RIGIDITY FOR ORIENTABLE FUNCTORS

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Introduction

Seventeen years ago Andrei Suslin published the following remarkable result concerning the algebraic K-theory of algebraically closed fields. [12]

Theorem 0.1. (Suslin) *Let $F_0 \subset F$ be an extension of algebraically closed fields. Then for $(p, \operatorname{char} F) = 1$ the natural map*

$$K_n(F_0, \mathbb{Z}/p) \to K_n(F, \mathbb{Z}/p)$$

is an isomorphism.

Later, in papers of H.Gillet, R.Thomason [4] and O.Gabber [3] similar results were established for the case of strict Henselization at a smooth point of an algebraic variety and for Henselian pairs.

Both the proofs are based on the concept of "rigidity" for K-theory, which appeared in [12].

Further developments in this field were made in a paper of Suslin-Voevodsky [13] in 1996. There the K-functor was replaced by a "homotopy invariant functor with transfers". It was shown that for such a class of functors similar rigidity property also holds. Unfortunately, it is sometimes very non-trivial question whether a functor is a "functor with transfers".

The aim of the present work is to prove the rigidity property (and, therefore, an analog of the Suslin's theorem as well) for "orientable" functors. Roughly speaking, we call a multiplicative cohomology theory on schemes orientable if it has Chern classes which satisfy the standard list of properties (including the "projective bundle" theorem). We actually prove that an orientable functor can be endowed with transfers (in a sense different from the one in [13]). In order to construct the desired transfers, we replant the topological construction [1, 11] into the soil of algebraic geometry.

The orientability seems to be a very natural and easy-to-check property, which holds for a rather wide class of theories. Besides the K-functor, one can see that étale cohomology [5], motivic cohomology [16], and algebraic cobordism [17, 18] are orientable in our sense. The latter example seems to be the most interesting one.

We learned the concept of orientable spectrum in the algebraic context from lectures given by Morel during the workshop in Münster in June 1999. We are very grateful to Fabian Morel and Andrei Suslin for inspiring discussions during the work.

1. Functors with weak transfers

We are starting with some basic definitions. Let Sch/k be a category of schemes of finite type over a field k. We denote by Sm/k a full subcategory of Sch/k, whose objects are smooth algebraic varieties over k.

Definition 1.1. Consider the Cartesian square

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[d, "f"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

of objects and morphisms in the category Sm/k with regular closed embeddings $X \hookrightarrow Y$ and $X' \hookrightarrow Y'$. We call this square *transversal* if the natural map of the corresponding normal bundles

$$f^*(\mathcal{N}_{Y/X}) \to \mathcal{N}_{Y'/X'}$$

is an isomorphism.

Example 1.2. Flatness of the morphism f in the square above implies the transversality of the square. (See [2] B 7.4.)

Example 1.3. Let

$$\begin{tikzcd} \coprod_{y_i/x} y_i \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ x \arrow[r] & X \end{tikzcd}$$

be a Cartesian square, where $f\colon Y \to X$ is a morphism of smooth curves, étale over the point x. Then the square above is transversal.

Let $\mathcal{F}\colon (Sm/k)^\circ \to \mathcal{A}b$ be a contravariant functor on the category Sm/k. Suppose that for every finite projective morphism $f\colon X \to Y$ and its decomposition $X \overset{\tau}{\hookrightarrow} \mathbb{P}^n \times Y \overset{p_Y}{\to} Y$ there exists corresponding map $f_*^\tau\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ called *the transfer map* and the family of the transfer maps satisfies three properties below.

Remark 1.4. From now on we accept the following convention. Let $X \overset{\tau}{\hookrightarrow} Y \times \mathbb{P}^n \to Y$ be a projective morphism with fixed decomposition. Let f' be a base-change of f with respect to a morphism g. We shall consider only the decompositions of the map f' which make all squares in the diagram below Cartesian.

$$\begin{tikzcd} X' \arrow[r, "\tau'"] \arrow[d] \arrow[rr, bend left, "f'"] & Y' \times \mathbb{P}^n \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "\tau"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{P}^n \arrow[r] & Y \end{tikzcd}$$

This actually means we should choose the embedding $\tau'\colon X' \hookrightarrow Y \times \mathbb{P}^n$ canonically with respect to morphisms τ and g. (Namely, $\tau' = \mathrm{id} \times_Y \tau$.)

Under this convention we usually omit mentioning the embedding maps in the notation for transfers.

Property 1.5. (Base change) *For any projective morphism $f\colon X \to Y$ endowed with a decomposition $X \overset{\tau}{\hookrightarrow} \mathbb{P}^n \times Y \to Y$ and a morphism $g\colon Y' \to Y$ such that the square*

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, hook, "\tau'"] \arrow[d, "g'"'] & Y' \times \mathbb{P}^n \arrow[d, "g \times \mathrm{id}"] \\ X \arrow[r, hook, "\tau"] & Y \times \mathbb{P}^n \end{tikzcd}$$

is transversal, the diagram:

$$\begin{tikzcd} \mathcal{F}(X') \arrow[r, "f'_*"] & \mathcal{F}(Y') \\ \mathcal{F}(X) \arrow[u] \arrow[r, "f_*"] & \mathcal{F}(Y) \arrow[u, "g^*"'] \end{tikzcd} \tag{1.1}$$

commutes.

Property 1.6. (Finite additivity) *Let $X = X_0 \coprod X_1$, $j_m\colon X_m \hookrightarrow X$ $(m = 0, 1)$ be the embedding maps, and $f\colon X \to Y$ be a projective morphism. Setting $f_m = fj_m$, we have:*

$$f_{0,*}j_0^* + f_{1,*}j_1^* = f_*. \tag{1.2}$$

Property 1.7. (Normalization) *If $X = Y = \mathrm{Spec}(k)$ and f is the identity map, then for any decomposition $X \overset{\tau}{\hookrightarrow} \mathbb{P}^n \to Y$ the map $f_*\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ is the identity.*

Definition 1.8. A functor $\mathcal{F}\colon (Sm/k)^\circ \to \mathcal{A}b$ called a *functor with weak transfers* if it can be endowed with the family of transfer maps $\{f_*\}$, satisfying properties 1.5-1.7.

Definition 1.9. A functor $\mathcal{F}\colon (Sm/k)^\circ \to \mathcal{A}b$ called a *homotopy invariant functor* if for every $X \in Sm/k$ the map $p_X^*\colon \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$ is an isomorphism.

We prove here one property of homotopy invariant functors, which will be used in further sections.

Proposition 1.10. *Let $X \in Sm/k$ and $\mathcal{E} \overset{p}{\to} X$ be a vector bundle over X of dimension n. Then the map: $\mathcal{F}(X) \overset{p^*}{\to} \mathcal{F}(\mathcal{E})$ is an isomorphism.*

Proof. Let $s\colon X \to \mathcal{E}$ be the zero-section of the considered bundle. We have: $s^*p^* = (ps)^* = \mathrm{id}_X$, which shows p^* is a monomorphism.

Consider now two maps $i_0, i_1\colon \mathcal{E} \to \mathcal{E} \times \mathbb{A}^1$, given by points 0, 1 of $\mathbb{A}^1$. Let $pr\colon \mathcal{E} \times \mathbb{A}^1 \to \mathcal{E}$ be the projection map. We get $pr \circ i_m = \mathrm{id}_{\mathcal{E}}$, therefore, $i_0^* \circ pr^* = i_1^* \circ pr^*$. Since pr^* is an isomorphism, $i_0^* = i_1^*$.

Let $\{X_i = \mathrm{Spec}\, U_i\}$ be an affine covering of the scheme X such that the bundle $\mathcal{E}$ is trivial over each chart X_i. Let us define a map $\bar{\varphi}_i\colon U_i[x_1, \dots, x_n] \to U_i[x_1, \dots, x_n, t]$, setting $\bar{\varphi}_i(x_k) = tx_k$ $(k = 1, \dots, n)$. These maps can be, obviously, patched together, forming the morphism of schemes $\varphi\colon \mathcal{E} \times \mathbb{A}^1 \to \mathcal{E}$.

One can easily check that $\varphi i_1 = \mathrm{id}$ and $\varphi i_0 = sp$. Therefore, we have: $p^* s^* = (sp)^* = (\varphi i_0)^* = i_0^* \varphi^* = (\varphi i_1)^* = \mathrm{id}_{\mathcal{E}}$. Summarizing the observations above, we see that p^* is an isomorphism inverse to s^*. $\square$

From now on we consider the case of algebraically closed base field k. Let $\mathcal{F}\colon (Sm/k)^\circ \to \mathcal{A}b$ be a contravariant functor and X be a smooth affine curve over k. We can construct a map $\Phi\colon \mathrm{Div}(X) \to \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(k))$ defined on canonical generators by the formula: $[x] \mapsto x^*$, where $x^*\colon \mathcal{F}(X) \to \mathcal{F}(k)$ is the pull-back map, corresponding to the point $x \in X(k)$.

Below we will need a notion of the relative Picard group $\mathrm{Pic}(\bar{X}, X_\infty)$, where $\bar{X}$ is the smooth compactification of X and X_∞ is the complement of X in $\bar{X}$. Let us recall that this group is generated by divisors D of the curve X and two divisors D and D' are considered to be equivalent iff there exists a rational function f on X regular in a neighborhood of X_∞ and such that $f|_{X_\infty} = 1$, $\mathrm{div}_0(f) = D$, and $\mathrm{div}_\infty(f) = D'$.

Theorem 1.11. *Let $\mathcal{F}$ be a homotopy invariant functor with weak transfers. Then, in the above notation, the map Φ can be decomposed in the following way:*

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[rr, "\Phi"] \arrow[dr, "\Omega"'] & & \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(k)) \\ & \mathrm{Pic}(\bar{X}, X_\infty), \arrow[ur, "\Psi"'] & \end{tikzcd}$$

where $\mathrm{Pic}(\bar{X}, X_\infty)$ *is the relative Picard group of the smooth compactification $\bar{X}$ of X, $X_\infty = \bar{X} \backslash X$, and the map Ω is the canonical homomorphism.*

Proof. Since the field k is algebraically closed, the group $\mathrm{Pic}(\bar{X}, X_\infty)$ is generated by unramified divisors on the curve X (i.e. the divisors which have the form $\sum_i \pm[x_i]$, where x_i are pairwise different points on X). All the relations between unramified divisors come from ones of the following type: "$D \sim D'$ iff there exists a rational function on X regular in a neighborhood of X_∞ and such that $f|_{X_\infty} = 1$, $\mathrm{div}_0(f) = D$, and $\mathrm{div}_\infty(f) = D'$ ". Let us define $\Psi(D) = \Phi[\Omega]^{-1}(D)$. In order to check this map is well-defined, we need to prove that $\Phi(D) = \Phi(D')$ for equivalent divisors $D \sim D'$. One may assume that $\mathrm{Supp}(D) \cap \mathrm{Supp}(D') = \emptyset$. Let us take a function $f \in k(X)$ such that $f|_{X_\infty} = 1$, $\mathrm{div}_0(f) = D$, and $\mathrm{div}_\infty(f) = D'$.

Denote by X^0 the open locus of $\bar{X}$ where $f \neq 1$. By the choice of the function f, we have: $X^0 \subset X$ and the morphism $f\colon X^0 \to \mathbb{A}^1 = \mathbb{P}^1 - \{1\} = Y$ is finite. Moreover, since the corresponding divisors are unramified, the map f is étale over points 0 and ∞. Consider the diagram:

$$\begin{tikzcd} \bigoplus\limits_{x/\infty} \mathcal{F}(x) \arrow[dr, "\sum\limits_{x/\infty} f_{x,*}"'] & \mathcal{F}(D') \arrow[l] \arrow[d, "f_{\infty,*}"] & \mathcal{F}(X^0) \arrow[l] \arrow[r] \arrow[d, "f_*"] & \mathcal{F}(D) \arrow[r] \arrow[d, "f_{0,*}"] & \bigoplus\limits_{x/0} \mathcal{F}(x) \arrow[dl, "\sum\limits_{x/0} f_{x,*}"] \\ & \mathcal{F}(k) & \mathcal{F}(Y) \arrow[l, "i_\infty^*"'] \arrow[r, "i_0^*"] & \mathcal{F}(k) & \end{tikzcd} \tag{1.3}$$

In this diagram $f_{0,*}, f_{\infty,*}$ are transfer maps corresponding to the morphisms f_0, f_∞, which are restrictions of the morphism f to divisors D, D', respectively. Maps i_0^*, i_∞^* are induced by points 0 and ∞ of $\mathbb{P}^1$.

Finally, all summations in the very right (left) part of the diagram are over the set (finite) of all points in the preimage of 0 (∞).

Example 1.3 and Property 1.5 of functors with weak transfers show that two central squares of the diagram above commute. Property 1.6 implies that the triangles commute as well. Finally, from Property 1.7 we can see that for each point x over ∞ or over 0 one has a relation: $f_{x,*} = \mathrm{id}$.

Since $Y = \mathbb{A}^1$, both the maps i_∞^* and i_0^* are isomorphisms which are inverse to the isomorphism $p^*\colon \mathcal{F}(k) \to \mathcal{F}(\mathbb{A}^1)$ (the homotopy invariance property). So that, we have:

$$\sum_{x/\infty} x^* = \sum_{x/0} x^* : \mathcal{F}(X^0) \to \mathcal{F}(k). \tag{1.4}$$

This clearly implies:

$$\Phi(D') = \sum_{x/\infty} x^* = \sum_{x/0} x^* = \Phi(D)\colon \mathcal{F}(X) \to \mathcal{F}(k). \tag{1.5}$$

□

Corollary 1.12. *Let us assume, in addition to the hypothesis of Theorem 1.11, that there exists an integer n relatively prime to* $\mathrm{char}(k)$ *such that $n\mathcal{F}(Y) = 0$ for every $Y \in Sm/k$. Then the map Ψ can be passed through the degree map* $\mathrm{Pic}(\bar{X}, X_\infty) \overset{deg}{\to} \mathbb{Z}$*. Namely, if D, D' are two divisors of the same degree, then $\Psi(D) = \Psi(D')\colon \mathcal{F}(X) \to \mathcal{F}(k)$.*

Proof. We have an exact sequence:

$$0 \longrightarrow \mathrm{Pic}(\bar{X}, X_\infty)^\circ \longrightarrow \mathrm{Pic}(\bar{X}, X_\infty) \xrightarrow{deg} \mathbb{Z} \longrightarrow 0 \tag{1.6}$$

Since the group $\mathrm{Pic}(\bar{X}, X_\infty)^\circ$ is n-divisible over an algebraically closed field of characteristic relatively prime to n, we have: $\Psi|_{\mathrm{Pic}(\bar{X}, X_\infty)^\circ} = 0$. The corollary now follows. □

Theorem 1.13 (The Rigidity Theorem). *Let T be a smooth affine variety over the algebraically closed field k. Let $t_1, t_2 \in T(k)$ be two k-points. Then, under the hypothesis of the previous corollary, two maps $t_1^*, t_2^*\colon \mathcal{F}(T) \to \mathcal{F}(k)$ coincide.*

Proof. There exists an irreducible curve $C \subset T$ joining the points t_1 and t_2 (see [9] p.56). The statement follows from the application of Corollary 1.12 to the normalization of C. □

Extend the functor $\mathcal{F}$ to pro-objects in the category Sm/k setting

$$\mathcal{F}(\varprojlim(X_i)) = \varinjlim \mathcal{F}(X_i) \tag{1.7}$$

Theorem 1.14. *Let $\mathcal{F}\colon (Sm/k)^\circ \to \mathcal{A}b$ be a homotopy invariant functor with weak transfers. Let n be an integer relatively prime to* $\mathrm{char}(k)$ *such that for every $Y \in Sm/k$ $n\mathcal{F}(Y) = 0$. Let $k \subset K$ be an extension of algebraically closed fields. Then the natural map* $\mathrm{Spec}\, K \overset{\pi}{\to} \mathrm{Spec}\, k$ *induces an isomorphism* $\pi^*\colon \mathcal{F}(\mathrm{Spec}\, k) \overset{\simeq}{\to} \mathcal{F}(\mathrm{Spec}\, K)$.

Remark 1.15. $\mathrm{Spec}\, K$ is a pro-object in the category Sm/k. Moreover, schemes in the pro-system defining $\mathrm{Spec}\, K$ can be chosen affine [4, 12].

Proof. Let us take $\alpha \in \mathcal{F}(\mathrm{Spec}\, k)$ such that $\pi^*(\alpha) = 0$. Since $\mathcal{F}(\mathrm{Spec}\, K) = \varinjlim \mathcal{F}(U)$, we can choose a smooth affine variety $U \overset{\varphi}{\to} \mathrm{Spec}\, k$ such that $\varphi^*(\alpha) = 0$. Since the field k is algebraically closed, there exists a k-point $\mathrm{Spec}\, k \overset{x}{\to} U$. Since the composition $\mathcal{F}(\operatorname{Spec} k) \xrightarrow{\varphi^*} \mathcal{F}(U) \xrightarrow{x^*} \mathcal{F}(\operatorname{Spec} k)$ is the identity map, we obtain: $\alpha = 0$. This shows that π^* is a monomorphism.

Let us now prove the surjectivity of π^*. Suppose we are given $\beta \in \mathcal{F}(\operatorname{Spec} K)$. Choose an affine smooth variety U together with a morphism $\operatorname{Spec} K \xrightarrow{y} U$ such that $\beta = y^*(\beta')$, where $\beta' \in \mathcal{F}(U)$. As before, U has to have a k-point $\operatorname{Spec} k \xrightarrow{x} U$. Consider the commutative diagram:

(1.8)
$$\begin{tikzcd} U_K \arrow[r, "\bar\pi"] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} K \arrow[u, bend left, "\tilde y"] \arrow[u, bend right, "\tilde x"'] \arrow[ur, "y"] \arrow[r, "\pi"] & \operatorname{Spec} k, \arrow[u, bend right, "x"'] \end{tikzcd}$$

where $U_K = \operatorname{Spec} K \times_{\operatorname{Spec} k} U$. Two K-points $x \circ \pi$ and y of U determine two K-points $\tilde x, \tilde y$ of U_K. We have: $\beta - \pi^* \circ x^*(\beta') = (y^* - \pi^* \circ x^*)(\beta') = (\tilde y^* - \tilde x^*) \circ \bar\pi^*(\beta') = 0$, because $\tilde y^* - \tilde x^* = 0$ due to the Rigidity Theorem 1.13. Thus, $\beta \in \operatorname{Im} \pi^*$. □

2. Orientable functors

The purpose of the current section is to introduce a class of "Orientable Functors" for which a family of transfers will be constructed in further sections.

Let us consider a category Sm^2/k whose objects are pairs (X, U) such that $X \in Sm/k$ and U is open in X. Morphisms in this category are usual morphisms of pairs.

Orientable functors are among contravariant functors on this category. Typical examples are the functors which are represented by commutative ring T-spectra and orientable in the sense of Morel [6]. Functors given by these examples are double-indexed and the product satisfies the usual skew-commutativity property. In order to get a notion of an orientable functor, we axiomatize these and some other properties of mentioned functors. In fact, we never really need double-indexation but we decided to keep it to be consistent with the typical examples.

Let $E^{p,q} \colon (Sm^2/k)^\circ \to \mathcal{A}b$ $(p, q \in \mathbb{Z})$ be a functor (family of functors).

Remark 2.1. Further, we often write $E^{*,*}_Z(X)$ for $E^{*,*}(X, X \backslash Z)$ and $E^{*,*}(X)$ for $E^{*,*}(X, \emptyset)$.

Consider the following list of axioms.

Eilenberg – Steenrod Type Axioms

Axiom 2.2 (Vanishing property). *For every X $E^{*,*}_\emptyset(X) = 0$.*

Axiom 2.3 (Additivity). $E^{*,*}(X_1 \coprod X_2) \xrightarrow{\cong} E^{*,*}(X_1) \oplus E^{*,*}(X_2)$.

Axiom 2.4 (Excision). *Let $X \overset{i}{\supseteq} X_0 \supseteq Z$, where X_0 is open in X and Z is closed in X. Then the induced map $i^* \colon E^{*,*}_Z(X) \xrightarrow{\cong} E^{*,*}_Z(X_0)$ is an isomorphism.*

Axiom 2.5 (Homotopy Invariance). *A functor $E^{*,*}$ is homotopy invariant (see 1.9).*

Axiom 2.6 (Localization). *Let $f \colon U \to X$ be an open embedding morphism. Then we have the following long exact sequence:*

$$\cdots \longrightarrow E^{*,*}(X) \xrightarrow{f^*} E^{*,*}(U) \longrightarrow E^{*+1,*}(X, U) \longrightarrow \cdots$$

Product structure

Axiom 2.7 (Multiplicativity). *There exists a pairing*

$$E^{p,q}(X) \otimes E_Z^{p',q'}(X) \overset{\smile}{\to} E_Z^{p+p',q+q'}(X),$$

which is:
a) functorial;
b) for every X there exists two sides unit $1_X \in E^{0,0}(X)$ and it is functorial;
c) associative;
d) skew-commutative: $a \smile b = (-1)^{p \cdot p'} b \smile a$.

Chern classes

Axiom 2.8 (Chern classes). *For a linear bundle L over a smooth variety X we can choose a class $c_1(L) \in E^{2,1}(X)$ (called the first Chern class of the bundle L) such that the family $c_1(L)$ satisfies the following properties:*
a) If $L \simeq L'$ then $c_1(L) = c_1(L')$;
b) For a morphism $f\colon X \to Y$ we have: $f^*(c_1(L)) = c_1(f^*(L))$;
c) $c_1(\mathbf{1}) = 0$;
and Properties 2.9,2.11 below.

Axiom 2.9 (Projective Bundle Theorem). *Let $\mathcal{E}$ be a vector bundle of rank r over X.*

Denote by $\mathbb{P}(\mathcal{E})$ the projective bundle over X associated to $\mathcal{E}$. Its fiber over a point x in X is the projective space of lines in the fiber $\mathcal{E}_x$. Let $\mathcal{O}(-1)$ be the tautological line bundle over $\mathbb{P}(\mathcal{E})$.

The map: $\oplus_{i=0}^{r-1} E^{*-2i,*-i}(X) \overset{\Sigma(\smile \xi^i)}{\to} E^{*,*}(\mathbb{P}(\mathcal{E}))$ is an isomorphism. In this formula $\xi = c_1(\mathcal{O}(-1))$ is the first Chern class of the tautological line bundle and $\smile \xi^i$ means the multiplication by ξ^i;

The latter axiom implies the following relation:

$$\xi^r - c_1(\mathcal{E})\xi^{r-1} + \cdots + (-1)^{r-1} c_{r-1}(\mathcal{E})\xi + (-1)^r c_r(\mathcal{E}) = 0 \tag{2.1}$$

Definition 2.10. The coefficients $c_i(\mathcal{E}) \in E^{2i,i}(X)$ called the Chern classes of the vector bundle $\mathcal{E}$. (We follow here the construction proposed by Morel [6]).

Axiom 2.11. *If $\mathcal{E}$ is a trivial vector bundle then $c_i(\mathcal{E}) = 0$ for $i > 0$.*

Blow-up property

Axiom 2.12. *If $B(X,Z) = (\mathbb{A}^1 \times X)\widehat{\ }_{\{0\}\times Z}$ is the deformation to the normal cone, $\mathcal{N}$ is the normal bundle over Z corresponding to the embedding $Z \subset X$, and $i_0\colon \mathcal{N} \hookrightarrow B(X,Z)$ and $i_1\colon X \hookrightarrow B(X,Z)$ are canonical embeddings over 0 and 1, respectively, then the induced maps:*

$$E_Z^{*,*}(\mathcal{N}) \overset{i_0^*}{\leftarrow} E_{\mathbb{A}^1 \times Z}^{*,*}(B(X,Z)) \overset{i_1^*}{\to} E_Z^{*,*}(X) \tag{2.2}$$

are isomorphisms. (For more details about the meaning of this property see [8],section 3.2.3).

Definition 2.13. We call a functor (family of functors) $E^{p,q}\colon (Sm^2/k)^\circ \to \mathcal{A}b$, satisfying axioms 2.2 – 2.12 an oriented functor.

Consider a functor $\mathfrak{G}\colon Sm/k \to Sm^2/k$ given as follows: $\mathfrak{G}(X) = (X, \emptyset)$. This functor embeds Sm/k into Sm^2/k as a full subcategory. Given a functor $E^{*,*}\colon (Sm^2/k)^{\circ} \to \mathcal{A}b$ we can consider the composite: $E^{*,*}\mathfrak{G}^{\circ}\colon (Sm/k)^{\circ} \to \mathcal{A}b$.

Theorem 2.14. *Let $E^{*,*}\colon (Sm^2/k)^{\circ} \to \mathcal{A}b$ be an oriented functor. Then we can correspond to every projective morphism $f\colon X \to Y$ and its decomposition $X \overset{\tau}{\hookrightarrow} Y \times \mathbb{P}^n \overset{p^Y}{\to} Y$ the push-forward map $f_*^{\tau}\colon E^{*,*}(X) \to E^{*+2r,*+r}(Y)$ ($r = \dim Y - \dim X$) such that the functor $E^{*,*}\mathfrak{G}^{\circ}$ becomes a homotopy invariant functor with weak transfers.*

The rest of the paper is devoted to the proof of Theorem 2.14.

Example 2.15. The following functors supply us with examples of oriented functors:

a) Motivic cohomology.
Most of the orientability axioms for motivic cohomology are checked in [14].

b) Étale cohomology.

c) Algebraic K-theory.
We put $K^{p,q}(X, X - Z) = K_{2q-p}(X \operatorname{on} Z)$, where the groups on the right-hand side are the Thomason-Trobaugh K-groups (see [15]). These groups coincide with the Quillen's groups K' [10], but they are obviously functorial with respect to morphisms of pairs.

d) algebraic cobordism $MGL^{p,q}$.
Algebraic cobordism admits a multiplicative structure and a theory of Chern classes (see [7]).

3. Construction of the Thom class

From now on $E^{*,*}$ denotes an orientable contravariant functor on the category Sm^2/k.

Let X be a smooth variety over k and let $\mathcal{E}$ be a vector bundle over X of a constant rank r. Let $\mathbf{1}$ be the trivial vector bundle over X, $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}\oplus\mathbf{1})$ be projectivizations of bundles $\mathcal{E}$ and $\mathcal{E}\oplus\mathbf{1}$, respectively. Consider the following diagram:

(3.1)
$$\begin{tikzcd}
X \arrow[d, equal] & \mathcal{E} \arrow[d, equal] & & & \\
\mathbb{P}(\mathbf{1}) \arrow[r, hook, "s_{\mathbf{1}}"] \arrow[rr, hook, bend right, "\bar{s}_{\mathbf{1}}"'] & \mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathcal{E}) \arrow[r, hook, "j_{\mathbf{1}}"] & \mathbb{P}(\mathcal{E}\oplus\mathbf{1}) & \mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathbf{1}) \arrow[l, hook', "j_{\mathcal{E}}"'] & \mathbb{P}(\mathcal{E}) \arrow[l, hook', "s_{\mathcal{E}}"'] \arrow[ll, hook', bend right, "\bar{s}_{\mathcal{E}}"']
\end{tikzcd}$$

where $s_{\mathbf{1}}$, $s_{\mathcal{E}}$, $\bar{s}_{\mathbf{1}}$, and $\bar{s}_{\mathcal{E}}$ are corresponding canonical embeddings.

Lemma 3.1. *The following induced maps are isomorphisms.*
a) $s_{\mathcal{E}}^*\colon E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathcal{E}))$.
b) $j_1^*\colon E^{*,*}_{\mathbb{P}(1)}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \to E^{*,*}_{\mathbb{P}(1)}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathcal{E})) = E^{*,*}_X(\mathcal{E})$.

Proof. a) Let us mention that $\mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathbf{1})$ is the total space of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(\mathcal{E})$ and the map $s_{\mathcal{E}}$ is its zero-section. The claim now follows from this observation and Proposition 1.10.
b) It follows from the excision property. □

Proposition 3.2. *The sequence:*

$$0 \to E^{*,*}_X(\mathcal{E}) \to E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \xrightarrow{\bar{s}^*_{\mathcal{E}}} E^{*,*}(\mathbb{P}(\mathcal{E})) \to 0$$

is exact.

Proof. Let us look at the following fragment of the localization long exact sequence corresponding to the open embedding $\mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathbf{1}) \subset \mathbb{P}(\mathcal{E}\oplus\mathbf{1})$:

$$\cdots \to E^{*,*}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \xrightarrow{i} E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \to E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})\backslash\mathbb{P}(\mathbf{1})) \to \cdots \tag{3.2}$$

Applying the statements of Lemma 3.1 to this sequence, we get:

$$\cdots \to E^{*,*}_X(\mathcal{E}) \to E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \xrightarrow{\bar{s}^*} E^{*,*}(\mathbb{P}(\mathcal{E})) \to \cdots \tag{3.3}$$

To complete the proof, it is sufficient to show that the map

$$E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) \xrightarrow{\bar{s}^*} E^{*,*}(\mathbb{P}(\mathcal{E})) \tag{3.4}$$

is an epimorphism. The Projective Bundle Theorem tells us that both $E^{*,*}$- algebras $E^{*,*}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1}))$ and $E^{*,*}(\mathbb{P}(\mathcal{E}))$ are generated by the first Chern classes of the corresponding tautological bundles. The restriction of the tautological line bundle on $\mathbb{P}(\mathcal{E}\oplus\mathbf{1})$ to $\mathbb{P}(\mathcal{E})$ coincides with the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Thus, Property 2.8.b of the first Chern class completes the proof of surjectivity. The proposition follows. □

Let ζ and $\xi \in E^{2,1}(X)$ be the first Chern classes of the tautological line bundles over projectivizations $\mathbb{P}(\mathcal{E}\oplus\mathbf{1})$ and $\mathbb{P}(\mathcal{E})$, respectively. As before, the functoriality of the first Chern class implies that $\bar{s}^*_{\mathcal{E}}(\zeta) = \xi$. Consider now the Chern polynomial:

$$c_t(\mathcal{E}) = \sum_{k=0}^{r} (-1)^{r-k} c_{r-k}(\mathcal{E}) t^k. \tag{3.5}$$

We have: $c_\zeta(\mathcal{E}) \in E^{2r,r}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1}))$. It follows from the Projective Bundle Theorem, that $\bar{s}^*_{\mathcal{E}}(c_\zeta(\mathcal{E})) = c_{\bar{s}^*_{\mathcal{E}}\zeta}(\mathcal{E}) = c_\xi(\mathcal{E}) = 0$. So that, from 3.2 we have: $c_\zeta(\mathcal{E}) \in \operatorname{Im} i(E^{2r,r}_X(\mathcal{E}))$. Let us mention that the excision axiom implies: $E^{2r,r}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathcal{E}\oplus\mathbf{1})) = E^{2r,r}_X(\mathcal{E})$.

Definition 3.3. (The Thom class) We set $\mathrm{th}(\mathcal{E}) = j^*_{\mathbf{1}}(i^{-1}c_\zeta(\mathcal{E})) \in E^{2r,r}_X(\mathcal{E})$ and call $\mathrm{th}(\mathcal{E})$ the Thom class of the vector bundle $\mathcal{E}$.

Proposition 3.4. *Let $f\colon Y \to X$ be a morphism of smooth varieties. Let $\mathcal{E}$ be a vector bundle of dimension r over X. Then, we have:*

$$\mathrm{th}(f^*\mathcal{E}) = f^*(\mathrm{th}(\mathcal{E}))$$

Proof. Let ξ_X (resp. ξ_Y) be the first Chern class of the bundle $\mathcal{O}(-1)$ over $\mathbb{P}_X(\mathcal{E})$ (resp. $\mathbb{P}_Y(f^*\mathcal{E})$). Since $\xi_Y = f^*\xi_X$, we have a relation:

$$\xi_Y^r + f^*(-c_1(\mathcal{E}))\xi_Y^{r-1} + \cdots + f^*((-1)^r c_r(\mathcal{E})) = 0 \tag{3.6}$$

which holds in $E^{*,*}(\mathbb{P}_Y(f^*\mathcal{E}))$.

On the other hand, one has a similar relation for the class ξ_Y^r with coefficients $(-1)^i c_i(f^*(\mathcal{E}))$. Since $E^{*,*}(\mathbb{P}_Y(f^*\mathcal{E}))$ is a free $E^{*,*}(Y)$-module with a base $1, \xi_Y, \ldots, \xi_Y^{r-1}$, this implies the functoriality of the higher Chern classes: $f^*c_i(\mathcal{E}) = c_i(f^*\mathcal{E})$.

Therefore, $f^*(c_t(\mathcal{E})) = c_t(f^*\mathcal{E})$. Since the inverse image of the tautological bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E}\oplus\mathbf{1})}(-1)$ equals to $\mathcal{O}_{\mathbb{P}_Y(f^*\mathcal{E}\oplus\mathbf{1})}(-1)$, we have the desired result. □

4. Gysin map for a closed embedding

Let $\tau\colon X \hookrightarrow Y$ be a closed embedding of codimension d of smooth k-varieties and let $\mathcal{N} = \mathcal{N}_{Y/X}$ be the corresponding normal bundle. We need to recall a well-known construction of the deformation to the normal cone. Let $B(Y,X)$ denote the blow-up of $Y \times \mathbb{A}^1$ with center at $X \times \{0\}$. Considering the fibers of the map $B(Y,X) \to Y \times \mathbb{A}^1 \to \mathbb{A}^1$ over the points 0 and 1 one easily obtains two embeddings: $i_0\colon \mathbb{P}(\mathcal{N} \oplus \mathbf{1}) \hookrightarrow B(Y,X)$ and $i_1\colon Y \hookrightarrow B(Y,X)$. The subvariety $X \times \mathbb{A}^1$ contains the center $X \times \{0\}$ of the blow-up as a divisor.Therefore, it lifts canonically to a subvariety in $B(Y,X)$. Since $X \times \mathbb{A}^1$ crosses $\mathbb{P}(\mathcal{N} \oplus \mathbf{1})$ along $\mathbb{P}(\mathbf{1})$ and crosses $i_1(Y)$ along $i_1(X)$, one has following embeddings of the pairs:

$$(\mathcal{N}, \mathcal{N} - X) \overset{i_0}{\hookrightarrow} (B, B - X \times \mathbb{A}^1) \overset{i_1}{\hookleftarrow} (Y, Y - X) \tag{4.1}$$

From Property 2.12, we have:

$$E^{*,*}_X(\mathcal{N}) \underset{\simeq}{\overset{i_0^*}{\longleftarrow}} E^{*,*}_{X\times\mathbb{A}^1}(B) \underset{\simeq}{\overset{i_1^*}{\longrightarrow}} E^{*,*}_X(Y) \tag{4.2}$$

Construction-Definition 4.1. We define the Gysin map τ_* as the following chain of maps:

$$E^{*-2d,*-d}(X) \xrightarrow{\smile \mathrm{th}(\mathcal{N})} E^{*,*}_X(\mathcal{N}) \xrightarrow{i_1^* \circ (i_0^*)^{-1}} E^{*,*}_X(Y) \xrightarrow{j^*} E^{*,*}(Y),$$

where $j\colon (Y,\emptyset) \hookrightarrow (Y, Y - X)$ and we mean under $\smile \mathrm{th}(\mathcal{N})$ the composition of the canonical isomorphism $(s^*)^{-1}\colon E^{*,*}(X) \overset{\simeq}{\to} E^{*,*}(\mathcal{N})$ induced by the zero-section of the bundle $\mathcal{N}$ and the multiplication with the Thom class $\mathrm{th}(\mathcal{N})$.

Let us prove some properties of the constructed Gysin map.

Proposition 4.2. (Base change property) *For a transversal square:*

$$\begin{tikzcd} X' \arrow[r, hook, "\tau'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "\tau"] & Y \end{tikzcd}$$

the square

$$\begin{tikzcd} E^{*,*}(X') \arrow[r, "\tau'_*"] & E^{*,*}(Y') \\ E^{*,*}(X) \arrow[u, "g^*"] \arrow[r, "\tau_*"] & E^{*,*}(Y) \arrow[u, "f^*"] \end{tikzcd}$$

commutes.

Proof. Consider the diagram:

$$\begin{tikzcd} E^{*-2d,*-d}(X') \arrow[r, "\smile \mathrm{th}(\mathcal{N}')"] & E^{*,*}_{X'}(\mathcal{N}') & E^{*,*}_{\mathbb{A}^1\times X'}(B') \arrow[l, "i_0'^*"'] \arrow[r, "j'^* i_1'^*"] & E^{*,*}(Y') \\ E^{*-2d,*-d}(X) \arrow[u, "g^*"] \arrow[r, "\smile \mathrm{th}(\mathcal{N})"] & E^{*,*}_X(\mathcal{N}) \arrow[u, "N(g)^*"] & E^{*,*}_{\mathbb{A}^1\times X}(B) \arrow[u, "B(f)^*"] \arrow[l, "i_0^*"'] \arrow[r, "j^* i_1^*"] & E^{*,*}(Y). \arrow[u, "f^*"] \end{tikzcd} \tag{4.3}$$

All the squares but the very left one are obviously commutative. Therefore, to obtain the relation $\tau'_* g^* = f^* \tau_*$, we just need to check the relation $f^*(\mathrm{th}(\mathcal{N})) =$

$\mathrm{th}(f^*(\mathcal{N})) = \mathrm{th}(\mathcal{N}')$, which holds due to Proposition 3.4, since the initial square is transversal. $\square$

Proposition 4.3. (Additivity) *Let us consider the commutative diagram:*

$$\begin{tikzcd}
Z_0 \arrow[dr, "j_0"] \arrow[ddr, bend right, "\tau^0"'] & & Z_1 \arrow[dl, "j_1"'] \arrow[ddl, bend left, "\tau_1"] \\
& Z = Z_0 \coprod Z_1 \arrow[d, hook, "\tau"] & \\
& X &
\end{tikzcd}$$

with canonical maps j_0, j_1 and the map τ to be a closed embedding. Then:

$$\tau_* = \tau_{0,*} j_0^* + \tau_{1,*} j_1^*$$

Proof. Consider commutative diagrams of natural inclusions (for $m = 0, 1$):

$$\begin{tikzcd}
(Z_m, \emptyset) \arrow[r, hook, "j_m"] \arrow[rr, hook, bend right, "\psi_m"'] & (Z, \emptyset) \arrow[r, hook, "\varphi_m"] & (Z, Z_{1-m})
\end{tikzcd} \tag{4.4}$$

Applying the functor $E^{*,*}$ to both diagrams and adding the results, we obtain the following commutative diagram:

$$\begin{tikzcd}
E^{*,*}(Z) \arrow[r, "j_0^* \oplus j_1^*", "\simeq"'] & E^{*,*}(Z_0) \oplus E^{*,*}(Z_1) \\
E^{*,*}_{Z_0}(Z) \oplus E^{*,*}_{Z_1}(Z) \arrow[u, "\varphi_0^* + \varphi_1^*"] \arrow[ur, "\simeq", "(\psi_0^*, \psi_1^*)"'] &
\end{tikzcd} \tag{4.5}$$

In the diagram above the horizontal arrow is an isomorphism since the functor $E^{*,*}$ is additive 2.3. The excision property 2.4 makes the diagonal arrow an isomorphism as well. Since two arrows in the considered diagram are isomorphisms, the last one $\varphi_0^* + \varphi_1^*$ is also an isomorphism. One can easily see that the composition: $\varphi_0^*[\psi_0^*]^{-1} j_0^* + \varphi_1^*[\psi_1^*]^{-1} j_1^*$ is the identity map on $E^{*,*}(Z)$. Let us now rewrite what we want to prove: $\tau_{0,*} j_0^* + \tau_{1,*} j_1^* = \tau_* = \tau_* \circ \left[\varphi_0^*[\psi_0^*]^{-1} j_0^* + \varphi_1^*[\psi_1^*]^{-1} j_1^*\right]$. Therefore, in order to prove the proposition, it is sufficient to check that $\tau_{m,*} = \tau_* \varphi_m^* [\psi_m^*]^{-1}$ for $m = 0, 1$. To verify this relation for $m = 1$ (the case of $m = 0$ is exactly the same), consider the diagram:

$$\begin{tikzcd}
(\mathcal{N}, \mathcal{N} - Z) \arrow[r] \arrow[d] & (B, B - Z \times \mathbb{A}^1) \arrow[d] & (X, X - Z) \arrow[l] \arrow[d] \\
(\mathcal{N}, \mathcal{N} - Z_1) \arrow[r] & (B, B - Z_1 \times \mathbb{A}^1) \arrow[d, "\sigma"] & (X, X - Z_1) \arrow[l] \arrow[d, equal] \\
(\mathcal{N}_1, \mathcal{N}_1 - Z_1) \arrow[r] \arrow[u] & (B_1, B_1 - Z_1 \times \mathbb{A}^1) & (X, X - Z_1) \arrow[l]
\end{tikzcd} \tag{4.6}$$

The top (bottom) row of this diagram corresponds to the morphism $Z \overset{\tau}{\hookrightarrow} X$ ($Z_1 \overset{\tau_1}{\hookrightarrow} X$), respectively. We also set: $B = B(X, Z)$ ($B_1 = B(X, Z_1)$) The morphism σ is the blow-down morphism corresponding to blowing up the component Z_0.

Since $\sigma^{-1}(Z_1 \times \mathbb{A}^1) = Z_1 \times \mathbb{A}^1$, the morphism σ gives us a well-defined morphism of pairs: $(B, B - Z_1 \times \mathbb{A}^1) \xrightarrow{\sigma} (B_1, B_1 - Z_1 \times \mathbb{A}^1)$.

Applying the functor $E^{*,*}$ to Diagram 4.6 and combining the result with the definition of Gysin map, we obtain the following commutative diagram (here and below we sometimes omit the indexes for the shortness):

$$
\begin{tikzcd}
E(Z) \arrow[r, "\smile \mathrm{th}(\mathcal{N})"] \arrow[rrrr, bend left, "\tau_*"] & E_Z(\mathcal{N}) & E_{Z\times\mathbb{A}^1}(B) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & E_Z(X) \arrow[r] & E(X) \\
E_{Z_1}(Z) \arrow[u, "\varphi_1^*"] \arrow[r, "\smile \mathrm{th}(\mathcal{N})"] \arrow[d, "\psi_1^*"', "\simeq"] & E_{Z_1}(\mathcal{N}) \arrow[u] \arrow[d, "N(\psi_1)^*"', "\simeq"] & E_{Z_1\times\mathbb{A}^1}(B) \arrow[u] \arrow[l] \arrow[r] & E_{Z_1}(X) \arrow[u] \arrow[r] \arrow[d, equal] & E(X) \arrow[u, equal] \arrow[d, equal] \\
E(Z_1) \arrow[r, "\smile \mathrm{th}(\mathcal{N}_1)"] \arrow[rrrr, bend right, "\tau_{1,*}"'] & E_{Z_1}(\mathcal{N}_1) & E_{Z_1\times\mathbb{A}^1}(B_1) \arrow[u, "\sigma^*"'] \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & E_{Z_1}(X) \arrow[r] & E(X)
\end{tikzcd}
\tag{4.7}
$$

This diagram proves the desired relation and the proposition as well. ☐

5. Transfer map and the proof of Theorem 2.14

We are now starting to construct the family of transfer maps. Let $f\colon X \to Y$ be a projective morphism of codimension d, with a decomposition:

$$
\begin{tikzcd}
X \arrow[r, hook, "\tau"] \arrow[rr, bend right, "f"'] & Y \times \mathbb{P}^n \arrow[r, "p^Y"] & Y,
\end{tikzcd}
\tag{5.1}
$$

where τ is a closed embedding and p^Y denotes the standard projection.

Definition 5.1. Define a map $p^Y_*\colon E^{*,*}(Y \times \mathbb{P}^n) \to E^{*-2n,*-n}(Y)$ as follows. By the Projective Bundle Theorem 2.9 one has a direct sum decomposition:

$$E^{*,*}(Y \times \mathbb{P}^n) = \bigoplus_{i=0}^{n} E^{*-2i,*-i}(Y) \times \xi^i, \tag{5.2}$$

where $\xi = c_1(\mathcal{O}(-1))$. Set p^Y_* be a projection on the summand $E^{*-2n,*-n}(Y)$.

Proposition 5.2. *The following properties hold for the map p^Y_*:*
a) base change property for an arbitrary morphism;
b) $p^Y_*(\xi^n) = 1$.

Proof. Both properties are direct corollaries of the definition above. ☐

Definition 5.3. For each diagram of form 5.1 we can define the map $f^\tau_*\colon E^{*,*}(X) \to E^{*+2d,*+d}(Y)$, setting $f^i_* = p^Y_* \circ \tau_*$. ($d = \dim Y - \dim X$).

To complete the proof of Theorem 2.14, we need to check that Properties 1.5-1.7 hold for the constructed transfer maps. We do it in Propositions 5.4-5.6.

Proposition 5.4. *(Compare 1.5). Let*

$$\begin{tikzcd} X' \arrow[r, "f'"'] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a Cartesian square where the projective morphism $X \xrightarrow{f} Y$ is decomposed like in 5.1 and the square

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, hook, "\tau'"] \arrow[d, "g'"'] & Y' \times \mathbb{P}^n \arrow[d, "G"] \\ X \arrow[r, hook, "\tau"] & Y \times \mathbb{P}^n \end{tikzcd}$$

is transversal. Then the diagram:

$$\begin{tikzcd} E^{*,*}(X') \arrow[r, "f'_*"] & E^{*,*}(Y') \\ E^{*,*}(X) \arrow[u, "g'^*"] \arrow[r, "f_*"] & E^{*,*}(Y). \arrow[u, "g^*"'] \end{tikzcd}$$

commutes.

Proof. Consider the diagram:

$$\begin{tikzcd} & Y' \times \mathbb{P}^n \arrow[rr, "G"] \arrow[ddl, "p^{Y'}"'] & & Y \times \mathbb{P}^n \arrow[ddl, "p^Y"] \\ X' \arrow[ur, hook, "\tau'"] \arrow[rr, "g'"] \arrow[d, "f'"'] & & X \arrow[ur, hook, "\tau"] \arrow[d, "f"] & \\ Y' \arrow[rr, "g"] & & Y & \end{tikzcd} \tag{5.3}$$

where $G = g \times \mathrm{id}$. Since Square 5.4 is transversal, Proposition 4.2 implies that: $G^*\tau_* = \tau'_* g'^*$. On the other hand, from Proposition 5.2, we have: $g^* p^Y_* = p^{Y'}_* G^*$. Therefore, one gets: $g^* f_* = g^* p^Y_* \tau_* = p^{Y'}_* G^* \tau_* = p^{Y'}_* \tau'_* g'^* = (f')_* g'^*$. The base change property is proven. $\square$

Proposition 5.5. *(Compare 1.6) Let $X = X_0 \coprod X_1$, $j_m \colon X_m \hookrightarrow X$ $(m = 0, 1)$ be embedding maps, and $f \colon X \to Y$. Setting $f j_m = f_m$ one has:*

$$f_{0,*} j_0^* + f_{1,*} j_1^* = f_* \tag{5.4}$$

Proof. The additivity property 1.6 gives us the relation: $\tau_* = \tau_{0,*} j_0^* + \tau_{1,*} j_1^*$. So, we have the equality: $f_* = p^Y_* \tau_* = p^Y_*(\tau_{0,*} j_0^* + \tau_{1,*} j_1^*) = f_{0,*} j_0^* + f_{1,*} j_1^*$, which proves the additivity property. $\square$

The last step is to check the normalization property.

Proposition 5.6. *(Compare 1.7) Let $X = Y = Spec(k)$ and $f \colon X \to Y$ be the identity morphism. Then for every decomposition $\mathrm{pt} \overset{\tau}{\hookrightarrow} \mathbb{P}^n \xrightarrow{p} \mathrm{pt}$ the map $f^\tau_* \colon E^{*,*}(\operatorname{Spec} k) \to E^{*,*}(\operatorname{Spec} k)$ is the identity.*

Proof. Without loosing of generality we may assume $\tau(Spec(k)) = [0\colon 0\colon \cdots : 1]$. Granting for a moment the equality $\tau_*(1) = \xi^n$, we easily see that the desired property follows from a chain of equalities: $f_*(1) = p_*\tau_*(1) = p_*(\xi^n) = 1$. The last one follows from Proposition 5.2. Since both maps τ_* and p_* are $E^{*,*}(\operatorname{Spec} k)$-linear, the normalization property follows. ☐

Lemma 5.7. $\tau_*(1) = \xi^n$, *where* $\xi = c_1(\mathcal{O}(-1)) \in E^{2,1}(\mathbb{P}^n)$.

Proof. Let $\mathcal{N}$ denote now the normal bundle corresponding to the embedding $\mathrm{pt} \overset{\tau}{\hookrightarrow} \mathbb{P}^n$. We have: $\mathcal{N} = \mathbf{1}^n$ and $\mathbb{P}^n = \mathbb{P}(\mathcal{N} \oplus \mathbf{1})$. Consider the transversal square:

(5.5)
$$\begin{tikzcd} \mathrm{pt} \arrow[r, hook] \arrow[d, equal] & \mathbb{A}^n \arrow[d, hook, "j"] \\ \mathrm{pt} \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

where j is the open embedding of $\mathbb{A}^n$ into $\mathbb{P}^n$ given by the formula: $j(x_1, \ldots, x_n) = (x_1\colon \cdots : x_n\colon 1)$. The corresponding base-change diagram for the Gysin map looks now as follows:

(5.6)
$$\begin{tikzcd} E(\operatorname{Spec} k) \arrow[r, "\smile \mathrm{th}(\mathcal{N})"] \arrow[d, equal] & E_{\{0\}}(\mathcal{N}) \arrow[d, equal] & E_{\{0\}\times\mathbb{A}^1}(B(\mathbb{P}^n, \{0\})) \arrow[l, "\bar{i}_0^*"'] \arrow[r, "\bar{i}_1^*"] \arrow[d] & E_{\{0\}}(\mathbb{P}^n) \arrow[r] \arrow[d, "\simeq"', "j^*"] & E(\mathbb{P}^n) \\ E(\operatorname{Spec} k) \arrow[r, "\smile \mathrm{th}(\mathcal{N})"] & E_{\{0\}}(\mathcal{N}) & E_{\{0\}\times\mathbb{A}^1}(B(\mathbb{A}^n, \{0\})) \arrow[l, "i_0^*"'] \arrow[r, "i_1^*"] & E_{\{0\}}(\mathbb{A}^n). & \end{tikzcd}$$

The map j^* in the diagram above is an isomorphism due to the excision property 2.4. Since the Chern classes of a trivial vector bundle vanish in positive dimensions (see 2.11), we have an equality: $c_t(\mathcal{N}) = t^n$.

From the definition of the Thom class we can see that $j^*(c_\zeta(\mathcal{N})) = \mathrm{th}(\mathcal{N})$. Therefore, in order to verify the desired relation we just need to check that $i_1^*[i_0^*]^{-1} = \mathrm{id}$. ☐

Lemma 5.8. *Let us consider the diagram:*

$$E^{*,*}_{\{0\}}(\mathbb{A}^n) \xleftarrow{\;i_0^*\;} E^{*,*}_{\{0\}\times\mathbb{A}^1}(B(\mathbb{A}^n, \{0\})) \xrightarrow{\;i_1^*\;} E^{*,*}_{\{0\}}(\mathbb{A}^n),$$

where maps i_0^*, i_1^* *are isomorphisms (see 2.12) induced by embeddings over the points* $\{0\}$ *and* $\{1\}$*, described above. Then* $i_1^*(i_0^*)^{-1} = \mathrm{id}$.

Proof. Let us mention, first, that $B(\mathbb{A}^n, \{0\}) = B = (\mathbb{A}^n \times \mathbb{A}^1)\widehat{\ }_{\{0\}}$ is the total space of the tautological line bundle over $\mathbb{P}^n$. Consider the following Cartesian square:

(5.7)
$$\begin{tikzcd} L \arrow[r, "J"] \arrow[d, "q"'] & B \arrow[d, "\bar{q}"] \\ \mathbb{A}^n \arrow[r, hook, "j"] & \mathbb{P}^n \end{tikzcd}$$

In this square $L = B \times_{\mathbb{P}^n} \mathbb{A}^n$ is the restriction of the vector bundle B to an open subset of the base. Evidently, the map $i_0\colon \mathbb{A}^n \to B$ can be decomposed as Jj_0, where j_0 is the zero-section of the bundle L over $\mathbb{A}^n$.

In the same way the embedding i_1 comes from the section $j_1 : \mathbb{A}^n \to L$ given by $x \mapsto (x, 1) \times x \in L$.

The projection map $q\colon L \to \mathbb{A}^n$ clearly induces a morphism of pairs $(L, L - \{0\} \times \mathbb{A}^1) \to (\mathbb{A}^n, \mathbb{A}^n - \{0\})$, which we also denote by q. Since L is a vector bundle over $\mathbb{A}^n$, the map q^* is an isomorphism (inverse to j_0^*). Thus, we have: $j_1^*[j_0^*]^{-1} = j_1^* q^* = (qj_1)^* = \mathrm{id}$. Since $i_k^* = j_k^* J^*$ $(k = 0, 1)$, the relation $i_1^*[i_0^*]^{-1} = \mathrm{id}$ follows. $\square$

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