to the relative representation variety, it follows that the hyperbolic structure $s(\rho)$ extends to a convex complete hyperbolic structure on the cusps. This argument also yields a ρ-equivariant diffeomorphism

$$Hull_\epsilon(\Lambda(\Gamma)) \to Hull_\epsilon(\Lambda(\Gamma'))$$

depending continuously on ρ. To get from the convex hulls to the domain of discontinuity one uses the existence of the canonical equivariant diffeomorphisms ("the nearest-point projections")

$$\Omega(\Gamma) \to \partial Hull_\epsilon(\Lambda(\Gamma)), \quad \Omega(\Gamma') \to \partial Hull_\epsilon(\Lambda(\Gamma')).$$

We refer the reader to [64] for the details.

Sullivan also had a converse to the Stability Theorem for (finitely-generated) subgroups on $\mathbf{Mob}(\mathbb{S}^2)$:

Theorem 10.15. *(D. Sullivan,* [193, Theorem A′]*) If a (finitely-generated) Kleinian subgroup of* $\mathbf{Mob}(\mathbb{S}^2)$ *is stable in the sense of Theorem 10.14, then it is geometrically finite or its identity embedding in* $\mathbf{Mob}(\mathbb{S}^2)$ *is locally rigid in* $X_2(\Gamma)$*.*

It was proved in [113] that every locally rigid Γ in the above theorem has to be geometrically finite. Now it, of course, follows from the positive solution of the Bers–Thurston density conjecture (geometrically finite groups are dense among Kleinian subgroups of $\mathbf{Mob}(\mathbb{S}^2)$).

Question 10.16. Does Theorem 10.15 hold for subgroups of $\mathbf{Mob}(\mathbb{S}^n)$, $n \geq 3$?

We expect the answer to be negative.

10.3. **Space of discrete and faithful representations.** Let $\mathcal{D}_n(\Gamma) \subset X_n(\Gamma)$ denote the subset corresponding to discrete, injective and nonelementary representations of Γ.

Theorem 10.17 (Chuckrow–Jørgensen–Wielenberg)**.** $\mathcal{D}_n(\Gamma) \subset X_n(\Gamma)$ *is closed. See for instance* [208, 143]*.*

It turns out that there exists another way to construct limits of sequences of Kleinian groups, by regarding them as *closed subsets* of $\mathbf{Mob}(\mathbb{S}^n)$. This leads to the topology of *geometric convergence* of Kleinian groups. With few exceptions, the space of Kleinian groups is again closed in this topology (see e.g. [197, 113]). In general, $\mathcal{D}_n(\Gamma)$ is not compact. Nevertheless, this space can be compactified by *projective classes* of nontrivial Γ-actions on real trees. This compactification generalizes

Thurston's compactification of the Teichmüller space. The compactification by actions on trees was first defined by J. Morgan and P. Shalen [159] and J. Morgan [158] using algebraic geometry. More flexible, geometric, definitions of this compactification were introduced by M. Bestvina [21] and F. Paulin [167]. See also [113] for the construction of this compactification using ultralimits of metric spaces.

This viewpoint provides a powerful tool for proving compactness of $\mathcal{D}_n(\Gamma)$ for certain classes of groups: If $\mathcal{D}_n(\Gamma)$ is non-compact then Γ admits nontrivial action on a certain $\mathbb{R}$-tree. One then proves that such action cannot exist. The tools for proving such non-existence theorems are originally due to Morgan and Shalen (but limited to the fundamental groups of 3-manifolds, see [159]); much more general method is due to E. Rips (Rips theory).

Theorem 10.18 (Rips–Thurston Compactness theorem)**.** *Suppose that Γ is a finitely-presented group which does not split as an amalgam over a virtually abelian group. Then $\mathcal{D}_n(\Gamma)$ is compact.*

For Γ isomorphic to the fundamental groups of 3-manifolds this theorem is due to J. Morgan and P. Shalen [159]. For more general groups, this theorem is a corollary of the Rips theory of groups acting on trees [24], see [113].

Unfortunately, none of the known proofs of this theorem gives explicit bounds on the "size" of $\mathcal{D}_(\Gamma)$.

Problem 10.19. *Find a "constructive" proof of Theorem 10.18. More precisely, consider a group Γ with a finite presentation $\langle g_1, .., g_k | R_1, .., R_m\rangle$. Given $[\rho] \in \mathcal{D}_{n-1}(\Gamma)$ define*

$$B_{n-1}([\rho]) := \inf_{x\in\mathbb{H}^n} \max_{i=1,\dots,k} d(x, \rho(g_i)(x)).$$

Find an explicit constant C, which depends on n, k, m and the lengths of the words R_i, so that the function $B_{n-1} : \mathcal{D}_{n-1}(\Gamma) \to \mathbb{R}$ is bounded from above by C.

Theorem 10.18 suggests that one should also look for *geometric bounds* on $[\rho] \in \mathcal{D}_n(\Gamma)$: Even if $\mathcal{D}_n(\Gamma)$ is noncompact (or its "size" is unknown), one can still try to find some natural functionals on $\mathcal{D}_n(\Gamma)$ and obtain explicit bounds (from below and from above) on these functionals.

Definition 10.20 (Diameter of a representation)**.** Given a discrete embedding $\rho : \Gamma \to \Gamma' = \rho(\Gamma) \subset \mathbf{Mob}(\mathbb{S}^n)$, consider the set S of connected subgraphs $\sigma \subset \mathbb{H}^{n+1}/\Gamma'$ with the property: The map $\pi_1(\sigma) \to \pi_1(M)$ is surjective.

Then the *diameter* of ρ is

$$\mathrm{diam}(\rho) := \inf\{\mathrm{length}(\sigma) : \sigma \in S\}.$$

Problem 10.21. *Given a group Γ as in Theorem 10.18, find explicit bounds on* $\mathrm{diam}(\rho)$ *(in terms of the presentation of Γ) for representations $[\rho] \in \mathcal{D}_n(\Gamma)$.*

Note that the positive *lower bound* on $\mathrm{diam}(\rho)$ is an easy corollary of Kazhdan-Margulis lemma.

Definition 10.22. (Volumes of a representation) Fix a homology class $[\zeta] \in Z_p(\Gamma)$, $2 \le p \le cd(\Gamma)$. For a representation $\rho \in \mathcal{D}_n(\Gamma)$ consider the quotient manifold

$M = \mathbb{H}^n/\rho(\Gamma)$. Define the set $E(\zeta)$ of singular p-cycles $\zeta' \in Z_p(M)$ which represent the homology class $[\zeta]$ under the isomorphism

$$H_p(\Gamma) \to H_p(M)$$

induced by the isomorphism $\rho : \Gamma \to \pi_1(M)$. Lastly, define the *$\rho$-volume* of the class $[\zeta]$ by

$$Vol_\rho(\zeta) := \inf\{Vol(\zeta') : \zeta' \in E(\zeta)\}.$$

Let $||\zeta||$ denote the Gromov-norm of the class $[\zeta]$ and let c_p denote the volume of the regular ideal geodesic p-simplex in $\mathbb{H}^p$. Then an easy application of Thurston's "chain-straightening" is the inequality

$$Vol_\rho(\zeta) \leq c_p||\zeta||$$

for all ρ, $[\zeta]$ and $p \geq 2$. However good lower bounds on the volume are considerably more difficult to get.

Given a hyperbolic manifold M define $H_p^{cusp}(M)$ to be the image in $H_p(M)$ of the p-th homology group of the union of all cusps of M. Then for every cuspidal class $[\zeta]$ and every ρ, we clearly have

$$Vol_\rho(\zeta) = 0.$$

However there exists a positive constant $\epsilon = \epsilon(p, n)$ such that for every $p > 0$, every non-cuspidal class $[\zeta]$ and every ρ, we obtain

$$Vol_\rho(\zeta) \geq \epsilon,$$

see [103]. Below are some nontrivial lower bounds on the volume:

Theorem 10.23. *(Follows directly from* [82, Theorem 5.38][9]*). Let Γ be isomorphic to the fundamental group of a compact aspherical k-manifold N and $[\zeta] = [N]$ be the fundamental class of M. Then there exists a universal (explicit) constant $c(p, n) > 0$ depending only on p and n, such that*

$$Vol_\rho(\zeta) \geq c(p, n)||N||.$$

One gets better estimates using the work of Besson, Courtois and Gallot [19] [10]:

Theorem 10.24. *Fix a representation $[\phi] \in \mathcal{D}_n(\Gamma)$. Then for every $[\rho] \in \mathcal{D}_n(\Gamma)$ and $p \geq 3$ we obtain*

$$\left(\frac{p}{n}\right)^{p+1} Vol_\phi(\zeta) \leq Vol_\rho(\zeta).$$

For instance, if $\Gamma' := \phi(\Gamma)$ happens to be a uniform lattice in $\mathrm{Isom}(\mathbb{H}^p)$, we obtain

Corollary 10.25. *For every $[\rho] \in \mathcal{D}_n(\Gamma)$ and $p \geq 3$ we have*

$$\left(\frac{p}{n}\right)^{p+1} Vol(M') \leq Vol_\rho(\zeta),$$

where $M' = \mathbb{H}^p/\Gamma'$ and $[\zeta]$ is the fundamental class.

[9] I am grateful to A. Nabutovsky for this reference.

[10] I am grateful to J. Souto for pointing this out.

10.4. **Why is it so difficult to construct higher-dimensional geometrically infinite Kleinian groups?** (i). The oldest trick for proving existence of geometrically infinite groups is due to L. Bers [17]:

Start with (say) a convex-cocompact subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$. Let $Q(\Gamma) \subset \mathcal{D}_n(\Gamma)$ be the (open) subset of representations induced by quasiconformal conjugation. Let $Q_0(\Gamma)$ denote the component of $Q(\Gamma)$ containing the (conjugacy class of) identity representation $[\rho_0]$. We assume that the closure of $Q_0(\Gamma)$ is not open in $X_n(\Gamma)$. Then there exists a curve $[\rho_t] \in X_n(\Gamma), t \in [0,1]$, so that ρ_1 is either nondiscrete or non-injective. Since $\mathcal{D}_n(\Gamma)$ is closed, it follows that there exists $s \in (0,1)$ such that $[\rho_s(\Gamma)]$ belongs to $\mathcal{D}_n(\Gamma)$ but $\rho_s(\Gamma)$ is not convex-cocompact. If $\Gamma' = \rho_s(\Gamma)$ contains no parabolic elements, it would follow that Γ' is isomorphic to Γ and is not geometrically finite. However it could happen that the frontier of $Q_0(\Gamma)$ consist entirely of the classes $[\rho]$ for which $\rho(\Gamma)$ contains parabolic elements.

The latter cannot occur if $n = 2$ for the dimension reasons: The set of parabolic elements of $PSL(2,\mathbb{C})$ has real codimension 2. However for all $n \neq 2$, the set of parabolic elements has real codimension 1 and this argument is inconclusive.

One can try to apply the above argument in the case of a codimension 1 fuchsian group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ which acts as a cocompact lattice on $\mathbb{H}^n \subset \mathbb{H}^{n+1}$. Suppose that $M = \mathbb{H}^n/\Gamma$ contains a totally-geodesic compact hypersurface S. Then we have the circle $\mathbb{S}^1$ worth of bending deformations ρ_t along S. As $t = \pi$, the image of ρ_t is again contained in $\mathbf{Mob}(\mathbb{S}^{n-1})$. Therefore ρ_π is either nondiscrete or non-injective. However, conceivably, in all such cases, for $[\rho_s] \in \partial Q_0(\Gamma)$ the representation ρ_s is geometrically finite (because its image may contain parabolic elements). It happens, for instance, if Γ is a reflection group.

Note that even when $n = 2$ and we are bending a 1-fuchsian group Γ, it is hard to predict which simple closed geodesics $\alpha \subset \mathbb{H}^2/\Gamma$ yield geometrically infinite groups (via bending along α).

(ii). One can try to construct explicit examples of fundamental domains, following, say, T. Jorgensen [100] or A. Marden and T. Jorgensen [101].

The trouble is that constructing fundamental polyhedra with infinitely many faces in $\mathbb{H}^4$ is quite a bit harder than in $\mathbb{H}^3$. One can try to find a lattice $\Delta \subset \mathbf{Mob}(\mathbb{S}^3)$ which contains a nontrivial finitely-generated normal subgroup Γ of infinite index.[11] This is, probably, the most promising approach, since it works for complex-hyperbolic lattices in $PU(2,1)$, cf. [112]. One can try to imitate Livne's examples, by constructing $\Gamma \subset \Delta$ such that Δ/Γ is isomorphic to a surface group. This would require coming up with a specific compact convex polyhedron in $\mathbb{H}^4$ such that the associated 4-manifold appears as a (singular) fibration over a surface.

(iii). One can try to use the combinatorial group theory. Note that there are plenty of examples of (mostly 2-dimensional) Gromov-hyperbolic groups Δ each of which contains a nontrivial finitely-generated normal subgroup Γ of infinite index. See e.g. [178, 23, 160, 38] for the examples which are not 3-manifold groups. However embedding a given hyperbolic group Δ in $\mathbf{Mob}(\mathbb{S}^n)$ is a nontrivial task, cf. [33, 114] and

[11] If Δ is a Kleinian group containing a nontrivial normal subgroup Γ of infinite index, then Γ is necessarily geometrically infinite.

discussion in Section 11. The groups considered in [114], probably provide the best opportunity here since most of them do not pass the *perimeter test* of J. McCammond and D. Wise [151].

(iv). *What would geometrically infinite examples look like?* Let $\Gamma \subset \mathbf{Mob}(\mathbb{S}^2)$ be a singly-degenerate group; assume for simplicity that the injectivity radius of $\mathbb{H}^3/\Gamma$ is bounded away from zero. Let S denote the boundary of

$$Hull(\Lambda(\Gamma))/\Gamma$$

and $\lambda \subset S$ be the ending lamination of Γ. Then every leaf of λ lifts to an exponentially distorted curve κ in $\mathbb{H}^3$: Given points $x, y \in \kappa$, their extrinsic distance $d(x, y)$ in $\mathbb{H}^3$ is roughly the logarithm of their intrinsic distance along κ.

One would like to imitate this behavior in dimension 4. Let M be a closed hyperbolic 3-manifold containing an embedded compact totally-geodesic surface $S \subset M$. Let $\lambda \subset S$ be an ending lamination from the above example. One would like to construct a complete hyperbolic 4-manifold N homotopy-equivalent to M, so that under the (smooth) homotopy-equivalence $f : M \to N$ we have:

For every leaf L of λ, $f(L)$ lifts to an exponentially distorted curve in $\mathbb{H}^4$.

Then $\pi_1(N)$ will necessarily be a geometrically infinite subgroup Γ of $\mathbf{Mob}(\mathbb{S}^4)$. At the moment it is not even clear how to make this work with a hyperbolic metric on N replaced by a complete Riemannian metric of negatively pinched sectional curvature, although constructing a Gromov-hyperbolic metric with this behavior is not that difficult. (Recall that a Riemannian metric is said to be *negatively pinched* if its sectional curvature varies between two negative numbers.) An example Γ of this type is likely to have two components of $\Omega(\Gamma)$: One contractible and one not.

More ambitiously, one can try to get a singly degenerate group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^3)$ (so that $\Omega(\Gamma)$ is contractible and $M^3(\Gamma)$ is compact). How such example would look like? One can imagine taking a 1-dimensional quasi-geodesic foliation λ of the 3-manifold M as above and then requiring that for every leaf $L \subset \lambda$, the curve $f(L)$ lifts to an exponentially distorted curve in $\mathbb{H}^4$. At the moment I do not see even a Gromov-hyperbolic model of this behavior. Another option would be to work with 2-dimensional laminations ν (with simply-connected leaves) in M and require every leaf $L \subset \nu$ to correspond to an exponentially distorted surface in $\mathbb{H}^4$ (or a negatively-curved simply-connected 4-manifold), which limits to a single point in $\mathbb{S}^3$.

Problem 10.26. *Construct a complete negatively pinched 4-dimensional Riemannian manifold N homotopy-equivalent to a hyperbolic 3-manifold M, so that the convex core of N either has exactly one boundary component or equals N itself.*

Question 10.27. Is there a geometrically infinite Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ whose limit is homeomorphic to the Menger curve? Is there a geometrically infinite Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^n)$ which is isomorphic to the fundamental group of a closed aspherical manifold of dimension ≥ 3? Are there examples of such groups acting isometrically on complete negatively pinched manifolds? Are there examples of hyperbolic (or even negatively curved) 4-manifolds M such that $\pi_1(M) = \Gamma$ fits into a short exact sequence

$$1 \to \pi_1(S) \to \Gamma \to \pi_1(F) \to 1,$$

where S, F are closed hyperbolic surfaces? Note that there are no complex-hyperbolic examples of this type, see [112].

11. Algebraic and topological constrains on Kleinian groups

Sadly, there are only few known algebraic and topological restrictions on Kleinian subgroups in $\mathbf{Mob}(\mathbb{S}^n)$ that do not follow from the *elementary* restrictions, which come from the restrictions on geometry of complete negatively curved Riemannian manifolds. Examples of the elementary restrictions on a Kleinian group Γ are:

1. Every solvable subgroup of a Kleinian group is virtually abelian.
2. The normalizer (in Γ) of an infinite cyclic subgroup of Γ is virtually abelian.
3. If $\Pi \in \Gamma$ is an infinite elementary subgroup and $\alpha \in \Gamma$ is such that the groups

$$\alpha\Pi\alpha^{-1}, \quad \Pi$$

are commensurable, then α belongs to the normalizer of Π in Γ.

4. Every Kleinian group has finite (virtual) cohomological dimension.

11.1. Algebraic constrains.

Definition 11.1. An *abstract Kleinian group* is a group Γ which admits a discrete embedding in $\mathbf{Mob}(\mathbb{S}^n)$ for some n. Such a group is called *nonelementary* if it is not virtually abelian.

In order to eliminate trivial restrictions on abstract Kleinian groups one can restrict attention to Gromov-hyperbolic Kleinian groups. Below is the list of known algebraic constrains on Kleinian groups under this extra assumption:

1. Kleinian groups are residually finite and virtually torsion-free.[12] (This, of course, holds for all finitely generated matrix groups.)

2. Kleinian groups satisfy the *Haagerup property*, in particular, infinite Kleinian groups do not satisfy property (T), see [50].

3. If a Kleinian group Γ is *Kähler*, then Γ is virtually isomorphic to the fundamental group of a compact surface. This is a deep theorem of J. Carlson and D. Toledo [47] who proved that every homomorphism of a Kähler group to $\mathbf{Mob}(\mathbb{S}^n)$ either factors through a virtually surface group, or its image fixes a point in $\mathbb{B}^{n+1}$.

Recall that a topological group G is said to satisfy the Haagerup property if it admits a (metrically) proper continuous isometric action on a Hilbert space H. An action of a metrizable topological group G on H is *metrically proper* if for every bounded subset $B \subset H$, the set

$$\{g \in G : g(B) \cap B \neq \emptyset\}$$

is a bounded subset of G. Since $\mathbf{Mob}(\mathbb{S}^n)$ satisfies the Haagerup property for every n (see e.g. [50]), all Kleinian groups also do.

[12]It is widely believed that there are Gromov-hyperbolic groups which are not residually finite.

A group π is called *Kähler* if it is isomorphic to the fundamental group of a compact Kähler manifold. For instance, every uniform lattice in $\mathbb{CH}^n$ is Kähler; therefore it cannot be an abstract Kleinian group unless $n = 1$.

Remark 11.2. A (finitely-generated) group satisfies the Haagerup property if and only if it admits an isometric (metrically) properly discontinuous action on the infinite dimensional hyperbolic space $\mathbb{H}^\infty$, see [81, 7.A.III]. The result of Carlson and Toledo shows that (for Gromov-hyperbolic groups) there are nontrivial obstructions to replacing these infinite-dimensional actions with finite-dimensional ones.

Observation 11.3. All currently known nontrivial restrictions on abstract Kleinian groups can be traced to 1, 2 or 3.

Problem 11.4. *Find other restrictions on abstract Kleinian groups.*

Potentially, some new restrictions would follow from Rips-Thurston compactness theorem. The difficulty comes from the following. Let Γ is a Gromov-hyperbolic group which admits no nontrivial isometric actions on $\mathbb{R}$-trees. Then (see [113]) there exists $C < \infty$ such that for every sequence $[\rho_j] \in \mathcal{D}_n(\Gamma)$, we obtain a uniform bound

$$B_n([\rho_j]) \le C \tag{11.5}$$

where $B_n : \mathcal{D}_n(\Gamma) \to \mathbb{R}$ is the minimax function defined in Problem 10.19. If n was fixed, then (ρ_j) would subconverge to a representation to $\mathbf{Mob}(\mathbb{S}^n)$ (for some choice of representations ρ_j in the classes $[\rho_j]$). However, since we are not fixing the dimension of the hyperbolic space on which our Γ is supposed to act, the inequality (11.5) does not seem to yield any useful information. By taking an ultralimit of ρ_j's we will get an action of Γ on an infinite-dimensional hyperbolic space. This action, however, may have a fixed point, since

$$\lim_{n\to\infty} \mu_n = 0,$$

where μ_n is the Margulis constant for $\mathbb{H}^n$.

Example 11.6. Let M^3 be a closed non-Haken hyperbolic 3-manifold, so that $\Pi := \pi_1(M)$ contains a maximal 1-fuchsian subgroup F. For each automorphism $\phi : F \to F$ we define the HNN extension

$$\Gamma_\phi := \Pi *_{F \cong_\phi F} = \langle \Pi, t | tgt^{-1} = \phi(g), \forall g \in F\rangle.$$

Then Γ_ϕ is Gromov-hyperbolic for all pseudo-Anosov automorphisms ϕ, see [24]. It is a direct corollary of Theorem 10.18 that for every n, only finitely many of the groups Γ_ϕ embed in $\mathbf{Mob}(\mathbb{S}^n)$ as Kleinian subgroups. Is it true that there exists a pseudo-Anosov automorphism ϕ such that Γ_ϕ is not an abstract Kleinian group?

Infinite finitely-generated Gromov-hyperbolic Coxeter groups are all linear, satisfy the Haagerup property and are not Kähler (except for the virtually surface groups).

Problem 11.7. *Is it true that every finitely-generated Gromov-hyperbolic Coxeter group Γ is an abstract Kleinian group?*

Note that there are Gromov-hyperbolic Coxeter groups Γ which do not admit discrete embeddings $\rho : \Gamma \to \mathbf{Mob}(\mathbb{S}^n)$ (for any n), so that the Coxeter generators of Γ act as reflections in the faces of a fundamental domain of $\rho(\Gamma)$, see [72].

The answer to the next question is probably negative, but the examples would be tricky to construct:

Question 11.8. Is it true that a group weakly commensurable to a Kleinian group is also a Kleinian group? Even more ambitiously: Is the property of being Kleinian a quasi-isometry invariant of a group?

Recall that two groups Γ and Γ' are called *weakly commensurable* if there exists a chain of groups and homomorphisms

$$\Gamma = \Gamma_0 \to \Gamma_1 \leftarrow \Gamma_2 \to \Gamma_3 .... \leftarrow \Gamma_{k-1} \to \Gamma_k = \Gamma'$$

where each arrow $\Gamma_i \to \Gamma_{i\pm1}$ is a homomorphism whose kernel and cokernel are finite.

There are few more known algebraic restictions on geometrically finite Kleinian groups. All such groups are relatively hyperbolic. T. Delzant and L. Potyagailo [58] obtained a characterization of geometrically finite Kleinian groups which are *cohopfian*.

Question 11.9. Is the isomorphism problem solvable within the class of all finitely-presented Kleinian groups? Note that the work of F. Dahmani and D. Groves [54] implies solvability of the isomorphism problem in the category of geometrically finite Kleinian groups.

It was proved by M. Bonk and O. Schramm [32] that every Gromov-hyperbolic group Γ embeds quasi-isometrically in the usual hyperbolic space $\mathbb{H}^n$ for some $n = n(\Gamma)$. A natural question is if one can prove an *equivariant* version of this result. Note that there are many Gromov-hyperbolic groups which are not Kleinian, e.g. groups with property (T) and Gromov-hyperbolic Kähler groups. Therefore one has to relax the *isometric* assumption. The natural category for this is the *uniformly quasiconformal* actions. Such action is an embedding

$$\rho : \Gamma \hookrightarrow QC(\mathbb{S}^n)$$

whose image consists of K-quasiconformal homeomorphisms with K depending only on ρ.

Problem 11.10. *Let Γ be a Gromov-hyperbolic group. Does Γ admit a uniformly quasiconformal discrete action on $\mathbb{S}^n$ for some n? For instance, is there such an action if Γ is a uniform lattice in $PU(n, 1)$ or satisfies the property (T)?*

T. Farrell and J. Lafont [69] proved the topological counterpart of this problem has positive solution. A corollary of their work is that every Gromov-hyperbolic group Γ admits a *convergence* action ρ on the closed n-ball, so that the limit set of $\Gamma' = \rho(\Gamma)$ is equivariantly homeomorphic to the ideal boundary of Γ and $\Omega(\Gamma')/\Gamma'$ is compact and connected. We refer the reader to [74] for the definition of a convergence action.

11.2. **Topological constrains.** The basic problem here is to find topological restrictions on the hyperbolic manifold $\mathbb{H}^{n+1}/\Gamma$ and the conformally-flat manifold $M^n(\Gamma) = \Omega(\Gamma)/\Gamma$, which do not follow from the algebraic restrictions on the group Γ and from the general algebraic topology restrictions (e.g. vanishing of the characteristic classes). There are only few nontrivial results in this direction.

For $n = 3$ we have:

Theorem 11.11. *(M. Kapovich [106]) There exists a function $c(\chi)$ with the following property. Let S be a closed hyperbolic surface. Suppose that M^4 is a complete hyperbolic 4-manifold which is homeomorphic to the total space of an $\mathbb{R}^2$-bundle $\xi : E \to S$ with the Euler number $e(\xi)$. Then*

$$|e(\xi)| \le c(\chi(S)).$$

More generally,

Theorem 11.12. *(M. Kapovich [106]) There exists a function $C(\chi_1, \chi_2)$ with the following property. Suppose that M^4 is a complete oriented hyperbolic 4-manifold. Let $\sigma_j : \Sigma_j \to M^4$ ($j = 1, 2$) be π_1-injective maps of closed oriented surfaces Σ_j. Then*

$$|\langle \sigma_1, \sigma_2 \rangle| \le C(\chi(\Sigma_1), \chi(\Sigma_2)).$$

Here $\langle , \rangle$ is the intersection pairing on $H_2(M^4)$. The bounds appearing in these theorems are explicit but astronomically high. The expected bounds are linear in $\chi(S)$ and $\chi(S_i)$, $i = 1, 2$, cf. [83].

Other known restrictions are applications of the compactness theorem 10.18 and therefore explicit bounds in the following theorems are unknown.

Theorem 11.13. *(M. Kapovich, [108].) Given a closed hyperbolic n-manifold B ($n \ge 3$) there exists a number $c(B)$ so that the following is true. Suppose that M^{2n} a complete hyperbolic $2n$-manifold which is homeomorphic to the total space of an $\mathbb{R}^n$-bundle $\xi : E \to B$ with the Euler number $e(\xi)$. Then*

$$|e(\xi)| \le c(B).$$

I. Belegradek greatly improved this result:

Theorem 11.14. *(I. Belegradek, [14].) Given a closed hyperbolic n-manifold B ($n \ge 3$) there exists a number $C(B, k)$ so that the number of inequivalent $\mathbb{R}^k$-bundles $\xi : E \to B$ whose total space admits a complete hyperbolic metric, is at most $C(B, k)$.*

Given a group π, let $\mathcal{M}_{\pi,n}$ denote the set of n-manifolds, whose fundamental group is isomorphic to π and which admit complete hyperbolic metrics.

Theorem 11.15. *(I. Belegradek, [15].) Suppose that π is a finitely-presented group with finite Betti numbers. Assume that π does not split as an amalgam over a virtually abelian subgroup. The set $\mathcal{M}_{\pi,n}$ breaks into finitely many intersection preserving homotopy types.*

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