Poincaré automorphisms for nondegenerate CR quadrics

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MPI / 93-55

POINCARÉ AUTOMORPHISMS FOR NONDEGENERATE CR QUADRICS

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ABSTRACT. In this paper we suggest a formula for holomorphic automorphisms of an arbitrary nondegenerate quadric CR manifold which comprises all of the formerly described automorphism groups for quadrics of codimension 2 and of RAQ quadrics. This formula is a generalization of the formula of H.Poincaré for Aut S^3.

1. INTRODUCTION

In 1907, Poincaré [9] proved that any germ of a holomorphic isotropic automorphism of the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = \bar{z}z\}$ is a fractional linear transformation of the form:

$$
\begin{aligned}
z^* &= \frac{c(z+aw)}{1-2i\bar{a}z-(r+i\bar{a}a)w}, \\
w^* &= \frac{\rho w}{1-2i\bar{a}z-(r+i\bar{a}a)w},
\end{aligned}
\tag{1}
$$

where $a, c \in \mathbb{C}$, $r \in \mathbb{R}$, and $\rho = |c|^2$.

In 1962, Tanaka [10] proved the analogous result for arbitrary nondegenerate hyperquadrics in $\mathbb{C}^{n+1}$: $\{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w = \langle z,z\rangle\}$, where $\langle\cdot,\cdot\rangle$ is a nondegenerate Hermitian form in $\mathbb{C}^n$.

Nondegenerate hyperquadrics serve as quadratic models of hypersurfaces in $\mathbb{C}^{n+1}$ with nondegenerate Levi form.

Nondegenerate quadrics in $\mathbb{C}^{n+k}$ are the quadratic models of surfaces with nondegenerate (in sense of Baouendi - Trèves - Beloshapka) vector-valued Levi form:

$$
Q = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^k : \operatorname{Im} w = \langle z,z\rangle\}, \tag{2}
$$

where $\langle z,z\rangle$ is a $\mathbb{R}^k$-valued Hermitian form in $\mathbb{C}^n$ with the properties:

1991 *Mathematics Subject Classification.* 14J50, 32F25.
Research of the first author was supported by Max-Planck-Institut Bonn.
Research of the second author was supported by Deutsche Forschungsgemeinschaft.

i) $\langle z, b\rangle = 0$, for all $z \in \mathbb{C}^n$, implies $b = 0$,
ii) If $f(\langle z, z\rangle) \equiv 0$, for some linear functional $f \in (\mathbb{R}^k)'$, then $f = 0$.

Beloshapka proved that these properties are necessary and sufficient for having a finite dimensional automorphism group [1].

Any quadric Q (not necassarily nondegenerate) can be equipped with a canonical group structure. If $(z, w) \in Q$, and $(p, q) \in Q$, then $(z+p, w+q+2i\langle z, p\rangle) \in Q$. The group Q will be called Heisenberg-group. Since this group operation is holomorphic with respect to the first argument, we obtain a transitive family of holomorphic automorphisms being parametrized by Q itself. Thus, Q is a homogeneous manifold. Therefore, it is sufficient to find the automorphisms which preserve a fixed point, say the origin. We denote the connected component of the unit of the group of local automorphisms of Q at 0 by $\mathrm{Aut}_0 Q$.

Any automorphism $\Phi \in \mathrm{Aut}_0 Q$ can be uniquely decomposed into a linear automorphism $\Phi_{C,\rho} \in \mathrm{Aut}_{lin} Q : z \mapsto Cz,\ w \mapsto \rho w$ (where $C \in \mathrm{GL}(n, \mathbb{C})$, $\rho \in \mathrm{GL}(k, \mathbb{R})$ with $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$, for all z) and an automorphism $\Phi_{\mathrm{id}} \in \mathrm{Aut}_{0,\mathrm{id}} Q$ with the property that the restriction of $d\Phi_{id}$ to the complex tangent space at 0 is the identical map.

Using the reflection principle, Henkin, and Tumanov [8] proved that the local automorphisms from $\mathrm{Aut}_{0,id} Q$ admit a birational extension to $\mathbb{C}^{n+k}$.

Beloshapka [2] obtained a description of the Lie algebra of the infinitesimal automorphisms of Q, and he proved also that the quadrics of codimension $k > 2$ in general position are rigid, i.e., their isotropy groups consist of trivial automorphisms $z \mapsto cz$, $w \mapsto |c|^2 w$, for some complex number c (see [3]).

Recently, Forstnerič [7] formulated the problem about the description of $Aut_0 Q$ once again.

The authors described the automorphisms in the case $k = 2$ (see [5]), and defined, in the case $n = k$, a class of quadrics with large automorphism groups being called real associative quadrics (RAQ), and wrote the explicit formula for their automorphisms [6].

Generalizing these results, we prove in the present paper the following

Theorem 1. *Let Q be a nondegenerate quadric in $\mathbb{C}^{n+k}$ and $a : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ be a linear operator, A be a $\mathbb{C}^n$-valued bilinear form on $\mathbb{C}^n \otimes \mathbb{C}^n$, r be an $\mathbb{R}^k$-valued Hermitian form on $\mathbb{C}^k$, and B be a $\mathbb{C}^n$-valued bilinear form on $\mathbb{C}^k \otimes \mathbb{C}^n$ which are connected by the relations*

$$\langle A(z, \zeta), \xi\rangle = \langle z, a\langle \xi, \zeta\rangle\rangle, \tag{3}$$
$$\langle B(w, \zeta), \xi\rangle = r(w, \langle \xi, \zeta\rangle), \tag{4}$$

for all $z, \zeta, \xi \in \mathbb{C}^n$ and $w \in \mathbb{C}^k$, then the map

$$(5)\qquad \begin{aligned} z^* &= (\mathrm{id} - 2iA(z,\cdot) - B(w,\cdot) - iA(aw,\cdot))^{-1}(z+aw), \\ w^* &= (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - r(w,\bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle)^{-1}w, \end{aligned}$$

is an automorphism from $\mathrm{Aut}_{0,id}\, Q$.

We call the automorphisms which can be written by formula (5) Poincaré-automorphisms.

We emphasize that we do not know any example of non-Poincaré automorphisms.

2. Algebras corresponding to quadrics

Let Q be a quadric in $\mathbb{C}^{n+k}$ as above (not necessarily nondegenerate).

Consider the set $\mathfrak{A}$ of pairs of matrices $(D,d) \in \mathfrak{gl}(n,\mathbb{C}) \times \mathfrak{gl}(k,\mathbb{C})$ with the property $\langle Dz,\zeta\rangle = d\langle z,\zeta\rangle$, for all $z,\zeta \in \mathbb{C}^n$.

Proposition 1. *The set $\mathfrak{A}$ is an algebra with a unit.*

Proof. It is clear that $\mathfrak{A}$ is a linear space containing (id, id). Let (D_1,d_1), $(D_2,d_2) \in \mathfrak{A}$ then, obviously, $\langle D_1D_2z,\zeta\rangle = d_1d_2\langle z,\zeta\rangle$. □

Proposition 2. *If Q is nondegenerate, then a pair (D,d) is uniquely determined by d as well as by D.*

Proof. Let (D_1,d), $(D_2,d) \in \mathfrak{A}$, then $\langle (D_1-D_2)z,\zeta\rangle = 0$, for all z,ζ. By (i) of the nondegeneracy condition follows that $D_1 - D_2 = 0$.

Since, by (ii) of the nondegeracy condition $\mathbb{R}^k$ is spanned by vectors of the form $\langle z,z\rangle$, D determines d. □

Therefore, we can interprete $\mathfrak{A}$ as a subalgebra of $\mathfrak{gl}(k,\mathbb{C})$, or of $\mathfrak{gl}(n,\mathbb{C})$.

Proposition 3. *For any $d_1,d_2 \in \mathfrak{A}$, we have $d_1\bar{d}_2 = \bar{d}_2d_1$.*

Proof. It follows from $\langle Dz,\zeta\rangle = d\langle z,\zeta\rangle$, for all $z,\zeta \in \mathbb{C}^n$, that $\langle z,D\zeta\rangle = \bar{d}\langle z,\zeta\rangle$, for all $z,\zeta \in \mathbb{C}^n$. Then, $d_1\bar{d}_2\langle z,\zeta\rangle = \langle D_1z,D_2\rangle = \bar{d}_2d_1\langle z,\zeta\rangle$. □

Remark. In general, $d \in \mathfrak{A}$ does not imply that $\bar{d} \in \mathfrak{A}$.

Definition 1. *Two quadrics Q_1 and Q_2 are equivalent, if there exist matrices $C \in \mathrm{GL}(n,\mathbb{C})$ and $\rho \in \mathrm{GL}(k,\mathbb{R})$ such that $\langle z,z\rangle_2 = \rho^{-1}\langle Cz,Cz\rangle_1$.*

Proposition 4. *If two quadrics Q_1 and Q_2 are equivalent, then the corresponding algebras $\mathfrak{A}$ are isomorphic.*

Proof. If $\langle z,\zeta\rangle_2 = \rho^{-1}\langle Cz,C\zeta\rangle_1$ and $\langle Dz,\zeta\rangle_2 = d\langle z,\zeta\rangle_2$, then $\rho^{-1}\langle CDz,C\zeta\rangle_1 = \langle Dz,\zeta\rangle_2 = d\langle z,\zeta\rangle_2 = d\rho^{-1}\langle Cz,C\zeta\rangle_1$.

Hence, $(CDC^{-1},\rho d\rho^{-1}) \in \mathfrak{A}_1$. □

Proposition 5. *A quadric Q of type (n,k) is the direct product of two quadrics $Q_1 \times Q_2$ of type (n_1,k_1) resp. $(n-n_1,k-k_1)$ if and only if the corresponding algebra $\mathfrak{A}$ splits into $\mathfrak{A}_1 \oplus \mathfrak{A}_2$.*

Proof. It is clear that $Q = Q_1 \times Q_2$ implies $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$.

Now, let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, and let (E_1, e_1), and (E_2, e_2) be the units in $\mathfrak{A}_1$ resp. $\mathfrak{A}_2$. Then, $(E_1 \oplus E_2, e_1 \oplus e_2)$ is the unit in $\mathfrak{A}$, and $e_i = \bar{e}_i$, $e_1e_2 = e_2e_1 = 0$. Set $z = z' + z''$, $w = w' + w''$, where $z' = E_1 z$, $z'' = E_2 z$, $w' = e_1 w$, $w'' = e_2 w$. Now, the equation of Q can be written

$$\begin{aligned} v' &= \langle z', z' \rangle, \\ v'' &= \langle z'', z'' \rangle. \end{aligned}$$

Thus, $Q = Q_1 \times Q_2$. □

It is easy to observe that $Q_1 \times Q_2$ is nondegenerate, if and only if Q_1 and Q_2 are nondegenerate. Beloshapka [3] proved that $\mathrm{Aut}_0(Q_1 \times Q_2) = \mathrm{Aut}_0 Q_1 \times \mathrm{Aut}_0 Q_2$.

Let Q be nondegenerate, and $\mathfrak{g}$ be the Lie algebra corresponding to the Lie group $\mathrm{Aut}_{lin} Q$. Then $\mathfrak{g}$ can be identified with some real subalgebra of $\mathfrak{gl}(\mathbb{C}, n)$, since, for any $(X, s) \in \mathfrak{g}$, s is uniquely determined by X.

Proposition 6. *For nondegenerate quadrics, $\mathfrak{A} = \mathfrak{g} \cap i\mathfrak{g}$.*

Proof. If (X, s), and $(iX, s') \in \mathfrak{g}$, then

$$\begin{aligned} \langle X\zeta, z \rangle + \langle \zeta, Xz \rangle &= s\langle \zeta, z \rangle, \\ \langle iX\zeta, z \rangle + \langle \zeta, iXz \rangle &= s'\langle \zeta, z \rangle. \end{aligned}$$

Hence, $\langle X\zeta, z \rangle = \frac{1}{2}(s - is')\langle \zeta, z \rangle$.
If $D \in \mathfrak{A}$, then

$$\begin{aligned} \langle D\zeta, z \rangle + \langle \zeta, Dz \rangle &= 2\,\mathrm{Re}\, d\langle \zeta, z \rangle, \\ \langle iD\zeta, z \rangle + \langle \zeta, iDz \rangle &= -2\,\mathrm{Im}\, d\langle \zeta, z \rangle. \end{aligned}$$

□

3. Poincaré automorphisms and chains

Using the terminology of the previous section, it is easy to prove Theorem 1:

Proof. (3), (4) imply that $(A(z,\cdot),\langle z, a\bar{\cdot}\rangle)$ and $(B(w,\cdot), r(w,\bar{\cdot}))$ take values in the algebra $\mathfrak{A}$.

Representing the operators

$$(\mathrm{id} - 2iA(z,\cdot) - B(w,\cdot) - iA(aw,\cdot))^{-1}, (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - r(w,\bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle)^{-1}$$

as geometric progression, one proves that they also take values in $\mathfrak{A}$.

Using this, Proposition 3, and the Hermitian symmetry of r, one directly verifies that (5) is indeed an automorphism. □

According to Chern-Moser [4], we introduce the notion of a chain.

Definition 2. *A chain is a k-dimensional real submanifold of the quadric Q which can be mapped by an holomorphic automorphism to the plane $\{z = 0, \operatorname{Im} w = 0\}$.*

We call a chain Poincaré chain if and only if it can be mapped to the plane $\{z = 0, \operatorname{Im} w = 0\}$ by means of some Poincaré automorphism.

Corollary 1. *Poincaré chains passing through the origin coincide with the intersections of Q with complex k-planes $\{z = aw\}$, where $a : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ is a linear map satisfying (3), for some bilinear form A.*

In the remaining part of the paper we give some arguments concerning the question whether any automorphism of a quadric is a Poincaré automorphism.

We begin with a description of the group $\mathrm{Aut}_{id,0}$ as Heisenberg group for some quadric.

Let $\mathcal{A}$ be the complex vector space of linear maps $\hat{a} : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ such that there exists a $\mathbb{C}^n$-valued quadratic form $\hat{A}$ on $\mathbb{C}^n$, satisfying

$$\langle \hat{A}(z), z\rangle = \langle z, \hat{a}\langle z, z\rangle\rangle, \tag{6}$$

and, let $\mathcal{R}$ be the real vector space of symmetric $\mathbb{R}^k$-valued bilinear forms $\hat{r}$ on $\mathbb{R}^k$ such that there exists a $\mathbb{C}^n$-valued bilinear form $\hat{B}$ on $\mathbb{C}^k \otimes \mathbb{C}^n$, satisfying

$$\operatorname{Re}\langle \hat{B}(u,z), z\rangle = \hat{r}(u, \langle z, z\rangle), \tag{7}$$

$$\operatorname{Im}\langle \hat{B}(\langle z, z\rangle, z), z\rangle = 0. \tag{8}$$

Now, Beloshapka's uniqueness theorem can be reformulated as follows: The map $\mathrm{Aut}_{\mathrm{id},0}\, Q \longrightarrow \mathcal{A} \times \mathcal{R}$, being defined by

$$\Phi = (F, G) \mapsto \left(\left. \frac{\partial F}{\partial w} \right|_0, \frac{1}{2} \operatorname{Re} \left. \frac{\partial^2 G}{(\partial w)^2} \right|_0 \right) = (\hat{a}, \hat{r}), \tag{9}$$

is bijective. This bijection induces the following group structure on $\mathcal{A} \times \mathcal{R}$:

$$(\hat{a}_1, \hat{r}_1) \circ (\hat{a}_2, \hat{r}_2) = (\hat{a}_1 + \hat{a}_2, \hat{r}_1 + \hat{r}_2 - 2 \operatorname{Im}\langle \hat{a}_1 \cdot, \hat{a}_2 \cdot \rangle).$$

It follows that $\langle \hat{a}_1 \cdot, \hat{a}_2 \cdot \rangle$ takes values in $\mathcal{R} \otimes \mathbb{C}$.

Therefore, the equation

$$\operatorname{Im} \hat{r}(u, u) = \langle \hat{a}(u), \hat{a}(u) \rangle,$$

defines a quadric in $\mathcal{A} \times \mathcal{R} \otimes \mathbb{C}$. The group $\mathcal{A} \times \mathcal{R} \cong \operatorname{Aut}_{0,\mathrm{id}} Q$ is then isomorphic to the Heisenberg group of this quadric via

$$(\hat{a}, \hat{r}) \mapsto (\hat{a}, \hat{r}(u, u) + i \langle \hat{a}(u), \hat{a}(u) \rangle).$$

The parameters $\hat{A}$ and $\hat{B}$ have the following interpretation:

$$\hat{A} = \frac{1}{4i} \left. \frac{\partial^2 F}{(\partial z)^2} \right|_0 .$$

Using the isomorphism from above, we see that any $\Phi \in \operatorname{Aut}_{0,\mathrm{id}} Q$, corresponding to $(\hat{a}, \hat{r})$, can be uniquely decomposed into $\Phi_{\hat{a}} \circ \Phi_{\hat{r}}$, corresponding to $(\hat{a}, 0)$, resp. $(0, \hat{r})$. Then,

$$\hat{B} := \left. \frac{\partial^2 F_{\hat{r}}}{\partial z \partial w} \right|_0$$

satisfies the equations (7) and (8).

If Φ is a Poincaré automorphism of Q, then we obtain, by direct computation, that $a = \hat{a}$, $B = \hat{B}$, $A(z, z) = \hat{A}(z, z)$, and $r(u, u) = \hat{r}(u, u)$.

We denote the subspaces of $\mathcal{A}$ and $\mathcal{R}$, consisting of (a, r) which define Poincaré automorphisms, by $\mathcal{A}_P$ and $\mathcal{R}_P$.

The example below shows that, on the contrary to $\hat{A}, \hat{r}$, the tensors A and r need not be symmetric:

Example 1. *Let Q be the quadric in $\mathbb{C}^6$:*

$$\begin{aligned} v^1 &= |z^1|^2, \\ v^2 &= |z^2|^2, \\ v^3 &= \operatorname{Re} z^1 \bar{z}^2, \\ v^4 &= \operatorname{Im} z^1 \bar{z}^2. \end{aligned}$$

The algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{gl}(2,\mathbb{C})$. We represent a vector $w \in \mathbb{C}^4$ as 2×2-matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} w^1 & w^3 + iw^4 \\ w^3 - iw^4 & w^2 \end{pmatrix}.$$

Set $\operatorname{Im}\Omega = \frac{1}{2i}(\Omega - \Omega^*)$, *where* Ω^* *is the transposed, conjugate matrix to* Ω. *Then the equation of* Q *takes the form*

$$\operatorname{Im}\Omega = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \begin{pmatrix} \bar{z}^1 & \bar{z}^2 \end{pmatrix}.$$

For any $a \in \mathbb{C}^2$, *and any Hermitian* 2×2-*matrix* Θ, *we introduce a map* $\Delta_{a,\Theta} : \mathbb{C}^6 \longrightarrow \mathfrak{gl}(2,\mathbb{C})$,

$$\Delta_{a,\Theta}(z,\Omega) = 2i \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \begin{pmatrix} \bar{a}^1 & \bar{a}^2 \end{pmatrix} + \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \left(\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} + i \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \begin{pmatrix} \bar{a}^1 & \bar{a}^2 \end{pmatrix} \right).$$

Then, any $\Phi \in \operatorname{Aut}_{0,\mathrm{id}} Q$ *has the form*

$$\begin{aligned} \tilde{z} &= (\mathrm{id} - \Delta_{a,\Theta}(z,\Omega))^{-1} \left(z + \Omega \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \right), \\ \tilde{\Omega} &= (\mathrm{id} - \Delta_{a,\Theta}(z,\Omega))^{-1}\, \Omega. \end{aligned}$$

Thus,

$$\begin{aligned} A(z,\zeta) &= \begin{pmatrix} \bar{a}^1 z^1 \zeta^1 + \bar{a}^2 z^1 \zeta^2 \\ \bar{a}^1 z^2 \zeta^1 + \bar{a}^2 z^2 \zeta^2 \end{pmatrix}, \\ r(w,\omega) &= \begin{pmatrix} w^1 & w^3 + iw^4 \\ w^3 - iw^4 & w^2 \end{pmatrix} \begin{pmatrix} r^1 & r^3 + ir^4 \\ r^3 - ir^4 & r^2 \end{pmatrix} \begin{pmatrix} \bar{\omega}^1 & \bar{\omega}^3 + i\bar{\omega}^4 \\ \bar{\omega}^3 - i\bar{\omega}^4 & \bar{\omega}^2 \end{pmatrix}. \end{aligned}$$

The linear automorphisms are

$$\begin{aligned} \tilde{z} &= Cz, \\ \tilde{\Omega} &= C\Omega C^*, \end{aligned}$$

for $C \in \mathrm{GL}(2,\mathbb{C})$.

The existence of non-Poincaré automorphisms is equivalent to the existence of solutions $(\hat{a}, \hat{r}, \hat{A}, \hat{B})$ of the system (6), (7), (8) such that the system (3) and (4) is unsolvable for $a = \hat{a}$, $B = \hat{B}$.

We will call a nondegenerate quadric $Q \subset \mathbb{C}^{n+k}$ regular if $\mathcal{A} = \mathcal{A}_P$, and $\mathcal{R} = \mathcal{R}_P$.

Corollary 2. *If* Q *is regular, then any* $\Phi \in \operatorname{Aut}_{\mathrm{id},0} Q$ *is a Poincaré automorphism.*

It follows that the real associative quadrics (RAQ), the quadrics of codimension ≤ 2, and of codimension n^2 are regular.

4. Fractional linear automorphisms and reduced quadrics

We show that Poincaré automorphisms generalize fractional linear automorphisms.

Definition 3. *We call a nondegenerate quadric Q strictly nondgenerate if, instead of the nondegeneracy condition i), the following stronger condition holds:*

i') There exists a linear functional $f \in (\mathbb{R}^k)'$ such that the scalar Hermitian form $f(\langle\cdot,\cdot\rangle)$ is nondegenerate.

Otherwise, Q is called nullquadric.

Proposition 7. *Let Q be a nondegenerate quadric and Φ :*

$$\begin{aligned} z^* &= \frac{1}{1-\phi(z)-\psi(w)}(z+aw), \\ w^* &= \frac{1}{1-\phi(z)-\psi(w)}w, \end{aligned}$$

a fractional linear automorphism of Q.
Then, Φ is a Poincaré automorphism.
If the codimension $k > 1$, then $\psi = \frac{1}{2}\phi(a\cdot)$.
If $k > 1$, and Q is strictly nondegenerate, then $\phi = \psi = 0$.

Proof. Since, in the case $k = 1$, the assertion follows from the explicit automorphism formula, we can restrict ourselves to the case $k > 1$.

From $2i\langle z, a\langle\xi,\zeta\rangle\rangle = \phi(z)\langle\zeta,\xi\rangle$, we obtain, that $A(z,\zeta) := \phi(z)\zeta$ is a solution of (3).

Set

$$\psi'(u) = \psi(u) - \frac{1}{2}\phi(au).$$

Then, it follows that $r(\omega, w) = \psi'(\omega)\bar{w} = \bar{\psi}'(\bar{w})\omega$. Since $k > 1$, then $\psi' \equiv 0$ and, hence, $r = 0$.

It remains to prove that $\phi = 0$, if Q is strictly nondegenerate.

Without loss of generality, we may assume that (z^μ), (w^κ) are coordinates such that

$$v^1 = \sum_{\mu=1}^{n} \epsilon_\mu |z^\mu|^2,$$

where $\epsilon_\mu \in \{-1, 1\}$.

For any $z \in \mathbb{C}^n$, we define the $k \times n$ matrix Z, having the property $\langle z, \zeta\rangle = Z\bar{\zeta}$.

From $\phi(z)w \equiv 2i\langle z, a\bar{w}\rangle$, we obtain $\phi(z)\,\mathrm{id} = 2iZ\bar{a}$. The first row of this matrix identity implies that all columns of a, except the first one, are zeroes.

If $k > 1$, then the second row of this identity implies that $\phi(z) \equiv 0$. □

Definition 4. *We call a quadric Q reduced if the corresponding algebra $\mathfrak{A} \cong \mathbb{C}$.*

Proposition 8. *For a reduced quadric Q, any Poincaré automorphism is fractional linear, and, therefore, linear in the corresponding projective space.*

Proof. Condition (3) implies that $\langle z, a\bar{\cdot}\rangle$ takes values in $\mathfrak{A} \cong \mathbb{C}$. Therefore, it equals $\phi(z)\,\mathrm{id}$, where ϕ is some linear functional. Analogously, we obtain $r(w, \bar{\cdot}) = \psi(w)\,\mathrm{id}$. Then Φ takes the form

$$\begin{aligned} z^* &= \frac{1}{1 - 2i\phi(z) - \psi(w) - i\phi(aw)}(z + aw) \\ w^* &= \frac{1}{1 - 2i\phi(z) - \psi(w) - i\phi(aw)}w. \end{aligned}$$

□

Corollary 3. *Let Q be a reduced strictly nondegenerate quadric. Then, either Q is a hyperquadric ($k = 1$), or any Poincaré automorphism of Q is identical.*

Proof. This follows from Propositions 7, and 8. □

5. Sums of quadrics

For two quadrics Q_1 in $\mathbb{C}^{n_1+k}$, and Q_2 in $\mathbb{C}^{n_2+k}$, with the same codimension we define the sum $Q_1 + Q_2$ by

$$Q_1 + Q_2 = \{(z, w) \in \mathbb{C}^{n_1+n_2} \times \mathbb{C}^k : \operatorname{Im} w = \langle z, z\rangle_1 + \langle z, z\rangle_2\}. \tag{10}$$

If Q_1, and Q_2 both satisfy (i), and, at least one of them, satisfies (ii) of the nondegeneracy condition then $Q_1 + Q_2$ is nondegenrate.

It is easy to verify that the algebra $\mathfrak{A}$, corresponding to $Q_1 + Q_2$ equals $\mathfrak{A}_1 \cap \mathfrak{A}_2$.

We consider now the following question: which automorphisms of Q_1 can be lifted to automorphisms of the sum $Q_1 + Q_2$.

Proposition 9. *Let Q_1 be a nondegerate quadric of codimension k, and Q_2 be a quadric of the same codimension, satisfying (i) of the nondegeneracy condition. If $\mathfrak{A}_1 \subset \mathfrak{A}_2$, then any Poincaré automorphism of Q_1 can be lifted to a Poincaré automorphism of $Q_1 + Q_2$.*

Proof. Let (a, r, A_1, B_1) be the parameters defining a Poincaré automorphism Φ_1 of Q_1. Then the operators $\langle z, a\cdot\rangle$, and $r(w, \cdot)$ are contained in $\mathfrak{A}_1$, and, therefore, also in $\mathfrak{A}_2$. Hence, for any z, w there exist uniquely determined $A_2(z, \cdot)$, $B_2(w, \cdot)$ from $\mathrm{GL}(n_2, \mathbb{C})$. Thus, $(a, r, A_1 \oplus A_2, B_1 \oplus B_2)$ defines a Poincaré automorphism on $Q_1 + Q_2$. □

Corollary 4. *Any fractional linear automorphism of Q_1 can be lifted to $Q_1 + Q_2$.*

Now, let Q be a nondegenerate quadric. Considering the system (6)-(8) for $Q + Q$ we obtain that $(a \oplus a', r)$ are the parameters of some automorphism if and only if (a, r) and (a', r) define automorphisms of Q, and a has a solution A of (3). Thus, the Poincaré property (3) for a is necessary for lifting an automorphism to $Q + Q$.

We illustrate the introduced calculus in the case of codimension $k = 1$.

If $k = 1$, then always $\mathfrak{A} \cong \mathbb{C}$. Moreover, hyperquadrics are sums of spheres in $\mathbb{C}^2$. By these reasons automorphisms of hyperquadrics have a quite simple structure. They can be lifted from automorphisms (1) of S^3, by means of the described construction.

6. Some questions and conjectures

At the end of the paper we list some open problems and conjectures:

1. Is any nondegenerate quadric regular?

It would be also interesting to know the answer in the following special cases:

1'. Is any automorphism of a reduced quadric fractional linear?

1". Is any automorphism of a reduced, strictly nondegenerate quadric of codimension $k > 1$ linear?

Conjecture 1. *The questions 1, 1', and 1" have an affirmative answer.*

One can give a rough estimate of $dim_{\mathbb{C}}\mathcal{A}$ by nk, and of $dim_{\mathbb{R}}\mathcal{R}$ by $k^2\frac{k+1}{2}$. However, in the cases when the explicit groups are known, the dimension of the first space does not exceed n and that of the second space does not exceed k.

2. Is always $dim_{\mathbb{C}}\mathcal{A} \leq n$, and $dim_{\mathbb{R}}\mathcal{R} \leq k$?

2'. Is $dim_{\mathbb{C}}\mathcal{A}_P \leq n$, and $dim_{\mathbb{R}}\mathcal{R}_P \leq k$?

Conjecture 2. *The questions 2 and 2' have an affirmative answer.*

Remark. If Q is reduced, then the answer to question 2' follows from Proposition 8.

We have showed above that, for any $a \in \mathcal{A}$, the quadratic form $\langle au, au\rangle \in \mathcal{R}$. In the cases of RAQ, as well as for quadrics of codimension ≤ 2, any $r \in \mathcal{R}$ is a linear combination of such forms.

3. Does there exist a nondegenerate quadric Q and some $r \in \mathcal{R}$ which cannot be represented as a linear combination of $\langle au, au\rangle \in \mathcal{R}$, where $a \in \mathcal{A}$?

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