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Abstract

Since a tropical Nullstellensatz fails even for tropical univariate polynomials we study a conjecture on a tropical *dual* Nullstellensatz for tropical polynomial systems in terms of solvability of a tropical linear system with the Cayley matrix associated to the tropical polynomial system. The conjecture on a tropical effective dual Nullstellensatz is proved for tropical univariate polynomials.

Keywords: dual Nullstellensatz, solving tropical polynomial systems

Introduction

Let T be a tropical semi-ring with operations $\oplus, \otimes$ (see e. g. [2], [3], [9], [12]). Typically $\oplus = \min$, $\otimes = +$. Examples of T are $\mathbb{Z}$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. A *tropical monomial* has a form $Q = a \otimes X_1^{\otimes i_1} \otimes \cdots \otimes X_n^{\otimes i_n}$, $a \in T$. The *tropical degree* $\mathrm{trdeg}(Q) := i_1 + \cdots + i_n$. From the point of view of the classical algebra a tropical monomial is a linear function. A point $x = (x_1, \dots, x_n) \in T^n$ (with some of $x_i \neq \infty$) is a *tropical zero* of a *tropical polynomial* $f = \bigoplus_l Q_l$ if the minimum $\min_l\{Q_l(x)\}$ is attained for at least two different tropical monomials Q_l.

We study the issue of a tropical Nullstellensatz. Its direct formulation fails even for tropical univariate polynomials: for example, two tropical polynomials $X \oplus 0$, $X \oplus 1$ have no common tropical zero, while the generated by them tropical ideal does not contain 1 or any other tropical monomial. That is why we consider a *tropical "dual" Nullstellensatz*.

One can treat the (customary) Hilbert's Nullstellensatz as a reduction of solvability of a polynomial system to solvability of a suitable linear system. Namely, solvability of a polynomial system is equivalent to that the Cayley matrix C associated to the system does not contain the vector $(1, 0, \dots, 0)$ in the linear hull of its rows. In its turn it is equivalent to that the linear system $C \cdot (a_0, a_1, \dots) = 0$ has a solution with $a_0 \neq 0$ (cf. Section 1). The latter rephrasing of the Nullstellensatz we call the "dual" Nullstellensatz. It holds also for the *infinite* matrix C (we call it the infinite "dual" Nullstellensatz) unlike the customary Nullstellensatz, and it holds for a finite submatrix of C with the size depending on n and on the degrees of the polynomials in the system (we call it the *effective* "dual" Nullstellensatz).

In Section 2 we formulate the conjecture on a tropical "dual" Nullstellensatz. In Section 3 we give a rephrasing of the conjecture in terms of the combinatorial convex geometry. Finally, in Section 4 we prove the tropical effective "dual" Nullstellensatz for univariate polynomials ($n = 1$).

Observe that the latter result in case of a system of two tropical polynomials f_1, f_2 follows from the approach of [11] which relies on the (classical) resultant of a pair of (classical) polynomials, so it fails for overdetermined systems in the tropical setting. We mention also that the problem of solvability of tropical polynomial systems is NP-complete even for tropical quadratic polynomials [12].

Solvability of tropical linear systems belongs to the complexity class $NP \cap co - NP$ [1], [5]. In [1], [5] two different algorithms for solving tropical linear systems were designed with the similar complexity bounds polynomial in $s. M$, where s is the size of the tropical linear system (so, of its matrix) and M majorates the absolute values of the finite (integer) coefficients of the system. We note that the algorithm from [5] possesses an extra feature that it has also a complexity bound polynomial in $\exp(s)$, $\log M$. The open question is whether it runs in fact, within complexity polynomial in s, $\log M$ (which would provide a polynomial complexity for the problem of solvability of tropical linear systems)?

In addition, the algorithm from [5] entails as a by-product the equivalence of solvability of a tropical linear system with the degeneration of its *tropical rank* and simultaneously with the degeneration of its *Kapranov rank*. The latter for systems with *finite* coefficients (say, from $\mathbb{Z}$) was shown in [3], also a part of this equivalence just for the tropical rank follows from [7].

Besides, we mention that in [6] the tropical (customary) Nullstellensatz was established for an introduced there a "ghost" tropical semi-ring. In [10] the radical of a tropical ideal was explicitly described.

1 "Dual" Nullstellensatz

Let $F_1, \dots, F_s \in K[X_1, \dots, X_n]$ be polynomials over an algebraically closed field K. Denote by $C := C(F_1, \dots, F_s)$ the (infinite size) Cayley matrix over K consisting of the coefficients of $F_1, \dots, F_s$. The columns of C correspond to all the monomials $X^I := X_1^{i_1} \cdots X_n^{i_n}$, $I = (i_1, \dots, i_n)$, and the rows of C correspond to all the polynomials of the form $X^I \cdot F_j, 1 \leq j \leq s$. Let the first column of C correspond to the monomial $X^0 = 1$. For an integer N denote by C_N the (finite size) submatrix of C formed by the rows $X^I \cdot F_j$, $1 \leq j \leq s$ with the degrees $\deg X^I = i_1 + \cdots + i_n \leq N$ and the corresponding columns which contain a non-zero entry in at least one of these rows.

Nullstellensatz states that a polynomial system

$$F_1 = \cdots = F_s = 0 \tag{1}$$

has a solution in K^n iff for any N the linear hull of the rows of C_N does not contain the vector $(1, 0, \dots, 0)$. An *effective* Nullstellensatz provides an upper bound on N for which the latter equivalence holds. The bound $N < (\max_j\{\deg(F_j)\})^{O(n)}$ close to optimal was obtained in [4], [8].

Thus, the effective Nullstellensatz is equivalent to the following. System (1) has a solution iff the linear system $C_N \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$ for an appropriate N depending on n and on $\max_j\{\deg(F_j)\}$. We call the latter statement the *effective dual Nullstellensatz*. The equivalence that (1) has a solution iff the linear system $C_N \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$ for any N, we call the *dual Nullstellensatz*. Finally, the statement (also equivalent to Nullstellensatz) that (1) has a solution iff the infinite linear system $C \cdot (y_1, y_2, \dots) = 0$ has a solution with $y_1 \neq 0$, we call the *infinite dual Nullstellensatz*. The latter infinite linear system makes sense because each row of C contains just a finite number of non-zero entries.

2 Conjecture on a tropical dual Nullstellensatz

Below we assume that the tropical semi-ring $T = \mathbb{R}_\infty$, but for the sake of simplifying the exposition we study tropical zeroes defined over $\mathbb{R}$ (although, one could also consider zeroes defined over $\mathbb{R}_\infty$). For each monomial $Q_l = a_l \otimes X_1^{\otimes i_{1,l}} \otimes \cdots \otimes X_n^{\otimes i_{n,l}}$ of a tropical polynomial $f = \bigoplus_l Q_l$ we plot the point $(a_l, i_{1,l}, \dots, i_{n,l}) \in \mathbb{R} \times \mathbb{Z}^n \subset \mathbb{R}^{n+1}$. Then a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a tropical zero of f iff the linear function $(a, i_1, \dots, i_n) \to a + i_1 \cdot x_1 + \cdots i_n \cdot x_n$ attains its minimum at the plotted points at least twice.

Therefore, without changing the set of tropical zeroes of f one can replace the plotted points by their convex hull. Moreover, w.l.o.g. for any point $(a, b_1, \dots, b_n) \in \mathbb{R}^{n+1}$ from this convex hull one can add the ray $\{(b, b_1, \dots, b_n) : b \geq a\}$. The resulting convex set $P(f) \subset \mathbb{R}^{n+1}$ we call the (extended) *Newton polyhedron* of f. Thus, w.l.o.g. one can modify f replacing it by a tropical polynomial whose plotted points are just the points of the form $(a, i_1, \dots, i_n) \in (\mathbb{R} \times \mathbb{Z}^n) \cap P(f)$ with the minimal possible a. Finally, so modified tropical polynomial has the same set of tropical zeroes as f, and (in abuse of notations) we keep for it the same notation. We say that the modified tropical polynomial is in the *convex form*, and from now on we consider tropical polynomials only in the convex form. Observe that x is a tropical zero of f iff for the maximal $b \in \mathbb{R}$ such that the hyperplane $\{(z_1, \dots, z_{n+1}) : z_1 + x_1 \cdot z_2 + \cdots + x_n \cdot z_{n+1} = b\} \subset \mathbb{R}^{n+1}$ has a non-empty intersection with $P(f)$, the hyperplane has at least two common points with $P(f)$.

Similarly to the classical algebra to a system of tropical polynomials

$$f_1, \dots, f_s \tag{2}$$

in n variables we associate the Cayley matrix $C := C(f_1, \dots, f_s)$ over $\mathbb{R}_\infty$ consisting of the coefficients of (2). The columns of C correspond to the tropical monomials of the form $X^{\otimes I}$, $I \in \mathbb{Z}^n$, and the rows of C correspond to the tropical polynomials of the form $X^{\otimes I} \otimes f_j$, $I \in \mathbb{Z}^n, 1 \leq j \leq s$. Note that unlike the classical algebra the tropical Cayley matrix is infinite in all 4 directions.

Conjecture 1 on a tropical infinite dual Nullstellensatz. System (2) has a tropical zero iff the matrix C has a tropical zero.

The latter statement is obvious in the direction that if (2) has a zero then C has a zero (the similar is true for two conjectures below as well).

Observe that being a particular case of tropical polynomials (of the tropical degree 1) matrix $C = (c_{i,I})$ (or in other words, a tropical linear system) has a tropical zero $(\dots, y_I, \dots)$ if for every row i of C (in the language of classical algebra) the minimum $\min_I\{c_{i,I} + y_I\}$ is attained at least for two different coordinates I. Note that a tropical zero of C makes sense because every row of C contains just a finite number of finite (so, from $\mathbb{R}$) entries.

Similarly to the classical algebra for an integer N denote by C_N a (finite) submatrix of C formed by the rows $X^{\otimes I} \otimes f_j$, $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, $1 \leq j \leq s$ with $|i_1| + \dots + |i_n| \leq N$, and by the columns of C which contain at least one finite entry at one of these rows.

Conjecture 2 on a tropical dual Nullstellensatz. System (2) has a tropical zero iff for any N the matrix C_N has a tropical zero.

Conjecture 3 on a tropical effective dual Nullstellensatz. There is a function N on n and on $\mathrm{trdeg}(f_j)$, $1 \leq j \leq s$ such that (2) has a tropical zero iff the matrix C_N has a tropical zero.

Clearly, Conjecture 3 implies Conjecture 2, which in its turn implies Conjecture 1.

3 Convex-geometric rephrasing of the tropical dual Nullstellensatz

In the present Section we give a rephrasing of Conjecture 1 (and similarly of Conjectures 2, 3) in terms of the convex geometry in $\mathbb{R}^{n+1}$. Thus, assume that Cayley matrix C has a tropical zero $(\dots, y_I, \dots)$, $I \in \mathbb{Z}^n$.

For any $I \in \mathbb{Z}^n$ consider the shift $P(f_j) + (0, I) \subset \mathbb{R}^{n+1}$, $1 \leq j \leq s$ of the Newton polyhedron. We say that a set $U \subset \mathbb{R}^{n+1}$ *lies above* (with respect to the first coordinate) a set $V \subset \mathbb{R}^{n+1}$ if for any pair of points $(u_1, w_1, \dots, w_n) \in U$, $(v_1, w_1, \dots, w_n) \in V$ we have $u_1 \geq v_1$.

Proposition 3.1 *The following statement is equivalent to Conjecture 1.*

For $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ take the minimal $a \in \mathbb{R}$ such that the polyhedron $P(f_j) + (a, I)$ lies above the set $Y := \{(-y_J, J) : J \in \mathbb{Z}^n\}$. Assume that for any $I \in \mathbb{Z}^n$, $1 \leq j \leq s$ the polyhedron and Y have at least two common points. Then there exists a hyperplane $H \subset \mathbb{R}^{n+1}$ defined by a linear equation $z_1 + b_2 \cdot z_2 + \dots + b_n \cdot z_n = 0$ such that for each $1 \leq j \leq s$ for the minimal $b \in \mathbb{R}$ with the property that the polyhedron $P(f_j)$ lies above the hyperplane $H - (b, 0)$, the intersection of $P(f_j)$ and $H - (b, 0)$ has at least two points.

For an equivalent statement to Conjecture 2 one has for any N to consider all $I = (i_1, \dots, i_n)$ such that $|i_1| + \cdots + |i_n| \leq N$. Respectively, for Conjecture 3 one has to take N as a suitable function in n and in $\mathrm{trdeg}(f_j)$, $1 \leq j \leq s$.

4 Tropical effective dual Nullstellensatz for univariate polynomials

Now let $n = 1$. In this case for a pair of tropical polynomials f_1, f_2 a tropical effective dual Nullstellensatz follows from [11] with the bound $N \leq \mathrm{trdeg}(f_1) + \mathrm{trdeg}(f_2)$, but since this approach relies on the (classical) resultant of a pair of (classical) polynomials being *liftings* of f_1, f_2, respectively, the approach fails for overdetermined tropical systems ($s \geq 3$).

Theorem 4.1 *A tropical effective dual Nullstellensatz for univariate tropical polynomials* $f_1, \dots, f_s$ *holds with* $N \leq 4 \cdot (\mathrm{trdeg}(f_1) + \cdots + \mathrm{trdeg}(f_s))$.

Proof. Fix $1 \leq j \leq s$ for the time being. For the convex polyhedron $P := P(f_j) \subset \mathbb{R}^2$ and $i \in \mathbb{Z}$ take the minimal $a_i \in \mathbb{R}$ such that the shifted polygon $P_i := P(f_j) + (a_i, i)$ lies above the set $Y = \{(-y_l, l) : l \in \mathbb{Z}\}$ (see Proposition 3.1). By the assumption for any $i \in \mathbb{Z}$ there exist at least two points $(u_1, l_1), (u_2, l_2) \in P_i \cap Y$, $l_1 < l_2$. Points from the latter intersection we call *extremal points* of P_i.

Lemma 4.2 *The function* $i \to a_i$ *is convex.*

Proof of Lemma 4.2. Suppose the contrary and let $2 \cdot a_i > a_{i-1} + a_{i+1}$ for a certain i. Let $(u_1, l_1), (u_2, l_2) \in P_i \cap Y$. Denote by

$$S = \{(v, w) : v - w \cdot (a_i - a_{i-1}) \leq u_1 - l_1 \cdot (a_i - a_{i-1}),\ v + w \cdot (a_i - a_{i+1}) \geq u_1 + l_1 \cdot (a_i - a_{i+1})\} \subset \mathbb{R}^2$$

the sector with the vertex at the point (u_1, l_1) between two rays $R_+ = (u_1, l_1) + \{\lambda \cdot (a_i - a_{i+1}, -1) : \lambda \geq 0\}$ and $R_- = (u_1, l_1) + \{\lambda \cdot (a_i - a_{i-1}, 1) : \lambda \geq 0\}$. We claim that $P_i \subset S$.

Indeed, consider a left adjacent to (u_1, l_1) point $(u_+, l_1 - 1) \in \partial P_i$ on the boundary of P_i (provided that such a point does exist). If $u_+ < u_1 + l_1 \cdot (a_i - a_{i+1})$ (in other words, the point $(u_+, l_1 - 1)$ lies strictly below the ray R_+, cf. the description of S) then the point $(u_+, l_1 - 1) + (a_{i+1} - a_i, 1) \in P_{i+1}$ lies strictly below Y, the achieved contradiction implies that $(u_+, l_1 - 1) \in S$. In a similar way a right adjacent to (u_1, l_1) point $(u_-, l_1 + 1) \in \partial P_i$ (provided that it does exist) belongs to S, which justifies the claim.

By the same token the parallel shift $S + (u_2, l_2) - (u_1, l_1)$ of the sector S (with its vertex at the point (u_2, l_2)) also contains P_i. This contradicts to the convexity of P_i and completes the proof of Lemma 4.2. ■

Denote by $E := E(f_j) \subset \mathbb{R}^2$ the polygon with the vertices in the extremal points of P_i for all $i \in \mathbb{Z}$. Below we enumerate the (finite) edges of the polygon P from the left to the right. Denote by $(b_r, 1)$ the vector parallel to the r-th edge of P.

Lemma 4.3 *Let $(u_1, l_1), \dots, (u_t, l_t) \in P_i$, $l_1 < \dots < l_t$ be all the extremal points of P_i. Let the point $(u_t, l_t) - (a_i, i) \in P$ lie in the r-th (finite) edge of P (when the latter point belongs to the r-th and to the $(r+1)$-th edges we agree that the point lies in the r-th edge). Then $a_{i+1} - a_i \geq b_r$.*

For any extremal point (v, k) of P_{i+1} the point $(v, k) - (a_{i+1}, i+1) \in P$ lying in the q-th edge of P either satisfies an inequality $q \geq r$ or $(v, k) - (a_{i+1}, i+1)$ is the vertex of the $(r-1)$-th and r-th edges of P (in the latter case (v, k) is the leftmost extremal point of P_{i+1}). There exists an extremal point (v, k) for which either $q = r$ and $(v, k) - (a_{i+1}, i+1)$ not being the vertex of the r-th and $(r+1)$-th edges of the polygon P or $(v, k) - (a_{i+1}, i+1)$ is the vertex of the $(r-1)$-th and r-th edges of P iff $a_{i+1} - a_i = b_r$. Moreover, when $a_{i+1} - a_i = b_r$ any extremal point (u_m, l_m) of P_i with $(u_m, l_m) - (a_i, i)$ lying in the r-th edge of P is also an extremal point of P_{i+1}.

Proof of Lemma 4.3. Consider the point $(u_t, l_t) - (b_r, 1) \in P_i$. Then the point $((u_t, l_t) - (b_r, 1)) + (a_{i+1} - a_i, 1) \in P_{i+1}$ should lie above the extremal point (u_t, l_t), this entails the inequality $a_{i+1} - a_i \geq b_r$.

Let (v, k) be an extremal point of P_{i+1} with $(v, k) - (a_{i+1}, i+1)$ lying in the q-th edge of P. The point $(v, k) - (a_{i+1} - a_i, 1)$ lies in the q-th edge of the polygon P_i. If $q < r$ and the point $(v, k) - (a_{i+1}, i+1)$ is not the vertex of the $(r-1)$-th and r-th edges of P then its shift $(v, k) = ((v, k) - (a_{i+1} - a_i, 1)) + (a_{i+1} - a_i, 1)$ lies *strictly* inside the polygon P_i, and therefore (v, k) can not be an extremal point. The achieved contradiction implies that either $q \geq r$ or $(v, k) - (a_{i+1}, i+1)$ is the vertex of $(r-1)$-th and r-th edges of P.

When $a_{i+1} - a_i > b_r$ a similar argument shows that either $q > r$ or $(v, k) - (a_{i+1}, i+1)$ is the vertex of the r-th and $(r+1)$-th edges of P. Finally, when $a_{i+1} - a_i = b_r$, for any extremal point (u_m, l_m) of P_i with $(u_m, l_m) - (a_i, i)$ lying in the r-th edge of P take the point $(u_m, l_m) - (a_{i+1} - a_i, 1) \in P_i$, then the point $(u_m, l_m) = ((u_m, l_m) - (a_{i+1} - a_i, 1)) + (a_{i+1} - a_i, 1) \in P_{i+1}$ is also an extremal point of P_{i+1}. ∎

Remark 4.4 *Lemma 4.3 is formulated for the shifts passing from the polygon P_i to P_{i+1} (so, from the left to the right). By the same token a similar statement holds while passing from P_{i+1} to P_i (so, from the right to the left).*

Lemma 4.5 *The polygon E is convex.*

Proof of Lemma 4.5. We prove by induction on i that the union of the extremal points of $P_0, \dots, P_i$ form a convex polygon E_i. At the inductive step we consider the polygon P_{i+1} (so, we move from the left to the right). By the same token one can alternatively consider the polygon P_{-1} (so, move from the right to the left). This would entail Lemma 4.5.

We say that a polygon with the vertices $\dots, w_{i-1}, w_i, w_{i+1}, \dots$ is convex at the vertex w_i if there is a line passing through w_i such that both points w_{i-1}, w_{i+1} lie above this line.

Let (u'_1, l'_1), $(u'_2, l'_2), \dots$, $l'_1 < l'_2 < \cdots$ be all the extremal points of P_{i+1} (if they exist) being not extremal points of P_i. Lemma 4.3 implies that the point $(u'_1, l'_1) - (a_{i+1}, i+1)$ either lies in the q-th edge of P with $q > r$ or it is the vertex of the r-th and $(r+1)$-th edges of P (we keep the notations from Lemma 4.3).

The inductive hypothesis and Lemma 4.3 entail that the polygon E_{i+1} is convex at all the vertices of E_i, perhaps, with the exception of the rightmost extremal point (u_t, l_t) of E_i (and simultaneously of P_i). The point (u_t, l_t) lies in the r-th edge of the polygon P_i, and both polygons P_i, P_{i+1} lie above the line L spanned by this edge (due to Lemma 4.3), whence E_{i+1} is convex at its vertex (u_t, l_t) as well.

Since the extremal points (u'_1, l'_1), $(u'_2, l'_2), \dots$ are located on the convex polygon P_{i+1} we get that E_{i+1} is convex at its vertices $(u'_2, l'_2), \dots$. Thus, it remains to verify that E_{i+1} is convex at its vertex (u'_1, l'_1).

Denote the vector $w := (u'_2, l'_2) - (u'_1, l'_1)$. The points $p := (u'_1, l'_1) - (a_{i+1} - a_i, 1)$, $(u'_2, l'_2) - (a_{i+1} - a_i, 1) \in P_i$. Therefore, the point p lies in a sector S_0 with the vertex (u_t, l_t) formed by the rays $(u_t, l_t) + \{\lambda \cdot (b_r, 1) : \lambda \geq 0\} \subset L$ and $(u_t, l_t) + \{\lambda \cdot w : \lambda \geq 0\}$. Now consider a sector $S_1 \subset S_0$ parallel to S_0 with the vertex p formed by the rays $p + \{\lambda \cdot (b_r, 1) : \lambda \geq 0\}$ and $p + \{\lambda \cdot w : \lambda \geq 0\}$. The point $(u'_1, l'_1) = p + (a_{i+1} - a_i, 1)$ is located in S_1 due to Lemma 4.3 and taking into the account that the point (u'_1, l'_1) is extremal in P_{i+1} and thereby, can not lie strictly inside P_i. Hence the polygon E_{i+1} is convex at its vertex (u'_1, l'_1). ∎

Remark 4.6 *The latter statement that E_{i+1} is convex at its vertex (u'_1, l'_1) becomes obvious when the point (u_t, l_t) is an extremal point of P_{i+1}, this is equivalent to the equality $a_{i+1} - a_i = b_r$ due to Lemma 4.3. In case when $a_{i+1} - a_i > b_r$ the polygon P_{i+1} has no common extremal points with E_i.*

Corollary 4.7 *Any edge $e = ((u, l), (u', l'))$ of the convex polygon E is one of the following three types:*

1) either $(u, l), (u', l') \in P_i$ for a certain $i \in \mathbb{Z}$ where the point (u, l) lies in the r-th edge of P_i, the point (u', l') lies in the r'-th edge of P_i for some $r < r'$, except the case when (u, l) is the vertex of the $(r-1)$-th and r-th edges of P_i and (u', l') lies in the r-th edge of P_i (in the latter case e is parallel to the r-th edge of P_i, cf. 3) below);

2) either the point (u, l) lies in the r-th edge of P_i for a certain $i \in \mathbb{Z}$, the point (u', l') lies in the r'-th edge of P_{i+1} for some r, r', and the point $(u', l') - (a_{i+1} - a_i, 1)$ lies in the r'-th edge of P_i, moreover either $r < r'$ or $(u', l') - (a_{i+1} - a_i, 1)$ is the vertex of the r-th and $(r+1)$-th edges of P_i. Case 2) occurs when $a_{i+1} - a_i > b_r$ (see Lemma 4.3 and Remark 4.6);

3) or e is parallel to an edge of P.

The edges e of E of types 1), 2) we call *intermediate* and the edges of types 3) we call *r-principal* when e is parallel to the r-th edge of P. For an edge of the type either 1) or 2) we define its *projection* (to the second coordinate) as the interval $(l - i, l' - i)$ for the type 1) and as $(l - i, l' - i - 1)$ for the type 2).

Lemma 4.8 *The polygon E lies above Y.*

Proof of Lemma 4.8. Consider a point $(-y_m, m) \in Y$. If $(-y_m, m)$ is a vertex of E or m is a projection of a point strictly inside an edge of E of a type either 1) or 3) then the claim of Lemma 4.8 is obvious.

Else if m is a projection of a point strictly inside an edge e of the type 2) (we keep the notations of 2) of Corollary 4.7) then the point $(-y_m, m)$ lies below the interval $((u, l), (u' - a_{i+1} + a_i, l' - 1))$ with its endpoints on the polygon P_i, and it lies also below the interval $((u + a_{i+1} - a_i, l + 1), (u', l'))$ with its endpoints on the polygon P_{i+1}. Hence the point $(-y_m, m)$ lies below the edge $((u, l), (u', l'))$ of E. ∎

Corollary 4.9 *i) For a pair of adjacent intermediate edges of E their projections are also adjacent (in the same order).*

ii) For each r all r-principal edges of E (if they exist) constitute an interval in E (parallel to the r-th edge of P). We call it r-interval. Among these intervals there are either two intervals infinite in one of directions or one interval infinite in both directions.

Let $e_- := ((u_-, l_-), (u'_-, l'_-))$ be an edge of E adjacent to the r-interval from the left (provided that the r-interval is not infinite to the left). Then e_- is intermediate. Assume for definiteness that $(u'_-, l'_-) \in P_i$ for a certain i, while either $(u_-, l_-) \in P_i$ in case of the type 1) (see Corollary 4.7) or $(u_-, l_-) \in P_{i-1}$ in case of the type 2). Then the point (u'_-, l'_-) lies in the r-th edge of P_i.

Similarly, let an edge $e_+ := ((u_+, l_+), (u'_+, l'_+))$ be an edge adjacent to the r-interval from the right (provided that the r-interval is not infinite to the right). Then e_+ is intermediate. Assume that $(u_+, l_+) \in P_i$ for a certain i and either $(u'_+, l'_+) \in P_i$ in case of the type 1) or $(u'_+, l'_+) \in P_{i+1}$ in case of the type 2). Then the point (u_+, l_+) either lies in the r-th edge of P_i or (u_+, l_+) is the vertex of the $(r-1)$-th and r-th edges of P_i.

iii) Denote by (d_1, k_1), (d_2, k_2) the endpoints of the r-th edge of P. Then for any pair of adjacent extremal points (d'_1, k'_1), (d'_2, k'_2) in the r-interval of E we have $k'_2 - k'_1 \leq k_2 - k_1$.

Finally, we complete the proof of Theorem 4.1. So far, we studied the convex polygon $E(f_j)$ for a fixed $1 \leq j \leq s$ (see Lemma 4.5). Now we consider the intersection $\mathcal{E} := \bigcap_{1 \leq j \leq s} E(f_j)$. Every edge ϵ of the convex polygon $\mathcal{E}$ is some subinterval of either an intermediate edge of $E(f_j)$ or an r-interval for certain $1 \leq j \leq s$ and r. The total sum of the lengths of the projections of the edges being subintervals of intermediate edges of $E(f_j)$, $1 \leq j \leq s$ does not exceed $3 \cdot \sum_{1 \leq j \leq s} \operatorname{trdeg} f_j$ due to i), ii) of Corollary 4.9.

Observe that if ϵ is a subinterval of an r-interval of $E(f_j)$ for a certain $1 \leq j \leq s$ and not all the points of ϵ belong to all polygons $E(f_{j_1})$, $1 \leq j_1 \leq s$ (the latter is equivalent to that any strictly inside point of ϵ does not belong to all $E(f_{j_1})$, $1 \leq j_1 \leq s$) then ϵ can not contain extremal points strictly inside itself according to Lemma 4.8. Hence the total sum of the lengths of the projections of all the edges of $\mathcal{E}$ being subintervals of some r-intervals of $E(f_j)$, $1 \leq j \leq s$ does not exceed $\sum_{1 \leq j \leq s} \operatorname{trdeg} f_j$ by virtue of iii) of Corollary 4.9.

Thus, a truncation $\mathcal{E}_N$ of $\mathcal{E}$ with the length of the projection to the second coordinate equal N, where $N \geq 4 \cdot \sum_{1 \leq j \leq s} \operatorname{trdeg} f_j$, contains an edge which is a common subinterval of r_j-intervals for appropriate r_j of all $E(f_j)$, $1 \leq j \leq s$. Taking into the account the Proposition 3.1 we conclude with Theorem 4.1. ∎

It would be interesting to improve the factor 4 in Theorem 4.1.

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