# On the method of exact linearization of autonomous ordinary differential equations

**L.M. Berkovich**

Abstract

The method of exact linearization for nonlinear autonomous ordinary differential equations (ODE), created by the author, is being considered in the preprint. A class of nonlinear autonomous equations depending on two arbitrary functions and few parameters, is being constructed, as well as nonlinear variable transformation ,nonlinear superposition principle (NSP), which transforms the above class to a linear autonomous form. A new class of $n$-order nonlinear evolutionary equations is developed as well, along with corresponding NSP. In conclusion, the question of finding of NSP and constructing of corresponding classes for nonlinear autonomous ODE having NSP was set.

# Introduction

The preprint is based on the lecture delivered by the author, at Mathematische Arbeitstagung 2001, Bonn on June 13 - 19, 2001,[1] the extended abstract of the lecture is in [1]. For a long time it was thought that there is no regular method for overcoming the man difficulty in intergration of differential equations, which is looking for correspondent variable substitutions. Each time such a substitution was found, it has stimulated the process of looking for the problems for which the found substitution maybe successfully done. Sophus Lie succeded to make an incredible change in integration, when he created the whole field of science known now under the name Theory of Lie groups and Lie algebras.

However, the classical S.Lie theorie turned to be not sufficiently applied to many important classes of nonlinear equations.

The last 35 years or so have seen an explosion of activity in the study of nonlinear phenomena. The motivation has largely come from the physical sciences and engineering and also from problems mathematical physics and mechanics.

Mathematical progress in the treatment of nonlinear differential equations has been swiftly and broadly based. The most spectacular developments have been in the field of exact analytic solutions of nonlinear equations.

This lecture describes the new method of exact linearizatiion of $n$-order nonlinear ODE, which was created by the author. Not only linearizable class of equations was found, but also the explicit form through the quadratures of linearizing transformation.

*Let us say that given nonlinear equation possesses nonlinear superposition principle (NSP), if its general solution may be formulated in a form of (arbitrary or concrete) function of transformed linear equation solutions.*

In another words, *in order to specify the NSP for given equation or class of equations, one needs to specify nonlinear variable transformations which transform nonlinear equations to their linear equivalents.*

It means that the NSP depends on both the type of the transformation being used and the form of the designated equation.

Such a meaning of the NSP differs in some way from the usual definition of this principle, but in more detail, it will be covered in the lecture.

[1]Arbeitstagung organiziers are Max-Planck-Institut für Mathematik Bonn and Mathematisches Institut der Universität Bonn.

**Plan of the lecture**.

*1. About some of nonlinear superposition principles.* NSPs, tight to such names as S.Lie, V.Ermakov etc., are being considered. By the way, the Russian mathematician Ermakov was possibly the first scientist which constructed a second order nonlinear equation, whose general solution is a nonlinear function of fundamental set of solutions (FSS) of the associated second order linear equation.

On the other hand, NSPs connected to Bäcklund transformation of nonlinear evolutionary equations, are not considered in the lecture.

*2. Linearization of second order autonomous equations.* Not only general class dependent on two arbitrary functions and three parameters, is constructed here, but also NSP is specified for it.

*3. Linearization of third order autonomous equations.* General class dependent on two arbitrary functions and four parameters, is constructed along with an NSP for it.

*4. Linearization of n-order autonomous equations.* Class of linearizable equations depended on two arbitrary functions and $n+1$ parameters is constructed on base on recurrent relations, along with the corresponding NSP.

*5. New classes of nonlinear evolutionary equations.* Nonlinear evolutionary equations of orders two, three and $n$ depending on two arbitrary functions and some number of parameters are constructed. For them, NSPs are specified correspondently to variable transformation, are being used. We also show that so-called Maslov's *correspondence principle in idempotent analysis* is a NSP.

*6. Additional comments on nonlinear superposition principles.*

(i) Fundamental set of solutions of nonlinear equation does not describe the equation it completely, what differs the nonlinear equation from the linear one. You need to know also the appropriate NSP.

(ii) The same equation may have different NSPs.

(iii) Nonlinear equations, that possess NSP, allow nonlocal symmetries.

(iv) Factorization of nonlinear differential operators as a method of getting NSP for nonlinear equations.

The lecture is accomponied by detailed examples, some of them were considered earlier using the Painlévé–Kovalevskaya test.

In conclusion, the following open question is arisen.

*How to find NSP for nonlinear and nonautonomous n-order equations,*
*transformable to linear autonomous form by sufficiently general transformation of dependent and independent variables ? How to construct the corresponding classes of equations ?*

# 1 About some nonlinear superposition principles

## 1.1 The theorem of S.Lie about fundamental system of solutions

The principle of linear superposition for linear ordinary differential equations (LODE) or systems of such equations allows to represent the general solution in terms of finite set of partial solutions. In particular, the theory of linear Schrödinger's equation is based on the principle. Are the mentioned property bounded by the linear equations ? Can it be generalized by some methods and prolonged on the case of nonlinear ODE ?

This question was necessarilly arized before mathematicians, which tried to construct the theory of integrability of ODE. Near 1893 S.Lie [2] and E.Vessiot [3] formed the question as follows.

Let us consider the system ODE of the first order

$$\frac{du}{dt} = \eta(u,t), \text{ or } \frac{du^k}{dt} = \eta^k(u,t), \quad 1 \le k \le n, \tag{1.1}$$

here $u, \eta \in \mathbf{R^n}$ or $\mathbf{C^n}$.

**Question**: *What are the conditions to represent general solution of this system by nonlinear function $F$,concrete or arbitrary function, of finite number $m$ partial solutions*

$$u_1(t), u_2(t), \cdots, u_m(t) \tag{1.2}$$

*and sufficient number $n$ of arbitrary constants $c_1, c_2, \cdots, c_n$:*

$$u(t) = F(u_1(t), u_2(t), \cdots, u_m(t); c_1, c_2, \ldots, c_n) \ ? \tag{1.3}$$

The answer to this question is in the following theorem.

**Theorem 1.1** (S.Lie [2]). *The system of equations (1.1) has a fundamental set of solutions (1.2), such that general solution of the system is given by nonlinear superposition principle*[2] *(1.3), if and only if the following conditions hold:*

*1) After a transformation possibly, the right-hand side of (1.1) can be factorized to give*

$$\frac{du}{dt} = Z_1(t)\xi_1(u) + Z_2(t)\xi_2(u) + \cdots + Z_r(t)\xi_r(u).$$

*2) The vector fields*

$$X_i = \sum_{s=1}^{n} \xi_i^s(u)\frac{\partial}{\partial u^s},$$

*generate a finite-dimensional Lie algebra*

$$[X_a, X_b] = \sum_{c=1}^{r} C_{ab}^c X_c, \quad 1 \le a, b, c \le r.$$

*3) The number $m$ of solutions in the fundamental set (1.2) satisfies the inequality*

$$n \cdot m \ge r.$$

**Example 1. The Riccati equation**. The superposition principle for the Riccati equation is well known. Indeed, any four solutions of

$$\frac{du}{dt} = Z_1(t) + Z_2(t)u + Z_3(t)u^2,$$

e.g. $u, u_1, u_2, u_3$, satisfy the property that their anharmonic ratio is constant:

$$\frac{u - u_1}{u - u_2} : \frac{u_3 - u_1}{u_3 - u_2} = c. \tag{1.4}$$

Considering $u(t)$ to be the general solution and $u_1, u_2, u_3$ to be particular solutions, we can solve (1.4) for $u$ and obtain the superposition law

$$u(t) = \frac{(u_1 - u_3)u_2 c + u_1(u_3 - u_2)}{(u_1 - u_3)c + (u_3 - u_2)},$$

where $c$ is a constant related to the initial value $u(t_0)$ of the solution $u(t)$.

Investigations in this direction are developed by Winternitz [4], Ibragimov [5] and others.

[2] law, rule, formula

## 1.2 The Ermakov's nonlinear superposition principle (ENSP)

The other way to NSP was proposed by V.Ermakov in 1880 [6] (see also Pinney [7], Berkovich [8,9]). The correspondence notion we will call the Ermakov's principle of nonlinear superposition in honour of Ermakov.

**Definition 1.** Let us consider the given nonlinear equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}). \tag{1.5}$$

Let the adjoint equation be also given

$$Y^{(m)} + a_1(x)Y^{(m-1)} + \dots + a_{m-1}(x)Y' + a_m(x)Y = 0, \tag{1.6}$$

here

$$Y_1(x), Y_2(x), \dots, Y_m(x) \tag{1.7}$$

they form fundamental set of solutions (FSS) for (1.6). We will say that (1.5) admits NSP if its general solution has the form

$$y = F(Y_1(x), \dots, Y_m(x); c_1, \dots, c_n). \tag{1.8}$$

**Example 2. Ermakov–Pinney equation** (Ermakov [6], Pinney [7]).

Consider the nonlinear equation

$$y'' + a_0(x)y - b_0 y^{-3} = 0,\ b_0 = const, \tag{1.9}$$

and adjoint linear equation

$$Y'' + a_0(x)Y = 0. \tag{1.10}$$

The general solution of (1.9) has the form

$$y(x) = \sqrt{AY_1^2 + BY_1Y_2 + CY_2^2},\ B^2 - 4AC = -4b_0,$$

here $Y_1(x), Y_2(x) = Y_1 \int dx/Y_1^2$ are FSS of equation (1.10).

**Example 3. Kummer–Schwarz's equation of the second order**

$$\frac{1}{2}\frac{y''}{y} - \frac{3}{4}\left(\frac{y'}{y}\right)^2 + b_0 y^2 = a_0(x). \tag{1.11}$$

The equation (1.11) has the general solution

$$y(x) = (AY_1^2 + BY_1Y_2 + CY_2^2)^{-1},\ B^2 - 4AC = -4b_0,$$

here $Y_1(x), Y_2(x)$ are FSS of the equation (1.10).

For (1.11) the self-conjugated equation of the third order

$$\eta''' + 4a_0(x)\eta' + 2a_0'(x)\eta = 0, \tag{1.12}$$

is also adjoint equation, here $\eta_1 = Y_1^2(x),\ \eta_2 = Y_1(x)Y_2(x),\ \eta_3 = Y_2^2(x)$ are FSS of the equation (1.12).

**Example 4. Kummer–Schwarz's equation of the third order** [8-10].

$$\{y, x\} + b_0 y'^2 = a_0, \tag{1.13}$$

where

$$\{y,x\} = \frac{1}{2}\frac{y'''}{y'} - \frac{3}{4}\left(\frac{y''}{y'}\right)^2$$

is Schwarz derivative. Let (1.10) be its adjoint equation and $Y_1, Y_2$ are FSS of the equation (1.10), then

$$y(x) = \frac{1}{\sqrt{\delta}} \log \frac{(B-\sqrt{\delta})Y_1 + 2CY_2}{(B+\sqrt{\delta})Y_1 + 2CY_2} + k$$

is the general solution of the equation (1.13), where $A, B, C, k$ are arbitrary constants, $B^2 - 4AC = -4b_0$.

## 1.3 The nonlinear superposition principle: general definition

**Definition 2.** The equation (1.5) admits NSP if its general solution has the form (1.8), here $c_1, c_2, \ldots, c_n$ are arbitrary constants and the system (1.7) are either

1) partial solutions of (1.5) or
2) partial linearly independent solutions of adjoint linear equation (1.6) or
3) partial solutions of adjoint nonlinear equation

$$Y^{(m)} = g(x, Y, Y', \ldots, Y^{(m-1)})$$

or

4) general solution of (1.5) is nonlinear function of arbitrary constants.

*NSP can be extended also to the partial differential equations.*

**Example 5.** Let us have *Burgers* equation

$$u_t + uu_x - \nu u_{xx} = 0. \tag{1.14}$$

By the transformation Cole–Hopf,

$$u = -2\nu\frac{\theta_x}{\theta},$$

the equation (1.14) is transformed to the linear *heat* equation

$$\theta_t - \nu\theta_{xx} = 0.$$

More precisely,

$$u_t + uu_x - \nu u_{xx} = -\frac{2\nu}{\theta}(\partial_x - \theta_x/\theta)(\theta_t - \nu\theta_{xx}).$$

We do not consider here such an important nonlinear superposition principle as the generation of solutions. For instance, this takes place in the soliton theory (see, for example, Calogero and Degasperis [11]).

# 2 Linearization of second order autonomous equations

**Lemma 2.1** [12,13]. *Let us have the equation of the form*

$$y'' + F(y, y') = 0. \tag{2.1}$$

*In order to transform (2.1) to the linear equation*

$$\ddot{z} + 2b_1\dot{z} + b_2 z + c = 0, \;\; b_1, \;\; b_2, \;\; c = const \tag{2.2}$$

*by the convertible transformation*[3] *,in some domain* $\Gamma(x,y)$

$$y = v(y)z, \qquad (i.e.\ z = \Phi(y),\ y = \Phi^{-1}(z)),\ dt = u(y)dx, \tag{2.3}$$

*(where Jacobian* $J\left(\frac{t,z}{x,y}\right) = t_x z_y - t_y z_x \neq 0$, *and* $v, u$ *are sufficiently smooth functions with* $v(y(x))u(y(x)) \neq 0,\ \forall (x,y) \in \Gamma$*), it is necessary and sufficient, that the noncommutative factorization ,up to additive term* $cu^2v$*:*

$$[D - (\frac{v^*}{v} + \frac{u^*}{u})y' - r_2 u][D - \frac{v^*}{v}y' - r_1 u]y + cu^2 v = 0,\ D = d/dx,\ (*) = d/dy, \tag{2.4}$$

*or the commutative factorization, up to additive term* $cv$*,*

$$[\frac{1}{u}D - \frac{v^*}{uv}y' - r_2][\frac{1}{u}D - \frac{v^*}{uv}y' - r_1]y + cv = 0$$

*hold, here* $r_1, r_2$ *are the roots of the characterisic equation*

$$r^2 + 2b_1 r + b_2 = 0. \tag{2.5}$$

• This fact is verified directly. For example, using the substitution (2.3) in (2.4) and getting the relation $u^2v(\ddot{z} + 2b_1\dot{z} + b_2 z + c)$, we obtain the linear equation (2.2). •

**Proposition 2.1.** *In order the equation (2.1) can be linearized by the transformation (2.3) it is necessary and sufficient that it has the following form*

$$y'' + fy'^2 + 2b_1\varphi y' + \varphi \exp(-\int f dy)[b_2 \int \varphi \exp(\int f dy)dy + c/\beta] = 0, \tag{2.6}$$

*with the condition that the equation is linearized by the transformation*

$$z = \beta \int \varphi \exp(\int f dy)dy,\ dt = \varphi dx, \tag{2.7}$$

*here* $\beta = const$ *is normalized factor.*

*Let* $c = 0$ *in the equation (2.6). Then it admits the following one-parameter families of solutions:*

$$r_k x + C = \int \frac{exp(\int f dy)dy}{\int \varphi exp(\int f dy)dy}, \tag{2.8}$$

*here* $r_k$ *are primitive roots of the characteristic equation (2.5), and* $C$ *is integration constant.*

*The special cases of the equation (2.6) and correspondently transformations (2.7) are obtained if* $\varphi = \exp(-\int f dy)$*, and* $\varphi = \exp(-\int f dy)(ay+b)^{-2},\ b \neq 0$.

• After the factorization in (2.4), we get the equation

$$y''(1 - \frac{v^*}{v}y) - y'^2[\frac{v^{**}}{v}y + (2\frac{v^*}{v} + \frac{u^*}{u})(1 - \frac{v^*}{v}y)] - (r_1 + r_2)u(1 - \frac{v^*}{v}y)y' + r_1 r_2 u^2 y + cu^2 v = 0 \tag{2.9}$$

with the leading coefficient

$$1 - \frac{v^*}{v}y.$$

To define the transformation (2.3) in the explicit form we introduce the denotation

$$-[\frac{v^{**}}{v}y + (2\frac{v^*}{v} + \frac{u^*}{u})(1 - \frac{v^*}{v}y)] = f(y)(1 - \frac{v^*}{v}y).$$

[3] We do not define the functional class exept the most general characteristics. But we get below the quadrature expressions of transformation functions.

According to the factor $v(y)$ of the transformation (2.3) we obtain the equation

$$v^{**} - 2\frac{v^{*2}}{v} + (\frac{2}{y} - \frac{u^*}{u} - f)v^* + \frac{1}{y}(\frac{u^*}{u} + f)v = 0,\ f = f(y), v \neq ay + b. \tag{2.10}$$

By the substitution $v = V^{-1}$ the equation (2.10) is reduced to the linear nonautonomous equation

$$V^{**} + \left(\frac{2}{y} - \frac{u^*}{u} - f\right)V^* - \frac{1}{y}\left(\frac{u^*}{u} + f\right)V = 0.$$

This equation admits the factorization

$$(D_y + \frac{1}{y} - \frac{u^*}{u} - f)(D_y + \frac{1}{y})V = 0.$$

Its general solution is

$$V = \frac{1}{y}(\alpha + \beta\int u\exp(\int f dy)dy).$$

Thus, the equation (2.10) has the general solution expressed by the quadratures

$$v(y) = \frac{y}{\alpha + \beta\int u\exp(\int f dy)dy}, \tag{2.11}$$

here $\alpha,\ \beta$ are the constants of the integration.

To obtain the transformation (2.7), it is sufficient to put $u = \varphi, \alpha = 0$ in (2.11).

We obtain the equation (2.6) substituting (2.11) in (2.9) and cancellating the term

$$1 - \frac{v^*}{v}y = \frac{\varphi\exp(\int f dy)y}{\int\varphi\exp(\int f dy)dy}. \tag{2.12}$$

Equation (2.6) with $c = 0$ is consistent with the first-order equations

$$(D - \frac{v^*}{v}y' - r_k u)y \equiv (1 - \frac{v^*}{v}y)y' - r_k uy = 0 \tag{2.13}$$

Taking $u = \varphi, \alpha = 0$ and substituting (2.11) in (2.13), we obtain (2.8).

In the equation (2.10) we put $v = ay + b,\ b \neq 0$. Then it takes the form

$$u^* + (\frac{2a}{ay + b} + f)u = 0,$$

from where the special cases for the kernel $\varphi$ of the transformation (2.3) follow. Let us note, that in these cases the equation (2.6) and the transformation (2.7) depend on an arbitrary function only. •

**Proposition 2.2. Nonlinear superposition principle.**

*The equation (2.6) has the general solution in the parametric form*

$$\beta\int\varphi e^{\int f dy}dy = C_1 z_1(t) + C_2 z_2(t) + C_0(t),\ dt = \varphi dx,$$

*or more precisely*

$$\beta\int\varphi e^{\int f dy}dy=\begin{cases} C_1 exp(r_1 t)+C_2 exp(r_2 t)-c/b_2, \ r_1\neq r_2\neq 0\\ exp(-2b_1 t)(C_1 t+C_2)-c/b_2, \ r_1=r_2=-b_1\neq 0\\ C_1+C_2 exp(-2b_1 t)-ct/(2b_1), \ r_1=0, \ r_2\neq 0\\ C_1+C_2 t+ct^2/2, \ r_1=r_2=0\\ A sin(\sqrt{b_2}t+B)-c/b_2, \ b_1=0, b_0>0\\ A\sinh(\sqrt{b_2}t+B)-c/b_2, \ b_1=0, b_2<0, \end{cases}$$

$$x=\int\frac{dt}{\varphi(y(t))},$$

*where $C_1, C_2, A, B$ are the constants of the integration.*

**Comment 2.1.** *The equations of the type*

$$y''+f_2(y)y'^2+f_1(y)y'+f_0(y)=0$$

*can be tested by the method of the exact linearization.*

**Proposition 2.3. The generalized nonlinear harmonic oscillator.**

*The equation*

$$y''+f(y)y'^2\pm a^2\psi(y)=0 \tag{2.14}$$

*by the transformation*

$$z=\sqrt{2\int\psi\exp(2\int f dy)dy}, \ dt=z^{-1}\psi\exp(\int f dy)dx \tag{2.15}$$

*is linearized:*

$$\ddot{z}\pm a^2 z=0,$$

*and it also admits the one-parameter families of solutions*

$$\int z^{-1}\exp(\int f dy)dy=\pm\sqrt{\mp a^2}x+C, \tag{2.16}$$

*where $C$ is a constant of integration.*

• We will test the equation (2.14) with the help of the proposition 2.1. Let us put

$$b_1=0, \ b_2=\pm a^2, \ c=0, \ \varphi\exp(-\int f dy)\int\varphi\exp(\int f dy)dy=\psi.$$

We have sequently:

$$\varphi\exp(\int f dy)dy\int\varphi\exp(\int f dy)dy=\psi\exp(2\int f dy)dy,$$

$$1/2(\int\varphi\exp(\int f dy)dy)^2=\int\psi\exp(2\int f dy)dy.$$

We have then

$$\int \varphi \exp(\int f dy) dy = \sqrt{2 \int \psi \exp(2 \int f dy) dy}.$$

The formulas (2.15), (2.16) follow. •

**Example 6.** (Painlévé ([14,15], Gambier [16], Golubev [17]):

$$y'' + 3yy' + y^3 = 0. \tag{2.17}$$

(In the papers cited, the equation (2.17) was studied by the test called now the test of Painlévé–Kovalevskaya.)

The equation (2.17) belongs to the class (2.6). Indeed, let us put

$$f = 0,\ 2b_1 = 3,\ \varphi = y,\ b_2\varphi \int \varphi dy = y^3,\ \text{from here } b_2 = 2,\ c = 0.$$

Since $z = \beta \int y dy$, then we get the formulas

$$z = y^2, \quad dt = y dx, \tag{2.18}$$

if we put $\beta = 2$. We make use the transformation (2.18) for linearization of the equation (2.17) to the form

$$\ddot{z} + 3\dot{z} + 2z = 0.$$

Consequently, equation (2.17) has the general solution represented in the parametric form

$$y = \pm\sqrt{C_1 \exp(-2t) + C_2 \exp(-t)},\ x = \pm \int \frac{dt}{\sqrt{C_1 exp(-2t) + C_2 exp(-t)}}. \tag{2.19}$$

Obtained integral can be expressed in finite terms. Let us eliminate the parameter $t$ from (2.19), so we obtain the general solution in the form

$$y = \frac{2x + a}{x^2 + ax + b}.$$

Since $u = y, v = 1/y$, then the equation (2.17) can be factorized in the following manner:

$$(D - r_2 y)(D + \frac{y'}{y} - r_1 y)y = 0,\ r_{1,2} \in \{-1, -2\}.$$

This implies the formulas for the one-parametric solutions:

$$y = (x + C)^{-1}, \quad y = 2(x + C)^{-1}.$$

# 3 Linearization of third order autonomous equations

**Lemma 3.1** [17,18]. *Let the equation be given*

$$y''' = F(y, y', y''). \tag{3.1}$$

*In order to reduce it to linear form*

$$\dddot{z} + 3b_1\ddot{z} + 3b_2\dot{z} + b_3 z + c = 0,\ b_1, b_2, b_3, c = const \tag{3.2}$$

*by the transformation of the type (2.3), it is necessary and sufficient, the noncommutative factorization must be,with regard of additive term $cu^3v$,*

$$[D-(\frac{v^*}{v}+2\frac{u^*}{u})y'-r_3u][D-(\frac{v^*}{v}+\frac{u^*}{u})y'-r_2u][D-\frac{v^*}{v}y'-r_1u]y+cvu^3=0, \tag{3.3}$$

*or, up to additive term $cv$, the commutative factorization*

$$[\frac{1}{u}D-\frac{v^*}{uv}y'-r_3][\frac{1}{u}D-\frac{v^*}{uv}y'-r_2][\frac{1}{u}D-\frac{v^*}{uv}y'-r_1]y+cv=0, (^*)=d/dy$$

*hold, where $r_k$ are roots of the characteristic equation*

$$r^3+3b_1r^2+3b_2r+b_3=0. \tag{3.4}$$

**Proposition 3.1** [18-20]. *The transformation of the type (2.3) reduces the equation (3.1) to the linear equation (3.2) if and only if (3.1) can be represented in the form*

$$y'''+3f(y)y'y''+\left(\frac{1}{3}\frac{\varphi^{**}}{\varphi}-\frac{5}{9}\frac{\varphi^{*2}}{\varphi^2}-\frac{1}{3}f\frac{\varphi^*}{\varphi}+f^2+f^*\right)y'^3+3b_1\varphi y''+b_1\varphi(f+\frac{\varphi^*}{\varphi})y'^2+$$

$$+3b_2\varphi^2y'+\varphi^{5/3}\exp(-\int fdy)[b_3\int\varphi^{4/3}\exp(\int fdy)dy+\frac{c}{\beta}]=0, \tag{3.5}$$

*which is reduced to (3.2) by the transformation*

$$z=\beta\int\varphi^{4/3}\exp(\int fdy)dy, \quad dt=\varphi(y)dx. \tag{3.6}$$

*We have the one-parameter solutions*

$$\int\frac{\varphi^{1/3}\exp(\int fdy)dy}{\int\varphi^{4/3}\exp(\ fdy)dy}=r_kx+C, \tag{3.7}$$

*at $c=0$, where $r_k$ are distinct roots of the characteristic equation (3.4), and $C$ is integration constant.*

*Special case is obtained at*

$$\varphi=\exp(-3/4\int fdy),\ \varphi=\exp(-3/4\int fdy)(ay+b)^{-3/2}, \tag{3.8}$$

*where we have in (2.3) $v=ay+b,\ b\neq 0$.*

- After the factorization in (3.3), we get the equation in extendent form

$$y'''(1-\frac{v^*}{v}y)-y''y'[3\frac{v^{**}}{v}y+(1-\frac{v^*}{v}y)(6\frac{v^*}{v}+4\frac{u^*}{u})]+y'^3[(1-\frac{v^*}{v}y)\times$$

$$\times(6\frac{v^{*^2}}{v^2}+6\frac{u^*v^*}{uv}+3\frac{u^{*^2}}{u^2}-2\frac{v^{**}}{v}-\frac{u^{**}}{u})+4\frac{v^*v^{**}}{v^2}y+3\frac{u^*v^{**}}{uv}y-\frac{v^{**}}{v}-\frac{v^{***}}{v}y]-$$

$$-u(r_1+r_2+r_3)y''(1-\frac{v^*}{v}y)+u(r_1+r_2+r_3)y'^2[\frac{v^{**}}{v}y+(1-\frac{v^*}{v}y)(2\frac{v^*}{v}+\frac{u^*}{u})]+$$

$$+(r_1r_2+r_1r_3+r_2r_3)u^2y'(1-\frac{v^*}{v}y)-r_1r_2r_3u^3y+cvu^3=0, \tag{3.9}$$

where $r_k,\ k=1,2,3,$ are the roots of the characteristic equation (3.4).

To determine the transformation (2.3) in the explicit form, we introduce the denotation

$$-[3\frac{v^{**}}{v}y+(6\frac{v^*}{v}+4\frac{u^*}{u})(1-\frac{v^*}{v}y)]=3f(y)(1-\frac{v^*}{v}y),$$

from where we get equation for the (2.3) transformation factor $v(y)$:

$$v^{**}-2\frac{v^{*2}}{v}+(\frac{2}{y}-\frac{4}{3}\frac{u^*}{u}-f)v^*+\frac{1}{y}(\frac{4}{3}\frac{u^*}{u}+f)v=0,\ \ f=f(y). \tag{3.10}$$

The equation (3.10) has the general solution

$$v(y)=\frac{y}{\alpha+\beta\int u^{4/3}\exp(\int f dy)dy}, \tag{3.11}$$

where $\alpha,\ \beta$ are integration constants.

Indeed, by the substitution $v=V^{-1}$, the equation (3.10) is reduced to linear nonautonomous equation

$$V^{**}+(\frac{2}{y}-\frac{4}{3}\frac{u^*}{u}-f)V^*-\frac{1}{y}(\frac{4}{3}\frac{u^*}{u}+f)V=0.$$

It admits the factorization

$$(D_y+\frac{1}{y}-\frac{4}{3}\frac{u^*}{u}-f)(D_y+\frac{1}{y})V=0.$$

Its general solution is $V=c_1V_1+c_2V_2$, where $V_1=\frac{1}{y},\quad V_2=\frac{1}{y}\int u^{4/3}exp(\int f dy)dy.$

Consequently,

$$V=\frac{1}{y}\left(\alpha+\beta\int u^{4/3}exp(\int f dy)dy\right),$$

where $\alpha,\ \beta$ are arbitrary constants, from where formula (3.11) follow.

To obtain the transformation (3.6), it is necessary and sufficient to put $u=\varphi,\alpha=0$ in (3.11).

Substituting (3.11) in (3.9) and cancellating the term

$$1-\frac{v^*}{v}y=\frac{\varphi^{4/3}\exp(\int f dy)y}{\int\varphi^{4/3}\exp(\int f dy)dy},$$

we obtain the equation (3.5).

Equation (3.5) with $c=0$ is consistent with the first-order equations of the form (2.13). Taking in (3.11) $u=\varphi,\alpha=0$ and substituting (3.11) in (2.13), we obtain (3.7).

In the equation (3.10) we put $v=ay+b, b\neq 0$. Then it takes the form

$$u^*=-(\frac{3}{2}\frac{a}{ay+b}+\frac{3}{4}f)u,$$

from where the special cases (3.8) for the kernel $\varphi$ of the transformation (3.6) follow. Note, that in these cases, the equation (3.5) and the transformation (3.6) depend on an arbitrary function only. •

**Proposition 3.2. Nonlinear superposition principle.** *General solution of the equation (3.5) can be represented in the form*

$$\beta\int\varphi^{4/3}\exp(\int f dy)dy=C_1z_1(t)+C_2z_2(t)+C_3z_3(t)+C_0(t),\ \ dt=\varphi(y)dx,$$

*where $z_i(t)$ are linearly independent partial solutions of linear homogeneous equation, corresponding to (3.2), and $C_0(t)$ is a partial solution of nonhomogeneous equation (3.2), and $C_1, C_2, C_3$ are arbitrary constants.*

**Comment 3.1.** *Equations of the type*

$$y''' + \varphi(y)y'y'' + \psi(y)y'' + \sum_{k=0}^{3} f_k(y)y'^k = 0$$

*can be tested by the method of exact linearization.*

**Example 7** (J.Chazy [21]):

$$y''' = ay^\lambda y'' + by^{\lambda-1}y'^2 + cy^{2\lambda}y' + dy^{3\lambda+1}, \;\; a, b, c, d = const. \tag{3.12}$$

Let us find the conditions under which the equation (3.12) belongs to the class (3.5). Compare (3.12) and (3.5), we get the following relations:

$$f = 0, \;\; 3b_1\varphi = -ay^\lambda, \;\; \frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} = 0, \;\; b_1\varphi^* = -by^{\lambda-1},$$

$$3b_2\varphi^2 = -cy^{2\lambda}, \;\; b_3\varphi^{5/3}\int \varphi^{4/3}dy = -dy^{3\lambda+1}. \tag{3.13}$$

The relations (3.13) reduce to the value $\lambda = -3/2$, and, also, to the expression of the parameter $b$ in terms of $a$ as follows: $b = -a/2$. Thus, the linearizable equation (3.12) takes the form:

$$y''' = ay^{-3/2}y'' - \frac{a}{2}y^{-5/2}y'^2 + cy^{-3}y' + dy^{-7/2}. \tag{3.14}$$

If we put the normalizing factor $\beta = -1$, the transformation (3.6) takes the form

$$z = y^{-1}, \;\; dt = y^{-3/2}dx.$$

Linearized equation of the type (3.2) takes the form $\dddot{z} - a\ddot{z} - c\dot{z} + dz = 0$. By the way, therefore the formulas (3.7) of the equation (3.14) has also the following one-parameter solutions

$$-\frac{2}{3}y^{3/2} = r_k x + C,$$

where $r_k$ are distinct roots of the characteristic equation $r^3 - ar^2 - cr + d = 0$, and $C$ is an integration constant.

**Example 8** (J.Chazy [21]):

$$y''' = b(y)y'y'' + c(y)y'^3. \tag{3.15}$$

The equation (3.15) belongs to the class (3.5) and by the transformation

$$z = \beta\int \varphi^{4/3}\exp(-\frac{1}{3}\int bdy)dy, \;\; dt = \varphi(y)dx,$$

where $\varphi(y)$ satisfies to the equation

$$\frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} + \frac{1}{9}b(y)\frac{\varphi^*}{\varphi} + \frac{1}{9}b^2(y) - \frac{1}{3}b^* + c(y) = 0,$$

is linearized:

$$\dddot{z} = 0. \tag{3.16}$$

Let, in particular, we have $b(y) = -3/y,\ c(y) = 4/y^2$. So, if we take $\varphi = y^3$, then we obtain the linearizing transformation

$$z = y^6,\ dt = y^3 dx.$$

The equation

$$y''' + \frac{3}{y} y' y'' - \frac{4}{y^2} y'^3 = 0 \tag{3.17}$$

admits the factorization

$$\frac{1}{6}(D - \frac{1}{y} y')(D + \frac{2}{y} y')(D + \frac{5}{y} y')y = 0.$$

Other linearizing transformation takes the form

$$z = y^{-6},\ dt = y^{-6} dx.$$

The equation (3.17) admits the factorization

$$-\frac{1}{6}(D + \frac{5}{y} y')(D - \frac{y'}{y})(D - \frac{7}{y} y')y = 0.$$

**Example 9.** (Generalization of example 6). Equation

$$y''' - \frac{1}{y} y' y'' + 3b_1 y y'' + 3b_2 y^2 y' + \frac{1}{2} b_3 y^4 + \frac{c}{2} y^2 = 0$$

by the substitution (2.18) is linearized to the form (3.2) and has one-parameter set of solutions

$$y = -\frac{2}{r_k x + C},$$

where $r_k$ are simple roots of the characteristic equation (3.4), and $C$ is integration constant.

**Example 10.** We consider ODE

$$y''' + \frac{1}{2} y'^3 = 0. \tag{3.18}$$

It corresponds to the generalized symmetry of the third order for *sin-Gordon* equation

$$u_{xt} = \sin u,$$

which takes the form $(u_{xxx} + \frac{1}{2} u_x^3)\partial_u$ (see, for example, Olver [22]).

The equation (3.18) belongs to the class (3.5), here $\varphi(y)$ satisfies to the equation

$$\varphi^{**} - \frac{5}{3\varphi} \varphi^{*2} - \frac{3}{2} \varphi = 0. \tag{3.19}$$

According to the proposition 2.3, the equation (3.19) by the substitution $z = 3/2\varphi^{-2/3}$ is reduced to the linear form $z^{**} + z = 0$ and it admits the one-parameter set of solutions of the form $\varphi = c\exp(\pm 3/2iy),\ i = \sqrt{-1}$, $c$ is an parameter. Consequently, the equation (3.18) by the complex-valued transformations

$$z = \exp(\pm 2iy),\ dt = \exp(\pm \frac{3}{2} iy)dx,$$

see formulas (3.6), is linearized, reduced to the form (3.16), and equation (3.18) admits the factorizations

$$\pm 2iy(y''' + \frac{1}{2} y'^3) \equiv [D - (\frac{1}{y} \pm i)y'][D - (\frac{1}{y} \mp \frac{1}{2} i)y'][D - (\frac{1}{y} \mp 2i)y']y = 0.$$

The general solution of (3.18) has the form

$$x = \int (c_1 + c_2 t + c_3 t^2)^{-3/4} dt, \quad y = -\frac{1}{2} i \log(c_1 + c_2 t + c_3 t^2).$$

But the equation (3.18) is linearized also by each of the following real-valued transformations

$$z = \tan y, \; dt = \cos^{-3/2} y dx, \tag{3.20}$$

$$z = \cot y, \; dt = \sin^{-3/2} y dx. \tag{3.21}$$

According to the transformations (3.20), (3.21) one can obtain the real factorizations of the equation (3.18). For example, the transformation (3.20) corresponds to the following factorization:

$$(\tan y + \cot y) y (y''' + \frac{1}{2} y'^3) \equiv$$

$$[D - (1/y + 2\tan y - \cot y) y'][D - (1/y + 1/2 \tan y - \cot y) y'][D - (1/y - \tan y - \cot y) y'] y = 0.$$

The general solution of the equation (3.18) in the real parametric form can be represented as follows

$$\tan y = C_1 + C_2 t + C_3 t^2, \; x = \int [1 + (C_1 + C_2 t + C_3 t^2)^2]^{-3/4} dt.$$

# 4 Linearization of $n$-order autonomous equations

Let we have given autonomous nonlinear ODE of $n$th order

$$F(y, y', y'', \ldots, y^{(n-1)}, y^{(n)}) = 0, \; (') = \frac{d}{dx}, \tag{4.1}$$

which left part is the polynomial of $y', y'', \ldots, y^{(n-1)}, y^{(n)}$ with the coefficients depending on $y$, and $y^{(n)}$ in $F$ is in the first power.

**Lemma 4.1** [23,24]. *In order (4.1) to be transformed by the convertible transformation of independent and dependent variables of the type*[4]

$$y = v(y) z, \quad dt = u(y) dx, \tag{4.2}$$

*where $v$ and $u$ are sufficiently smooth nonvanishing functions in some domain $\Gamma(x, y)$, into the linear autonomous form*

$$M_n z \equiv z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) + c = 0, \; b_k, c = const, \tag{4.3}$$

*it is necessary and sufficient that (4.1) admits, in general, the following noncommutative, up to additive term $cu^n v$, factorization*

$$F \sim \prod_{k=n}^{1} \left[ D - \left( \frac{v_y}{v} + (k-1) \frac{u_y}{u} \right) y' - r_k u \right] y + c u^n v = 0, \; D = \frac{d}{dx}, \; v_y = \frac{dv}{dy}, \; u_y = \frac{du}{dy}, \tag{4.4}$$

[4] see the footnote 3.

*or, up to additive term cv, the following commutative factorization*

$$F \sim \prod_{k=1}^{n}\left[\frac{1}{u}D - \frac{v_y}{vu}y' - r_k\right]y + cv = 0,$$

*here $r_k$ are the roots of the characteristic equation*

$$r^n + \sum_{k=1}^{n}\binom{n}{k}b_k r^{n-k} = 0. \tag{4.5}$$

**Lemma 4.2.** *If the equation (4.1) is linearized by (4.2) then it admits the following additive representation*

$$F \equiv \sum_{k=0}^{n}\binom{n}{k}b_k u^k \psi_{n-k},\ b_0 = 1, \tag{4.6}$$

*here $\psi_k$ is homogeneous polynomial of symbols $y', y'', \ldots, y^{(k)}$, which consists of all possible monomials of differential order $k$ with coefficients depending on $y$.*

**Theorem 4.1** [23]. *In order ODE (4.1) to be transformed (4.2) to the linear equation (4.3), it is necessary and sufficient that (4.1) can be in the form of noncommutative factorization*

$$F \sim \prod_{k=n}^{1}\left[D - \left(\frac{1}{y} - \frac{\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy} + (k-1)\frac{u^*}{u}\right)y' - r_k u\right]y +$$

$$+\frac{c}{\beta}\frac{\varphi^n y}{(\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy)} = 0, \tag{4.7}$$

*or represented in the form of commutative factorization*

$$F \sim \prod_{k=1}^{n}\left[\frac{1}{\varphi}D - \left(\frac{1}{y\varphi} - \frac{\varphi^{\frac{n^2-3n+2}{2n}}\exp(\int f dy)}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy}\right)y' - r_k\right]y +$$

$$+\frac{c}{\beta}y(\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy)^{-1} = 0, \tag{4.8}$$

*where $\beta$ is normalizing factor, and $r_k$ are the roots of the characteristic equation (4.6).*

*The lexicographic form of the equation (4.7) is following:*

$$y^{(n)} + nf(y)y'y^{(n-1)} + \ldots + nb_1\varphi(y)y^{(n-1)} + \ldots +$$

$$+\varphi^{\frac{n^2+n-2}{2n}}\exp(-\int f dy)dy\left(b_n\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy + \frac{c}{\beta}\right) = 0 \tag{4.9}$$

*after the cancellation of the term*

$$(1 - \frac{v^*}{v}y) = \frac{\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)y}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy}. \tag{4.10}$$

*And the linearizing transformation (4.2) takes the form*

$$z = \beta\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy,\ ds = \varphi dx,\ \beta = const \text{ is normalizing factor.} \tag{4.11}$$

*At $c=0$, the equations (4.7)-(4.9) have one-parameter families of solutions*

$$\int \frac{\varphi^{\frac{n^2-3n+2}{2n}} \exp(\int f dy)}{\int \varphi^{\frac{n^2-n+2}{2n}} \exp(\int f dy) dy} = r_k x + C, \tag{4.12}$$

*where $r_k$ are distinct roots of characteristic equation (4.5), and $C$ is an integration constant.*
*Special cases are obtained at*

$$\varphi = \exp(-\frac{2n}{n^2-n+2}\int f dy),\ \varphi = \exp(-\frac{2n}{n^2-n+2}\int f dy)(ay+b)^{-\frac{4n}{n^2-n+2}}. \tag{4.13}$$

• Equation (4.1) admits the factorization (4.4).
Expanding (4.4), we first obtain the equation of the form

$$(1-\frac{v^*}{v}y)y^{(n)} - [n\frac{v^{**}}{v}y + (1-\frac{v^*}{v}y)(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u})]y'y^{(n-1)} + \dots$$
$$\dots + b_n u^n y + cvu^n = 0.$$

This can be proved by induction for $n \geq 3$.
Let us introduce the notation

$$n\frac{v^{**}}{v}y + (1-\frac{v^*}{v}y)\left(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u}\right) = -nf(y)(1-\frac{v^*}{v}y).$$

With respect to $v(y)$ we have a nonlinear nonautonomous equation of the second order

$$v^{**} - \frac{2}{v}v^{*2} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right)v^* + \left(\frac{n^2-n+2}{2n}\frac{u^*}{u} + f\right)\frac{1}{y}v = 0 \tag{4.14}$$

By the substitution $v = V^{-1}$ the equation (4.14) is reduced to linear nonautonomous equation

$$V^{**} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right)V^* - \left(\frac{n^2-n+2}{2n}\frac{u^*}{u} + f\right)\frac{1}{y}V = 0. \tag{4.15}$$

Equation (4.15) admits the factorization

$$\left(D_y + \frac{1}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right)\left(D_y + \frac{1}{y}\right)V = 0.$$

Its general solution is $V = C_1V_1 + C_2V_2$, where

$$V_1 = \frac{1}{y},\quad V_2 = V_1\int u^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy.$$

Consequently, general solution of linear equation (4.15) is

$$V = \frac{1}{y}(\alpha + \beta\int u^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy),$$

where $\alpha, \beta$ are arbitrary constants. Then nonlinear equation (4.14) has general solution

$$v(y) = y/(\alpha + \beta\int u^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy). \tag{4.16}$$

Substituting (4.16) into (4.4) yields (4.7). Setting $u = \varphi(y), \alpha = 0$ and using (4.16), we obtain (4.11). Expanding (4.7) after the cancellation of the term (4.10), it gives (4.9). Equation (4.4) with $c = 0$ is consistent with the first-order equations

$$(1 - v^* v^{-1} y) y' - r_k u y = 0. \tag{4.17}$$

Taking $u = \varphi(y)$ and substituting (4.16) into (4.17), we obtain (4.12).

As special cases we get the formulas (4.13). In fact, in the equation (4.13) we put $v = ay+b, b \neq 0$. Then after simplifications it takes the form

$$\frac{n^2 - n + 2}{2n} \frac{u^*}{u} = -\left( \frac{2a}{ay+b} + f \right),$$

from where formulas (4.13) follow. Let us note, that in these cases the equations (4.7),(4.8),(4.9) and the transformation (4.11) depend on an arbitrary function only. •

**Theorem 4.2. Nonlinear superposition principle.**

*Equation (4.1), represented in the form (4.7)-(4.9), admits the following general solution*

$$\beta \int \varphi^{\frac{n^2-3n+2}{2n}} \exp\left(\int f dy\right) dy = \sum_{k=1}^{n} c_k z_k(t) + \tilde{z}(t), \;\; dt = \varphi(y) dx,$$

*where $z_k(t), k = \overline{1,n}$, are linearly independent solutions of linear homogeneous equation*

$$M_n z \equiv z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \;\; b_k = const,$$

*and $\tilde{z}$ is partial solution of nonhomogeneous equation (4.3), and $c_k$ are arbitrary constants.*

**Comment 4.1.** *The equations of the type (4.1),(4.6) can be tested by the method of exact linearization.*

# 5 New classes of nonlinear evolutionary equations

## 5.1 Second order nonlinear evolutionary equation (NEE)

**Lemma 5.1** *In order to NEE of second order*

$$\frac{\partial y}{\partial t} = F\left(y, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}\right) \tag{5.1}$$

*by the substitution of the form*

$$y = v(y) z, \quad \partial s = u(y) \partial x, \;\; \tau = t \tag{5.2}$$

*reduced to the linear evolutionary equation (LEE)*

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial s^2} + 2b_1 \frac{\partial z}{\partial s} + b_2 z, \tag{5.3}$$

*it is necessary and sufficient that (5.1) to be expressed as the equation*

$$u^2 \left(1 - \frac{v_y}{v} y\right) \frac{\partial y}{\partial t} = \left[ \frac{\partial}{\partial x} - \left( \frac{v_y}{v} + \frac{u_y}{u} \right) \frac{\partial y}{\partial x} - r_2 u \right] \left[ \frac{\partial}{\partial x} - \frac{v_y}{v} \frac{\partial y}{\partial x} - r_1 u \right] y, \tag{5.4}$$

*the right-hand side of which represents the noncommutative factorization, or to be expressed as the equation*

$$\left(1-\frac{v_y}{v}y\right)\frac{\partial y}{\partial t}=\left(\frac{1}{u}\frac{\partial}{\partial x}-\frac{v_y}{vu}\frac{\partial y}{\partial x}-r_2\right)\left(\frac{1}{u}\frac{\partial}{\partial x}-\frac{v_y}{vu}\frac{\partial y}{\partial x}-r_1\right)y,$$

*the right-hand side of which represents the commutative factorization, here $r_1, r_2$ are the roots of the characteristic equation*

$$r^2+2b_1r+b_2=0. \tag{5.5}$$

• We apply the substitution (5.2) to (5.4). We obtain the expression $u^2v(\partial_s-r_2)(\partial_s-r_1)z$ on the right-hand side , corresponding to the right-hand side of equation (5.3). We transform the left-hand side of equation (5.4). Since $y(t,x)=v(y(t,x))z(t,s)$, we get

$$\frac{\partial z}{\partial t}=\frac{1}{v}\left(1-\frac{v_y}{v}y\right)\frac{\partial y}{\partial t}.$$

Thus the left-hand side of the equation (5.4) is transformed in to the expression $u^2v\frac{\partial z}{\partial t}$. As a consequence we get LEE (5.3). •

**Theorem 5.1** [25]. *In order the NEE (5.1) to be reduced LEE (5.3) by the substitution*

$$z=\beta\int\varphi\exp(\int f dy)dy,\ \partial s=\varphi\partial x,\ t=\tau,\ \textit{where } \beta \textit{ is normalizing factor,}$$

*it is necessary and sufficient that (5.1) to be expressed as the equation*

$$\frac{\varphi^3\exp(\int f dy)y}{\int\varphi\exp(\int f dy)dy}\frac{\partial y}{\partial t}=$$

$$=\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\varphi\exp(\int f dy)}{\int\varphi\exp(\int f dy)dy}-\frac{\varphi_y}{\varphi}\right)\frac{\partial y}{\partial x}-r_2\varphi\right]\times$$

$$\times\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\varphi\exp(\int f dy)}{\int\varphi\exp(\int dy)dy}\right)\frac{\partial y}{\partial x}-r_1\varphi\right]y,$$

*the right-hand side of which represents the noncommutative factorization, or to be expressed as the equation*

$$\frac{\varphi\exp(\int f dy)y}{\int\varphi\exp(\int f dy)dy}\frac{\partial y}{\partial t}=\left[\frac{1}{\varphi}\frac{\partial}{\partial x}-\left(\frac{1}{y\varphi}-\frac{\exp(\int f dy)}{\int\varphi\exp(\int f dy)dy}\right)\frac{\partial y}{\partial x}-r_2\right]\times$$

$$\times\left[\frac{1}{\varphi}\frac{\partial}{\partial x}-\left(\frac{1}{y\varphi}-\frac{\exp(\int f dy)}{\int\varphi\exp(\int dy)dy}\right)\frac{\partial y}{\partial x}-r_1\right]y,$$

*the right-hand side of which represents the commutative factorization, here $r_1, r_2$ are the roots of the characteristic equation (5.5).*

*Lexicographic form of the NEE (5.1) is*

$$\varphi^2y_t=y_{xx}+fy_x^2+2b_1\varphi y_x+b_2\varphi\exp(-\int f dy)\int\varphi\exp(\int f dy)dy.$$

• We apply the lemma 5.1, the formula (2.12) and the proposition 2.1. •

**Example 12.** (Maslov et al., see, for example, [26]).

The equation

$$y_t=\frac{h}{2}y_{xx}-\frac{1}{2}y_x^2 \tag{5.6}$$

by transformation $z = \exp(-y/h)$ is reduced to the form

$$z_t = -hz_{xx}. \tag{5.7}$$

Let $z_1(t,x),\ z_2(t,x)$ be linearly independed solutions of (5.7). Then the total integral $z = c_1 z_1 + c_2 z_2$, where $c_1, c_2$ are arbitrary constants, is a *linear superposition principle* (LSP), and the formula

$$y = -h \log\left[\exp\left(-\frac{1}{h}y_1 - \frac{\lambda_1}{h}\right) + \exp\left(-\frac{1}{h}y_2 - \frac{\lambda_2}{h}\right)\right],$$

is NSP for (5.6), where $\lambda_1, \lambda_2$ are arbitrary constants, $y_1(x,t), y_2(x,t)$ are particular solutions, and $y(x,t)$ is a total integral of (5.6).

We note that the equation (5.6) by the transformation $z = \exp(-y/h),\quad ds = \sqrt{2/h}dx$ is linearized to $z_t = z_{ss}$.

**Comment 5.1.** *In idempotent analysis correspondence principle* (in the sense [26]) *is NSP.*

## 5.2 NEE of $n$-order

**Lemma 5.2.** *In order NEE*

$$\frac{\partial y}{\partial t} = F(y, \frac{\partial y}{\partial x}, \ldots, \frac{\partial^n y}{\partial x^n}) \tag{5.8}$$

*to be reduced to LEE*

$$\frac{\partial z}{\partial t} = \frac{\partial^n z}{\partial s^n} + \sum_{k=1}^{n} \binom{n}{k} b_k \frac{\partial^{n-k} z}{\partial s^{n-k}},\ \ b_k = const, \tag{5.9}$$

*by the substitution of the form (5.2), it is necessary and sufficient that (5.8) can be represented in the form of the equation*

$$u^n\left(1 - \frac{v_y}{v}y\right)\frac{\partial y}{\partial t} = \prod_{k=n}^{1}\left[\frac{\partial}{\partial x} - \left(\frac{v_y}{v} + \frac{u_y}{u}\right)\frac{\partial y}{\partial x} - r_k u\right],$$

*where the right-hand side represents the noncommutative factorization, or, NEE be expressed as the equation*

$$\left(1 - \frac{v_y}{v}y\right)\frac{\partial y}{\partial t} = \prod_{k=1}^{n}\left(\frac{1}{u}\frac{\partial}{\partial x} - \frac{v_y}{vu}\frac{\partial y}{\partial x} - r_k\right),$$

*the right-hand side of which represents the commutative factorization, here $r_k, k = \overline{1,n}$, are the roots of the characteristic equation (4.5).*

**Theorem 5.2** [25]. *In order NEE (5.8) to be transformed by substitution (5.2), where*

$$z = \beta\int \varphi^{\frac{n^2-n+2}{2n}} \exp(\int f dy)dy,\ \partial s = \varphi\partial x,\ t = \tau,\ \beta \text{ is normalizing factor},$$

*to the LEE (5.9), it is necessary and sufficient that (5.8) can be represented in the form of the equation*

$$\frac{\varphi^{\frac{3n^2-n+2}{2n}}\exp(\int f dy)y}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy}\frac{\partial y}{\partial t} =$$

$$= \prod_{k=n}^{1}\left[\frac{\partial}{\partial x} - \left(\frac{1}{y} - \frac{\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)dy} + (k-1)\frac{u^*}{u}\right)\frac{\partial y}{\partial x} - r_k u\right] y,$$

*the right-hand side of which represents the noncommutative factorization, or in the form of the equation*

$$\frac{\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy) y}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy) dy}\frac{\partial y}{\partial t}=\prod_{k=1}^{n}\left[\frac{1}{\varphi}\frac{\partial}{\partial x}-\left(\frac{1}{y\varphi}-\frac{\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy)}{\int \varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy) dy}\right)\frac{\partial y}{\partial x}-r_k\right]y,$$

*where the right-hand side represents the commutative factorization, here $r_k$, $k=\overline{1,n}$, are the roots of the characteristic equation (4.5), or in lexicographic form*

$$\varphi^n\frac{\partial y}{\partial t}=\frac{\partial^n y}{\partial x^n}+nf(y)\frac{\partial^{n-1}y}{\partial x^{n-1}}\frac{\partial y}{\partial x}+\ldots+nb_1\varphi\frac{\partial^{n-1}y}{\partial x^{n-1}}+\ldots$$

$$\ldots+b_n\varphi^{\frac{n^2+n-2}{2n}}\exp(-\int f dy)\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy) dy. \tag{5.10}$$

• We apply the lemma 5.2, the formula (4.10) and the theorem 4.1. •

For example, the general form of NEE of order $n=3$, belonging to the class (5.8),(5.10), is

$$\varphi^3 y_t=y_{xxx}+3fy_xy_{xx}+(\frac{1}{3}\frac{\varphi_{yy}}{\varphi}-\frac{5}{9}\frac{\varphi_y^2}{\varphi^2}-\frac{1}{3}f\frac{\varphi_y}{\varphi}+f^2+f_y)y_x^3+$$

$$+3b_1\varphi[y_{xx}+(f+\frac{1}{3}\frac{\varphi_y}{\varphi})y_x^2]+3b_2\varphi^2y_x+b_3\varphi^{5/3}\exp(-\int f dy)\int\varphi^{4/3}\exp(\int f dy) dy$$

(see also the proposition 3.1).

**Theorem 5.3. Nonlinear superposition principle.**

*Let $z_1(s,t),\ldots,z_n(s,t)$ are partial linearly independent solutions of (5.9). Then total integral of (5.9) (depending on arbitrary constants) is*

$$z=c_1z_1(s,t)+\ldots+c_nz_n(s,t),$$

*where $c_1,\ldots,c_n$ are arbitrary constants, and this formula is LSP. We obtain NSP for (5.8),(5.10) according to the formulas*

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\exp(\int f dy) dy=\sum_{k=1}^{n}c_kz_k(s,t),\quad ds=\varphi dx,$$

*where $\beta=const$ is normalizing factor.*

# 6 Additional comments on nonlinear superposition principles

(i) *A FSS does not determine the nonlinear equation completely; to determine the equation one has to know both the FSS and the NSP.*

Thus, Ermakov-Pinney equation and Kummer-Schwarz's equations of second and third order has the same FSS (see the examples 2, 3, 4).

(ii) *The same equation may have different NSPs, i.e. the same equation can be linearized by means of different transformations.*

**Example 13.** Consider the equation (see Ex. 6)

$$y''+3yy'+y^3=0. \tag{6.1}$$

Equation (6.1) can be linearized to the form $\ddot{z}+3\dot{z}+2z=0$ by the substitution $z=y^2,\ dt=ydx$. But, applying the point transformations

$$X=x-\frac{1}{y},\quad Y=\frac{1}{2}x^2-\frac{x}{y},$$

we get another linear equation $Y''(X)=0$.

The equation (6.1) can be linearized also to the form $z'''=0$ by the following substitution $y=z'/z$.

(iii) *Nonlinear equations that possess NSP, allow nonlocal symmetries.*

**Example 14.** The equation

$$y''-y^2y'=0 \tag{6.2}$$

be linearized by the substitution $z=y^3,\ dt=y^2dx$ to the equation $\ddot{z}-\dot{z}=0$. The equation (6.2) admits the nonlocal exponential symmetry $X=\exp(\int y^2dx)\frac{\partial}{\partial y}$.

(iv) *Factorization of nonlinear differential operators as a method of getting NSP for nonlinear equations.*

Let an equation

$$F(y,y',\dots,y^{(n)})=0$$

admits a factorization

$$\prod_{k=1}^{n}(\gamma D-\beta y'-\alpha-r_k)y=0,\ D=(')=d/dx,$$

where $\gamma=\gamma(y),\ \beta=\beta(y),\ \alpha=\alpha(y),\ r_k=const.$

This factorization generates Bäcklund transformation

$$\frac{y'}{y}\left(1-\frac{\beta}{\gamma}y\right)=\frac{\alpha}{\gamma}+\frac{z'}{z},\quad dt=\frac{1}{\gamma}dx$$

that reduces to a linear equation

$$\prod_{k=1}^{n}(D_t-r_k)z\equiv z^{(n)}(t)+\sum_{k=1}^{n}\binom{n}{k}b_kz^{(n-k)}(t)=0,\ b_k=const,\ D_t=d/dt.$$

# 7 Open question

In this lecture the question of finding of nonlinear superposition principle is studied. We also study the question of the construction of autonomous equations of the form

$$F(y,y',\dots,y^{(n)})=0, \tag{7.1}$$

which can be linearized to the form

$$M_nz\equiv z^{(n)}(t)+\sum_{k=1}^{n}\binom{n}{k}b_kz^{(n-k)}(t)+c=0,\ b_k,c=const, \tag{7.2}$$

by the transformations

$$y=v(y)z,\ dt=u(y)dx. \tag{7.3}$$

The next question is the natural generalization.

*How to find the nonlinear superposition principles for nonlinear nonautonomous equations*

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{7.4}$$

*and construct the equations itself, which can be linearized to the form (7.2) by the convertible transformations of the form*

$$y = v(x, y)z, \; dt = u_1(x, y)dx + u_2(x, y)dy \; ? \tag{7.5}$$

The transformations (7.5) include many famous transformations of variables. Answeging the question, one can integrate the equations of the constructed class (7.4) in the quadratures or in terms of ODE of the first order.

To solve this problem one can use the following result:

**Theorem 7.1** (Berkovich [23]). *Consider the equation (7.4).*

*The convertible transformation of the form (7.5) where $v$, $u_1$ and $u_2$ are sufficiently smooth nonvanishing functions of two variables $x$ and $y$ in some domain $\Gamma(x, y)$, transforms (7.4) into the linear autonomous form (7.2) (for $c = 0$), if and only if (7.4) admits the following factorization through first-order nonlinear differential operators:*

$$\prod_{k=1}^{n} \left[ \frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_k \right] y = 0$$

*(commutative factorization), or*

$$\prod_{k=n}^{1} \left[ D - \frac{v_x + v_y y'}{v} - (k-1) \frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - r_k (u_1 + u_2 y') \right] y = 0$$

*(noncommutative factorization), where $D = d/dx$ and $r_k$ are the roots of the characteristic equation*

$$M_n(r) \equiv r^n + \sum_{k=1}^{n} \binom{n}{k} b_k r^{n-k} = 0, \; b_k = const.$$

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Lev M. Berkovich: Samara State University,
Department of Mathematics and Mechanics,
Acad. Pavlov Str., 1,
443011, Samara, RUSSIA.
e-mail: berk@ssu.samara.ru
URL: http://www.ssu.samara.ru/Campus/Teachers/Berkovich