An analogue of metaplectic representation on the sheaves of abelian variety

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0.1 Introduction

In the past few decades the subject of algebro-geometric Fourier functors have been studied in several directions. Deligne [KL] studied a geometric transformation on l - adic sheaves on the affine space over a finite field. Brylinski, Malgrange and Verdier ([Bryl],[La]) studied the Fourier transform on $\mathcal{D}$- modules on a vector space over $\mathbf{C}$.

More than a decade back the Fourier functor on abelian varieties and sheaves of modules on them was developed by Shigeru Mukai [Muk1]. This functor proved to be very useful for investigating the moduli scheme of some locally free sheaves on an abelian variety. In gauge theory, this functor is related to Nahm's transformation ([BB],[Na]) introduced by Warner Nahm [Na] in connection with the application of ADHM construction [AHMD] for monopoles (one translation invariant instanton).

Let X be an abelian variety over the field of characteristic zero and suppose L_A is the line bundle of even Chern class A over X i.e.

$$A \in H^2(X, Z) \cap H^{1,1}(X)$$

and $c_1(L_A) = A$. So we fix the complex structure of the line bundle Chern class of a line bundle depends on the origin of the abelian variety, so Thus in the process of fixing the Chern class we have fixed up the base of the abelian variety. We assume that the Chern class of the line bundle over the abelian variety is even.

Hence when the Chern class of the line bundle is even then there is a prefered choice of L_A.

Let $\widehat{X}$ be the dual of X and L_α be the corresponding line bundle of even Chern class, satisfies

$$\alpha \in H^2(\widehat{X}, Z) \cap H^{1,1}(\widehat{X}).$$

Let $X \times \widehat{X}$ be the symplectic torus. Then there exist a natural line bundle, called Poincare bundle $\mathcal{P}$ ([Mum],[LB]) over it. Let $\Omega \in H^2(X \times \widehat{X}, Z)$ be the first Chern class of the Poincare bundle. Let Γ be the automorphism group of symplectic torus $X \times \hat{X}$, fixing a base point; i.e. the group of automorphisms which preserve the complex structure and the closed two form Ω and which are induced by linear maps on the covering vector space. Let $\Gamma_0 \subset \Gamma$ be the finite-index subgroup of elements $\gamma \in \Gamma$ such that $\gamma = 1 \bmod 2$ (i.e. those which act trivially on the half lattice in the covering space).

Let $D(X)$ and $D(\widehat{X})$ be the derived categories of $\mathcal{O}_X$ and $\mathcal{O}_{\hat{X}}$ modules $Mod(X)$ and $Mod(\widehat{X})$ respectively. Mukai [Muk1] showed that when $X = \widehat{X}$ i.e X is a principally polarized abelian variety, the derived category of $\mathcal{O}_X$ modules over X has a natural action of $SL(2, Z)$. In this chapter we extend the results of Mukai and show the following few things :

(1) a new interpretation of Mukai's action using the analogous method of metaplectic representation in ordinary geometric quantization method ([Fol],[Wo]).

(2) Also we generalize the action of $SL(2, Z)$ to the action of Aut $(X \times \widehat{X})$ on $D(X)$. This result is applicable to non principally polarized abelian varieties too.

We organise this paper as follows:
In the first section we shall discuss a few basic things regarding abelian variety and Fourier functors. In the second section we will give an explicit picture about how $SL(2, Z)$ acts. Although this appears in the Mukai's theorem it is not made explicitly clear in his paper. In the next section some elementary features of metaplectic representations are discussed. In the final section we will show that how the ordinary metaplectic representation of $SL(2, R)$ exactly match up the $SL(2, Z)$ action on the moduli space. In the final section we prove some partial results towards conjecture.

Unfortunately we fail to give the direct proof of our main result hence we end up with the following conjecture.

Conjecture .1 *Suppose X is an abelian variety and $\hat{X}$ be its dual. Then $X \times \widehat{X}$ is the symplectic torus and Ω be the canonical two form on it. Let $\Gamma = Aut(X \times \widehat{X})$ be the automorphism group of symplectic torus which preserves the two forms Ω. Then there exist an action of $\Gamma_0 \subset \Gamma$, on the derived category $D(X)$ of $\mathcal{O}_X$ modules on abelian variety X and it acts on $D(X)$ modulo shift in degree.*

In the next section we shall recapitulate some basic materials about the abelian variety and sheaves of modules on it and after that we will discuss briefly Mukai's Fourier functor on sheaves of modules on the abelian variety.

Comment : (1) Throughout the paper we only consider the line bundles over complex tori to be even Chern classes. (2) In general we will specify the line bundles over X and $\widehat{X}$ by L_A and L_α respectively. (3) A sheaf on X is a coherent $\mathcal{O}_X$ module.

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0.2 Background

We split up this section into two parts, in the first part we consider all the relevant definition and theorems about complex tori (see [Mum],[LB]) and in the second half we consider a pedogological introduction to Mukai's Fourier functor on the sheaves over complex tori. Here we have mainly consulted the papers of Shigeru Mukai (see [Muk1],[Muk2]).

0.2.1 Complex tori: a quick survey

The complex torus X is by defination the quotient space of V/Λ where $V \cong \mathbf{C}^m$ is the vector space of dimension m and Λ is a discrete subgroup of rank $2m$ of V.

Theorem .2 *All the holomorpic line bundles on V are trivial.*

Proof: We have

$$0 \longrightarrow Z \longrightarrow \mathcal{O}_V \stackrel{exp(2\pi i\cdot)}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

Since $H^1(V, \mathcal{O}) = 0$ by $\bar{\partial}$ - Poincaŕe lemma and also $H^2(V, Z) = 0$ (for proof see Lange and Birkenhake [LB]) then from the long exact sequence $H^1(V, \mathcal{O}^*) = (1)$. So the result follows from this. $\square$

Suppose $m : V \longrightarrow X$, then the pullback bundle $m^*(L)$ of L to V is trivial. If we choose an isomorphism

$$\mu : m^*(L) \longrightarrow V \times \mathbf{C}$$

The canonical action of Λ on $m^*(L)$ is carried by μ into a linear action of Λ on $V \times \mathbf{C}$. It acts on V by translation. For $z \in V$ and $\lambda \in \Lambda$, the fibres of $m^*(L)$ at z and $z + \lambda$ are both identified with the fibre L at $m(z)$. By comparing the local trivialization at z and $z + \lambda$ we yield a linear automorphism of $\mathbf{C}$. Since the only holomorphic automorphisms of a line bundle fixing the base are given by non-vanishing holomorphic function. Therefore the action of Λ on $V \times \mathbf{C}$ is given by

$$(z, \alpha) \longrightarrow (z + \lambda, e_\lambda(z).\alpha)$$

The functions e_λ necessarily satisfy the compatibility relation

$$e_\lambda(z + \lambda')e_{\lambda'}(z) = e_{\lambda'}(z + \lambda)e_\lambda(z) = e_{\lambda'+\lambda}$$

This collection of non-zero holomorphic functions

$$\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda \in \Lambda}$$

are called a set of multipliers for L. Given a set of multiplier $\{e_\lambda\}$ one can write any line bundle $L = (\mathbf{C} \times \mathbf{C}^m)/ \sim$ where $(z, \alpha) \sim (z + \lambda, e_\lambda(z)\alpha)$.

Let us consider a group of Hermitian forms (called the Neŕon - Severi group) $H : V \times V \longrightarrow V$, where $V \cong \mathbf{C}^m$, such that the imaginary part ImH is the integral when restricted to given $\Lambda \times \Lambda$. We define a semi- character for H to be a map

$$\beta : \Lambda \longrightarrow S^1$$

which satisfies

$$\beta(\lambda_1 + \lambda_2) = exp(\pi i ImH(\lambda_1, \lambda_2))\beta(\lambda_1)\beta(\lambda_2) \text{ for } \lambda_i \in \Lambda$$

If $P(\Lambda)$ denotes the set of all pairs (H, β) where $H \in NS(X)$ and β is the semi-character for H. Then we obtain the following short exact sequence

$$1 \longrightarrow Hom(\Lambda, S^1) \xrightarrow{i} P(\Lambda) \xrightarrow{\pi} NS(X)$$

where $i(\beta) = (0, \beta)$ and $\pi(H, \beta) = H$. $P(\Lambda)$ is a group with respect to the composition

$$(H_1, \beta_1) \circ (H_2, \beta_2) = (H_1 + H_2, \beta_1\beta_2)$$

We have a map (for details see [LB, chap.2] or [Mum, chap.1]) $P(\Lambda) \longrightarrow Pic(X)$ such that for every (H, β) we have

$$e_\lambda(z) = \beta(\lambda)exp(\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda).........(*)$$

We have the following commutative diagram

$$\begin{tikzcd}
P(\Lambda) \arrow[dr, "\pi"] \arrow[dd] & \\
& NS(X) \\
Pic(X) \arrow[ur, "c_1"'] &
\end{tikzcd}$$

To show the second map, we write $\beta(\lambda) = exp(2\pi i\xi(\lambda))$ then we obtain from $(*)$

$$e_\lambda(z) = exp(2\pi i[\xi - \frac{i}{2}H(z, \lambda) - \frac{i}{4}H(\lambda, \lambda)]) = exp(2\pi i w)$$

Lemma .3 *There exists a canonical isomorphism $H^2(X, Z) \longrightarrow Alt^2(\Lambda, Z)$ such that the Chern class of the line bundle L on X with factor automorphy $e_\lambda = exp(2\pi i w)$ is related to the alternating form*

$$A_L(\lambda, \mu) = w(\mu, z + \lambda) + w(\lambda, z) - w(\lambda, z + \mu) - w(\mu, z)$$

for all $\lambda, \mu \in \Lambda$ and $z \in V$.

[for proof and detail see [LB] chapter 2.]

Finally we obtain $A = ImH$.

Hence we have proved the following statement.

Proposition .4 *Let X be a complex torus and V be its covering space. Let H be a Hermitian form on V such that if $A = ImH$ then $A(\Lambda \times \Lambda) \subset \mathbf{Z}$. Let $\beta : \Lambda \longrightarrow S^1$ be a map which satisfies*

$$\beta(\lambda_1 + \lambda_2) = exp(i\pi A(\lambda, \lambda)\beta(\lambda_1)\beta(\lambda_2) \ , \ \lambda_i \in \Lambda$$

These maps β are called semicharacters of H. If we put

$$e_\lambda(z) = \beta(\lambda)exp(\pi H(z, \lambda) + \frac{1}{2}\pi H(\lambda, \lambda)$$

then $\lambda \longmapsto e_\lambda$ is a 1 cocycle on Λ with coefficient in $H^0(V, \mathcal{O}_V^*) = H^*$ and these determines a line bundle with the Chern class being A.

Corollary .5 *If the Chern class of the line bundle over the complex tori is even then the semicharacters β of the Néron-Severi group satisfy*

$$\beta(\lambda_1 + \lambda_2) = \beta(\lambda_1)\beta(\lambda_2) \ , \ \lambda_i \in \Lambda$$

Next we shall focus on the translation on the complex tori.

Proposition .6 *For any $y \in X$, the translation τ_y acts on any line bundle $L(H, \beta) \in Pic(X)$ on X and satisfies*

$$\tau_y^* L(H, \beta) = L(H, \beta(e(2\pi i ImH(y, .)))$$

Proof:: The translation τ_Y on the covering space V, where $Y \in V$, induces the translation τ_y on X for $y \in X$. We recall

$$e_\lambda(y) = \beta(\lambda)exp(\pi H(y, \lambda) + \frac{\pi}{2}H(\lambda, \lambda))$$

The induced map τ_{y*} on the fundamental group Λ of X is the identity. So if e_λ denotes the canonical factor ([Mum],[LB]) of L then $(id_\Lambda \times \tau_Y)^* e_\lambda$ is a factor for $\tau_y^* L$. But it is not a unique canonical factor. Let $k(w) = exp(-\pi H(w, y))$ then,

$$(id_\Lambda \times \tau_y)^* e_\lambda(w) k(w+\lambda) k(w)^{-1}$$

$$= \beta(\lambda) exp(\pi H(w+y, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)) exp(-\pi H(w+\lambda, y) + \pi H(w, y))$$

$$= \beta(\lambda) exp(\pi H(y, \lambda)) exp(-\pi H(\lambda, y)) exp(\pi H(w, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$$

$$= \beta(\lambda) exp(2\pi i Im H(y, \lambda)) exp(\pi H(w, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$$

and this is equivalent to $(id_\Lambda \times \tau_y)^* L$ and certainly this is a canonical factor of $\tau_y^* L$ whose semi-character is $\beta(\lambda) exp(2\pi i Im H(y, \lambda))$. □

Let (X, L_A) be a complex torus having line bundle L_A whose first Chern class is A. Let us recall V is the covering space of X. Consider the **C**-vector space, then $\bar{V}^* = Hom_{\bar{\mathbf{C}}}(\bar{V}, C)$ of **C**-antilinear forms $\nu : V \longrightarrow \mathbf{C}$. The underlying real vector space of $\bar{V}^*$ is canonically isomorphic to $Hom_R(V, R)$, such that $\nu \longmapsto Im\nu$. Hence canonically we obtain **R**-bilinear form

$$<,>: \bar{V}^* \times V \longrightarrow \mathbf{R}$$

such that the isomorphism given by $< \nu, v >= Im\nu(v)$. This bilinear form is non-degenerate. This implies that

$$\widehat{\Lambda} := \{\nu \in \bar{V}^* | < \nu, \Lambda > \subseteq \mathbf{Z}\}$$

is a lattice in $\bar{V}^*$, called dual lattice of Λ. Then $\widehat{X} := \bar{V}^*/\widehat{\Lambda}$ be the dual of X and L_α be the line bundle over it.

An isogeny of a complex torus X to complex X' is by definition a surjective homomorphism

$$X \longrightarrow X'$$

with a finite kernel.

Proposition .7 *If $f : X_1 \longrightarrow X_2$ is an isogeny of complex tori, the dual map $\hat{f} : \hat{X}_2 \longrightarrow \hat{X}_1$ is also an isogeny and its kernel is isomorphic to $Hom(ker f, \mathbf{C}_1)$. It also satisfies $deg\ \hat{f} = deg\ f$.*

Proof: The proof is given in ([LB], chapter 2).

Given a line bundle L_A and L_α on X and $\hat{X}$ respectively we get a pair of maps

$$\phi_{L_A} : X \longrightarrow \widehat{X} \text{ and } \phi_{\widehat{L_\alpha}} : \widehat{X} \longrightarrow X$$

Let τ_x be the translation on X and it satisfies

$$\tau_x : X \longrightarrow X \ , y \longmapsto x + y$$

Given an ample line bundle L on X for any $x \in X$, we a construct line bundle $t_x^* L \otimes L^{-1}$ whose first Chern class is zero. So we can identify the dual variety of X as $\widehat{X} := Pic^0(X)$, the group of line bundles on X which are algebraically equivalent to zero. Hence we obtain a map

$$\phi_L : X \longrightarrow \widehat{X}$$

such that

$$y \longmapsto t_y^* L \otimes L^{-1}$$

is a homomorphism and this follows from the theorem of square.

Theorem .8 *(Theorem of square)*

$$\tau_{x+y}^* L = \tau_x^* L \otimes \tau_y^* L \otimes L^{-1}$$

for all $x, y \in X$ *and* $L \in Pic(X)$.

For a proof see [Mum] or [LB].

Let us consider one of the map, say ϕ_L, suppose the kernel of this map is $K(L)$. In order to describe $K(L)$, define

$$\Lambda(L) = \{v \in V | ImH(v, \Lambda) \subseteq Z\},$$

this yields

$$K(L) = \Lambda(L)/\hat{\Lambda}.$$

It is known that $K(L)$ is finite if and only if L_A is non-degenerate. Hence the map $\phi_L : X \longrightarrow \hat{X}$ is an isogeny. Similarly it is true for the dual also.

By abusing the notation we will often write

$$A : X \longrightarrow \widehat{X} \text{ and } \alpha : \widehat{X} \longrightarrow X$$

The isogeny map ϕ_L is an isomorphism when X is a principally polarized abelian variety i.e. when $X = \widehat{X}$. In fact $\widehat{X} \cong Pic^0(X)$, where $Pic^0(X)$ denotes the isomorphism group of holomorphic line bundles on X with first Chern class zero which is known as the Picard group of degree zero on X. The $Pic^0(X)$ arises from the exponential sheaf exact sequence as

$$Pic^0(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, Z)} \cong \frac{\bar{V}^*}{\widehat{\Lambda}}$$

since $H^1(X, \mathcal{O}_X) \cong \bar{V}^* = Hom_{\mathbf{C}}(\bar{V}, \mathbf{C})$ and $\widehat{\Lambda} = Hom_R(\Lambda, Z)$.

Let us once again recall V and $\bar{V}$, covering spaces of X and $\widehat{X}$.Then $V \times \bar{V}^*$ is the covering space of $X \times \widehat{X}$. We can identify $\bar{V}^*$ with the dual of V as a real vector space, then we have the familiar real symplectic form on $U \times U^*$, where U is a real vector space. The variety $X \times \widehat{X}$ is again a complex torus (we shall call symplectic torus) and one can define bundle $\mathcal{P}$ over $X \times \widehat{X}$ by

$$H((x, \hat{x}), (y, \hat{y})) = \hat{y}(x) + \hat{x}(y)$$

$$\beta((\lambda, \lambda^*)) = exp(-\pi i Im < \lambda^*, \lambda >$$

where $x, y \in X$, $\hat{x}, \hat{y} \in \widehat{X}$, $\lambda \in \Lambda$ and $\lambda^* \in \Lambda^*$. Then $\mathcal{P}$ coincides with Poincaŕe bundle if X is algebraic.

Definition .9 *The two form Ω on $X \times \widehat{X}$ is a type $(1, 1)$ form and it is the curvature form of a canonical connection of the Poincaŕe bundle.*

We end this part here and in the next part we shall discuss Mukai's Fourier functor.

0.2.2 Mukai's Fourier functor

Let X be an abelian variety and suppose $\widehat{X}$ be its dual. Then the following projections

$$X \xleftarrow{\pi} X \times \widehat{X} \xrightarrow{\hat{\pi}} \widehat{X}$$

are flat and projective. Let $\mathcal{F}$ be the functor between the category of $\mathcal{O}_X$ modules $M(X)$ into the category of $\mathcal{O}_{\widehat{X}}$ module $M(\widehat{X})$ and it is given by

$$\mathcal{F}(M) = \pi_{\widehat{X}*}(\mathcal{P} \otimes \pi_X^* M)$$

Here $\mathcal{P}$ is the normalized *Poincare' bundle* on $X \times \widehat{X}$ and it is a flat $\mathcal{O}_{X\times\widehat{X}}$ module. Normalized means that both $\mathcal{P}|_{X\times\widehat{0}}$ and $\mathcal{P}|_{0\times\widehat{X}}$are trivial, also $\widehat{x} \in \widehat{X}$(respectively $x \in X$) $\mathcal{P}_{\widehat{x}}$ (respectively $\mathcal{P}_x$) denotes $\mathcal{P}|_{X\times\widehat{x}}$ (resp. $\mathcal{P}|_{x\times\widehat{X}}$) If the category of $\mathcal{O}_X$ module $M(X)$ have enough injectives then the functor $\mathcal{F}$ descends down to derived functor ([Ha1]) between the two derived categories $D(X)$ and $D(\widehat{X})$ of $M(X)$ and $M(\widehat{X})$ respectively ([Muk1],[Muk2]).

Theorem .10 *Suppose*

$$R\widehat{\mathcal{F}} : D(X) \longrightarrow D(\widehat{X})$$

and

$$R\mathcal{F} : D(\widehat{X}) \longrightarrow D(X)$$

be the derived functor of $\mathcal{F}$ and $\hat{\mathcal{F}}$, then these satisfy

$$R\mathcal{F} \circ R\widehat{\mathcal{F}} \cong (-1_X)^*[-m]$$

and

$$R\widehat{\mathcal{F}} \circ R\mathcal{F} \cong (-1_{\widehat{X}})^*[-m]$$

where $[-m]$ denotes the shifting of the complex m places to the right.

Proof: We shall carry out this proof here. We shall use the following algebraic tools which can be found in Hartshorne's book ([Har1],[Har2]).(1) Base change. (see III.9.3 in [Har2].
(2) Projection formula. (see II.5.6 in [Har1].
(3) Spectral sequence: If $\mathbf{R}G : D(A) \longrightarrow D(B)$ and $\mathbf{R}H : D(B) \longrightarrow D(C)$ are two suitable derived functors, then there is a natural isomorphism of

functors $\mathbf{R}(H \circ G) \cong \mathbf{R}H \circ \mathbf{R}G$. On the level of cohomology this gives rise to a spectral sequence:

$$E_2^{p,q} = R^pH \circ R^qG \Rightarrow E_\infty^{p+q} = R^{p+q}(H \circ G).$$

(see [Hart1] for details)

$$\begin{array}{ccccc} \widehat{X} & \xleftarrow{\hat{\pi}} & X \times \widehat{X} & \longrightarrow & \widehat{X} \\ \hat{\pi}\uparrow & & p_1\uparrow & & \uparrow \\ X \times \widehat{X} & \xleftarrow{p_2} & X \times X \times \widehat{X} & \xrightarrow{p_1} & X \times \widehat{X} \\ \pi\downarrow & & r\downarrow & & \pi\downarrow \\ X & \xleftarrow{q_2} & X \times X & \xrightarrow{q_1} & X \end{array}$$

Looking at the commutative diagram

$$R\mathcal{F} \circ R\widehat{\mathcal{F}}(!)$$

$$\equiv R\pi_*(\hat{\pi}^* R\hat{\pi}_*(\pi^*(!) \otimes \mathcal{P}) \otimes \mathcal{P})$$

$$\equiv R\pi_*(Rp_{1*}p_2^*(\pi^*(!) \otimes \mathcal{P}) \otimes \mathcal{P})$$

$$\equiv R\pi_*Rp_{1*}((\pi p_2)^*(!) \otimes (p_2^*\mathcal{P} \otimes p_1^*\mathcal{P})$$

$$\equiv R(\pi p_1)_*((\pi p_2)^*(!) \otimes (p_2^*\mathcal{P} \otimes p_1^*\mathcal{P}))$$

$$\equiv R(q_1 r)_*((q_2 r)^*(!) \otimes (p_2^*\mathcal{P} \otimes p_1^*\mathcal{P}))$$

$$\equiv Rq_{1*}Rr_*((q_2 r)^*(!) \otimes (p_2^*\mathcal{P} \otimes p_1^*\mathcal{P}))$$

Now $(m \times 1)^*\mathcal{P} = p_2^*\mathcal{P} \otimes p_1^*\mathcal{P}$, where $m : X \times X \longrightarrow X$. This is true because

$$p_2^*\mathcal{P} \otimes p_1^*\mathcal{P}|_{\{(x,y)\} \times \hat{X}} = \mathcal{P}_x \times \mathcal{P}_y = \mathcal{P}_{x+y} = (m \times 1)^*\mathcal{P}|_{\{x,y\} \times \hat{X}}$$

Hence, we obtain

$$R\mathcal{F} \circ R\widehat{\mathcal{F}}(!) \equiv Rq_{1*}(q_2^*(!) \otimes Rr_*(m \times 1)^*\mathcal{P})$$

$$= Rq_{1*}(q_2^*(!) \otimes m^* R\pi_*\mathcal{P})$$

Since we know [Mum] $H^i(X \times \widehat{X}, \mathcal{P}) = \mathbf{C}$ for $i = m$ and zero otherwise, hence we obtain

$$R\pi_*\mathcal{P} \equiv \mathbf{C}[-m]$$

We shall use another fact, let $\lambda : X \longrightarrow X \times X$ such that $x \longrightarrow (-x, x)$ and Λ be the image of λ, with $\mathcal{O}_\Lambda = \mathcal{O}_X|_\Lambda$. Thus we have

$$
\begin{aligned}
&Rq_{1*}(q_2^*(!) \otimes \mathcal{O}_\Lambda[-m]) \\
&\equiv Rq_{1*}R\lambda_*\lambda^* q_2^*(!)[-m] \\
&R(q_1\lambda)_*(q_2\lambda)^*(!)[-m] \\
&\equiv (-1_X)^*(!)[-m]
\end{aligned}
$$

□

The following definitions of W.I.T. and I.T. [Muk1] are important to define Fourier functor.

Definition .11 *Let M be a coherent $\mathcal{O}_X$ module. We say W.I.T. holds for M if*

$$R^i\widehat{\mathcal{F}}(M) = 0 \tag{0.1}$$

for all but one i, say $i(M)$, and this $i(M)$ is called index of M. We say

$$R^{i(M)}\widehat{\mathcal{F}}(M)$$

is a Fourier transform of the $\mathcal{O}_X$ module M. We say I.T. holds for M if it satisfies

$$H^i(X, M \otimes L) = 0 \tag{0.2}$$

for all but one i and for all $L \in Pic^0(X)$.

When I.T. holds for M ,then W.I.T. also holds for M and $\widehat{M}$.

Claim .12 *If a sheaf M satisfies the condition that $H^j(X, M \otimes \mathcal{P}_{\widehat{x}}) = 0$ for all $j \neq i(M)$ and all $\mathcal{P}_{\widehat{x}}$, then M satisfies W.I.T.*

Proof: For the proof we need the following things; If $f : X \longrightarrow Y$ is a map of projective varieties over any arbitrary closed field k and M is the sheaf on X then there is a natural map [Ha2]

$$R^i f_* M \otimes k(y) \longrightarrow H^i(X_y, M_y)$$

where $X \times_Y spec[k(y)]$ is the fibre of f over $y \in Y$ and M_y is the pullback bundle to X_y. If $H^i(X_y, M_y) = 0$ for all $y \in Y$ and $i > r$ then

$$R^r f_* M \otimes k(y) \cong H^r(X_y, M_y)$$

and since the Euler number $\chi(M_y)$ are independent of y, so if $H^j(X_y, M_y) = 0$ for all $j \neq r$ then dimension of $H^r(X_y, M_y)$ is independent of y and in that case $R^r f_* M$ is locally free.

Our claim is the simple application only of the fact that M satisfies I.T. so $H^j(X, M \otimes \mathcal{P}_x) = 0$ for all $j \neq i(M)$ then M satisfies W.I.T. Then $\hat{M}$ is also locally free.

$$R^i \widehat{\mathcal{F}}(M) \otimes k(\hat{x}) \cong H^i(X, (\pi^* M \otimes \mathcal{P})|_{X \times \hat{x}}) \cong H^j(X, M \otimes \mathcal{P})$$

□

If W.I.T. holds for any coherent sheaf $M(X)$ then we are in the position to define the Fourier functor

Theorem .13 *If W.I.T. holds for any coherent sheaf M, then $\widehat{M}$ i.e. the Fourier transform of M, also satisfies W.I.T. and*

$$i(\widehat{M}) = m - i(M)$$

Moreover $\widehat{\widehat{M}}$ is isomorphic to $(-1_X)^* M$ where $(-1_X)^* : X \longrightarrow X$ is the map sending $x \longrightarrow -x$.

As an example of this Fourier transform we shall see the next proposition

Example

Suppose E be the semi-stable vector bundle of rank r and degree $d \neq 0$ over X. Then E has a Fourier transform which is also a vector bundle.

Proof :

The proof is very simple, it suffices to show E is I.T. Suppose $d > 0$ then by Serre duality

$$H^1(X, E \otimes L) \cong H^0(X, E^* \otimes L^*)^*$$

Now E is semi-stable so also $E^* \otimes L^*$. Since

$$degE^* \otimes L^* = -d < 0$$

So we obtain

$$H^0(X, E^* \otimes L^*) = 0$$

Therefore E is $(I.T.)_0$ Similarly this will hold for degree $d < 0$,there E would $(I.T.)_1$.

Mukai [Muk1] stated the action of different functors, for example, Fourier functor, isogeny functor, twisting with line bundle etc on the derived category $D(X)$ of X. When it is a principally polarized abelian variety then clearly isogeny functor is an identity but we have action of Fourier functor and twisting of line bundles on $D(X)$. Mukai also gave the exchange relation between isogeny functor and the Fourier functor.

Lemma .14 *Suppose X and Y are two abelian varieties and $\varphi : X \longrightarrow Y$ an isogeny and $\hat{\varphi} : \widehat{Y} \longrightarrow \widehat{X}$ be the corresponding dual isogeny of φ then the exchange relation between the Fourier functor and isogeny and dual isogeny functors are*

$$\varphi^* \circ R\mathcal{F}_{\hat{Y} \to Y} \cong R\mathcal{F}_{\hat{X} \to X} \circ \hat{\varphi}_*$$
$$\varphi_* \circ R\mathcal{F}_{\hat{X} \to X} \cong R\mathcal{F}_{\hat{Y} \to Y} \circ \hat{\varphi}^*$$

Here we have kept the indices different intentionally and for proof one can consult Mukai[Muk1]. In Mukai's case the line bundle used here is assumed to be non-degenerate one i.e. the Euler characteristic of the line bundle is 1 in such a case it is loosely called principal polarization.

From the following observation Mukai asserted the action of $SL(2, Z)$ on the moduli space.

Proposition .15 *Let (X, L_A) be a principally polarized abelian variety of dimension m, the automorphisms $\otimes L_A$, $R\mathcal{F}$ on the derived category of X and satisfy*

$$(R\mathcal{F})^2 \cong (-1_X)^*[-m] \tag{0.3}$$

$$(\otimes L_A \circ R\mathcal{F})^3 \;\cong\; [-m] \tag{0.4}$$

$SL(2, Z)$ is generated by the two elements

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with the relations $P^2 = -1$ and $(QP)^3 = 1$. Mukai deduces from above proposition:

Theorem .16 *There exists an action of $SL(2, Z)$ on the derived category of principally polarized abelian variety, in which P acts as $R\mathcal{F}$ and Q as $\otimes L_A$.*

In the following section we shall lay out the picture of this $SL(2, Z)$ action on $D(X)$ explicitly. We will see shortly how the geometry of the ordinary metaplectic representation help us to interpret this action. Subsequently we shall give a new proof of this theorem.

0.3 Finding $SL(2, Z)$ action on $D(X)$

In this section our goal is to understand the action of $SL(2, Z)$ group on the derived category $D(X)$ of complex torus X. In Mukai's paper [Muk1] this action appears as something of an oddity.

Let X be a m dimensional complex torus and $\widehat{X}$ be its dual. Suppose $X \times \widehat{X}$ be a complex symplectic torus. Let $V \cong \mathbf{C}^m$ be the covering space of X and Λ be the lattice inside V so that $X = V/\Lambda$, X is identified to a m dimensional complex torus C^m/Λ. Suppose $f : X \longrightarrow X$ be a complex analytic automorphism then it induces a map $df : X \longrightarrow C^m$ and we know from the theory of maximum modular principle it is a locally constant map. So given any such holomorphic map f we can lift to locally $\mathbf{C}$- linear map on the covering space

$$g : \mathbf{C}^m \longrightarrow \mathbf{C}^m$$

fixing the lattice $\Lambda \subset \mathbf{C}^m$, since f takes L to itself (see [GH]). We have the following diagram commutative.

$$\begin{array}{ccc} \mathbf{C}^m & \longrightarrow & \mathbf{C}^m \\ \downarrow & & \downarrow \\ \mathbf{C}^m/\Lambda & \longrightarrow & \mathbf{C}^m/\Lambda \end{array}$$

Now $U(m,m)$ be automorphism between the covering spaces. Let Ω be the canonical closed $(1,1)$ form, Kähler form on the symplectic torus $X \times \widehat{X}$. Then the holomorphic automorphism $\mathrm{Aut}(X \times \widehat{X}, \Omega)$ be the automorphism group of symplectic torus which preserves the symplectic $(1,1)$- form Ω, induced by the linear action of the covering spaces. This group $\mathrm{Aut}(X \times \widehat{X})$ sits inside $U(m,m)$. So we have

$$U(m,m) \supset Aut(X \times \widehat{X})$$

In order to see how the $SL(2,Z)$ appearing we can define a map

$$SL(2,Z) \longrightarrow Aut(X \times \widehat{X})$$

such that for any $\begin{pmatrix} g & A \\ \alpha & g^T \end{pmatrix} \in SL(2,Z)$ we can define an element

$$\left(\begin{array}{c|c} g\,\mathbf{I_m} & A\mathbf{I_m} \\ \hline \alpha\mathbf{I_m} & g^T\mathbf{I_m} \end{array}\right) \in Aut(X \times \widehat{X})$$

where $\mathbf{I_m}$ is the $m \times m$ unit matrix. The elements of the matrix denote the set of maps; e,g,

$$g : X \longrightarrow X \ , \ A : X \longrightarrow \widehat{X};$$

and

$$\alpha : \widehat{X} \longrightarrow X \ , \ g^T : \widehat{X} \longrightarrow \widehat{X};$$

Since the determinant of the matrix $\begin{pmatrix} g & A \\ \alpha & g^T \end{pmatrix} \in SL(2,Z)$ is one so the action of the corresponding representative matrix of $\mathrm{Aut}(X \times \widehat{X})$ preserves the canonical $(1,1)$ form on the symplectic torus.

Putting down everything together in this section we obtain the following two results.

Lemma .17 *If X be the m dimensional complex torus and $\widehat{X}$ be its dual. Then $X \times \widehat{X}$ is a symplectic torus. Suppose $\mathbf{C}^m$ and $\mathbf{C}^{*m}$ be the corresponding covering spaces of X and $\widehat{X}$ respectively. Then the automorphism group $U(m,m)$ of the covering space of symplectic torus induced a holomorphic automorphism $U(m,m) \supset Aut(X \times \widehat{X})$ of the symplectic torus which preserves the closed $(1,1)$ form on it.*

Proposition .18 *If $Aut(X \times \widehat{X})$ is the automorphism group of the symplectic torus preserving the closed $(1,1)$ form, then there exists a map*

$$\Xi : SL(2,Z) \longrightarrow Aut(X \times \widehat{X})$$

which gives the action of $SL(2,Z)$.

In the next programme our aim is to establish explicit nature of the $SL(2,Z)$ action on the moduli space of vector bundles on abelian variety X, which is an analogue of the ordinary metaplectic representation on the Hilbert space $L^2(\mathbf{R}^n)$. In otherwords we shall imitate the classical case and want to match up with the bundle case. So it is worth to recapitulate the classical case of metaplectic representation on the $L^2(\mathbf{R}^n)$.

0.4 The metaplectic representation of $SL(2,R)$

In this section we will focus primarily on two things. In the first half we have given a rapid introduction of geometric quantization (for details see [Fol] and [Wo]) and in the later half we have discussed the construction of metaplectic representation.

0.4.1 Basic idea of geometric quantization

Let V be a $2n$ dimensional symplectic manifold with a closed non-degenerate two form ω, called symplectic form. The group $Sp(V)$ is the group of automorphism preserving ω. In particular when V is a phase space $\mathbf{R}^{2n}$ then the group $Sp(R,2n)$ can be realized explicitly as the subgroup of $GL(V)$ consisting of matrices of the block form

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)$$

where A_1 , A_2 , A_3, A_4 are $n \times n$ matrices satisfying

$$A_4^t A_1 - A_2^t A_3 = I \ , \ A_1^t A_3 = A_3^t A_1 \ , \ A_2^t A_4 = A_4^t A_2$$

Let us consider symplectic manifold V, the non-degeneracy means that if we consider ω as a map from

$$\omega : TV \longrightarrow T^*V$$

then there exist an inverse map

$$\omega^{-1} : T^*V \longrightarrow TV$$

If v^i be the local co-ordinates then ω and ω^{-1} can be expressed as follows

$$\omega = \omega_{ij} dv_i \wedge dv^j$$

and

$$\omega^{-1} = \omega^{ij} \frac{\partial}{\partial v^i} \otimes \frac{\partial}{\partial v^j}$$

then the matrices ω_{ij} and ω^{ij} are inverses

$$\omega_{ij} \omega^{jl} = \delta_i^l .$$

Suppose the image of the cohomology class $[\omega]$ lies in the integral cohomology $H^2(V, Z)$ then there exists a corresponding line bundle L. The line bundle L has some special structures , viz. (1) a Hermitian metric on each fibre, a Hermitian form on L is denoted by $(.,.)$ and it is conjugate linear with the second variable. (2) A unitary connection ∇ whose curvature form is $-i\omega$. The section of the line bundle forms pre-Hilbert space and ∇ corresponding to each vector field ξ gives an endomorphism

$$\nabla_\xi : \Gamma(L) \longrightarrow \Gamma(L)$$

All these together form prequantum data for quantization. Quantization of symplectic manifold means the construction of an unitary Hilbert space representation of the Poisson Lie algebra $C^\infty(V)$.

In order to quantize V in addition to the prequantum data we need to introduce polarization which foliates the V by n-dimensional Lagrangian submanifolds P which are maximal isotropic subspaces of V. The maximal isotropic subspace means $\omega|_{TP} = 0$. There are two equivalent ways one can do that, one by real polarization method and the other one is called complex polarization. We take those polarized sections which are constant along the leaves of foliation. Then we denote the Hilbert space as H_P of the prequantum Hilbert space H, viewed as space of sections of a line bundle L over V which are covariant constant $\nabla_P s = 0$ along leaves. We denote $\mathcal{X}_P$ as the set vector field along the leaf.

Definition .19 *A smooth section $s : V \longrightarrow L$ is said to be polarized if*

$$\nabla_X s = 0 \textit{ for } X \in \mathcal{X}_P$$

Locally polarized sections exist because the curvature of ∇ vanishes on the restriction to the direction in P. So locally we can represent them in terms of coordinates.

Let P and Q are two polarization and H_P and H_Q are the corresponding Hilbert spaces. If $s \in H_P$ then we can define $s' \in H_Q$ which is covariantly constant along the leaves of Q as

$$s'(x) = \int_{\text{leaf of Q through x}} s(y)dy$$

This integral makes invariant sense when we tensor with half forms since the action of any $g \in sp(2n, R)$ on the sections given by

$$\phi^g(x) = \frac{1}{\sqrt{detg}}\phi(g(x))$$

preserves the L^2 - norm.

Let $\pi_{PQ} : H_P \longrightarrow H_Q$ be the above projection map between the two polarized Hilbert space. Given three maps for three different polarization then the maps between three different spaces in the commutative diagram agree upto some scale factor. Now we have to assume $V = \mathbf{R}^{2n}$ and P, Q are linear which follows from Schur's lemma.

$$\begin{tikzcd} H_P \arrow[dr, "\pi_{PQ}"] \arrow[dd] & \\ & H_R \\ H_Q \arrow[ur, "\pi_{QR}"'] & \end{tikzcd}$$

The isomorphism class of the representation of $Sp(2n, R)$ on the Hilbert space does not change when twisted by a diffeomorphism. If $U \in Sp(2n, R)$ be the automorphism such that $U \circ P = Q$ then U induced a map

$$U : H_P \longrightarrow H_{UPU^{-1}}$$

The map $U \circ \pi_{PQ}$ is the automorphism map of H_P is called metaplectic representation of U on H_P.

0.4.2 Construction of the metaplectic representation

In particular, we take phase space $\mathbf{R}^{2n}$ is our symplectic manifold. The metaplectic representation (see [Wo],[Fol]) is the double valued unitary representation of $Sp(2n, R)$ on the Hilbert space $L^2(\mathbf{R}^n)$. Also one can equally define the metaplectic representation is a unitary representation of the double cover group $Mp(2n, R)$ of the symplectic group $Sp(2n, R)$ on $L^2(\mathbf{R}^n)$.

One way to define this representation is to consider first the representation of the Heisenberg group $\mathbf{R}^{2n} \times S^1$ on $L^2(\mathbf{R}^n)$. It should be noted that Heisenberg algebra is the underlying algebraic structure of the Poisson bracket relations for the canonical coordinates in Hamiltonian mechanics and the commutation relations for their quantum analogues. The multiplication [Fol] of the Heisenberg group is defined by

$$(x, \lambda_1)(y, \lambda_2) = (x + y, \lambda_1\lambda_2 e^{i\omega(x,y)})$$

The Lie algebra of Heisenberg group may be equivalently viewed as

$$[p_j, p_k] = [q_j, q_k] = 0 \text{ and } [p_j, q_k] = \delta_{jk}$$

A representation of these generators on $L^2(\mathbf{R}^n)$ becomes

$$q_k \longrightarrow ix^k$$

$$p_k \longrightarrow \frac{\partial}{\partial x_k}$$

These give the representation of $Sp(2n, R)$. This can be argued in the following way.

Corresponding to the Lie algebra of $Sp(2n, R)$ we obtain the Poisson brackets of homogeneous quadratic polynomials on $\mathbf{R}^{2n}$. The Hamiltonian vector fields they generate are the infinitesimal linear symplectic maps on $\mathbf{R}^{2n}$. If the operator $\{\hat{g}_i\} \equiv (\hat{p}_i, \hat{q}_i)$ on $L^2(\mathbf{R})$ represent $\{g_i\}$. Then one represents the element of $sp(2n, R)$ corresponding to $g_i g_j$ by the operator $\frac{1}{2}(\hat{g}_i\hat{g}_j + \hat{g}_j\hat{g}_i)$ and these operator really satisfies commutation relations of $sp(2n, R)$.

Since $Sp(2n, R)$ acts projectively by unitary transformation on $L^2(\mathbf{R}^n)$ intertwining the action of $Sp(2n, R)$. This asserts ρ, the Schrödinger representation of Heisenberg group $\mathcal{H}_n$ i.e. the map from $\mathcal{H}_n$ to the group of unitary operators on the $L^2(\mathbf{R}^n)$, defined by

$$\rho(p, q)g(x) = e^{2\pi i qx + \pi i pq} g(x + p)$$

If $S \in Sp(2n, R)$, we can compose another new representation $\rho \circ S$ of the Heisenberg group on $L^2(\mathbf{R}^n)$. By the Stone - von Neumann theorem ρ and $\rho \circ S$ are equivalent. Hence there exist a unitary operator U_S on H_P such that U_S satisfies the following relation:

$$\rho(S(p, q)) = U_S \rho(p, q) U_S^{-1}$$

for any $S \in Sp(2n, R)$. By Schur's lemma $U(S)$ is determinded upto a phase factor ± 1 and satisfies

$$U_{S_1 S_2} = \pm U_{S_1} U_{S_2},$$

where $S_1, S_2 \in Sp(2n, R)$.Thus U_S is a double valued unitary representation of the symplectic group.

Now we give the metaplectic representation of some elements of the symplectic group, notice that these elements are the automorphism group of the Heisenberg Lie algebra that leaves the centre pointwise fixed. Let

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, R)$$

be the two elements viz. B and J,say. It is easy to find out U_S upto a phase factor for these elements rather than general element of $SL(2,R)$.

$$[\rho\circ S(p,q)]g(x)=\rho[(p,q).\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}]g(x)=[p,bp+q]g(x)$$

The Schrödinger representation of $\rho(p,bp+q)$ on $L^2(\mathbf{R}^n)$ is

$$exp(2\pi i[bp+q]x+\pi ip[bp+q])g(x+p)$$

$$=e^{-\pi ixbx}e^{2\pi iqx+\pi ipq}e^{\pi i(x+p)b(x+p)g(x+p)}$$

$$=U\rho(p,q)U^{-1}g(x),$$

where $Ug(x)=e^{-\pi i(xbx)}g(x)$.................$(I)$

Remark .20 *Notice that when b is positive then the corresponding gaussian factor is negative. This will be important when we shall propose the action of $Aut\ (X\times\hat{X})$ on $D(X)$.*

Similarly when $S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, the Schrödinger representation is

$$[\rho\circ\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}]g(x)=\rho(-q,p)g(x)=\mathcal{F}\rho(p,q)\mathcal{F}^{-1}......................(II)$$

where $\mathcal{F}$ stands for Fourier transform and intertwines $\rho(p,q)$ and $\rho(-q,p)$. So here the metaplectic representation of element $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is Fourier transform.

It is easy to see that if the metaplectic representation of these elements are known then one can easily construct the metaplectic representation of $\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix}$. We must notice that this matrix can be expressed as the product of the above two matrices, hence

$$\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix}=JSJ^{-1}.$$

The corresponding metaplectic representation is

$$U(\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}) g(x) = \mathcal{F}^{-1}(e^{\pi i \xi b \xi^{-1}} \hat{g})$$

where $\hat{g}$ denotes the Fourier transform of g.

In this way we can calculate the metaplectic representation of the full fledge $SL(2, R)$ group [Fol]. Let $\begin{pmatrix} d & b \\ c & 1 \end{pmatrix}$ be an element of $SL(2, R)$ whose diagonal entries are 1 and $d = 1 + bc$. Then if we decompose the matrix in the following way

$$\begin{pmatrix} d & b \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

We already know the metaplectic representation of $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. So the metaplectic representation of $\begin{pmatrix} d & b \\ c & 1 \end{pmatrix}$ would be

$$U \begin{pmatrix} d & b \\ c & 1 \end{pmatrix} g(x) = e^{-\pi i x b x} \mathcal{F}^{-1}[e^{\pi i \xi c \xi} \widehat{g}(x)]$$

Similarly one can easily obtain the metaplectic representation of $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ on $L^2(\mathbf{R}^n)$.

$$U \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} g(x) = \mathcal{F}[e^{\pi i \xi c \xi} \mathcal{F}^{-1}(e^{-\pi i x b x} g(x))]$$

So in this section we have given an account of metaplectic representation of different elements of $SL(2, R)$ on $L^2(\mathbf{R}^n)$. Next we will show how the actions of different elements of $SL(2, Z)$ group on the moduli space of vector bundle over complex tori X, as stated by Mukai, nicely match up with the metaplectic representation discussed in this section for the general case.

0.5 The metaplectic action of Γ_0 on the derived category $D(X)$

We arrange this section in the following way, in the first part we will describe Mukai's action and the conjectural action of $Aut(X \times \widehat{X})$ on the derived category $D(X)$ of complex tori. In the final part we will outline the proof of this conjecture.

0.5.1 Action of Γ_0 on derived category

The plan of this part is to give first a new interpretation of Mukai's action of $SL(2, Z)$ on the derived category $D(X)$ over complex tori and then establish the conjectural action on $D(X)$.

New interpretation of $SL(2, Z)$ action

To begin with, let us compare the results of the action of different elements of $SL(2, Z)$ on the $D(X)$ and the corresponding elements of $SL(2, R)$ ordinary metaplectic representation on $L^2(\mathbf{R}^n)$ in the ordinary case. When the action of generator is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then from the previous section we obtain the metaplectic representation to be multiplication by a Gaussian in the ordinary case. This really match up with the Mukai's definition of tensoring with line bundle $\otimes L$ on $D(X)$ corresponding to the same generator. Since multiplication by Gaussian is equivalent to twisting with line bundle in the module framework. Similarly the action of the generator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ also match with the general (symplectic) case. Here we obtain Fourier transform in the ordinary situation and Fourier functor in the bundle (or module) case. Also note that the shifting of the complex in the bundle in the right or left direction can be compared to the $\pm$ phase discrepency in the original metaplectic case.

Hence we can say explicitly the actual nature or type of $SL(2, Z)$ action which is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, on the derived category

$D(X)$ of complex tori. This gives a new interpretation of Mukai's action on $D(X)$.

Proposition .21 *Let X be an principally polarized abelian variety, then the group $SL(2, Z)$ acts on the derived category of X modulo shift.*

Next we want to show that a much bigger group Γ_0 than $SL(2, Z)$ acts on $D(X)$ and also this action is even true for non principally polarized abelian variety. In fact this is really the outcome of this new interpretation.

Establishing the conjectural action

Uptil now we have established an analogous picture of classical metaplectic representation in the derived category framework. In this set up the automorphism group Γ_0 acts on the symplectic torus $X \times \widehat{X}$ preserving the two form Ω. We lift this action of Γ_0 on the derived category $D(X)$. Here $D(X)$ is playing the role of $L^2(\mathbf{R}^n)$ in the classical case. Hence by comparing the results of ordinary situation [(I) in 4.4] one can immediately write down the action of $\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}$ where A is a matrix and $A : X \longrightarrow \widehat{X}$. The action of this element is equivalent to the action of $\otimes L_A$ on $\mathcal{M}(X)$, where $A \in H^2(X, Z)$ is the first Chern class of the line bundle L_A on X.

Remark .22 *We have noticed earlier [(I) in 4.4] that when b is negative the gaussian factor is positive. Likewise we will take that convention in the derived category case also, i.e. when A is "negative" we get a twisting of L_A and when A is positive we will twist with its dual L_A^*. This will be important when we shall give the partial proof of this conjecture.*

Since we already know from our previous section about the metaplectic representation of the different elements (also see [Fol]) of the $SL(2, R)$ or $Sp(2n, R)$ on $L^2(\mathbf{R}^n)$. So using the correspondence dictionary, which is, multiplication by Gaussian will go to twisting of line bundle and Fourier and inverse Fourier will go to Fourier functor and inverse Fourier functor respectively we can state the action of the different elements of Γ_0 on $D(X)$.

To see this let us consider the action of the $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ on the derived category $D(X)$ where $\alpha : \widehat{X} \longrightarrow X$. So in this case the conjectural action would be $R\mathcal{F} \circ (\otimes L_\alpha \circ R\widehat{\mathcal{F}})$ where L_α is the line bundle on the dual torus $\widehat{X}$ of X whose first Chern class is $\alpha \in H^2(\hat{X}, Z)$. Recall

$$R\widehat{\mathcal{F}} : D(X) \longrightarrow D(\widehat{X})$$

and

$$R\mathcal{F} : D(\widehat{X}) \longrightarrow D(X).$$

It should be noted that this action holds good for any arbitrary polarized abelian varieties.

When $\alpha = 1$, the principally abelian variety case then the operator must be reduced to $R\widehat{\mathcal{F}} \circ (\otimes L \circ R\mathcal{F})$.

Similar techniques can be used to calculate the action of different elements of Γ i.e. finite index subgroup of $Aut(X \times \widehat{X})$ on $D(X)$. Every time we use the action of $\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to calculate out the action of different elements of Γ on the derived category of $\mathcal{O}_X$ module. Finally we can write down a dictionary:

elements of $SL(2, R)$	metaplectic action on $L^2(R^n)$	proposed action on $D(X)$
$\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}$	$exp(\pi i < x, Ax >)$	$\otimes L_A$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Fourier transform	Fourier functor

We give more examples of action of different elements of Γ, viz. $\begin{pmatrix} d & -A \\ \alpha & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -A \\ \alpha & d^T \end{pmatrix}$ would be

$$\otimes \widehat{L}_\alpha \circ [R\widehat{\mathcal{F}} \circ (\otimes L_A \circ R\mathcal{F})]$$

and

$$R\mathcal{F} \circ [\otimes \widehat{L}_\alpha \circ (R\widehat{\mathcal{F}} \circ \otimes L_A)]$$

Here d stands for endomorphism of X i.e. $d : X \longrightarrow X$ and d^T stands for endomorphism of $\hat{X}$ i.e. $d^T : \hat{X} \longrightarrow \hat{X}$. Now we are in the position to state the conjectural action of Γ on the moduli space of vector bundles on the abelian variety.

Conjecture .23 *Let $D(X)$ be the derived category of $\mathcal{O}_X$ module $M(X)$ of complex tori X. Suppose $\hat{X}$ be the dual of X. Let L_A and L_α be the line bundles of even first Chern classes (A and α resp.) over X and $\hat{X}$ respectively. Let $X \times \widehat{X}$ be the symplectic torus and Γ_0 acts on it, preserving the two closed form Ω. This induces the action of Γ on $D(X)$. It acts modulo a shift on the $D(X)$. The action of different elements of Γ like*

$$\begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

$$\begin{pmatrix} d & A \\ \alpha & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & A \\ \alpha & d \end{pmatrix}$$

on $D(X)$ are $\otimes L_A$; $R\mathcal{F} \circ (\otimes L_\alpha \circ R\widehat{\mathcal{F}})$, $\otimes \widehat{L}_\alpha \circ [R\widehat{\mathcal{F}} \circ (\otimes L_A \circ R\mathcal{F})]$ and $R\mathcal{F} \circ [\otimes \widehat{L}_\alpha \circ (R\widehat{\mathcal{F}} \circ L_A)]$ respectively.

0.5.2 On the way to proof

In this section we are going to present two partial results along the line of proof of our conjecture.

Let $D(X)$ be the derived category of $\mathcal{O}_X$ module $M(X)$ on the complex tori X. We know that corresponding to any $\xi \in \widehat{X}$ we can associate a line bundle on X given by

$$L_\xi \longrightarrow X$$

Let $\iota \in X \times \widehat{X}$ and suppose τ_ι be the translation in the symplectic torus then this has an induced action τ_ι^* on the line bundle L of degree zero over X. Then the 'X' part of the translation acts on L trivially but the '$\widehat{X}$' part acts on L by twisting with flat line bundle. We want to see the action of group Γ_0 on translation.

Let h be the identity element of the automorphism group Γ_0 and suppose it is a product of different elements of Γ_0 such that $h = h_1 h_2 h_3 \ldots\ldots\ldots h_{n-1} h_n$. All these h_i are of special forms and their actions are known from the previous section.

Suppose T_{h_1} , T_{h_2} ,.......,T_{h_n} be the corresponding representations of h_1, h_2 .... h_n respectively on $D(X)$. We denote $T_h = T_{h_1}T_{h_2}T_{h_3}......T_{h_{n-1}}T_{h_n}$ and every T_{h_i} map

$$T_{h_i} : D(X) \longrightarrow D(X).$$

We want to proof our main conjecture but unfortunately we fail to give a complete proof. Instead of that we will give a partial proof of this conjecture and replace it by some other conjecture.

Conjecture .24

$$T_{h_1}T_{h_2}........T_{h_n}(E) = E$$

This is obviously the equivalent to our main conjecture.

The automorphism group Γ_0 is generated by elements of three kinds and these are $g_0 = \begin{pmatrix} g & 0 \\ 0 & g^T \end{pmatrix}$, $g_1 = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$

Our next task is to investigate the action of these generators on translation.

Lemma .25 *Let $\iota \in X \times \widehat{X}$ and τ_ι be the translation on symplectic torus $X \times \widehat{X}$. Then τ_ι^* satisfies*

$$T_{g_i}\tau_\iota^* = \tau_{g_i(\iota)}^* T_{g_i}$$

Proof: Case 1 : g_0 acts on $D(X)$ by simple automorphism. So the lemma is trivially satisfied.
Case 2 : Given $\iota \in X \times \hat{X}$ the translation τ_ι in the symplectic torus has an induced action τ_ι^* on the line bundle and "$\hat{X}$" part acts on line bundle by twisting with flat line bundle. We know from section 4.3:

$$A : X \longrightarrow \hat{X}$$

We know from the previous discussion action of g_1 on symplectic torus, if $\iota = (\hat{x}, x)$ then $g_1\iota = (\hat{x} - Ax, x)$. Also we know from our previous discussion about the proposed action of T_{g_1} on an arbitrary line bundle.

$$T_{g_1}L = L_A \otimes L$$

where L_A is the degree 2 line bundle of Chern class A

$$L.H.S. = T_{g_1}(\tau_\iota^* L)$$

$$= T_{g_1}(\tau_x^* L \otimes L_{\hat{x}})$$

$$= L_A \otimes \tau_x^* L \otimes L_{\hat{x}})$$

If we consider the right hand side of the above expression would be

$$R.H.S. = \tau_{g_1 \iota}^* T_{g_1} L$$

$$= \tau_{g_1 \iota}^* (L_A \otimes L)$$

$$= L_{\hat{x}-Ax} \otimes \tau_x^* L_A \otimes \tau_x^* L$$

$$= L_{\hat{x}-Ax} \otimes L_A \otimes L(0, exp(2\pi i A(v, \dot{}))) \otimes \tau_x^* L$$

$$= L_{\hat{x}} \otimes L(0, exp(-2\pi i A(v, \dot{}))) \otimes L_A \otimes L(0, exp(2\pi i A(v, \dot{}) \otimes \tau_x^* L$$

$$= L_{\hat{x}} \otimes L_A \otimes \tau_x^* L$$

$$= L.H.S.$$

Case 3 g_2 acts on L by

$$L \longrightarrow L_\alpha \widehat{\otimes} L$$

Again we have to consider line bundle L_α on $\widehat{X}$ to have even Chern class and rest of the calculation similar to case 2.

□

It follows that operation commutes with translation.

Corollary .26

$$T_h \tau_\iota^* = \tau_\iota^* T_h$$

Proof:: L.H.S.

$$T_h \tau_\iota^* = T_{h_1} T_{h_2} \ldots\ldots T_{h_{n-1}} T_{h_n} \tau_\iota^*$$

$$= T_{h_1} T_{h_2} \ldots\ldots T_{h_{n-1}} \tau_{h_n(\iota)}^* T_{h_n}$$

$$= T_{h_1} T_{h_2} \ldots\ldots \tau_{h_{n-1} h_n(\iota)}^* T_{h_{n-1}} T_{h_n}$$

$$\ldots\ldots = \tau^*_{h_1 h_2 \ldots h_n(\iota)} T_{h_1} T_{h_2} T_{h_3} \ldots\ldots T_{h_n}$$
$$= \tau_\iota^* T_h$$

□

Next we will investigate the action of group Γ (strictly speaking $\Gamma mod 2$) on the Chern character of the bundle ([Ful], [GH]). Let us start with some definitions.

Definition .27 *The Chern character $ch(E)$ of a vector bundle E of rank r is defined by the formula*

$$ch(E) = \sum_{i=1}^{r} exp(t_i)$$

where $t_1, t_2, \ldots\ldots\ldots, t_r$ are the Chern roots of E.

The first few terms are

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \ldots\ldots$$

For the tensor product of the bundles

$$ch(E \otimes F) = ch(E).ch(F)$$

We know $V \times \bar{V}^*$ be the covering space of the symplectic torus $X \times \widehat{X}$ where $V \cong \mathbf{C}^m$. $SU(m,m)$ be the automorphism of the covering space $V \times \bar{V}^*$.

$$\left(\begin{array}{c|c} g & A \\ \hline \alpha & g^T \end{array}\right) \in SU(m,m)$$

acts on $V \times \bar{V}^*$ such that

$$g : V \longrightarrow V \ , \ g^T : \bar{V}^* \longrightarrow \bar{V}^*$$

$$A : V \longrightarrow \bar{V}^* \ , \ \alpha : \bar{V}^* \longrightarrow V$$

We first define an action of $SU(m,m)$ on the differential forms $\wedge_R^* V$.

Theorem .28 *Let $V \times \bar{V}^*$ be the covering space of $X \times \hat{X}$. Suppose $su(m,m)$ is the automorphism group of the covering space. Then there exists an action of $su(m,m)$ on $\wedge^*_R V$ such that its generators act on $\wedge^*_R V$ in the following way.*

(1) The generator $\tilde{g}_0$ acts on $\wedge^*_R V$ through standard action of g.

(2) The second generator A acts on $\wedge^*_R V$ through wedging by the two form A.

(3) The third generator α acts on $\wedge^*_R V$ through contraction by two form α.

Proof:

Let V be a m-dimensional vector space which is equipped with an inner product $<,>$ and suppose $\wedge V$ is the exterior algebra of V.

Recall $V \times \bar{V}^*$ is the covering space of the symplectic torus $X \times \widehat{X}$ where $V \cong \mathbf{C}^m$. We can identify $\bar{V}^*$ with the dual of V as a real vector space. We have the familiar symplectic $(1,1)$ form $V \otimes \bar{V}^*$ on the covering space.

The automorphism group of the $V \times \bar{V}$ is $su(m,m)$ and this preserves the canonical two form on it. We can lift this action on the exterior algebra or space of covectors of V. Since $su(m,m)$ is generated by three generators $(\tilde{g}_0, \tilde{g}_1, \tilde{g}_2)$ so we shall consider the action of the generators of $su(m,m)$ one by one on the space of exterior algebras of V.
Let us consider $\tilde{g}_0$ acts on $\wedge^*_R V$ through standard action of g.

Next we consider action of $\tilde{g}_1 = A$ on $\wedge^* V$. This action

$$T_{\tilde{g}_1} : \wedge^* V \longrightarrow \wedge^* V$$

is defined by $A(v) = A \wedge v$, where $v \in \wedge^* V$. Since A is a two form so it acts on the exterior algebra by wedging through two form. Hence we obtain

$$A : \wedge^p V \longrightarrow \wedge^{p+2}(V).$$

Moreover

$$A : \wedge^{p,q} V \longrightarrow \wedge^{p+1,q+1} V$$

and A is bi-homogeneous of bidegree $(1,1)$.

Finally we consider the action of $\tilde{g}_2 = \alpha$.

$$T_{\tilde{g}_2} : \wedge^* V \longrightarrow \wedge^* V$$

In this case α is a two form and taking values in the dual space, so it acts on the exterior algebra $\wedge^* V$ by contraction.

$$\alpha : \wedge^p V \longrightarrow \wedge^{p-2} V$$

and it is also bi-homogeneous and bidegree.

$$\alpha : \wedge^{p,q} V \longrightarrow \wedge^{p-1,q-1} V$$

So we define a representation

$$\Re : su(m,m) \longrightarrow End(\Lambda V)$$

such that $\Re(\tilde{g}_2)$, $\Re(\tilde{g}_1)$ and $\Re(\tilde{g}_0)$ satisfy

$$[\Re(\tilde{g}_0), \Re(\tilde{g}_1] = -2\Re(\tilde{g}_1)$$

$$[\Re(\tilde{g}_0), \Re(\tilde{g}_2] = 2\Re(\tilde{g}_2)$$

$$[\Re(\tilde{g}_1, \Re(\tilde{g}_2] = \Re(\tilde{g}_0$$

□

Remark .29 *Under the inclusion $su(1,1) \subset su(m,m)$ we obtain an action of $su(1,1)$ and hence $sl(2,C)$ on the extendable of V, as in Wells [We].*

For details see Wells [We, chap.V , sec. 3].

□

Our next step will be the restricted case of this theorem and we look for the action of $\Gamma_0 \subset SU(m,m)$ on the Chern character, which lives in $\wedge^* V$ having integral coefficient. By the above we have an action of $\Gamma_0 \subset SU(m,m)$ on the Chern character. Explicitly

(1) g_0 acts on the Chern character in the obvious way.

(2) $T_{g_1} ch(E) = e^A ch(E)$ where A is a two form and it acts on $ch(E)$ by wedging.

(3) $T_{g_2} ch(E) = e^{\alpha} ch(E)$ where α is a two form on the dual of the abelian variety. α acts on $ch(E)$ via contraction.

Since the group action commutes with the Chern character and also we know the tangent bundle of a torus is trivial, then by the Grothendieck - Riemann - Roch [Ful] theorem the following diagram is commutative.

$$\begin{array}{ccc} K(X) & \xrightarrow{\Gamma} & K(\widehat{X}) \\ ch \downarrow & & ch \downarrow \\ H^*(X,Q) & \xrightarrow{s} & H^*(\widehat{X},Q) \end{array}$$

The second row operation is $s(x) = \pi_{\hat{X}*}(e^{\Omega} \cup \pi_X^*(x))$ for every $x \in H^*(X,Q)$ where $\Omega \in H^2(X \times \hat{X}, Z)$ is the Chern class of a Poincare bundle on $X \times \hat{X}$.

Lemma .30 *If W.I.T. holds for the coherent sheaf $\mathcal{E}$ and its index is equal to j, then*

$$ch^n(\mathcal{E}) = (-1)^{j+n} \eta^{2m-2n}(ch^{m-n}(\mathcal{E}))$$

in $H^{2n}(\hat{X}, Z)$ and η is the map $\eta^n : H^n(X,Z) \longrightarrow H^{2m-n}(\hat{X}, Z)$.

Proof: Let e_1, e_2,.....,e_{2m} be the basis of $H^1(X,Z)$ and e_1^*,e_2^*,.......,e_{2m}^* are the dual basis of $H^1(\hat{X},Z)$. Let $\Omega \in H^2(X \times \hat{X}, Z)$ is the Chern class of the Poincare bundle and it is given by

$$\Omega = \sum_{i=1}^{i=2m} e_i \wedge e_i^*$$

The p-fold wedge product

$$\Omega \wedge \Omega \wedge \ldots\ldots\ldots\ldots \wedge \Omega = (-1)^{p(p-1)/2} p! \sum_{j_1 < j_2 < \ldots\ldots\ldots < j_p} e_{j_1} \wedge \ldots\ldots\ldots \wedge e_{j_p} \wedge e_{j_1}^* \wedge \ldots\ldots \wedge e_{j_p}^*$$

So

$$e^{\Omega} = \sum_{p=0}^{2m} (-1)^{p(p-1)} \delta_p$$

where $\delta_p \in H^p(X, Z) \otimes H^p(\hat{X}, Z)$. We have the following canonical projections

$$X \xleftarrow{\pi_X} X \times \hat{X} \xrightarrow{\pi_{\hat{X}}} \hat{X}$$

Hence

$$e^{\Omega} \cup \pi_X^*(?) = \sum_{i=1}^{2m} (-1)^{p(p-1)/2} \delta_p \wedge \pi_X^*(?)$$

for every $? \in H^{\cdot}(X, Z)$. The direct image is the natural projection

$$H^{\cdot}(X \times \hat{X}, Z) \longrightarrow \oplus_i H^{2m}(X, Z) \otimes H^i(\widehat{X}, Z)$$

and

$$\oplus_i H^{2m}(X, Z) \otimes H^i(\widehat{X}, Z) \xrightarrow{\kappa \otimes 1} \oplus_i Z \otimes H^i(\widehat{X}, Z) \cong H^{\cdot}(\widehat{X})$$

where κ is the orientation of X. We know

$$s : H^{\cdot}(X, Z) \longrightarrow H^{\cdot}(\hat{X}, Z)$$

Hence

$$s(?) = \pi_{X,*}(e^{\Omega} \cup \pi_X^*(?))$$

$$= (-1)^{(2m-n)(2m-n-1)/2} \pi_{X,*}(\delta_{2m-p} \wedge \pi_X^*(?))$$

$$(-1)^{n(n+1)/2+m} \pi_{X,*}((e_{i_1} \wedge \ldots\ldots \wedge e_{i_n}) \wedge (e_{j_1} \wedge \ldots\ldots \wedge e_{j_{2m-n}}) \wedge (e^*_{j_1} \wedge \ldots\ldots \wedge e^*_{j_{2m-n}}))$$

$$\kappa(e_{j_1} \wedge \ldots\ldots \wedge e_{i_n} \wedge e_{j_1} \wedge \ldots\ldots \wedge e_{j_{2m-n}} \wedge e^*_{j_1} \wedge \ldots\ldots \wedge e^*_{j_{2m-n}}$$

$$= (-1)^{n(n+1)/2+m} \eta^n(?)$$

where $? = e_{j_1} \wedge \ldots\ldots \wedge e_{j_n}$. So the result follows immediately from this. □

Proposition .31

$$ch(T_{g_i}(E)) = T_{g_i} ch(E)$$

Proof: Case 1 : g_0 acts trivially on the Chern character.
Case 2 : T_{g_1} acts on $E \in \mathcal{M}$ by twisting the line bundle $T_{g_1}(E) = E \otimes L_A$, where A is the Chern class (degree 2) of the line bundle Then

$$ch(E \otimes L_A) = ch(E).ch(L_A)$$
$$= e^A ch(E)$$
$$= T_{g_1} ch(E)$$

Case 3 : We know T_{g_2} acts on E by

$$E \longrightarrow \widehat{\hat{E} \otimes L_\alpha}$$

Now we apply the previous lemma.

$$ch^p T_{g_2}(E) = ch\widehat{\hat{E} \otimes L_\alpha}$$
$$= (-1)^{j+p}\eta^{2m-2p}(ch^{m-p}(\hat{E} \otimes L_\alpha))$$
$$= (-1)^{j+p}\eta^{2m-2p}[e^\alpha ch^{m-p}(\hat{E})]$$
$$= (-1)^{j+p}(-1)^{2m-j-p}\eta^{2m-2p}\eta^{2p}[e^\alpha ch^p(E)]$$
$$= T_{g_2} ch(E)$$

Thus we proved the proposition.

Next one follows immediately from this result:

Corollary .32

$$ch(T_h(E)) = ch(E)$$

This corollary shows that the Chern character of $T(L)$ is the same as that of L and it seems likely that we can deduce that from this that $T(L)$ is induced a line bundle of zero Chern class.

We are finishing up this chapter with some interesting observations follow from the earlier parts of the chapter.

0.6 Concluding remarks

Let us compare the derived category case with the function case. Our problem is to prove that certain map $T : D(X) \longrightarrow D(X)$ is identity. In the function case we would have same problem (i.e. in defining the metaplectic representation). Let $\Im$ be a linear map from function on $\mathbf{R}$ to function on $\mathbf{R}$. In that case it suffices to prove $\Im$ commutes with translation and multiplication by $exp(ix\theta)$. This actually follows from the following:

Proposition .33 *Let*

$$\Im : L^2(\mathbf{R}) \longrightarrow L^2(\mathbf{R})$$

commutes with the translation and the multiplication by $exp(ix\theta)$ then $\Im = \lambda.id$ for $\lambda \in \mathbf{C}$.

This analogy suggest us to something similar in the derived category case.

In the other approach also we have a resemblance with the 'function' case.

Proposition .34 *If $\mathcal{T}(e^{ix\theta}) = e^{ix\theta}$ for all θ, then by Fourier inversion theorem, we know $\mathcal{T} = id$.*

This suggest us to prove in our case

$$T(L) = L \text{ for } L \in Pic^0(X).$$

From the corollary (54) we obtain

$$T(L) = L'$$

where L' is another line bundle with first Chern class $c_1(L') = 0$.

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