ON RATIONAL CUSPIDAL CURVES I. SHARP ESTIMATE FOR DEGREE VIA MULTIPLICITIES

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Abstract. Let $C \subset \mathbf{P}^2$ be a rational curve of degree d which has only one analytic branch at each point. Denote by m the maximal multiplicity of singularities of C. It is proven in [**MS**] that $d < 3m$. We show that $d < \alpha m + \text{const}$ where $\alpha = 2.61...$ is the square of the "golden section". We also construct examples which show that this estimate is asymptotically sharp. When $\bar\kappa(\mathbf{P}^2 - C) = -\infty$, we show that $d > \alpha m$ and this estimate is sharp.

The main tool used here, is the logarithmic version of the Bogomolov-Miyaoka-Yau inequality. For curves as above we give an interpretation of this inequality in terms of the number of parameters describing curves of a given degree and the number of conditions imposed by singularity types.

Let C be an algebraic curve in $\mathbf{P}^2$ (over $\mathbf{C}$) which is homeomorphic to $\mathbf{CP}^1$.

This means that C is rational and *cuspidal*, i.e. all its singularities are cusps where *cusp* means a singularity at which the curve has only one analytic branch. Such curves were studied, for instance, in [**Y1-Y3**], [**Ka**], [**MS**], [**FZ2**], [**OZ3**].

Denote by m_p the multiplicity of C at a point p (i.e. $m_p = (C \cdot L)_p$ for a generic line L). Let d be the degree of C and $m = \max_{p\in C} m_p$.

Put $\alpha = (3+\sqrt5)/2 = 2.6180...$ (α is a root of $\alpha + \frac{1}{\alpha} = 3$).

Theorem A. $d < \alpha(m+1) + 1/\sqrt5 = \alpha m + 3.0652...$

For $m > 10$ this estimate is stronger than the estimate $d < 3m$ obtained by T. Matsuoka and F. Sakai [**MS**] though the method of the proof is very similar. Denote by $\bar\kappa(Y)$ the logarithmic Kodaira dimension of a non-complete surface Y (see [**F**]).

Theorem B.

(a). If $\bar\kappa(\mathbf{P}^2 \setminus C) = -\infty$ *then* $d < \alpha m$.

(b). If $\bar\kappa(\mathbf{P}^2 \setminus C) = 2$ *then* $d < \alpha(m+1) - 1/\sqrt5 = \alpha m + 2.1708...$

(c). $\bar\kappa(\mathbf{P}^2 \setminus C) \ne 0$.

Denote by $\varphi_0, \varphi_1, \varphi_2, \dots$ the Fibonacci numbers: $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_{k+2} = \varphi_k + \varphi_{k+1}$.

Theorem C. *For any* $j > 0$, $j \not\equiv 2 \mod 4$ *there exists a rational cuspidal curve* C_j *of degree* $d_j = \varphi_{j+2}$ *which has a single cusp of multiplicity* $m_j = \varphi_j$. *Thus,* $\lim d_j/m_j = \alpha$.

*(a). ([**Ka**]; see Remark 3) If* j *is odd then* $\bar\kappa(\mathbf{P}^2 \setminus C_j) = -\infty$ *and the cusp of* C_j *has one characteristic pair* (m_j, n_j) *where* $n_j = \varphi_{j+4}$.

(b). If j *is even (and hence, divisible by 4) then* $\bar\kappa(\mathbf{P}^2 \setminus C_j) = 2$. *The cusp of* C_j *for* $j = 8, 12, ...$ *has two characteristic pairs*[1] $(\varphi_j, \varphi_{j+4})$ *and* $(3,1)$*; the cusp of* C_4 *has one characteristic pair* $(\varphi_4, \varphi_8 + 1) = (3, 22)$.

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[1] See the definition in §3.

(c). For any $j > 0$, $j \equiv 0 \mod 4$ there exist a rational cuspidal curve C^*_j of degree $d^*_j = 2\varphi_{j+2}$ which has a single cusp of multiplicity $m^*_j = 2\varphi_j$. $\bar\kappa(\mathbf{P}^2 \setminus C_j) = 2$. The cusp of C^*_j for $j = 8, 12, \ldots$ has two characteristic pairs $(2\varphi_j, 2\varphi_{j+4})$ and $(6,1)$; the cusp of C^*_4 has one characteristic pair $(2\varphi_4, 2\varphi_4 + 1) = (6,43)$.

Since $\{d_j/m_j\}$ with odd j is the sequence of the upper convergents of the continuous fraction of α, Theorem C.a shows that the estimate in Theorem B.a is "sharp" in the following sense: the convex hull of all pairs $(m,d) \in \mathbf{Z}^2$ satisfying $m+1 \le d < \alpha m$ coincides with the convex hull of all pairs (m,d) realizable by rational cuspidal curves C with $\bar\kappa(\mathbf{P}^2 \setminus C) = -\infty$ (obviously, the opposite bound $m+1 \le d$ is sharp due to the curves $y = x^d$).

In contrary, the curves C_{4k} and C^*_{4k} provide examples where $d > \alpha m$.

Conjecture. *C_{4k} and C^*_{4k} are the only rational cuspidal curves with $d > \alpha m$.*

It is possible to prove that this conjecture follows from Conjecture 2.3. The first six pairs (m,d) with $d > \alpha m$ which are not realized by Theorem C and which are neither forbidden by Theorem A nor by the theorem of Matsuoka—Sakai [**MS**] are $(4,11)$, $(5,14)$, $(6,17)$, $(7,19)$, $(7,20)$, and $(\varphi_6, \varphi_8) = (8,21)$. Using successive birational quadratic transformations the author checked that the cases $(4,11)$ and $(8,21)$ are not realizable. For example, in the case $(8,21)$ the only candidate is the curve of degree 21 with two cusps of types $(8,55)$ and $(2,3)$. After 8 quadratic transformations such a curve would be transformed into the cuspidal cubic with two inflection points.

Remark 1. The curve C_1 is a conic. C_3 is a quintic with the cusp A_{12} (see [**DG**], [**Y1**]).

Remark 2. The idea how to pass from the construction of C_{4k} to the construction of C^*_{4k} belongs to E. Artal.

Remark 3. A curve C is rational cuspidal and satisfies the condition $\bar\kappa(\mathbf{P}^2 \setminus C) = -\infty$ if and only if C is a fiber of a rational function on $\mathbf{P}^2$ all whose fibers (with the indeterminacy point removed) are isomorphic to $\mathbf{C}^1$. H. Kashiwara [**Ka**] has classified all such functions. Our curve C_{2l+3} $(l \ge 0)$ is one of the multiple fibers[2] of a function of the type $II(l)^*$. The degree is computed in [**Ka**; 7.2]; It is possible to compute by induction the multiplicity of the cusp using the recurrence relations (written in [**Ka**; 11.1]) for the defining polynomial.

Remark 4. It is shown in [**MS**] that each of the rational cuspidal curves known up to that moment, can be mapped onto a line by a birational transformation of $\mathbf{P}^2$. It is clear from the construction that the curves C_j from Theorem C satisfy this property. The only known to the author new series of examples appeared in literature after the paper [**MS**], is the series of rational tricuspidal curves constructed by H. Flenner and M. Zaidenberg [**FZ2**]. It is shown in [**A**] that they also can be mapped onto a line by Cremona transformations.

The main tool used, is the logarithmic version of Bogomolov — Miyaoka — Yau inequality. In the case of rational cuspidal curves we give (see the end of §2) an interpretation of this inequality in the form $\#equ \le 5 + \#var$ where $\#var$ and $\#equ$ are respectively the number of variables and the number of the equations in the system of simultaneous equations which appears if one wants to construct a rational curve of a degree d with the given list of types of singular branches using a parameterization by polynomials with indeterminate coefficients.

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[2] The left one on the picture in [**Ka**; 6.1] where the graphs of resolution are described.

§1. M-number of a singularity of a plane curve

Let $(C,0)$ be a germ of an analytically irreducible curve in a neighborhood U of $0 \in \mathbf{C}^2$ and let $\sigma : (X, D, E) \to (U, C, 0)$ be the minimal resolution of singularity of C at 0, i.e. σ is birational, $D = \sigma^{-1}(C)$ is a curve with simple normal crossings (SNC-curve) and (X, D) is minimal possible under these conditions.

Let V_E be the vector space over $\mathbf{Q}$ formally generated by the irreducible components $E_1, ..., E_n$ of E and $V_E^+ = \{\Sigma x_i E_i \,|\, x_i \ge 0\}$. Define D_E and K_E as the elements of V_E such that $E_i D_E = E_i D$ and $E_i(E_i + K_E) = -2$ for any $i = 1, ..., n$. Since the intersection form on E is non-degenerate, these elements exist and are unique.

Following [**F**], we define a *twig* of D as $T = E_{i_1} + ... + E_{i_m}$ such that $E_{i_j} E_{i_k} = 1$ for $|j-k| = 1$, $E_{i_j} E_{i_k} = 0$ for $|j-k| > 1$, $E_{i_1}(D - T) = 1$ and $E_{i_j}(D-T) = 0$ for $j > 1$. By the *local Zariski–Fujita decomposition near E* we mean $K_E + D_E = H_E + N_E$ where $H_E \in V_E$ and $N_E \in V_E^+$ are such that

(i). $\operatorname{supp} N_E$ is contained in the union of all twigs of D;

(ii). $H_E \cdot E_i \ge 0$ for all $i = 1, ..., n$; (iii). $H_E \cdot E_i = 0$ for $E_i \subset \operatorname{supp} N_E$.

Let $\mu = \mu_{(C,0)}$ be the Milnor number of $(C,0)$ and put

$$M = M_{(C,0)} = \mu + H_E^2; \qquad \overline{M} = \overline{M}_{(C,0)} = \mu + (K_E + D_E)^2.$$

Since the intersection form is negatively definite, we have $\mu > \overline{M} > M$. In §4 we prove some other inequalities for μ and M.

We shall call the number $\overline{M}$ (resp. M) the *rough* (resp. *fine*) *M-number* of the singularity $(C,0)$. When the singularity is analytically irreducible, the rough M-number $\overline{M}$ will be called also the *parametric codimension* of $(C,0)$ (see Proposition 3.4 for the motivation of this term).

In this paper we are going to use the language of Puiseux characteristic pairs but the authors of [**MS**] studying similar problems use an alternative language of multiplicity sequences. To compare these approaches, we give an expression of $\overline{M}$ in terms of the multiplicity sequence. It will not be used in the sequel. Let $(m_1, m_2, \dots, m_n)$ be the *multiplicity sequence* of $(C,0)$, i.e. the sequence of multiplicities of C in all the points which were blown up during the resolution process. Following [**MS**], we say that a blow-up is *subdivisional* if it is performed at an intersection point of two exceptional curves. Otherwise, it is called *sprouting*. By convention, the first blow-up is subdivisional. Let $\omega = \omega_{(C,0)}$ denote the number of subdivisional blow-ups. Then

$$\mu = \sum m_i(m_i - 1) \quad \text{(see [\textbf{Mi}])}; \qquad \overline{M} = \omega - 2 + \sum (m_i - 1) \quad \text{(see [\textbf{OZ2}; (37)])}.$$

§2. BMY-inequalities

Let X be a smooth projective surface and $D \subset X$ a reduced (maybe, reducible) SNC-curve. As usual, by $\bar\kappa$ we denote the logarithmic Kodaira dimension. Let $K = K_X$ be the canonical class of X. If $\bar\kappa(X \setminus D) \ge 0$, i.e. $|m(K+D)| \ne 0$ for infinitely many m, then (see [**F**]) there exists the Zariski decomposition $K + D = H + N$ where $H, N \in \operatorname{Pic} X \otimes \mathbf{Q}$ are such that

(i). N is effective and the intersection form is negatively definite on the subspace $V_N \subset \operatorname{Pic} X \otimes \mathbf{Q}$ generated by the irreducible components of N;

(ii). $HC \ge 0$ for any effective $C \in \operatorname{Pic} X$; (iii). H is orthogonal to V_N.

By (iii) we have $(K+D)^2 = H^2 + N^2$ where $N^2 \le 0$. Thus, $H^2 \ge (K+D)^2$. We shall use the following logarithmic form of the Bogomolov – Miyaoka – Yau (log-BMY) inequality.

Theorem 2.1. *(a).* (Miyaoka [**My**]) *If* $\bar\kappa(X\setminus D)\ge 0$ *then* $(K+D)^2\le 3e(X\setminus D)$*;*
(b). (Kobayashi—Nakamura—Sakai [**KNS**]) *If* $\bar\kappa(X\setminus D)=2$ *then* $H^2\le 3e(X\setminus D)$*;*
where $e()$ *denotes the topological Euler characteristic.*

Let $C\subset\mathbf{P}^2$ be an algebraic curve of degree d and $\sigma:(X,D)\to(\mathbf{P}^2,C)$ be the minimal resolution of the singular points $p_1,...,p_s$ of C. Let $E_i=\sigma^{-1}(p_i)$ and denote by V_i, $i=1,...,s$ the subspace of $V=\operatorname{Pic}X\otimes\mathbf{Q}$ generated by the irreducible components of E_i. Let $V_0=\sigma^*((\operatorname{Pic}\mathbf{P}^2)\otimes\mathbf{Q})$, i.e. V_0 is generated by the transform of a generic line. Then $V=V_0\oplus...\oplus V_s$ is a direct sum of pairwise orthogonal subspaces.

Suppose that $\bar\kappa(X\setminus D)\ge 0$ and let $K+D=H+N$ be the Zariski decomposition. Denote by K_i, D_i, H_i the orthogonal projections of K, D, H onto V_i $(i=0,...,s)$. Obviously, that K_i and D_i $(i>0)$ coincide with K_{E_i}, D_{E_i} introduced in §1 and it follows from [**F**, Theorem (6.20)] that $H_i=H_{E_i}$ under some additional conditions which are satisfied in out case when $\bar\kappa(\mathbf{P}^2\setminus C)=2$. Clearly, that $(K_0+D_0)^2=H_0^2=(d-3)^2$, hence,

$$(K+D)^2=(d-3)^2+\sum_{i=1}^{s}(K_i+D_i)^2;\qquad H^2=(d-3)^2+\sum_{i=1}^{s}H_i^2;\tag{1}$$

If C is rational and cuspidal (see Introduction) then by genus formula we have

$$\sum\mu_i=(d-1)(d-2)\tag{2}$$

and $e(X\setminus D)=1$. Combining (1) and (2), we rewrite the log-BMY inequalities as follows.

Corollary 2.2. *Let* $C\subset\mathbf{P}^2$ *be a rational cuspidal curve of degree* d *and* $\overline{M}_1,...,\overline{M}_s$ *(resp.* $M_1,...,M_s$*) be the rough (resp. fine)* M*-numbers of its singularities (see §1). Then*

$$\overline{M}_1+\dots+\overline{M}_s\le 3d-4,\qquad \text{if } \bar\kappa(\mathbf{P}^2-C)\ge 0;\tag{3}$$

$$M_1+\dots+M_s\le 3d-4,\qquad \text{if } \bar\kappa(\mathbf{P}^2-C)=2.\tag{4}$$

The factor of the family of rational curves in $\mathbf{P}^2$ by the action of the group $PGL(3)$ is $(3d-9)$-dimensional. Thus, according to Proposition 3.4 below, the inequality (3) has a very natural interpretation as follows.

Given a list of singularity types satisfying (2), let us try to realize this list by a rational curve of a given degree. To this end let us write down its polynomial parameterization $(f_1(t):f_2(t):f_3(t))$. Fix some coefficients to cancel the action of $PGL(3)$ and $PGL(2)$ and consider the others as indeterminates. Add also indeterminates $t_1,...,t_s$ which are the values of t where we are going to put the singularities. Then each singularity type imposes some number of equations for the indeterminates. Thus, (3) means:

If (*the total number of the equations*) $>5+$ (*the number of the indeterminates*) *then the system of simultaneous equations has no solution.*

This speculation leads us to the following

Conjecture 2.3. *If* $C\in\mathbf{P}^2$ *is a rational (not necessarily cuspidal) curve then* $\sum\overline{M}_i\le 3d-9$ *where the sum is taken over all irreducible analytical branches of* C*.*

The question "What singularities may have a plane rational curve of a given degree?" is discussed in [**SS**] for affine curves parametrized by two polynomials in one variable. In this case, 2.3 can be formulated as a generalization of a Davenport theorem which estimates $\deg(p(t)^a-q(t)^b)$ via $\deg p$ and $\deg q$.

§3. Puiseux characteristic sequence

Let $(C, 0)$ be a germ of an analytically irreducible curve in $\mathbf{C}^2$. of multiplicity m and let $x = t^m$, $y = a_m t^m + a_{m+1}t^{m+1} + \dots$ be its analytic parameterization. Put $k_0 = d_1 = m$ and define recursively

$$k_i = \min\{k \,|\, a_k \neq 0 \;\; \& \;\; k \not\equiv 0 \mod d_i\}, \qquad d_{i+1} = \gcd(d_i, k_i), \qquad i = 1, 2, ...$$

Then $m = k_0 < k_1 < k_2 < \dots$ and $m = d_1 > d_2 > \dots$. Let h is defined by the conditions $d_h > 1$, $d_{h+1} = 1$ and put $q_1 = k_1$, $q_i = k_i - k_{i-1}$, $i = 2, ..., h$. The sequence $\mathrm{Ch}(C, 0) = (m; q_1, q_2, ..., q_h)$ will be called the *characteristic sequence* of the singularity $(C, 0)$. and the pairs (d_i, q_i) will be called the *characteristic pairs*[3]. When C is smooth at p we put $h = 0$, $\mathrm{Ch}(C, p) = (1; \,)$. The following statement is well-known.

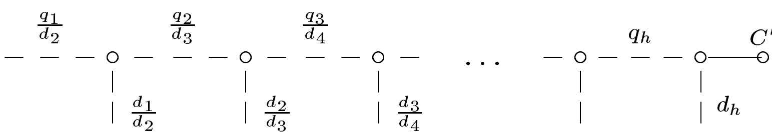
Proposition 3.1. *Let $\sigma : X \to \mathbf{C}^2$ be the blowing up of the origin, $E = \sigma^{-1}(0)$, and p a point on E. Let (C, p) be the germ at p of an analytically irreducible curve on X. Let $k = (C \cdot E)_p$ and $\mathrm{Ch}(C, p) = (m; q_1, q_2, ..., q_h)$. Then*

$$\mathrm{Ch}(\sigma(C), 0) = \begin{cases} (k; k + m, q_2, \dots, q_h) & \text{if } k = q_1, \\ (k; k + m, q_1 - k, q_2, \dots, q_h) & \text{if } k < q_1. \end{cases}$$

If a curve $C_1 \subset X$ is transversal to E and $(C_1 \cdot C)_p = k_1$ then $(\sigma(C_1) \cdot \sigma(C))_0 = k + k_1$.

For a complete SNC-curve E on a smooth surface X let us denote by $d(E)$ the *discriminant* of its dual weighted graph which is defined as $d(E) = \det A_E$ where $A_E = || - E_i \cdot E_j||_{i,j=1}^n$ is the intersection matrix of the irreducible components of E. Put by definition $d(\varnothing) = 1$.

Proposition 3.2. (see, for instance, [**EN**]). *Let $(X, D, E) \to (\mathbf{C}^2, C, 0)$ be the minimal resolution of the singularity of C (like in §1). Then the dual graph of $E \cup C$ looks like*



where the dashed lines mean linear chains of vertices, the numbers written near them are their discriminants and C' denotes the proper transform of C.

Denote by $\lceil a \rceil$ the minimal integer which is $\geq a$.

Proposition 3.3. *Let μ be the Milnor number and $\overline{M}$, M the rough and fine M-numbers (see §1) of $(C, 0)$. Then one has*

$$\mu = 1 - d_1 + \sum_{i=1}^{h} q_i(d_i - 1); \tag{5}$$

$$\overline{M} = d_1 - 2 + \sum_{i=1}^{h} \Big(q_i - \lceil \frac{q_i}{d_i} \rceil\Big); \tag{6}$$

[3]This definition differs from the standard one.

$$M = d_1(1 - q_1^{-1}) - 1 + \sum_{i=1}^{h} q_i(1 - d_i^{-1}). \tag{7}$$

The formula (5) is classical (see [**Mi**, p.93]); a proof of (6) (and of (5) as well) is written in [**OZ1**; (11), (13)]; (7) can be proved the same way as [**OZ1**; 5.2(ii)] (see also [**OZ2**; p.169]);

The next statement (as well as its proof) is not quite rigorous and we shall not use it in the sequel. We formulate it only as a motivation of the Conjecture 2.3.

Proposition 3.4. *$\overline{M}$ coincides with the codimension of the stratum of topological equisingularity of a generic family of mappings $\Delta \to \mathbf{C}^2$ where $\Delta \subset \mathbf{C}^1$ is a disk (parametrical analogue of the $\mu = \mathrm{const}$ stratum).*

Proof. Let $\mathcal{F} = \{f_u\}_{u \in U}$, $f_u(\tau) = (x_u(\tau), y_u(\tau))$, $\tau \in \mathbf{C}^1$, $u = (u_1, u_2, ...)$ be a given family and m the multiplicity of f_0. Let $\mathcal{F}_e$ be the equisingularity stratum of f_0 and $\mathcal{F}_0 = \{f_u \mid \exists \tau_0 \text{ such that } x_u'(\tau_0) = ... = x_u^{(m-1)}(\tau_0) = 0\}$. Then $\mathcal{F}_e \subset \mathcal{F}_0 \subset \mathcal{F}$ and $\operatorname{codim} \mathcal{F}_0 = m - 1$. Let $v = (v_1, v_2, ...)$ be coordinates on $\mathcal{F}_0$. For $f_v \in \mathcal{F}_0$ there exists a unique (up to multiplication by $\sqrt[m]{1}$) reparametrization $\tau = \tau(t)$ such that $x_v(\tau(t)) - x_v(\tau_0) = t^m$. Put $y_v(\tau(t)) = \sum a_k t^k$. Then a_k are analytic functions of v. The fact that the characteristic sequence of f_v coincides with that of f_0 is equivalent to vanishing of certain finite number of coefficients a_k. The number of them is easy to calculate from the definition of the characteristic sequence. It remains to compare the answer with (6).

§4. Some estimates for M-numbers

Let the notation be the same as in §3.

Lemma 4.1.

$$\overline{M} - \frac{\mu}{m} > m - 3. \tag{8}$$

Proof. By (6), using that $d_1 = m$ and $\lceil q_i/d_i \rceil \le (q_i + d_i - 1)/d_i$, we obtain

$$\overline{M} \ge m - 2 + \sum_{i=1}^{h} (q_i - 1)(1 - d_i^{-1}). \tag{9}$$

Dividing (5) by m, we get

$$\frac{\mu}{m} = \frac{1}{m} - 1 + \sum_{i=1}^{h} \frac{q_i d_i}{m}\left(1 - \frac{1}{d_i}\right), \qquad \text{hence,} \tag{10}$$

$$\begin{aligned}
\overline{M} - \frac{\mu}{m} &= m - 2 + \sum_{i \ge 2} \left(q_i\left(1 - \frac{d_i}{m}\right) - 1\right)\left(1 - \frac{1}{d_i}\right) && \text{subtract (10) from (9)} \\
&\ge m - 2 - \sum_{i \ge 2} d_i m^{-1}(1 - d_i^{-1}) && \text{use } q_i \ge 1 \\
&\ge m - 2 - \sum_{i \ge 2} d_i m^{-1}.
\end{aligned}$$

It remains to note that $d_{i+1} \le d_i/2$, hence, $d_i \le m/2^{i-1}$, thus, $\sum_{i \ge 2} d_i/m < 1$. $\square$

Corollary 4.2. [**OZ1**; 6.2].

$$\overline{M} \geq \frac{\mu}{m} \tag{11}$$

Proof. For $m > 2$ apply (8). If $m = 2$ then $\overline{M} = \mu/m$. $\square$

Lemma 4.3.

$$M \geq \frac{\mu - 1}{m} + m - \frac{m(m-1)}{\mu + m - 1} \tag{12}$$

and one has "=" in (12) if and only if $h = 1$.

Proof. Denote $\mu + m - 1$ by A. Since $d_1 = m$, we have $A = \sum_{i=1}^{h} q_i(d_i - 1)$. Then by (5), (7) we have

$$M - \frac{\mu}{m} - m + \frac{1}{m} = -\frac{d_1}{q_1} + \sum_{i\geq 1} \frac{q_i}{d_i}(d_i - 1) - \sum_{i\geq 1} \frac{q_i}{d_1}(d_i - 1) = B - \frac{d_1}{q_1}\ ,$$

where $B = \sum_{i\geq 2} q_i(d_i - 1)(d_i^{-1} - d_1^{-1})$. In this notation (12) is equivalent to $B \geq d_1/q_1 - m(m-1)/A$, and transforming the right hand side as

$$\frac{d_1}{q_1} - \frac{m(m-1)}{A} = \frac{d_1(A - q_1(d_1 - 1))}{q_1 A} = \frac{d_1}{q_1 A} \sum_{i\geq 2} q_i(d_i - 1),$$

we rewrite (12) in the form

$$\sum_{i\geq 2} q_i(d_i - 1)(d_i^{-1} - d_1^{-1}) \geq \frac{d_1}{q_1 A} \sum_{i\geq 2} q_i(d_i - 1)$$

To prove this inequality, it is sufficient to verify that for each $i \geq 2$ one has $d_i^{-1} - d_1^{-1} > d_1/(q_1 A)$. Indeed, $d_1 \geq 2d_2 \geq 2d_i$, hence $d_1 - d_i \geq d_i$. We have also $q_1 > d_1$ and $A \geq q_1(d_1 - 1) \geq q_1 > d_1$. The product of the last three inequalities yields $(d_1 - d_i)q_1 A > d_i d_1^2$. It remains to divide the both sides by $d_1 d_i q_1 A$. $\square$

Corollary 4.4.

$$M \geq \frac{\mu}{m} + m - 1 - \frac{1}{m(m+1)} \tag{13}$$

Proof. Apply to the μ in the denominator in (12) the estimate (see (7)). $\mu \geq 1 - d_1 + q_1(d_1 - 1) = (q_1 - 1)(d_1 - 1) \geq m(m-1)$. $\square$

§5. Proofs of Theorems A and B.

Let $C \subset \mathbf{P}^2$ be a rational cuspidal curve such that $\bar{\kappa}(\mathbf{P}^2 \setminus C) = -\infty$. According to [**W**], in this case C has only one cusp (denote it by p). Let $(X, D) \to (\mathbf{P}^2, C)$ be the minimal resolution of the singularity of C at p. Denote by C' the proper transform of C.

Lemma 5.1. $(C')^2 \geq -1$.

Proof. X is a smooth surface and D is an SNC-curve on it such that $X \setminus D$ is $\mathbf{Q}$-acyclic and $\bar\kappa(X \setminus D) = -\infty$. All such pairs (X, D) are described more or less explicitely in [**FZ1**; 4.9, 4.10]. One can see from this description that if such a pair is minimal (D contains no (-1)-curve of valence 2) then D contains a twig (see §1) whose intersection matrix is not negatively definite.

In our case the dual graph of D is shown in Proposition 3.2 and all the linear chains denoted by dashed lines are negatively definite. Thus, $(C')^2$ can not be ≤ -2 because this would imply that the graph is minimal and all linear chains are negatively definite. □

Let $d = \deg(C)$ and (d_i, q_i), $i = 1, ..., h$, $d_1 = m$ be the characteristic pairs of C at p (see §3). By (2) and (5) (recall that C has only one cusp) we have

$$d^2 - 3d + 2 = 1 - m + \sum q_i(d_i - 1). \tag{15}$$

It is not difficult to see that $(C')^2 = d^2 - \sum q_i d_i$ (see, for instance, the last formula on p.109 in [**OZ1**]). Hence, by Lemma 5.1, we have $d^2 - \sum q_i d_i \geq -1$. Subtracting (15) from this formula, we obtain

$$3d - m \geq \sum q_i. \tag{16}$$

Thus, $(d^2 - 3d + 2) + (m - 1) \overset{(15)}{=} \sum q_i(d_i - 1) \leq \sum q_i(m - 1) \overset{(16)}{\leq} (3d - m)(m - 1)$. Hence, $d^2 - 3dm + m^2 \leq -1$ and the part (a) of Theorem B is proven.

Remark. The curves from Theorem C.a provide "=" in all the above inequalities.

Proof of the part (c) of Theorem B. The classification of pairs (X, D) where X is a smooth surface and D a minimal SNC-curve on X such that $\bar\kappa(X \setminus D) = 0$ under some additional conditions is obtained by Fujita [**F**, (8.64)] and those of them which are $\mathbf{Q}$-acyclic are listed in [**FZ1**, 5.14 and Fig. 16]. The dual graphs of four of them are shown on Fig. 1(a) – (d) and all the others are obtained from the graph on Fig. 1(e) by the operation which is called in [**FZ1**] a *comb-attachment*. It is a sequence of blow-ups such that the center of the next one lies on the exceptional curve of the previous one and the last exceptional curve is not included into D. Thus, if $C \subset \mathbf{P}^2$ is a rational cuspidal curve with $\bar\kappa(\mathbf{P}^2 \setminus C) = -\infty$ and $(X, D) \to (\mathbf{P}^2, C)$ is its minimal resolution then either D coincides with one of the graphs listed above (case 1), or D is obtained from one of them by a sequence of blow-ups and C' (the proper transform of C) is the is the last exceptional curve (case 2).

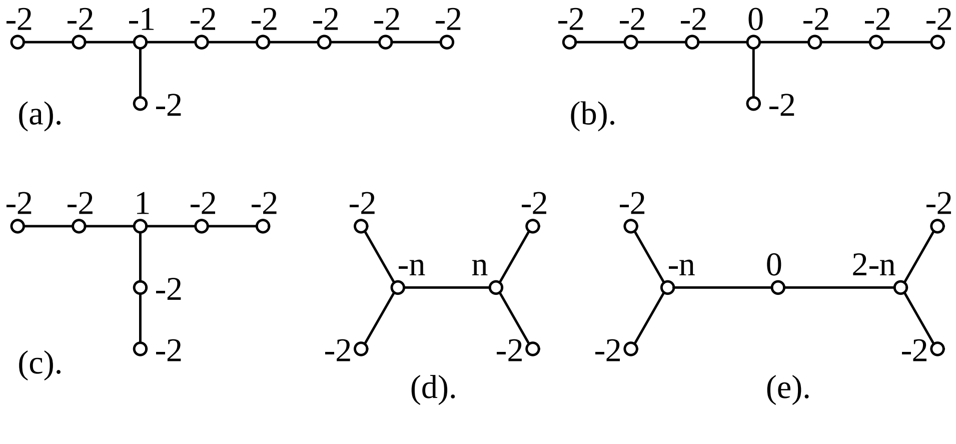


FIG. 1.

Case 1. Easy to see that none of the vertices on the graphs (a) – (d) can be chosen as the curve C', because each vertex has a connected component Γ of the complement such that $d(\Gamma)\neq 1$. Let us show that the graph (e) is also impossible. Denote by p the point where the comb-attachment was done and by D_0 the central component. If $C'\neq D_0$ and $p\notin D_0$ then the complement of C' contains a vertex with weight 0. Otherwise, for each choice of C', one of the connected components of its complement contains a vertex incident to two twigs, each of them is $\!-\!\circ^{-2}$. But the Proposition 3.2 shows that two twigs with equal discriminants at the same vertex can not exist.

Case 2. Since $D\setminus C'$ is obtained as a minimal resolution of singularities, each vertex connected to C' with an edge is a (-1)-curve of valence 3 (in D). This is possible only for the graph (e) when C' is connected with the central vertex and we can apply the same arguments as in the case 1. $\square$

Proof of Theorem A. If $\bar\kappa(\mathbf{P}^2\setminus C)=-\infty$ then the part (a) of Theorem B provides even a stronger estimate, so, we suppose that $\bar\kappa(\mathbf{P}^2\setminus C)\geq 0$ and we may apply (3). Let $p_1,...,p_s$ be the singular points of C. Denote by m_i, μ_i, $\overline{M}_i$ respectively the multiplicity, the Milnor number and the rough M-number (see §1) of p_i. Without loss of generality we may suppose that $m=m_1\geq m_2\geq\dots\geq m_s$. By (2) we have $\mu_1=(d-1)(d-2)-(\mu_2+\mu_3+\dots)$, hence, by (3) and (8),

$$3d-4\geq\sum\overline{M}_i>\frac{\mu_1}{m}+m-3+\sum_{i\geq 2}\overline{M}_i=\frac{d^2-3d+2}{m}+m-3+\sum_{i\geq 2}\Big(\overline{M}_i-\frac{\mu_i}{m}\Big).$$

By (11) the last sum is non-negative, hence, $f(d)<0$ where $f(x)=(x^2-3x+2)m^{-1}-3x+m+1$. But $f\big(\alpha(m+1)+5^{-1/2}\big)=11/(5m)>0$ and $f'(x)>0$ for $x\geq 3(m+1)/2$. $\square$

Proof of the part (b) of Theorem B. Repeating word by word the proof of Theorem A (replacing $\overline{M}_i$ with M_i and using (13) instead of (8), we arrive to $f(d)+2-(m^2+m)^{-1}\leq 0$. Evaluating this expression for $d=\alpha(m+1)-5^{-1/2}$ we obtain $(1/5)(m-4)(m^2+m)^{-1}$. This quantity is positive for $m>4$. It remains to note that for $m\leq 4$ the required estimate is weaker than the estimate $d<3m$ proven in [**MS**]. $\square$

§6. Examples (proof of Theorem C)

Let, as in Introduction, $\varphi_0=0$, $\varphi_1=1$, $\varphi_{i+2}=\varphi_i+\varphi_{i+1}$ be the Fibonacci numbers. Put also $\varphi_{-a}=(-1)^{a+1}\varphi_a$. We shall need the following identities:

$$\begin{array}{llll}\text{a)}. & \varphi_{a-2}+\varphi_{a+2}=3\varphi_a; & \text{c)}. & \varphi_a^2-\varphi_{a-2}\varphi_{a+2}=(-1)^a;\\ \text{b)}. & \varphi_a^2-\varphi_{a-1}\varphi_{a+1}=(-1)^{a+1}; & \text{d)}. & \varphi_{a-3}\varphi_{a+3}-\varphi_{a-1}\varphi_{a+1}=3\cdot(-1)^a.\end{array}\tag{17}$$

Proof. (a) is just the sum of the identities $\varphi_{a-2}+\varphi_{a-1}=\varphi_a$, $\varphi_{a+1}-\varphi_{a-1}=\varphi_a$ and $\varphi_{a+2}-\varphi_{a+1}=\varphi_a$; (b) can be easily proven by induction [**V**; (1.8)].

(c). $(-1)^a\overset{\text{by (b)}}{=}\varphi_{a-1}\varphi_{a+1}-\varphi_a^2=(\varphi_a-\varphi_{a-2})(\varphi_{a+2}-\varphi_a)-\varphi_a^2=(\varphi_{a-2}+\varphi_{a+2})\varphi_a-2\varphi_a^2-\varphi_{a-2}\varphi_{a+2}\overset{\text{by (a)}}{=}\varphi_a^2-\varphi_{a-2}\varphi_{a+2}$.

(d). By (a) we have $\varphi_{a\pm3}=3\varphi_{a\pm1}-\varphi_{a\mp1}$. Putting this into (d), we transform (d) into (b) multiplied by 3. $\square$

Fix a nodal cubic $N\subset\mathbf{P}^2$. Let p_0 be its node and $\mathcal{P}_1$, $\mathcal{P}_2$ the analytic branches of N at p_0. Define a birational transformation $f:\mathbf{P}^2--\to\mathbf{P}^2$ as follows. Blow up

7 infinitely close points of $\mathcal{P}_1$ at p_0. Denote by $E_1, \dots, E_7$ the exceptional curves and by E_0 the proper transform of N. Then $E_0^2 = E_7^2 = -1$, $E_1^2 = \dots = E_6^2 = -2$, $E_0 \cdot E_1 = E_1 \cdot E_2 = \dots = E_6 \cdot E_7 = E_7 \cdot E_0 = 1$ and the other intersections are zeros. E_0 and E_7 are symmetric to each other in the obtained configuration. Hence, we can blow down $E_0, \dots, E_6$ and the image of E_7 will be again a nodal cubic. Since all nodal cubics are projectively equivalent, we may assume that E_7 is mapped onto N and $E_0, \dots, E_6$ onto infinitely close points of $\mathcal{P}_2$.

Denote by C_{-3} the tangent to $\mathcal{P}_1$, by C_{-1} the tangent to $\mathcal{P}_2$, and by C_0 the tangent to N at a flex point. Let C_0^* be a conic[4] which intersects N only in one point which is a smooth point of N. To find such a conic, note that $v = u(1-u)^2$ and $v = u - 2u^2$ have the contact of order 3 at $(0,0)$, hence $v^2 = u(1-u)^2$ and $v^2 = u - 2u^2$ have the contact of order 6.

Now, let us define recurrently:

$$C_j = f(C_{j-4}), \quad j > 0, \quad j \not\equiv 2 \mod 4; \qquad\qquad C_{4k}^* = f(C_{4k-4}^*), \quad k = 1, 2, \dots$$

Since f is biregular on $\mathbf{P}^2 \setminus N$, these curves are rational and unicuspidal. The fact that their characteristic sequences at p_0 are like it was described in Theorem C, can be easily proven by the simultaneous induction together with the assertion that

$$\begin{aligned} k_{j,1} &= \varphi_j, & k_{j,2} &= \varphi_{j+4}, & j &= -3,\ -1, 0, 1,\ 3, \dots \\ k_{j,1}^* &= 2\varphi_j, & k_{j,2}^* &= 2\varphi_{j+4}, & j &= 4, 8, 12, \dots \end{aligned}$$

where $k_{j,\nu} = C_j \cdot \mathcal{P}_\nu$ and $k_{j,\nu}^* = C_j^* \cdot \mathcal{P}_\nu$. Indeed, using Proposition 3.1, easy to check that $m_{j+4} = k_{j+4,1} = k_{j,2}$ and $k_{j+4,2} = 7k_{j,2} - m_j$. It remains to apply the evident identity $\varphi_{j-4} - 7\varphi_j + \varphi_{j+4} = 0$ which is the sum of the identities (17.a) for $a = j \pm 2$ and three times for $a = j$. To find $d_j = \deg(C_j)$, note that $(d_j - 1)(d_j - 2) \overset{(2),(5)}{=} (\varphi_j - 1)(\varphi_{j+4} - 1) + 1 + (-1)^j \overset{(17a,c)}{=} (\varphi_{j+2} - 1)(\varphi_{j+2} - 2)$. Hence, $d_j = \varphi_{j+2}$. Analogously, $\deg(C_j^*) = 2\varphi_{j+2}$.

The fact that $\bar\kappa(\mathbf{P}^2 \setminus C_{2k+1}) = -\infty$ follows from [**Ka**] (see Remark 3 in Introduction). Let us show that $\bar\kappa(\mathbf{P}^2 \setminus \mathbf{C}_{4k}) = 2$. The fact that $\bar\kappa \neq 0$ and $\bar\kappa \neq -\infty$, follows from Theorem B.c and Lemma 5.1 respectively. To show that $\bar\kappa \neq 1$, we shall use the explicit classification of **Q**-acyclic surfaces with $\bar\kappa = 1$ obtained in [**GM**] and exposed in [**FZ1**; 5.7–5.11] in a form convenient for our purposes. We shall use the terminology and notation of [**FZ1**]. The pair $(\bar X, \bar D)$ is called *pre-***Q***-acyclic* if $\bar D$ is an SNC-divisor on a smooth surface $\bar X$ and maybe after some comb-attachments (see [**FZ1**] or the proof of B.c in §5) one can obtain a pair (X, D) such that $X \setminus D$ is **Q**-acyclic. All the pre-**Q**-acyclic pairs with $\bar\kappa = 1$ belong to four classes which are denoted in [**FZ1**] by (A1), (A2), (B1) and (B2).

A computation of the discriminant of the dual graphs (it is done, for instance, in [**O**]) easily shows that the surfaces in the cases (A2) and (B2) are already **Q**-acyclic and the surfaces (A1) and (B1) become **Q**-acyclic after a single comb-attachment.

Hence, the surfaces from (A2) and (B2) are impossible for our curve C_{4k} because their graphs do not coincide with the graph of the resolution (In the case (B2) the number of the broken chains must be 2, but then the sum of the weights of the vertices of valence 3 is -2).

The case (A1) is impossible because any choice of the comb-attachment preserves a vertex incident to two twigs of the form $\text{——}\circ^{-2}$. It remains to consider the case (B1).

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[4]The idea to use this conic belongs to E. Artal

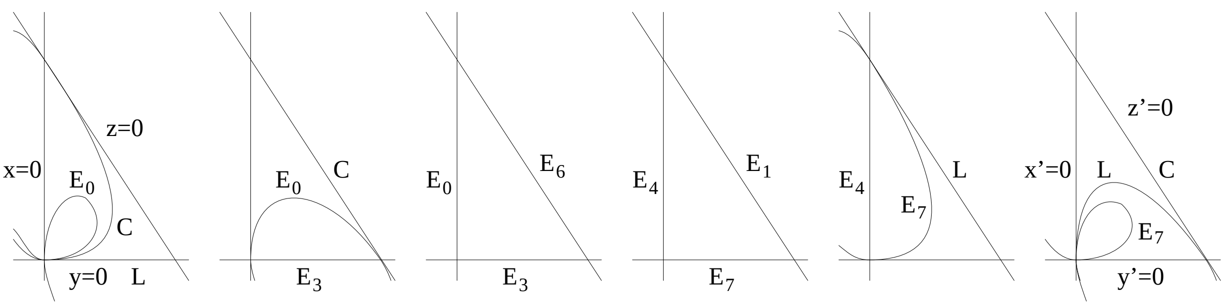
By [**FZ1**; 5.11] the number of the broken chains must be two. Hence, the graph of the resolution is obtained by a single comb-attachment from a graph $\overset{-n}{>\!\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{n-2}{\circ\!<}$. The resulting graph has only two vertices of valence ≥ 3 and they are connected by an edge. It is possible only if a single blow-up was performed and if it was done at the central vertex. The result of this operation is $\overset{1-n}{>\!\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{n-1}{\circ\!<}$. Such a graph is impossible because the weights of the vertices of valence 3 must be negative. The case of C^*_{4k} is treated the same way.

Now let us write the formulas for the constructed curves. To this end we decompose f into a product of quadratic transformations. Let $f = \pi_2 \circ \pi_1^{-1}$, where $\pi_i : X \to \mathbf{P}^2$ are the birational morphisms. Denote by L and C the proper transform on X (with respect to π_1) of C_{-3} and $f^{-1}(C_{-1})$ — the line and the conic which have the maximal possible contact with $\mathcal{P}_1$. Then f can be presented as

$$[E_0, L, C] \to [E_0, E_3, C] \to [E_0, E_3, E_6] \to [E_7, E_4, E_1] \to [E_7, E_4, L] \to [E_7, C, L].$$

This means that the result of each quadratic transformation is the variety obtained from X by blowing down all the curves L, C, $E_0, \ldots, E_7$ except the three curves listed in the brackets. Each of these quadratic transformation can be easily written by formulas:

$$\begin{array}{lllll}
x_1 = xy, & x_2 = x_1 z_1 - y^2, & x_2' = y_2 z_2, & x_1' = x_2' z_2', & x' = {x_1'}^2; \\
y_1 = y^2, & y_2 = y_1 z_1, & y_2' = x_2 z_2, & y_1' = y_2' z_2' + {x_2'}^2, & y' = x_1' y_1'; \\
z_1 = yz - x^2, & z_2 = z_1^2, & z_2' = x_2 y_2, & z_1' = {z_2'}^2, & z' = x_1' z_1' + {y_1'}^2.
\end{array}$$



The last two transformations are chosen to be symmetric to the first two, to guarantee that $N \to N$ and $\mathcal{P}_1 \to \mathcal{P}_2$. Performing all the substitutions we obtain $f(x : y : 1) = (x' : y' : z')$ where

$$x' = p_1 q^2, \quad y' = q p_3, \quad z' = p_1^3 - xy(4y - x^2)p_1^2 + y^4(2y + x^2)p_1 - xy^7,$$

and $p_1 := y - x^2$, $q := xy - x^3 - y^3$, and $p_3 := p_1^2 - 2xy^2 p_1 + y^5$. The equations $p_1 = 0$, $p_3 = 0$, $q = 0$ define C_1, C_3, and N. Put $p_0 := 3x + 3y + 1$. Then $p_0 = 0$ can be chosen as C_0 and substituting $u = -4(x+y)/p_0$, $v = 4(x-y)/p_0$ into the above equation of C_0^* we get $21x^2 - 22xy + 21y^2 - 6x - 6y + 1 = 0$.

The multiplicity sequences of C_j at p_0 is $(\varphi_j, S_j, S_{j-4}, \ldots, S_\nu)$ and where $j = 4k + \nu$, $\nu = 3, 4, 5$, $k = 0, 1, \ldots$ and S_i denotes the subsequence $(\varphi_i, \varphi_i, \varphi_i, \varphi_i, \varphi_i, \varphi_i - \varphi_{i-4})$. In the case of C_j^* all the multiplicities should be multiplied by 2.

§7. Open fans and linear chains of rational curves

To deal with linear chains of rational curves on a smooth surface X, it is convenient to use the following language. We shall call an *open fan* a sequence of vectors $c = (v_0, \dots, v_{n+1})$, $v_i \in \mathbf{Z}^2$ such that $v_i \wedge v_{i+1} = 1$ for $i = 0, \dots, n$. We use here the natural identification $\wedge^2 \mathbf{Z}^2 = \mathbf{Z}$ assuming $(a, b) \wedge (c, d) = ad - bc$. Given any sequence of integers $(a_1, \dots, a_n)$, one can construct an open fan c as above such that $v_i \wedge v_{i+1} = 1$ for $i = 0, \dots, n$ and $v_{i-1} \wedge v_{i+1} = -a_i$ for $i = 1, \dots, n$. Clearly that c is uniquely defined up to the action of $SL_2(\mathbf{Z})$ and we shall write $c = c(a_1, \dots, a_n)$.

Let X be a smooth surface and $D = D_1 + \dots + D_n$ a linear chain of rational curves on X (i.e. $D_i D_j = 0$ for $|i - j| > 1$, $D_i D_j = 1$ for $|i - j| = 1$). Put $c(D) = c(D_1^2, \dots, D_n^2)$. It is convenient to denote by D_0 (resp. D_{n+1}) the germ of a generic analytic curve which meets D_1 (resp. D_{n+1}) transversally.

Given an open fan $c = (v_0, \dots, v_{n+1})$, we define its *rotation number* as $\operatorname{rot}(c) = \sum_{i=0}^{n} a_i$ where a_i is the oriented angle from v_i to v_{i+1} (all a_i are positive by the definition of open fan). This definition depends on the choice of a base in $\mathbf{Z}^2$.

Given a vector $v \in \mathbf{Z}^2$, let A_v be the triangular automorphism of $\mathbf{Z}^2$ defined by $A_v u = u + (v \wedge u)v$. Recall that $d(D)$ denotes the discriminant of D (see §3).

Proposition 7.1. *(a). If $\tilde{X}$ is obtained by blowing up a point $D_i \cap D_{i+1}$ $(i = 0, ..., n)$ and $\tilde{D}$ is the total transform of D then $c(\tilde{D}) = (v_0, \dots, v_i, v_i + v_{i+1}, v_{i+1}, \dots, v_{n+1})$.*

(b). If X' is obtained by blowing up a smooth point of D_i $(i = 1, ..., n)$ and D' is the strict transform of D then $c(D') = (v_0, \dots, v_i,\ A_{v_i} v_{i+1}, \dots, A_{v_i} v_{n+1})$.

(c). $d(D) = v_0 \wedge v_{n+1}$.

(d). Let n_+ be the number of positive squares in a diagonalization over $\mathbf{Q}$ of the intersection matrix $A_D = ||D_j \cdot D_j||$. Then $n_+ = \lceil \operatorname{rot}(c)/\pi \rceil - 1$.

(e). D can be blown down to a smooth point iff $v_0 \wedge v_{n+1} = 1$ and $\operatorname{rot}(c) < \pi$.

Proof. (a,b) evident; (c) induction by n; (d) follows from (c) and the Sylvester formula.

(e). D can be blown down iff A_D is negatively definite and $\det A_D = \pm 1$ (see [**Mu**]). Thus, (e) follows from (c) and (d). $\square$

Let Σ be a primitive fan in $\mathbf{Z}^2$ and $X = X_\Sigma$ the corresponding toric variety (see [**D**]). Let $v_0, \dots, v_{n+1}$ be a sequence of generators of one-dimensional cones of Σ such that all $v_0, \dots, v_n$ are distinct but maybe $v_{n+1} = v_0$. Suppose that each pair (v_i, v_{i+1}) forms the positively oriented base of some two-dimensional cone of Σ. Let D_i be the closure of the one-dimensional orbit of X corresponding to v_i. It is not difficult to show that $v_{i-1} \wedge v_{i+1} = -D_i^2$, thus, $c(D_1 + \dots + D_n) = (v_0, \dots, v_{n+1})$.

§8. Another construction of the curves C_j

Fix an integer $j \geq 3$, $j \not\equiv 2 \mod 4$.

Case (a): j is odd. Consider three vectors $v_0 = -(\varphi_j^2, \varphi_{j+2}^2)$, $v_1 = (\varphi_{j-2}, \varphi_{j+2})$, $v_2 = (\varphi_j, \varphi_{j+4})$ in $\mathbf{Z}^2$. Since the determinants

$$v_0 \wedge v_1 \overset{(17.c)}{=} \varphi_{j+2}, \qquad v_1 \wedge v_2 \overset{(17.d)}{=} 3, \qquad v_2 \wedge v_0 \overset{(17.c)}{=} \varphi_j, \tag{18}$$

are positive, we can consider the complete fan Σ spanned by v_0, v_1, v_2.

Let X_Σ be the smooth two-dimensional toric variety associated with the minimal primitive subdivision of Σ. Denote by D_i the closure of the one-dimensional orbit corresponding to v_i $(i = 0, 1, 2)$ and by D the closure of $X_\Sigma - (X_0 \cup D_0)$ where X_0 is the open orbit of X_Σ. Let us blow up two generic points $p_1 \in D_1$ and $p_2 \in D_2$. Denote by E_1 and E_2 the exceptional curves and by D' (resp. D'_i) the strict preimage of D (resp. D_i).

As we pointed out above, $c(D) = (v_0, \ldots, v_1, \ldots, v_2, \ldots, v_0)$. Hence, by 7.1(b), $c(D') = (v_0 - (v_1 \wedge v_0)v_1, \ldots, v_1, \ldots, v_2, \ldots, v_0 + (v_2 \wedge v_0)v_2 \overset{(18)}{=} (e_1, \ldots, v_1, \ldots, v_2, \ldots, e_2)$ where v_i corresponds to D'_i and $e_1 = (1, 0)$, $e_2 = (0, 1)$ is the base of $\mathbf{Z}^2$.

Hence, by 7.1(e), D' can be blown down to a smooth point, denote it by p. Counting blow-ups and blow-downs, we see that the resulting surface is $\mathbf{P}^2$. By 3.2, 7.1(c), E_2 is mapped onto a rational unicuspidal curve C_2 with one characteristic pair $(\varphi_j, \varphi_{j+4})$).

By construction $\mathbf{P}^2 \setminus C$ contains the affine two-dimensional toric variety isomorphic to $\mathbf{C} \times (\mathbf{C} \setminus 0)$ which corresponds to the vector v_0. Hence, $\bar{\kappa}(\mathbf{P}^2 \setminus C) = -\infty$.

Case (b): j is even. Recall that $j \not\equiv 2 \mod 4$, hence, j is divisible by 4. Clearly that the complement of a rational cuspidal curve is a $\mathbf{Q}$-acyclic surface. We shall apply a general method due to T. tom Dieck and T. Petri [**tDP**] to construct a $\mathbf{Q}$-acyclic surface starting with a line arrangement.

Let us consider the arrangement of six lines $L_1, \ldots, L_6$ on $\mathbf{P}^1 \times \mathbf{P}^1$ where L_1, L_3, L_5 are horizontal and L_2, L_4, L_6 are vertical. Let $p_{ij} = L_i \cap L_j$ and $X \setminus D$ be the $\mathbf{Q}$-acyclic surface obtained from this line arrangement by cutting cycles at p_{16}, p_{25}, p_{14} and p_{36} (see [**tDP**]) according to Fig. 2. This means that X is the result of blowing up at these four points and at some of their infinitely close points and D is the total preimage of $L_1 \cup \cdots \cup L_6$ with the last (-1)-curves excluded. The strict transforms of L_i are denoted on Fig. 2by D_i. We cut the cycles at p_{16}, p_{25} and p_{36} as it is shown on Fig. 2 where the strict transforms of L_i are denoted by D_i. We cut the cycle at p_{14} with the multiplicities $(\frac{1}{3}\varphi_{j+4} - \varphi_j,\ \frac{1}{3}\varphi_j)$.

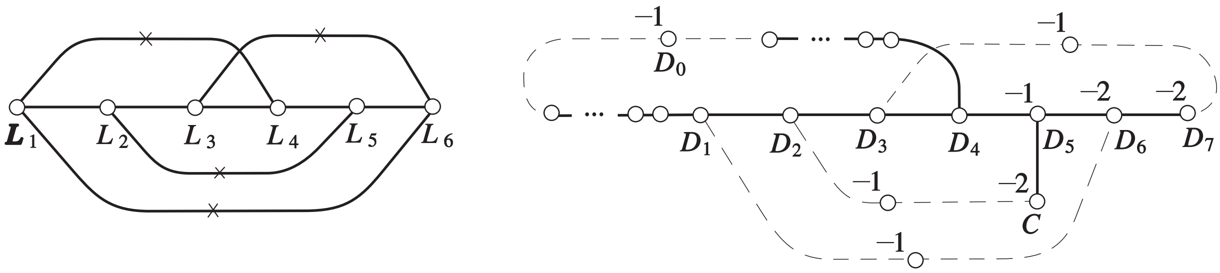


FIG. 2.

We are going to show that $D - C$ (see Fig. 2) can be blown down to a smooth point p and the image of C is the required curve.

Let $\sigma' : X' \to \mathbf{P}^1 \times \mathbf{P}^1$ cuts the cycle at p_{14} and $\sigma : X \to X'$ cuts the other three cycles. Denote by D'_i the strict transform of D_i on X' . Consider $\mathbf{P}^1 \times \mathbf{P}^1$ as the toric variety associated with the complete fan spanned on the vectors $v_4 = (1, 0)$, $v_3 = (0, 1)$, $v_2 = (-1, 0)$, $v_1 = (0, -1)$ whose orbits are $L_1, \ldots, L_4$.

Put $v_0 = (\frac{1}{3}\varphi_{j+4} - \varphi_j,\ -\frac{1}{3}\varphi_j)$. Then X' is the toric variety associated with the minimal primitive subdivision of the fan spanned on the vectors $v_0, \ldots, v_4$ and D'_0 correspond to v_0. Put $D'_{14} = {\sigma'}^{-1}(D_1 + \cdots + D_4) - D'_0$. Then $c(D'_{14}) = (v_0, \ldots, v_4, v_3, v_2, v_1, \ldots, v_0)$. Let D_{14} be the strict transform of D'_{14} on X. Then by 7.1(b) we have $c(D_{14}) = (v_0, \ldots, v_4, \ldots,\ Av_0)$ where $A = A_{v_3}^2 A_{v_2}^2 A_{v_1}$.

Blowing down successively D_5, D_6 and D_7 we map $D - C$ onto a linear chain $\bar{D}_{14}$ and by 7.1(e) it suffices to prove that $d(\bar{D}_{14}) = 1$. Since $\bar{D}_{14}$ is the result of the inverse of

the operation described in Proposition 7.1(b) applied three times to D_{14} at D_4, we have $c(\bar{D}_{14}) = (v_0, \dots, v_4, \dots, Bv_0)$ where $B = A_{v_4}^{-3} A$. Clearly that
$A_{v_1} = A_{v_3} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $A_{v_2} = A_{v_4} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, hence, $A = \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 11 \\ 1 & -3 \end{pmatrix}$, hence, $Bv_0 = \frac{1}{3}(-4\varphi_{j+4} + \varphi_j,\, \varphi_{j+4})$. Thus, by 7.1(c) we have
$d(\bar{D}_{14}) = v_0 \wedge Bv_0 = \frac{1}{9}(\varphi_{j+4} + \varphi_j)^2 - \varphi_j\varphi_{j+4} \overset{\text{by (17.a)}}{=} \varphi_{j+2}^2 - \varphi_j\varphi_{j+4} \overset{\text{by (17.c)}}{=} 1.$

Using 3.2 and 7.1(c) we find the characteristic pairs of the image of C on $\mathbf{P}^2$. For $j \geq 8$ we have $(q_1, d_1) = d_2 \cdot (v_4 \wedge Bv_0\,,\ v_0 \wedge v_4) = (\varphi_{j+4}, \varphi_j)$.

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