INVARIANT DIFFERENTIAL OPERATORS ON $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$

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1. Introduction

We let

$$SL_{2,1}(\mathbb{R}) := SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}$$

be the semidirect product of the special linear group $SL(2,\mathbb{R})$ of degree 2 and the commutative group $\mathbb{R}^{(1,2)}$ equipped with the following multiplication law

$$(g,\alpha) * (h,\beta) := (gh,\, \alpha\, {}^t h^{-1} + \beta), \quad g,h \in SL(2,\mathbb{R}), \quad \alpha,\beta \in \mathbb{R}^{(1,2)}. \tag{1.1}$$

Throughout this paper, for two positive integers m and n, we denote by $F^{(m,n)}$ the set of all $m \times n$ matrices with entries in a commutative ring F. We let

$$S\mathcal{P}_2 := \left\{ Y \in \mathbb{R}^{(2,2)} \mid Y = {}^tY > 0, \quad \det Y = 1 \right\}$$

be the symmetric space consisiting of all positive symmetric 2×2 real matrices with $\det Y = 1$. Then it is easy to see that $SL_{2,1}(\mathbb{R})$ acts on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ transitively by

$$(\, g,\, \alpha\,) \cdot (\, Y,\, V\,) := (\, gY\, {}^tg,\, (\, V \,+\, \alpha\,)\, {}^tg\,), \tag{1.2}$$

where $g \in SL(2,\mathbb{R})$, $\alpha \in \mathbb{R}^{(1,2)}$, $Y \in S\mathcal{P}_2$ and $V \in \mathbb{R}^{(1,2)}$. Let

$$K := \{\, (\, k,\, 0\,) \in SL_{2,1}(\mathbb{R}) \mid \ k \in SO(2)\,\}$$

be the subgroup of $SL_{2,1}(\mathbb{R})$, where $SO(2)$ denotes the special orthogonal group of degree 2. Then it is easy to see that K is a maximal compact subgroup of $SL_{2,1}(\mathbb{R})$ stabilizing the origin $(E_2,\, 0)$. Thus $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ may be identified with the homogeneous space $SL_{2,1}(\mathbb{R})/K$ as follows:

$$SL_{2,1}(\mathbb{R})/K \ni (g,\alpha)K \longmapsto (g,\alpha)\cdot (E_2,0) \in S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}, \tag{1.3}$$

where $g \in SL(2,\mathbb{R})$ and $\alpha \in \mathbb{R}^{(1,2)}$.

The motivation that we study the action (1.2) of $SL_{2,1}(\mathbb{R})$ on the homogeneous space $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ is described in [Y7]. However for the convenience of the reader, we explain the motivation roughly.

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Let n and m be two fixed positive integers. Let

$$Sp(n,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \right\}$$

be the real symplectic group of degree n, where

$$J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Here we denote by ${}^tA$ the transpose of any matrix A. Let

$$H_n := \left\{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \ \ \operatorname{Im} Z > 0 \right\}$$

be the Siegel upper half plane of degree n. Then $Sp(n,\mathbb{R})$ acts on H_n transitively by

$$M < Z > := (AZ+B)(CZ+D)^{-1}, \tag{1.4}$$

wher $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z \in H_n$. We let

$$\mathcal{P}_n := \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be the open convex cone of positive definite matrices of degree n in the Euclidean space $\mathbb{R}^{n(n+1)/2}$. Then the action (1.4) gives a natural action of the general linear group $GL(n,\mathbb{R})$ on $\mathcal{P}_n$ defined by

$$(g,Y) \longmapsto g < Y > := gY\,{}^tg, \qquad g \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n. \tag{1.5}$$

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} := \left\{ (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric} \right\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa') := (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G_{n,m}^J(\mathbb{R}) := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \tag{1.6}$$

endowed with the following multiplication law

$$\begin{aligned} &(M,(\lambda,\mu;\kappa))\cdot(M',(\lambda',\mu';\kappa')) \\ &:= (MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',;\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')), \end{aligned}$$

where $M,\ M' \in Sp(n,\mathbb{R})$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. The group $G_{n,m}^J(\mathbb{R})$ is called the *Jacobi group* of degree (n,m). It is easy to see that the Jacobi group $G_{n,m}^J(\mathbb{R})$ acts on the homogeneous space $H_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(M,(\lambda,\mu;\kappa))\cdot(Z,W) := (M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{1.7}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_n \times \mathbb{C}^{(m,n)}$. The action (1.7) arises from the study of Jacobi forms (cf. [E-Z], [Y1]-[Y6]) and plays an important role in the study of the moduli space of principally polarized abelian varieties. We let

$$G_{n,m}(\mathbb{R}) := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)} \tag{1.8}$$

be the semidirect product of $GL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,a) * (h,b) := (gh, a\,{}^t h^{-1} + b), \tag{1.9}$$

where $g, h \in GL(n,\mathbb{R})$ and $a, b \in \mathbb{R}^{(m,n)}$. Then the action (1.7) of $G^J_{n,m}(\mathbb{R})$ on $H_n \times \mathbb{C}^{(m,n)}$ gives a *canonical* action of $G_{n,m}(\mathbb{R})$ on the *reductive* homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ given by

$$(g,a)\cdot(Y,V) := (gY\,{}^t g,\, (V+a)\,{}^t g), \tag{1.10}$$

where $g \in GL(n,\mathbb{R}),\ a \in \mathbb{R}^{(m,n)},\ Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$.

We let

$$SL_{n,m}(\mathbb{R}) := SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

be the semidirect product of the special linear group $SL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ with the multiplication law (1.9). Obviously $SL_{n,m}(\mathbb{R})$ is a subgroup of $G_{n,m}(\mathbb{R})$.

We let

$$S\mathcal{P}_n := \{\, Y \in \mathcal{P}_n \mid \det Y = 1 \,\}$$

be the symmetric space. Clearly $SL_{n,m}(\mathbb{R})$ acts on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ transitively by (1.10). Using this action, the author introduced a new notion of *Maass-Jacobi forms* ([Y7-8]) generalizing that of Maass wave forms (cf. [G], [I], [H] and [T]).

In this paper, we consider only the simplest case $n = 2$ and $m = 1$. The main purpose of this paper is to find all the differential operators on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ invariant under the action (1.2) explicitly. This paper is organized as follows. In section 2, we prove the main theorem, that is, we give a complete description of the algebra of differential operators on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ invariant under the action (1.2). And we provide a $SL_{2,1}(\mathbb{R})$-invariant Riemannian metric on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ and compute its Laplacian. In the final section, we calculate the derivatives for functions on $SL_{2,1}(\mathbb{R})$ and $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ explicitly.

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Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ denotes the set of all positive integers. The symbol ":=" means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix A, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n. $\mathbb{H}$ denotes the Poincaré upper-half plane.

2. Proof of Main Theorem

From now on, we write $G := SL_{2,1}(\mathbb{R})$ for brevity. We recall that $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$g<\tau>:=(a\tau+b)(c\tau+d)^{-1},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{R}),\quad \tau\in\mathbb{H}. \tag{2.1}$$

Now we define the action of G on $\mathbb{H}\times\mathbb{C}$ by

$$(g,\alpha)\circ(\tau,z) := \left({}^tg^{-1}<\tau>,\ (z+\alpha_1\tau+\alpha_2)(-b\tau+a)^{-1} \right), \tag{2.2}$$

where $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{R})$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}$, and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$. Since the action (2.2) is transitive and K is the stabilizer of this action at the origin $(i,0)$, $\mathbb{H}\times\mathbb{C}$ can be identified with the homogeneous space G/K as follows:

$$G/K\ni(g,\alpha)K\longmapsto(g,\alpha)\circ(i,0). \tag{2.3}$$

We observe that we can express an element Y of $S\mathcal{P}_2$ uniquely as

$$Y=\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}\left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right]=\begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1}+y \end{pmatrix} \tag{2.4}$$

with $x,y\in\mathbb{R}$ and $y>0$.

Proposition 2.1. We define the mapping $T: S\mathcal{P}_2\times\mathbb{R}^{(1,2)}\longrightarrow\mathbb{H}\times\mathbb{C}$ by

$$T(Y,V) := (x+iy,\ v_1(x+iy)+v_2), \tag{2.5}$$

where Y is of the form (2.4) and $V=(v_1,v_2)\in\mathbb{R}^{(1,2)}$. Then the mapping T is a bijection which is compatible with the above two actions (1.2) and (2.2).

Proof. It is easy to prove the above proposition and so we omit its proof. □

Now we give a complete description of the algebra $\mathbb{D}(S\mathcal{P}_2\times\mathbb{R}^{(1,2)})$ of all differential operators on $S\mathcal{P}_2\times\mathbb{R}^{(1,2)}$ invariant under the action (1.2) of G. First we note that the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g}=\left\{ (X,Z)\mid X\in\mathbb{R}^{(2,2)},\ \sigma(X)=0,\ Z\in\mathbb{R}^{(1,2)} \right\}$$

equipped with the following Lie bracket

$$[(X_1,Z_1),(X_2,Z_2)] := ([X_1,X_2]_0,\ Z_2\,{}^tX_1-Z_1\,{}^tX_2),$$

where $[X_1,X_2]_0 := X_1X_2-X_2X_1$ denotes the usual matrix bracket and (X_1,Z_1), $(X_2,Z_2)\in\mathfrak{g}$. And $\mathfrak{g}$ has the following decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}\qquad(\text{direct sum}),$$

where

$$\mathfrak{k} := \left\{ (X,0)\in\mathfrak{g}\mid X=\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},\quad x\in\mathbb{R} \right\}$$

and

$$\mathfrak{p} := \Big\{ (X,Z) \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(2,2)},\ \sigma(X) = 0,\ Z \in \mathbb{R}^{(1,2)} \Big\}.$$

We observe that $\mathfrak{k}$ is the Lie algebra of K and that we have the following realtions

$$[\mathfrak{k},\, \mathfrak{k}] \subset \mathfrak{k} \qquad \text{and} \qquad [\mathfrak{k},\, \mathfrak{p}] \subset \mathfrak{p}.$$

Thus the coset space $G/K \cong S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ is a *reductive* homogeneous space in the sense of [H2], p. 284. It is easy to see that the adjoint action Ad of K on $\mathfrak{p}$ is given by

$$\mathrm{Ad}\,(k)((X,Z)) \,=\, (\, kX\,{}^tk,\ Z\,{}^tk\,), \tag{2.6}$$

where $k \in K$ and $(X,Z) \in \mathfrak{p}$ with $X = {}^tX$, $\sigma(X) = 0$. The action (2.6) extends uniquely to the action ρ of K on the polynomial algebra $\mathrm{Pol}\,(\mathfrak{p})$

$$\rho\,:\,K \longrightarrow \mathrm{Aut}\,(\,\mathrm{Pol}(\mathfrak{p}\,)). \tag{2.7}$$

Let $\mathrm{Pol}\,(\mathfrak{p})^K$ be the subalgebra of $\mathrm{Pol}\,(\mathfrak{p})$ consisting of all invariants of the action ρ of K. Then according to [H2], Theorem 4.9, p. 287, there exists a canonical linear bijection $\lambda\,(\,P \longmapsto D_{\lambda(P)})$ of $\mathrm{Pol}\,(\mathfrak{p})^K$ onto $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$. Indeed, if $(\xi_k)\;(\,1 \leq k \leq 4\,)$ is any basis of $\mathfrak{p}$ and $P \in \mathrm{Pol}\,(\mathfrak{p})^K$, then

$$\big(D_{\lambda(P)}f\big)\,(\tilde g \circ (i,0)) \,=\, \left[P\left(\frac{\partial}{\partial t_k}\right) f((\tilde g * \exp\,(\sum_{k=1}^{4} t_k\xi_k)) \circ (E_2,0))\right]_{(t_k)=0}, \tag{2.8}$$

where $\tilde g \in G$ and $f \in C^\infty(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$.

We put

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad f_1 := (1,0) \ \text{ and } \ f_2 := (0,1).$$

Then $e_1,\, e_2,\, f_1,\, f_2$ form a basis of $\mathfrak{p}$. We write for coordinates (X,Z) by

$$X \,=\, \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \quad \text{and} \quad Z \,=\, (z_1, z_2)$$

with real variables x, y, z_1 and z_2. Without difficulty we can show that the following polynomials

$$\begin{aligned} P(X,Z) &:= \frac{1}{8}\,\sigma(X^2) \,=\, \frac{1}{4}\,(x^2+y^2),\\ \xi(X,Z) &:= Z\,{}^tZ \,=\, z_1^2 + z_2^2,\\ P_1(X,Z) &:= -\,\frac{1}{2}\,ZX\,{}^tZ \,=\, \frac{1}{2}\,(z_2^2 - z_1^2)x \,-\, z_1z_2y \end{aligned}$$

and

$$P_2(X,Z) := \frac{1}{2}\,(z_2^2 - z_1^2)\,y \,+\, z_1z_2\,x$$

are algebraically independent generators of $\mathrm{Pol}\,(\mathfrak{p})^K$.

Now we are ready to compute the G-invariant differential operators D, Ψ, D_1 and D_2 corresponding to the K-invariants P, ξ, P_1 and P_2 respectively under the canonical linear bijection (2.8). For real variables $t=(t_1,t_2)$ and $s=(s_1,s_2)$, we have

$$\begin{aligned}&\exp\left(\,t_1e_1+t_2e_2+s_1f_1+s_2f_2\,\right)\\&=\left(\begin{pmatrix} a_1(t,s) & a_3(t,s)\\ a_3(t,s) & a_2(t,s)\end{pmatrix},\ \left(\,b_1(t,s),\,b_2(t,s)\,\right)\right),\end{aligned}$$

where

$$\begin{aligned}
a_1(t,s):&=\,1\,+\,t_1\,+\,\frac{1}{2!}\,(\,t_1^2+t_2^2\,)\,+\,\frac{1}{3!}\,(\,t_1^3+t_1t_2^2\,)\,+\,\frac{1}{4!}\,(\,t_1^4\,+\,2\,t_1^2\,t_2^2\,+\,t_2^4\,)\,+\,\cdots\\
a_2(t,s):&=\,1\,-\,t_1\,+\,\frac{1}{2!}\,(\,t_1^2+t_2^2\,)\,-\,\frac{1}{3!}\,(\,t_1^3\,+\,t_1t_2^2\,)\,+\,\frac{1}{4!}\,(\,t_1^4\,+\,2\,t_1^2\,t_2^2\,+\,t_2^4\,)\,-\,\cdots,\\
a_3(t,s):&=\,t_2\,+\,\frac{1}{3!}\,(\,t_1^2\,t_2\,+\,t_2^3\,)\,+\,\frac{1}{5!}\,(\,t_1^4\,t_2\,+\,2\,t_1^2\,t_2^3\,+\,t_2^5\,)\,+\,\cdots,\\
b_1(t,s):&=\,s_1\,-\,\frac{1}{2!}\,(s_1t_1+s_2t_2)\,+\,\frac{1}{3!}\,s_1(t_1^2+t_2^2)\,-\,\frac{1}{4!}\,(s_1t_1+s_2t_2)(t_1^2+t_2^2)\,+\cdots,\\
b_2(t,s):&=\,s_2\,-\,\frac{1}{2!}\,(s_1t_2-s_2t_1)\,+\,\frac{1}{3!}\,s_2(t_1^2+t_2^2)\,-\,\frac{1}{4!}\,(s_1t_2-s_2t_1)(t_1^2+t_2^2)\,+\cdots.
\end{aligned}$$

For brevity, we write a_j, b_k for $a_j(t,s)$, $b_k(t,s)$ $(\,j=1,2,3,\ k=1,2\,)$ respectively. We now fix an element $(g,\alpha)\in G$ and write

$$g\,=\,\begin{pmatrix} g_1 & g_{12}\\ g_{21} & g_2\end{pmatrix}\in SL(2,\mathbb{R})\quad\text{and}\quad \alpha\,=\,(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}.$$

We put

$$(\,Y(t,s),\,V(t,s)\,):=\,(\,(g,\alpha)*\exp\,(\,t_1e_1\,+\,t_2e_2\,+\,s_1f_1\,+\,s_2f_2\,))\circ(E_2,0)$$

with

$$Y(t,s):=\begin{pmatrix} y(t,s)^{-1} & -\,x(t,s)\,y(t,s)^{-1}\\ -\,x(t,s)\,y(t,s)^{-1} & x(t,s)^2\,y(t,s)^{-1}\,+\,y(t,s)\end{pmatrix}$$

and $V(t,s):=\,(\,v_1(t,s),\,v_2(t,s)\,)$. By an easy calculation, we obtain

$$\begin{aligned}
x(t,s)\,&=\,-(\,\tilde a\tilde c+\tilde b\tilde d\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\
y(t,s)\,&=\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\
v_1(t,s)\,&=\,\tilde a\,\tilde\alpha_1+\tilde b\tilde\alpha_2,\\
v_2(t,s)\,&=\,\tilde c\,\tilde\alpha_1+\tilde d\,\tilde\alpha_2,
\end{aligned}$$

where

$$\begin{aligned}
\tilde a:&=\,g_1a_1\,+\,g_{12}a_3,\\
\tilde b:&=\,g_1a_3\,+\,g_{12}a_2,\\
\tilde c:&=\,g_{21}a_1\,+\,g_2a_3,\\
\tilde d:&=\,g_{21}a_3\,+\,g_2a_2,\\
\tilde\alpha_1\,:&=\,\alpha_1a_2\,-\,\alpha_2a_3\,+\,b_1,\\
\tilde\alpha_2\,:&=\,-\alpha_1a_3\,+\,\alpha_2a_1\,+\,b_2.
\end{aligned}$$

By an easy calculation, at $t = s = 0$, we have

$$\begin{aligned}
\frac{\partial x}{\partial t_1} &= 4\, g_1\, g_{12}\, (\, g_1^2 \,+\, g_{12}^2\,)^{-2},\\
\frac{\partial y}{\partial t_1} &= -2\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-2},\\
\frac{\partial v_1}{\partial t_1} &= \frac{\partial v_2}{\partial t_1} \,=\, 0,\\
\frac{\partial^2 x}{\partial t_1^2} &= -\, 16\, g_1\, g_{12}\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3},\\
\frac{\partial^2 y}{\partial t_1^2} &= 8\, (\, g_1^2 \,-\, g_{12}^2\,)^2\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3} \,-\, 4\, (\, g_1^2 \,+\, g_{12}^2\,)^{-1},\\
\frac{\partial^2 v_1}{\partial t_1^2} &= \frac{\partial^2 v_2}{\partial t_1^2} \,=\, 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial x}{\partial t_2} &= -\, 2\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-2},\\
\frac{\partial y}{\partial t_2} &= -\, 4\, g_1\, g_{12}\, (\, g_1^2 \,+\, g_{12}^2\,)^{-2},\\
\frac{\partial v_1}{\partial t_2} &= \frac{\partial v_2}{\partial t_2} \,=\, 0,\\
\frac{\partial^2 x}{\partial t_2^2} &= 16\, g_1\, g_{12}\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3},\\
\frac{\partial^2 y}{\partial t_2^2} &= 32\, g_1^2\, g_{12}^2\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3} \,-\, 4\, (\, g_1^2 \,+\, g_{12}^2\,)^{-1},\\
\frac{\partial^2 v_1}{\partial t_2^2} &= \frac{\partial^2 v_2}{\partial t_2^2} \,=\, 0.
\end{aligned}$$

We note that

$$\tilde a \tilde d \,-\, \tilde b \tilde c \,=\, 1, \quad a_1 a_2 - a_3^2 = 1 \quad \text{and} \quad g_1 g_2 - g_{12} g_{21} = 1.$$

We also note that since

$$(\, g,\, \alpha\,) \cdot (E_2,\, 0) := \left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right],\, (\, v_1,\, v_2\,) \right),$$

we have

$$y^{-1} \,=\, g_1^2 \,+\, g_{12}^2, \quad x^2\, y^{-1} \,+\, y \,=\, g_{21}^2 \,+\, g_2^2, \quad -\, xy^{-1} \,=\, g_1\, g_{21} \,+\, g_{12}\, g_2$$

and

$$v_1 \,=\, \alpha_1\, g_1 \,+\, \alpha_2\, g_{12}, \qquad v_2 \,=\, \alpha_2\, g_{21} \,+\, \alpha_2\, g_2.$$

Using the above facts and applying the chain rule, we can easily compute the differential operators D, Ψ, D_1 and D_2. It is known that the images of generators P, ξ, P_1 and P_2 under λ are generators of $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ (cf. [H1] or [Y7]).

Summarizing, we have the following.

Main Theorem. The algebra $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ is generated by the following differential operators

$$D := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \tag{2.9}$$

$$\Psi := y^{-1} \left\{ \frac{\partial^2}{\partial v_1^2} - 2\,x\,\frac{\partial^2}{\partial v_1 \partial v_2} + (\,x^2 + y^2\,)\,\frac{\partial^2}{\partial v_2^2} \right\}, \tag{2.10}$$

$$\begin{aligned} D_1 := & \left\{ \frac{\partial^2}{\partial v_1^2} - 2x\,\frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 - y^2)\,\frac{\partial^2}{\partial v_2^2} \right\} \frac{\partial}{\partial y} \\ & + 2y \left(\frac{\partial^2}{\partial v_1 \partial v_2} - x\,\frac{\partial^2}{\partial v_2^2} \right) \frac{\partial}{\partial x} - \Psi \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} D_2 := & \left\{ \frac{\partial^2}{\partial v_1^2} - 2x\,\frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 - y^2)\,\frac{\partial^2}{\partial v_2^2} \right\} \frac{\partial}{\partial x} \\ & - 2y \left(\frac{\partial^2}{\partial v_1 \partial v_2} - x\,\frac{\partial^2}{\partial v_2^2} \right) \frac{\partial}{\partial y}, \end{aligned} \tag{2.12}$$

where Y is of the form (2.4) and $V = (\,v_1,\, v_2\,)$ with real variables x, y, v_1, v_2 and $y > 0$. Moreover, we have

$$\begin{aligned} [D,\,\Psi] : &= D\,\Psi - \Psi\,D \\ &= -\,4\,y\,\frac{\partial^3}{\partial x \partial v_1 \partial v_2} + 4\,x\,y\,\frac{\partial^3}{\partial x \partial v_2^2} - 2\,\frac{\partial^3}{\partial y \partial v_1^2} + 4\,x\,\frac{\partial^3}{\partial y \partial v_1 \partial v_2} \\ &\quad + 2\,(\,y^2 - x^2\,)\,\frac{\partial^3}{\partial y \partial v_2^2} + \frac{2\,x}{y}\,\frac{\partial^2}{\partial v_1 \partial v_2} + 2\,\Psi. \end{aligned}$$

In particular, the algebra $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ is not commutative.

Now we present a G-invariant metric on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$.

Theorem 2.2. The following Riemannian metric ds^2 on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ defined by

$$ds^2 := \frac{1}{y^2}\,(\,dx^2 + dy^2\,) + \frac{1}{y}\,\left\{\,(\,x^2 + y^2\,)\,dv_1^2 + 2\,x\,dv_1\,dv_2 + dv_2^2\,\right\}$$

is invariant under the action (1.2) of G, where we used the coordinates $(\,Y, V\,)$ such that Y is of the form (2.4) and $V = (\,v_1,\, v_2\,)$ with $v_1,\, v_2$ real variables. The Laplace-Beltrami operator Δ of the Riemannian space $(\,S\mathcal{P}_2 \times \mathbb{R}^{(1,2)},\, ds^2\,)$ is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{y} \left\{ \frac{\partial^2}{\partial v_1^2} - 2\,x\,\frac{\partial^2}{\partial v_1 \partial v_2} + (\,x^2 + y^2\,)\,\frac{\partial^2}{\partial v_2^2} \right\}.$$

That is, $\Delta = D + \Psi$. In particular, G is a subgroup of the isometry group of $(\,S\mathcal{P}_2 \times \mathbb{R}^{(1,2)},\, ds^2\,)$.

Proof. We put

$$dY := \begin{pmatrix} -\,y^{-2}\,dy & -\,y^{-1}\,dx \,+\, x\,y^{-2}\,dy \\ -\,y^{-1}\,dx \,+\, x\,y^{-2}\,dy & 2\,x\,y^{-1}\,dx \,+\, (\,1 \,-\, x^2\,y^{-2}\,)\,dy \end{pmatrix}.$$

Then since

$$Y^{-1} \,=\, \begin{pmatrix} y \,+\, x^2\,y^{-1} & x\,y^{-1} \\ x\,y^{-1} & y^{-1} \end{pmatrix},$$

we can easily show that

$$ds^2 \,=\, \frac{1}{2}\,\sigma(Y^{-1}dYY^{-1}dY) \,+\, dV\,Y^{-1}\,{}^t(dV).$$

For an element $(\,g,\,\alpha\,) \in G$ with $g \in SL(2,\mathbb{R})$ and $\alpha \in \mathbb{R}^{(1,2)}$, we put

$$(\,Y^*,\,V^*\,) := (\,g,\,\alpha\,)\cdot(Y,V) \,=\, (\,gY\,{}^tg,\,(V+\alpha)\,{}^tg\,).$$

Since

$$Y^* \,=\, gY\,{}^tg \qquad \text{and} \qquad V^* \,=\, (\,V \,+\, \alpha\,)\,{}^tg,$$

we get

$$dY^* \,=\, g\,dY\,{}^tg \qquad \text{and} \qquad dV^* \,=\, (\,V \,+\, \alpha\,)\,{}^tg.$$

Therefore by a simple calculation, we can show that

$$\begin{aligned} &\sigma\left(\,Y^{*-1}\,dY^*\,Y^{*-1}\,dY^*\,\right) \,+\, dV^*\,Y^{*-1}\,{}^t(dV^*) \\ &= \sigma(\,Y^{-1}dY\,Y^{-1}\,dY\,) \,+\, dV\,Y^{-1}\,{}^t(dV). \end{aligned}$$

Hence the metric ds^2 is invariant under the action (1.2) of G.

Since the matrix form $(\,g_{ij}\,)$ of the metric ds^2 is given by

$$(\,g_{ij}\,) \,=\, \begin{pmatrix} y^{-2} & 0 & 0 & 0 \\ 0 & y^{-2} & 0 & 0 \\ 0 & 0 & y^{-1}\,(\,x^2 \,+\, y^2\,) & x\,y^{-1} \\ 0 & 0 & x\,y^{-1} & y^{-1} \end{pmatrix},$$

the inverse matrix $(\,g^{ij}\,)$ of $(\,g_{ij}\,)$ is given by

$$(\,g^{ij}\,) \,=\, \begin{pmatrix} y^2 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & y^{-1} & -\,x\,y^{-1} \\ 0 & 0 & -\,x\,y^{-1} & (x^2+y^2)\,y^{-1} \end{pmatrix}.$$

Hence by an easy calculation, we can show that the Laplace-Beltrami operator Δ is equal to $D \,+\, \Psi$. $\square$

3. The Lie derivatives on G

In this section, we compute the Lie derivatives for functions on G and $S\mathcal{P}_2\times\mathbb{R}^{(1,2)}$ explicitly.

We put

$$W_1 := \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (0,0) \right), \ \ W_2 := \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (0,0) \right), \ \ W_3 := \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0) \right)$$

and

$$W_4 := \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right), \qquad W_5 := \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1) \right).$$

Clearly $W_1,\cdots,W_5$ form a basis of $\mathfrak{g}$.

We define the differential operators $L_k,\ R_k\ (\,1\leq k\leq 5\,)$ on G by

$$L_k f(\tilde{g}) := \frac{d}{dt}\bigg|_{t=0} f(\tilde{g}*\exp tW_k) \tag{3.1}$$

and

$$R_k f(\tilde{g}) := \frac{d}{dt}\bigg|_{t=0} f(\exp tW_k*\tilde{g}), \tag{3.2}$$

where $f\in C^\infty(G)$ and $\tilde{g}\in G$.

We also put

$$H := \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (0,0) \right) \ \text{ and } \ V := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (0,0) \right).$$

We define the differential operators $L_H,\ L_V,\ R_H$ and R_V by

$$L_H f(\tilde{g}) :\, = \frac{d}{dt}\bigg|_{t=0} f(\tilde{g}*\exp tH),$$
$$L_V f(\tilde{g}) :\, = \frac{d}{dt}\bigg|_{t=0} f(\tilde{g}*\exp tV),$$
$$R_H f(\tilde{g}) :\, = \frac{d}{dt}\bigg|_{t=0} f(\exp tH*\tilde{g})$$

and

$$R_V f(\tilde{g}) := \frac{d}{dt}\bigg|_{t=0} f(\exp tV*\tilde{g}).$$

By an easy calculation, we get

$$\exp tW_1 \ = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, (0,\,0) \right), \qquad \exp tW_2 \ = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, (\,0,\,0\,) \right)$$
$$\exp tW_3 \ = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, (\,0,\,0\,) \right), \qquad \exp tW_4 \ = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (\,t,\,0\,) \right)$$

and

$$\exp tW_5 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0, t) \right).$$

Now we use the following coordinates (g, α) in G given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.3}$$

and

$$\alpha = (\alpha_1, \alpha_2), \tag{3.4}$$

where $x, \alpha_1, \alpha_2 \in \mathbb{R},\ y > 0$ and $0 \le \theta < 2\pi$. By an easy computation, we have

$$\begin{aligned}
L_1 &= y\cos 2\theta \frac{\partial}{\partial x} + y\sin 2\theta \frac{\partial}{\partial y} + \sin^2\theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1}, \\
L_2 &= y\cos 2\theta \frac{\partial}{\partial x} + y\sin 2\theta \frac{\partial}{\partial y} - \cos^2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
L_3 &= -2y\sin 2\theta \frac{\partial}{\partial x} + 2y\cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2}, \\
L_4 &= \frac{\partial}{\partial \alpha_1}, \\
L_5 &= \frac{\partial}{\partial \alpha_2}, \\
R_1 &= \frac{\partial}{\partial x}, \\
R_2 &= (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - y \frac{\partial}{\partial \theta}, \\
R_3 &= 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\
R_4 &= y^{-1/2}\cos\theta \frac{\partial}{\partial \alpha_1} + y^{-1/2}\sin\theta \frac{\partial}{\partial \alpha_2}, \\
R_5 &= -y^{-1/2}(x\cos\theta + y\sin\theta) \frac{\partial}{\partial \alpha_1} + y^{-1/2}(y\cos\theta - x\sin\theta) \frac{\partial}{\partial \alpha_2}.
\end{aligned}$$

Since

$$H = W_1 - W_2 \quad \text{and} \quad V = W_1 + W_2,$$

we have

$$\begin{aligned}
L_H &= L_1 - L_2 = \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1} + \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
L_V &= L_1 + L_2 \\
&= 2y\cos 2\theta \frac{\partial}{\partial x} + 2y\sin 2\theta \frac{\partial}{\partial y} - \cos 2\theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
R_H &= R_1 - R_2 \\
&= (1 - y^2 + x^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + y \frac{\partial}{\partial \theta}
\end{aligned}$$

and

$$\begin{aligned} R_V \,&=\, R_1 \,+\, R_2 \\ &=\, (\,1\,+\,y^2\,-\,x^2\,)\,\frac{\partial}{\partial x}\,-\,2\,xy\,\frac{\partial}{\partial y}\,-\,y\,\frac{\partial}{\partial \theta}. \end{aligned}$$

For instance, we present a calculation for L_3 and R_5. For a simplicity, we denote the form (3.3) by $[x,\,y,\,\theta]$. Let $(g,\,\alpha)$ be an element of $SL_{2,1}(\mathbb{R})$ with $g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [\,x,\,y,\,\theta\,]$ and $\alpha := (\alpha_1,\,\alpha_2) \in \mathbb{R}^{(1,2)}$. Then by an easy computation, we have

$$d\,-\,i\,c \,=\, y^{-1/2}\,e^{i\theta} \quad \text{and} \quad d\,+\,i\,c \,=\, y^{-1/2}\,e^{-i\theta}. \tag{3.5}$$

So we get

$$|\,d\,+\,i\,c\,| \,=\, y^{-1/2} \quad \text{and} \quad e^{i\theta} \,=\, \frac{d\,-\,i\,c}{|\,d\,+\,i\,c\,|}.$$

We put

$$(\,g(t),\,\alpha(t)\,) := [\,x_*,(t),\,y_*(t),\,\theta_*(t)\,] :=\, (g,\alpha) * \exp tW_3, \quad t \in \mathbb{R}$$

and

$$\tau_*(t) :=\, g(t) < i >=\, x_*(t)\,+\,i\,y_*(t), \quad t \in \mathbb{R}.$$

Then

$$g(t) \,=\, \begin{pmatrix} a\,e^t & b\,e^{-t} \\ c\,e^t & d\,e^{-t} \end{pmatrix} \quad \text{and} \quad \alpha(t) \,=\, (\,\alpha_1 e^{-t},\,\alpha_2\,e^t\,).$$

Thus

$$\tau_*(t) \,=\, \frac{i\,a\,e^t\,+\,b\,e^{-t}}{i\,c\,e^t\,+\,d\,e^{-t}} \,=\, \frac{i\,a\,e^{2t}\,+\,b}{i\,c\,e^{2t}\,+\,d} \,=\, x_*(t)\,+\,i\,y_*(t).$$

Differentiating $\tau_*(t)$ with respect to t and evaluating at $t = 0$, by (3.5), we have

$$\tau_*'(0) \,=\, 2\,i\,y\,e^{2i\theta}.$$

Thus

$$x_*'(0) \,=\, -\,2\,y\sin 2\theta \quad \text{and} \quad y_*'(0) \,=\, 2\,y\cos 2\theta.$$

And since

$$i\,\theta_*(t) \,=\, \log \frac{d\,e^{-t}\,-\,i\,c\,e^t}{(\,c^2\,e^{2t}\,+\,d^2\,e^{-2t}\,)^{1/2}},$$

we obtain

$$\theta_*'(0) \,=\, 2\sin\theta\cdot\cos\theta \,=\, \sin 2\theta.$$

We set

$$\alpha(t) :=\, (\,\alpha_{1,*}(t),\,\alpha_{2,*}(t)\,), \quad t \in \mathbb{R}.$$

Since

$$\alpha_{1,*}(t) \,=\, \alpha_1\,e^{-t} \quad \text{and} \quad \alpha_{2,*}(t) \,=\, \alpha_2\,e^t,$$

we get

$$\alpha_{1,*}'(0) \,=\, -\,\alpha_1 \quad \text{and} \quad \alpha_{2,*}'(0) \,=\, \alpha_2.$$

For a smooth function f on $SL_{2,1}(\mathbb{R})$, using the chain rule, we get the desired formula for L_3.

Since
$$\exp tW_5 \,=\, (\,E_2,\,(0,\,t)\,),$$
we have
$$\exp tW_5 * (g,\alpha) \,=\, (\,g,\,(0,t)\,{}^t g^{-1} \,+\, \alpha\,).$$
We put
$$(\,g(t),\,\alpha(t)\,) \,=\, \exp tW_5 * (g,\alpha)$$
and
$$g(t) \,=\, [\,x_*(t),\,y_*(t),\,\theta_*(t)\,], \quad \alpha(t) \,=\, (\,\alpha_{1,*}(t),\,\alpha_{2,*}(t)\,).$$
Then we have
$$g(t) \,=\, g, \quad \alpha_{1,*}(t) \,=\, -b\,t \,+\, \alpha_1 \;\text{ and }\; \alpha_{2,*}(t) \,=\, a\,t \,+\, \alpha_2.$$
From (3.3), we get
$$a \,=\, y^{1/2}\cos\theta \,-\, x\,y^{-1/2}\sin\theta \;\text{ and }\; b \,=\, y^{1/2}\sin\theta \,+\, x\,y^{-1/2}\cos\theta.$$
Finally we obtain the desired explicit formula for R_5. We leave the other formulas to the reader.

We define the differential operators $\mathbb{L}_k\ (\,1 \le k \le 5\,)$ on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ by
$$\mathbb{L}_k f(Y,V) := \frac{d}{dt}\bigg|_{t=0} f(\exp tW_k \cdot (Y,V)), \qquad 1 \le k \le 5,$$
where $f \in C^\infty(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ and $(Y,V) \in S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$.

Proposition 3.1. The differential operators $\mathbb{L}_k$'s are given by
$$\begin{aligned}
\mathbb{L}_1 &= (\,x^2 \,-\, y^2\,)\,\frac{\partial}{\partial x} \,+\, 2\,x\,y\,\frac{\partial}{\partial y} \,+\, v_2\,\frac{\partial}{\partial v_1}\\
\mathbb{L}_2 &= -\,\frac{\partial}{\partial x} \,+\, v_1\,\frac{\partial}{\partial v_2},\\
\mathbb{L}_3 &= -\,2\,x\,\frac{\partial}{\partial x} \,-\, 2\,y\,\frac{\partial}{\partial y} \,+\, v_1\,\frac{\partial}{\partial v_1} \,-\, v_2\,\frac{\partial}{\partial v_2},\\
\mathbb{L}_4 &= \frac{\partial}{\partial v_1},\\
\mathbb{L}_5 &= \frac{\partial}{\partial v_2}.
\end{aligned}$$

Proof. It is easy to prove it and so we omit its proof. ∎

References

[**E-Z**] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math. vol. 55, Birkhäuser, 1985.

[**G**] S. Gelbart, *Automorphic forms on adele groups*, Annals of Math. Studies, Princeton Univ. Press, No. 83, 1975.

[**H1**] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239-299.

[**H2**] ______, *Groups and geometric analysis*, Academic Press, 1984.

[**I**] H. Iwaniec, *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Mathemática Iberoamericana, Madrid, 1995.

[**M**] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichlescher Reihen durch Funktionalgleichung*, Math. Ann. **121** (1949), 141-183.

[**T**] A. Terras, *Harmonic analysis on symmetric spaces and applications I*, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[**Y1**] J.-H. Yang, *Some results on Jacobi forms of higher degree*, Proc. RIMS Symposium on Automorphic Forms and Associated Zeta Functions, Kyoto (1992), 35-50.

[**Y2**] ______, *The Siegel-Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135-146.

[**Y3**] ______, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.

[**Y4**] ______, *Singular Jacobi forms*, Trans. of American Math. Soc. **347, No. 6** (1995), 2041-2049.

[**Y5**] ______, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. **47 (6)** (1995), 1329-1339.

[**Y6**] ______, *A geometrical theory of Jacobi forms of higher degree*, Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147.

[**Y7**] J.-H. Yang, *Maass-Jacobi forms*, preprint, MPI (1997).

[**Y8**] ______, *Notes on Maass-Jacobi forms*, preprint, MPI (1997).

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