MODULAR FORMS, SHIMURA SUMS AND ARITHMETIC OF QUADRATIC FIELDS

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ABSTRACT

In this article we study Shimura sums related to modular forms with multiplicative coefficients which are products of Dedekind $\eta-$functions of various arguments. These modular forms are called multiplicative $\eta-$products. The author proves several families of identities involving Shimura sums. The type of identity obtained depends on the splitting of primes in certain imaginary quadratic number fields.

Key words: modular forms, Dedekind $\eta-$function, Shimura sums, quadratic fields.

1. Introduction.

In this article, we study the properties of multiplicative $\eta-$products. These modular forms are products of Dedekind $\eta-$functions of various arguments corresponding to partitions of the number 24 with multiplicative coefficients. They were discovered by D. Dummit, H. Kisilevsky, J. McKay in 1985 [1]. Also these functions can be completely described by the following conditions:

1. they are cusp forms of integral weight with characters;
2. they are eigenforms for all Hecke operators;
3. they have zeroes only in the cusps and every zero has the multiplicity 1.

This fact was proved in [2]. The condition that the multiplicity is equal to 1 is essential. These functions have been studied from various points of view in works [1] - [20].

Shimura sums have been used in investigations of relations between modular forms of integral and half-integral weights. Ken Ono in his article [3] has considered Shimura sums related to the multiplicative $\eta-$product $\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$. In this article we study Shimura sums related to other multiplicative $\eta-$products. The author proves several families of identities involving Shimura sums. The type of identity obtained depends on the splitting of primes in certain imaginary quadratic number fields.

We can express by Shimura sums the connections between coefficients of various modular forms. It is very useful because the calculation of coefficients of modular forms of weight 1 is easier then the calculation of coefficients of modular forms of weights greater then 1 . These expressions help to calculate the coefficients.

2. Multiplicative $\eta-$products.

Here we give the complete list of multiplicative $\eta-$products with weights, levels and characters.

The Dedekind $\eta-$function $\eta(z)$ is defined by the formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i z},$$

z belongs to the upper complex half-plane.

Table 1.

$f(z)$	k	N	$\chi(d)$
$\eta(23z)\eta(z)$	1	23	$\left(\frac{-23}{d}\right)$
$\eta(22z)\eta(2z)$	1	44	$\left(\frac{-11}{d}\right)$
$\eta(21z)\eta(3z)$	1	63	$\left(\frac{-7}{d}\right)$
$\eta(20z)\eta(4z)$	1	80	$\left(\frac{-5}{d}\right)$
$\eta(18z)\eta(6z)$	1	108	$\left(\frac{-3}{d}\right)$
$\eta(16z)\eta(8z)$	1	128	$\left(\frac{-2}{d}\right)$
$\eta^2(12z)$	1	144	$\left(\frac{-1}{d}\right)$
$\eta^4(6z)$	2	36	1
$\eta^2(8z)\eta^2(4z)$	2	32	1
$\eta^2(10z)\eta^2(2z)$	2	20	1
$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$	2	24	1
$\eta(15z)\eta(5z)\eta(3z)\eta(z)$	2	15	1
$\eta(14z)\eta(7z)\eta(2z)\eta(z)$	2	14	1
$\eta^2(9z)\eta^2(3z)$	2	27	1
$\eta^2(11z)\eta^2(z)$	2	11	1
$\eta^3(6z)\eta^3(2z)$	3	12	$\left(\frac{-3}{d}\right)$
$\eta^6(4z)$	3	16	$\left(\frac{-1}{d}\right)$
$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$	3	8	$\left(\frac{-2}{d}\right)$
$\eta^3(7z)\eta^3(z)$	3	7	$\left(\frac{-7}{d}\right)$
$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$	4	6	1
$\eta^4(5z)\eta^4(z)$	4	5	1
$\eta^8(3z)$	4	9	1
$\eta^4(4z)\eta^4(2z)$	4	8	1
$\eta^4(4z)\eta^2(2z)\eta^4(z)$	5	4	$\left(\frac{-1}{d}\right)$
$\eta^6(3z)\eta^6(z)$	6	3	1
$\eta^{12}(2z)$	6	4	1
$\eta^8(2z)\eta^8z)$	8	2	1
$\eta^{24}(z)$	12	1	1

We add to this list two cusp forms of half-integral weight, $\eta(24z), \eta^3(8z)$. Their coefficients are also multiplicative.

3. Shimura sums. Theorems Cipra and K. Ono.

Definition.
Let a(n) be an arithmetic function and c a positive integer.
Then for $m \geq 1$ the Shimura sum $Sh(m, a, c)$ is defined by the formula:

$$Sh(m, a, c) = \sum_{j=1}^{m-1} a\left(\frac{m^2 - j^2}{c}\right).$$

If the argument of the function $a(n)$ is fractional then its value is equal to 0.
Let $f(z)$ be a cusp form of integral weight k,

$$\Theta(z) = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

$$F(z) = f(4z)\Theta(z) = \sum_{n=1}^{\infty} b(n)q^{n^2} \in S_{k+\frac{1}{2}}(4N, \chi).$$

Let t be a square-free positive integer.
We define $A_t(n)$ by the formal product:

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = L(s-k+1, \chi_t^{(k)}) \sum_{m=1}^{\infty} \frac{b(tm^2)}{m^s}.$$

Here $\chi_t^{(k)}(m) = \chi(m)\left(\frac{-1}{m}\right)^k \left(\frac{t}{m}\right)$ is a Dirichlet character *mod* $4Nt$,
$\chi_1^{(k)}(m) = \chi(m)\left(\frac{-1}{m}\right)^k$.
The image of $F(z)$ under the Shimura lift is defined by the formula

$$S_t(F) = \sum_{n=1}^{\infty} A_t(n)q^n.$$

G.Shimura proved [4] that $S_t(F) \in S_{2k}(2N, \chi^2)$ *if* $k > 1$,
$S_t(F) \in M_{2k}(2N, \chi^2)$ *if* $k = 1$.
Theorem Cipra. *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \chi)$$

be an integral weight newform, and

$$F(z) = f(4z)\Theta(z) = \sum_{n=1}^{\infty} b(n)q^{n^2}.$$

Then

$$S_1(F) = f^2(z) - 2^{k-1}\chi(2)f^2(2z).$$

In this case

$$A(n) = A_1(n) = \sum_{d|n} d^{k-1}\chi_1^{(k)}(d)b(\frac{n^2}{d^2}),$$

$$b(m^2) = a(\frac{m^2}{4}) + 2\sum_{j=1}^{m-1} a\left(\frac{m^2-j^2}{4}\right) = a(\frac{m^2}{4}) + 2Sh(m, a, 4).$$

This theorem was proved in [5].
Also we see that $Sh(2n, a, 4) = Sh(n, a, 1)$.
Theorem Ono.

$$If \quad \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_3(8, \chi),$$

then

1) *If p is inert in* $K = \mathbf{Q}(\sqrt{-2})$, *then*

$$p = \sqrt{-\ \frac{Sh(4p, a, 4)}{2}}\ .$$

2) *If p splits or ramifies in* $K = \mathbf{Q}(\sqrt{-2})$, *then*

$$p = \sqrt{a^2(p) + \ \frac{Sh(4p, a, 4)}{2}}\ .$$

This theorem was proved in [16]. In this case $\chi(2) = 0$ and $a(4) \neq 0$. This condition is essential. For many other multiplicative $\eta-$products if $\chi(2) \neq 0$ then $a(2n) = 0$, and we cannot use the similar arguments in our considerations. Also we cannot find the expressions for p from the Cipra theorem when the weight $k = 1$.

We shall study the Shimura sums related to some other multiplicative $\eta-$products without the Cipra theorem in the sections 4 and 5 and discuss the application of this theorem in the section 6.

4. The coefficients of multiplicative $\eta-$products and Shimura sums.

Here we prove the following

Theorem 1.

$$If \quad f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \chi) \ is\ such\ a\ multiplicative\ \eta - product\ that$$

$$h(z) = f^2(\frac{z}{2}) = \sum_{n=1}^{\infty} c(n)q^n \in S_{2k}(N) \ is\ also\ a\ multiplicative\ \eta - product\ then$$

$c(p) = 2Sh(p, a, 1) + a^2(p);$ (4.1.)

$c(p^2) = 2Sh(p^2, a, 1) + 2\chi(p)p^{k-1}Sh(p, a, 1) + a^2(p^2);$ (4.2.)

$p^{2k-1} = (2Sh(p, a, 1) + a^2(p))^2 - 2Sh(p^2, a, 1) - a^2(p^2) - 2\chi(p)p^{k-1}Sh(p, a, 1) =$
$(2Sh(p, a, 1) + a^2(p))^2 - 2Sh(p^2, a, 1) - a^2(p^2) - 2(a^2(p) - a(p^2))Sh(p, a, 1).$ (4.3.)

Proof.

There are 13 pairs of such functions (f(z),h(z)). We point them out in the following table.

Table 2.

$f(z)$	k	N	$\chi(p)$	$h(z)$
$\eta(22z)\eta(2z)$	1	44	$\left(\frac{-11}{p}\right)$	$\eta^2(11z)\eta^2(z)$
$\eta(20z)\eta(4z)$	1	80	$\left(\frac{-5}{p}\right)$	$\eta^2(10z)\eta^2(2z)$
$\eta(18z)\eta(6z)$	1	108	$\left(\frac{-3}{p}\right)$	$\eta^2(9z)\eta^2(3z)$
$\eta(16z)\eta(8z)$	1	128	$\left(\frac{-2}{p}\right)$	$\eta^2(8z)\eta^2(4z)$
$\eta^2(12z)$	1	144	$\left(\frac{-1}{p}\right)$	$\eta^4(6z)$
$\eta^4(6z)$	2	36	1	$\eta^8(3z)$
$\eta^2(8z)\eta^2(4z)$	2	32	1	$\eta^4(4z)\eta^4(2z)$
$\eta^2(10z)\eta^2(2z)$	2	20	1	$\eta^4(5z)\eta^4(z)$
$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$	2	24	1	$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$
$\eta^3(6z)\eta^3(2z)$	3	12	$\left(\frac{-3}{p}\right)$	$\eta^6(3z)\eta^6(z)$
$\eta^6(4z)$	3	16	$\left(\frac{-1}{p}\right)$	$\eta^{12}(2z)$
$\eta^4(4z)\eta^4(2z)$	4	8	1	$\eta^8(2z)\eta^8(z)$
$\eta^{12}(2z)$	6	4	1	$\eta^{24}(z)$

In all these cases $a(2m) = 0, a(1) = 1$. So we consider only $a(n)$ for odd n in all our sums. Of course we mean that $\chi(2) = 0$ in all these cases because levels N are even.

$$1)\quad c(p) = \sum_{j=1}^{2p-1} a(j)a(2p-j) = 2\sum_{j=1}^{p-1} a(j)a(2p-j) + a^2(p) = [t = p-j] =$$

$$= 2\sum_{t=1}^{p-1} a(p-t)a(p+t) + a^2(p) = 2\sum_{t=1}^{p-1} a(p^2-t^2) + a^2(p) = 2Sh(p,a,1) + a^2(p).$$

If $p-t$ and $p+t$ are odd and $1 \le t \le p-1$ then $d = gcd(p-t, p+t) = 1$, because $d|\ 2p$. We use the multiplicativity of the coefficients $a(n)$.

$$2)\quad c(p^2) = \sum_{j=1}^{2p^2-1} a(j)a(2p^2-j) = 2\sum_{j=1}^{p^2-1} a(j)a(2p-j) + a^2(p^2) =$$

$$= 2\sum_{j=1, j\neq pl}^{p^2-1} a(j)a(2p^2-j) + 2\sum_{l=1}^{p-1} a(pl)a(2p^2-pl) + a^2(p^2) =$$

$$= 2\sum_{j=1, j\neq pl}^{p^2-1} a(j)a(2p^2-j) + 2a^2(p)\sum_{l=1}^{p-1} a(l)a(2p-l) + a^2(p^2) = \left[t = p^2 - j\right] =$$

$$= 2\sum_{t=1, (t,p)=1}^{p^2-1} a(p^2-t)a(p^2+t) + 2a^2(p)Sh(p,a,1) + a^2(p).$$

$$2Sh(p^2,a,1) = 2\sum_{t=1,(t,p)=1}^{p^2-1} a(p^4-t^2) + 2\sum_{l=1}^{p-1} a(p^4-p^2l^2) =$$

$$= 2\sum_{t=1,(t,p)=1}^{p^2-1} a(p^2-t^2)a(p^2+t^2) + 2a(p^2)\sum_{l=1}^{p-1} a(p^2-l^2) =$$

$$= 2 \sum_{t=1,(t,p)=1}^{p^2-1} a(p^2-t)a(p^2+t) + 2a(p^2)Sh(p,a,1).$$

If p^2-t and p^2+t are odd and $(t,p)=1$ then $d = gcd(p^2-t, p^2+t) = 1$, because $d|\ 2p^2$. We use the multiplicativity of the coefficients $a(n)$. Comparing these two expressions we obtain:

$c(p^2) = 2Sh(p^2,a,1) + 2(a^2(p^2) - a(p^2))Sh(p,a,1) + a^2(p^2)$.

Multiplicative $\eta-$products are Hecke eigenforms and $\chi(p)p^{k-1} + a(p^2) = a^2(p)$.

From this relation we have

$c(p^2) = 2Sh(p^2,a,1) + 2\chi(p)p^{k-1}Sh(p,a,1) + a^2(p^2)$.

3) From the condition $p^{2k-1} = c^2(p) - c(p^2)$ we obtain

$p^{2k-1} = (2Sh(p,a,1) + a^2(p))^2 - 2Sh(p^2,a,1) - a^2(p^2) - 2\chi(p)p^{k-1}Sh(p,a,1)$.

In many cases we can simplify these basic expressions.

Example 1.

$f(z) = \eta(20z)\eta(4z), \quad \chi(p) = \left(\frac{-5}{p}\right), \chi(2) = 0, \quad p = 7, \quad k = 1.$

$\chi(7) = 1, a(49) = 0, Sh(7,a,1) = a(45) = 1, Sh(49,a,1) = a(801) = a(9)a(89) = -2.$

Other summands are equal to 0.

In the expression (4.3.) in theorem 1 we have the values: $7 = 4+4+1-2\cdot 1$.

Corollary.

Let $f(z)$ and $h(z)$ be multiplicative$\eta-$products and let K be a quadratic number field as given in Table 3. If p is inert in K then $c(p) = 2Sh(p,a,1)$.

Table 3.

$f(z)$	$h(z)$	K
$\eta(22z)\eta(2z)$	$\eta^2(11z)\eta^2(z)$	$\mathbf{Q}(\sqrt{-11})$
$\eta(20z)\eta(4z)$	$\eta^2(10z)\eta^2(2z)$	$\mathbf{Q}(\sqrt{-5})$
$\eta(18z)\eta(6z)$	$\eta^2(9z)\eta^2(3z)$	$\mathbf{Q}(\sqrt{-3}))$
$\eta(16z)\eta(8z)$	$\eta^2(8z)\eta^2(4z)$	$\mathbf{Q}(\sqrt{-2})$
$\eta^2(12z)$	$\eta^4(6z)$	$\mathbf{Q}(\sqrt{-1})$
$\eta^4(6z)$	$\eta^8(3z)$	$\mathbf{Q}(\sqrt{-3})$
$\eta^2(8z)\eta^2(4z)$	$\eta^4(4z)\eta^4(2z)$	$\mathbf{Q}(\sqrt{-1})$
$\eta^3(6z)\eta^3(2z)$	$\eta^6(3z)\eta^6(z)$	$\mathbf{Q}(\sqrt{-3})$
$\eta^6(4z)$	$\eta^{12}(2z)$	$\mathbf{Q}(\sqrt{-1})$

Proof. We know [1] that in the cases we consider, if p is inert in K, then $a(p) = 0$. The corollary follows from the formula (4.1.).

5. The arithmetic of quadratic fields and Shimura sums.

Theorem 2.

$$\textit{Let } f(z) \textit{ be } \ \eta(18z)\eta(6z) = \sum_{n=1}^{\infty} a(n)q^n \in S_1(108,\chi).$$

1) *If p is inert in $K = \mathbf{Q}(\sqrt{-3})$, then*

$$p = -2Sh(p^2,a,1) - 1.$$

2) *If p splits in* $K = \mathbf{Q}(\sqrt{-3})$, *then*

$$p = (2Sh(p,a,1) + a^2(p))^2 - 2Sh(p^2,a,1) - a^2(p^2) - 2Sh(p,a,1).$$

In particular, if $|a(p)| = 1$, *then*

$$p = 4Sh^2(p,a,1) - 2Sh(p^2,a,1) + 2Sh(p,a,1) + 1;$$

if $|a(p)| = 2$, *then*

$$p = 4Sh^2(p,a,1) - 2Sh(p^2,a,1) + 14Sh(p,a,1) - 5.$$

Proof.

In this case we have

$$h(z) = \eta^2(9z)\eta^2(3z) = \sum_{n=1}^{\infty} c(n)q^n \in S_2(27).$$

1) It is known [1] that if p is inert in $\mathbf{Q}(\sqrt{-3})$ then $a(p) = 0,\ c(p) = 0,\ a^2(p^2) = 1$. We have $2Sh(p,a,1) + a^2(p) = c(p)$. Hence, $Sh(p,a,1) = 0$. From the formula (4.3.) we have $p = -2Sh(p^2,a,1) - 1$.

2) In this case $\chi(p) = 1$. From the relation $\chi(p) + a(p^2) = a^2(p)$ we obtain: if $|a(p)| = 1$ then $a^2(p^2) = 0$, and if $|a(p)| = 2$ then $a^2(p^2) = 9$. The formulas for p are obtained from (4.3).

Example 2.

p= 7.

$a(7) = -1,\ Sh(7,a,1) = a(13) = -1,$

$Sh(49,a,1) = a(2257) + a(2077) + a(1825) + a(1501) + a(97) =$

$= a(37)a(61) + a(31)a(67) + a(25)a(73) + a(19)a(79) + a(97) = 1 - 2 - 1 + 1 - 1 = -2.$

Other summands are equal to 0.

In the expression $p = 4Sh^2(p,a,1) - 2Sh(p^2,a,1) + 2Sh(p,a,1) + 1$ in theorem 2 we have the values: $7 = 4 + 4 - 2 + 1$.

p= 5.

$Sh(25,a,1) = a(19)a(31) + a(13)a(37) + a(7)a(43) = -2 + 1 - 2 = -3.$

Other summands are equal to 0.

In the expression $p = -2Sh(p^2,a,1) - 1$ in theorem 2 we have the values:

$5 = (-2)\cdot(-3) - 1.$

Theorem 3.

$$\textit{Let } f(z) \textit{ be } \eta^2(12z) = \sum_{n=1}^{\infty} a(n)q^n \in S_1(144,\chi).$$

1) *If p is inert in* $K = \mathbf{Q}(\sqrt{-3})$, *then*

$$p = -2Sh(p^2,a,1) - 1.$$

2) *If p splits in* $K = \mathbf{Q}(\sqrt{-3})$, *then* $p = l^2 + 3m^2$, *where*

$\left(\frac{l}{3}\right)\, l = Sh(p,a,1) + 2.$

$$p = (2Sh(p,a,1) + a^2(p))^2 - 2Sh(p^2,a,1) - a^2(p^2) - 2Sh(p,a,1).$$

Proof.

In this case we have

$$h(z) = \eta^4(6z) = \sum_{n=1}^{\infty} c(n)q^n \in S_2(36).$$

1) It is known [1] that if p is inert in $\mathbf{Q}(\sqrt{-3})$ then $a(p) = 0,\ c(p) = 0,\ a^2(p^2) = 1$. We have $2Sh(p,a,1) + a^2(p) = c(p)$. Hence, $Sh(p,a,1) = 0$. From the formula (4.3.) we have $p = -2Sh(p^2,a,1) - 1$.

2) From the known formula

$$\eta^4(6z) = \sum_{l=1}^{\infty}(-1)^{l-1}(\frac{l}{3})lq^{l^2} + \sum_{l,m=1}^{\infty}(-1)^{l+m-1}(\frac{l}{3})lq^{l^2+3m^2}.$$

we see that

$$c(p) = \begin{cases} 2(\frac{l}{3})\ l,\ if\ p = l^2 + 3m^2. \\ 0, \qquad\qquad otherwise. \end{cases}$$

If $p \cong 1(3)$ then $|a(p)| = 2$, and we obtain from the formula (4.1.) the relation $(\frac{l}{3})\ l = Sh(p,a,1) + 2$.

The last formula in our theorem is proved as in the theorem 2.

Example 3.

p= 5.

$Sh(25,a,1) = a(1)a(49) + a(13)a(37) = 1 - 4 = -3.$

$p = -2Sh(p^2,a,1) - 1 = 5 = (-2)\cdot(-3) - 1.$

p = 37.

$37 = 5^2 + 3\cdot 2^2.$

$Sh(37,a,1) = -7$, and $5 = l = -Sh(37,a,1) - 2 = 7 - 2.$

Theorem 4.

Let us consider

$$\eta^4(6z) = \sum_{n=1}^{\infty} a(n)q^n \in S_2(36).$$

1) *If p is inert in $K = \mathbf{Q}(\sqrt{-3})$, then*

$$p^3 = -2Sh(p^2,a,1) - a^2(p^2).$$

2) *If p splits in $K = \mathbf{Q}(\sqrt{-3})$, then*

$$p^3 = (2Sh(p,a,1) + a^2(p))^2 - 2Sh(p^2,a,1) - a^2(p^2) - 2(a^2(p) - a(p^2))Sh(p,a,1).$$

Proof.

In this case we have

$$h(z) = \eta^8(3z) = \sum_{n=1}^{\infty} c(n)q^n \in S_4(9).$$

1) It is known [1] that if p is inert in $\mathbf{Q}(\sqrt{-3})$ then $a(p) = 0,\ c(p) = 0,\ a^2(p^2) = 1$. We have $2Sh(p,a,1) + a^2(p) = c(p)$. Hence, $Sh(p,a,1) = 0$. From the formula (4.3.) we have $p = -2Sh(p^2,a,1) - 1$.

2) The formula for p^3 is obtained from (4.3) in case $k=2$.

Theorem 5.

$$Let\ f(z)\ be\ \ \eta(16z)\eta(8z)=\sum_{n=1}^{\infty}a(n)q^n \in S_1(128,\chi).$$

1) *If p splits in $\mathbf{Q}(\sqrt{-1})$ and in $\mathbf{Q}(\sqrt{-2})$ then*

$$p=4Sh^2(p,a,1)-2Sh(p^2,a,1)+14Sh(p,a,1)-5.$$

2) *If p is inert in $\mathbf{Q}(\sqrt{-1})$, then*

$$p=-2Sh(p^2,a,1)-1.$$

3) *If p splits in $\mathbf{Q}(\sqrt{-1})$, and p is inert in $\mathbf{Q}(\sqrt{-2})$, then*

$$p=4Sh^2(p,a,1)-2Sh(p^2,a,1)+2Sh(p,a,1)-1.$$

Proof.

The character $\chi(p)=(\frac{-2}{p}),\ p-odd,\ \chi(2)=0$.

1) In this case $\chi(p)=1,\ p\cong 1(8)$. It can be proved by elementary methods that if $a(p)\neq 0$ then $|a(p)|=2$. From the relation $\chi(p)+a(p^2)=4$ we obtain $a(p^2)=3$. The formula for p follows from (4.3.).

2) In this case $p\cong 3(8)$ and $p\cong 7(8)$. We see that $a(p)=0$ and $c(p)=0$ for such p. we obtain from (4.1.) $Sh(p,a,1)=0$ and from (4.3.) $p=-2Sh(p^2,a,1)-1$.

3) In this case $p\cong 5(8),\ \chi(p)=-1$. In this case $a(p)=0,\ a^2(p^2)=1$. The formula for p follows from (4.3.).

6. The application of the Cipra theorem.

For the modular forms listed in the table 1 $\chi(2)=0$ (N is even), $a(2)=0,\ \ c(2p)=A(4p)$.

Also we have $Sh(2n,a,4)=Sh(n,a,1),\ Sh(4,a,4)=Sh(2,a,1)=a(3),\ c(2)=2a(3)$.

From the Cipra theorem we obtain:

$$A(4p)=a(4p^2)+\chi(p)p^{k-1}a(4)+2p^{k-1}\chi(p)Sh(4,a,4)+2Sh(4p,a,4)=c(2)c(p)=2a(3)c(p).$$

We obtain the formula

$$a(3)\chi(p)p^{k-1}=-Sh(2p,a,1)+a(3)(c(p). \qquad (6.1.)$$

For the functions

$\eta(20z)\eta(4z),\ \eta(18z)\eta(6z),\ \eta(16z)\eta(8z),\ \eta^2(12z),\ \eta^4(6z),\eta^2(8z)\eta^2(4z),\ \eta^6(4z)$

the coefficient $a(3)=0$ and $Sh(2p,a,1)=0$.

For other six modular forms from the table 1 $a(3)\neq 0$. In Table4, we give analogous results for five of them. The remaining form, $\eta^3(6z)\eta^3(2z)$, is considered in Theorem 5.

Table 4.

$f(z)$	$\chi(p)p^{k-1}$
$\eta(22z)\eta(2z)$	$(\frac{-11}{p})=Sh(2p,a,1)+2Sh(p,a,1)+a^2(p)$
$\eta^2(10z)\eta^2(2z)$	$p=\frac{1}{2}\cdot Sh(2p,a,1)+2Sh(p,a,1)+a^2(p)$
$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$	$p=Sh(2p,a,1)+2Sh(p,a,1)+a^2(p)$
$\eta^4(4z)\eta^4(2z)$	$p^3=\frac{1}{4}\cdot Sh(2p,a,1)+2Sh(p,a,1)+a^2(p)$
$\eta^{12}(2z)$	$p^5=\frac{1}{12}\cdot Sh(2p,a,1)+2Sh(p,a,1)+a^2(p)$

Theorem 6.

$$If \quad \eta^3(6z)\eta^3(2z) = \sum_{n=1}^{\infty} a(n)q^n \in S_3(12,\chi),$$

then

1) *If p is inert in $\mathbf{Q}(\sqrt{-3})$, then*

$$p = \sqrt{- \frac{Sh(2p,a,1)}{3} - 2Sh(p,a,1)} \ .$$

2) *If p splits in $\mathbf{Q}(\sqrt{-3})$, then*

$$p = \sqrt{\frac{Sh(2p,a,1)}{3} + 2Sh(p,a,1) + a^2(p)} \ .$$

In particular, $3|Sh(2p,a,1)$.

Proof.

We know [1] that $a(p) = 0$ if p is inert in $\mathbf{Q}(\sqrt{-3})$, $a(3) = -3$.
The theorem follows from the formulas (4.1.) and (6.1.).

Example 4 .

p = 5.

In this case $\chi(5) = -1$, $a(5) = 0$, $Sh(10,a,1) = -93$, $Sh(5,a,1) = 3$.
$p = \sqrt{- \frac{Sh(2p,a,1)}{3} - 2Sh(p,a,1)} = 5 = \sqrt{- \frac{(-93)}{3} - 2\cdot 3}$.

p = 7.

In this case $\chi(7) = 1$, $a(7) = 2$, $Sh(14,a,1) = 267$, $Sh(7,a,1) = -22$.
$p = \sqrt{\frac{Sh(2p,a,1)}{3} + 2Sh(p,a,1) + a^2(p)} = 7 = \sqrt{\frac{267}{3} + 2\cdot(-22) + 4}$.

7. The relations between Shimura sums for different multiplicative $\eta-$products .

Proposition 1.

$$If \ f(z) = \eta(21z)\eta(3z) = \sum_{n=1}^{\infty} a(n)q^n \in S_1(63,\chi),$$

$$g(z) = \eta(16z)\eta(8z) = \sum_{n=1}^{\infty} c(n)q^n \in S_1(128,\chi),$$

$$h(z) = \eta(14z)\eta(7z)\eta(2z)\eta(z) = \sum_{n=1}^{\infty} d(n)q^n \in S_2(14),$$

then

$$1) \ \ d(n) = \sum_{2j+l=3n} a(j)a(l) = \sum_{7j+l=8n} c(j)c(l),$$

$$2) \ \ d(p) = Sh(3p,a,8) + \chi(p) = Sh(4p,c,7) + \tilde{\chi}(p).$$

In these sums $j > 1$, $l > 1$, p is prime.
$\chi(p) = (\frac{-7}{p})$, $\tilde{\chi}(p) = (\frac{-2}{p})$, $\tilde{\chi}(2) = 0$.

In particular, if $(\frac{-7}{p}) = (\frac{-2}{p}),\ p > 2,$ *then*

$$Sh(3p, a, 8) = Sh(4p, c, 7).$$

Proof.

1) Let us consider

$$f(2z)f(z) = \eta(42z)\eta(6z)\eta(21z)\eta(3z) = \sum_{n=1}^{\infty} u(n)q^n.$$

We have

$$u(3n) = d(n);\ u(3n) = \sum_{2j+l=3n} a(j)a(l).$$

For $p = 2$ we calculate $Sh(6, a, 8) = 0,\ Sh(8, c, 7) = -1,\ d(2) = -1,$ and $d(p) = Sh(3p, a, 8) + \chi(p) = Sh(4p, c, 7) + \tilde{\chi}(p).$
In the case $p > 2.$

$$Sh(3p, a, 8) = \sum_{t=1,(t,p)=1}^{3p-1} a\left(\frac{(3p)^2 - t^2}{8}\right) + a(p^2) = [t = 3p - 2l] =$$

$$= \sum_{l=1,(l,p)=1}^{\frac{3p-1}{2}} a\left(\frac{(6p-l)(2l)}{8}\right) + a(p^2) =$$

$$= \sum_{l=1,(l,p)=1}^{\frac{3p-1}{2}} a\left(\frac{(3p-l)l}{2}\right) + a(p^2) = \frac{1}{2} \cdot (\sum_{l=1,(l,p)=1}^{3p-1} a\left(\frac{(3p-l)l}{2}\right) + 2a(p^2)) =$$

$$\frac{1}{2} \cdot (\sum_{l=1,(l,p)=1}^{3p-1} a\left(\frac{(3p-l)l}{2}\right) + 2a(p^2)) =$$

$$\frac{1}{2} \cdot (\sum_{l=1,(l,p)=1,l-odd}^{3p-1} a\left(\frac{(3p-l)l}{2}\right) + \frac{1}{2} \cdot (\sum_{l=1,(l,p)=1,l-even}^{3p-1} a\left(\frac{(3p-l)l}{2}\right) + 2a(p^2)) =$$

$$\frac{1}{2} \cdot (\sum_{l=1,(l,p)=1,l-odd}^{3p-1} a\left(\frac{(3p-l)}{2}\right) a(l) +$$

$$\frac{1}{2} \cdot (\sum_{l=1,(l,p)=1,l-even}^{3p-1} a(3p-l) a\left(\frac{l}{2}\right) + 2a(p^2)) =$$

$$\frac{1}{2} \cdot (d(p) - a^2(p) + d(p) - a^2(p) + 2a(p^2)) = d(p) - \chi(p).$$

We have $d(p) = Sh(3p, a, 8) + \chi(p).$

2) Now let us consider

$$g(7z)g(z) = \eta(112z)\eta(56z)\eta(16z)\eta(8z) = \sum_{n=1}^{\infty} v(n)q^n.$$

We have

$$v(8n) = d(n);\ v(8n) = \sum_{7j+l=8n} c(j)c(l).$$

$$Sh(8p,c,7)=\sum_{t=1,(t,p)=1}^{8p-1} c\left(\frac{(4p)^2-t^2}{7}\right)+c(p^2)=[t=4p-l]=$$

$$=\sum_{l=1,(l,p)=1}^{8p-1} c\left(\frac{(8p-l)l}{8}\right)+c(p^2)=$$

$$\frac{1}{2}\cdot(\sum_{l=1,(l,7p)=1}^{8p-1} c\left(\frac{(8p-l)}{7}\right)c(l)+\frac{1}{2}\cdot(\sum_{l=1,(l,p)=1,7|l}^{8p-1} c(8p-l)c\left(\frac{l}{7}\right)+2c(p^2))=$$

$$\frac{1}{2}\cdot(d(p)-c^2(p)+d(p)-c^2(p)+2c(p^2))=d(p)-\tilde{\chi}(p).$$

We have $d(p)=Sh(8p,c,7)+\tilde{\chi}(p)$.
In the proof we use the the multiplicativity of the coefficients $a(n)$ and $c(n)$ and the properties: $a(n)=0$ except $n\cong 1(3)$, $c(n)=0$ except $n\cong 1(8)$.

Example 5.
p = 5.
$\chi(5)=-1,\quad \tilde{\chi}(5)=-1,\quad d(5)=0,$
$Sh(15,a,8)=a(7)+a(25)+a(28)=-1+1+1=1,\quad Sh(20,c,7)=c(25)=1.$
$d(5)=0=1-1=1-1.$
Analogously we can prove the following propositions.

Proposition 2.

$$If\ f(z)=\eta(18z)\eta(6z)=\sum_{n=1}^{\infty}a(n)q^n\in S_1(108,\chi),$$

$$g(z)=\eta(16z)\eta(8z)=\sum_{n=1}^{\infty}c(n)q^n\in S_1(128,\tilde{\chi}),$$

$$h(z)=\eta(12z)\eta(6z)\eta(4z)\eta(z)=\sum_{n=1}^{\infty}d(n)q^n\in S_2(12),$$

then

$$1)\ \ d(n)=\sum_{2j+l=3n}a(j)a(l)=\sum_{3j+l=4n}c(j)c(l),$$

$$2)\ \ d(p)=Sh(3p,a,8)+\chi(p)=Sh(2p,c,3)+\tilde{\chi}(p).$$

In these sums $j>1,\ l>1,\ p$ *is prime.*
$\chi(p)=(\frac{-3}{p}),\ \tilde{\chi}(p)=(\frac{-2}{p})$ *for* $p>2;\ \chi(2)=\ \tilde{\chi}(2)=0.$
In particular, if $(\frac{-3}{p})=(\frac{-2}{p}),\ p>2,$ *then*

$$Sh(3p,a,8)=Sh(2p,c,3).$$

Example 6.
p = 5.
$\chi(5)=-1,\quad \tilde{\chi}(5)=-1,\quad d(5)=-2,$
$Sh(15,a,8)=a(7)+a(13)+a(25)=-1-1+1=-1,$
$Sh(10,c,3)=c(17)+c(25)=-2+1=-1.$
$d(5)=-2=-1-1=-1-1.$

Proposition 3.

$$If\ \ f(z) = \eta(20z)\eta(4z) = \sum_{n=1}^{\infty} a(n)q^n \in S_1(80, \chi),$$

$$g(z) = \eta(18z)\eta(6z) = \sum_{n=1}^{\infty} c(n)q^n \in S_1(108, \tilde{\chi}),$$

$$h(z) = \eta(15z)\eta(5z)\eta(3z)\eta(z) = \sum_{n=1}^{\infty} d(n)q^n \in S_2(15),$$

then

$$1)\ \ d(n) = \sum_{3j+l=4n} a(j)a(l) = \sum_{5j+l=6n} c(j)c(l),$$

$$2)\ \ d(p) = Sh(2p, a, 3) + \chi(p) = Sh(3p, c, 5) + \tilde{\chi}(p).$$

In these sums $j > 1,\ l > 1,\ p$ *is prime.*
$\chi(p) = (\frac{-5}{p}),\ \tilde{\chi}(p) = (\frac{-3}{p})$ *for* $p > 2;\ \chi(2) = \tilde{\chi}(2) = 0.$
In particular, *if* $(\frac{-5}{p}) = (\frac{-3}{p}),\ p > 2,$ *then*

$$Sh(2p, a, 3) = Sh(3p, c, 5).$$

Example 7.
p = 7.
$\chi(7) = 1, \quad \tilde{\chi}(7) = 1, \quad d(7) = 0,$
$Sh(14, a, 3) = a(9) + a(25) + a(49) = -1 + 1 - 1 = -1, \quad Sh(21, c, 5) = c(37) = -1.$
$d(7) = 0 = -1 + 1 = -1 + 1.$

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