On the Diophantine Approximations of logarithms in cyclotomic fields.

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To 100th birthday of
Professor A.O.Gelfond.

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Let $T \in \mathbb{R}$, $\{\Delta, m, n\} \in \mathbb{N}, \Delta \geq 2$, $K_m = \mathbb{Q}[\exp(2\pi i/m)]$ is a cyclotomic field, $\mathbb{Z}_{K_m}$ is the ring of all the integers in K_m, $\Lambda(n)$ is the Mangold's function, $\epsilon^2 = \epsilon$. Let $\Lambda_0(m) = 0$, if m is odd and $\Lambda_0(m) = \Lambda(m/2)$, if m is even. Let $\omega_1(m) = (m-1)/2$, if m is odd, $\omega_1(m) = m/2 - 2$, if $m \equiv 2(\mod 4)$ and $\omega_1(m) = m/2 - 1$, if $m \equiv 0(\mod 4)$. Let

$$(1) \qquad w_\Delta(T) = \sqrt{\frac{\sqrt{(\Delta^2(3-T^2)+1)^2 + 16\Delta^4 T^2} + \Delta^2(3-T^2)+1}{2}},$$

$$(2) \qquad V_\Delta^* = (\Delta+1) + \log((\Delta-1)^{(\Delta-1)/2}(\Delta+1)^{(\Delta+1)/2}\Delta^{-\Delta}) +$$

$$\frac{\pi}{2}\sum_{\mu=0}^{1}(1-2\mu)\sum_{\kappa=1}^{[(d-1)/2]+\mu}\cot\left(\frac{\pi\kappa}{d-1+2\mu}\right),$$

$$(3) \qquad V_\Delta(m) = V^* + (\Delta+1)\Lambda_0(m)/\phi(m),$$

$$(4) \qquad l_\Delta(\epsilon, T) = -log\left(4(\Delta+1)^{\Delta+1}(1-1/\Delta)^{(\Delta-1)}\right) +$$

$$\frac{1}{2}\log\left((2\Delta+(-1)^{\epsilon}w_{\Delta}(T)+(\Delta+1))^2+T^2\Delta^2\left(1+\frac{(-1)^{\epsilon}2\Delta}{w_{\Delta}(T)}\right)^2\right)+$$

$$\frac{1}{2}\log\left((2\Delta+(-1)^{\epsilon}w_{\Delta}(T)-(\Delta+1))^2+T^2\Delta^2\left(1+\frac{(-1)^{\epsilon}2\Delta}{w_{\Delta}(T)}\right)^2\right)+$$

$$\frac{(\Delta-1)}{2}\log\left((2\Delta+(-1)^{\epsilon}w_{\Delta}(T))^2+T^2\Delta^2\left(1+\frac{(-1)^{\epsilon}2\Delta}{w_{\Delta}(T)}\right)^2\right),$$

$$g_{\Delta,\epsilon}(m)=(-1)^{\varepsilon}(l_{\Delta}(\epsilon,\tan(\pi\omega_1(m)/m)+V_{\Delta}(m))), \tag{5}$$

$$h_{\Delta}(m)=-V_{\Delta}(m)-l_{\Delta}(1,\tan(\pi/m)), \tag{6}$$

where $m\neq 2$, $k=0,1$. Let

$$\beta(\Delta,m)=g_{d,0}(m)/h_{\Delta}(m),\ \alpha(\Delta,m)=\beta(\Delta,m)-1+g_{\Delta,1}(m)/h_{\Delta}(m).$$

Theorem. *Let* $m\in\mathbb{N}\diagdown\{1,\,2,\,6\}\ \Delta\in\{5,7\}$. *Then*

$$h_{\Delta}(m)>0 \tag{7}$$

and for each $\varepsilon>0$ *there exists* $C_{\Delta,m}(\varepsilon)>0$ *such that*

$$\max_{\sigma\in Gal(K/\mathbb{Q})}(|q^{\sigma}\log((2+\exp(2\pi i/m))^{\sigma})-p^{\sigma}|)\geq \tag{8}$$

$$C_{\Delta,m}(\varepsilon)(\max_{\sigma\in Gal(K_m/\mathbb{Q})}(|q^{\sigma}|)^{-\alpha(\Delta,m)-\varepsilon},$$

where $p\in\mathbb{Z}_{K_m}$ *and* $q\in\mathbb{Z}_{K_m}\diagdown\{0_{K_m}\}$; *moreover, for any* $q\in\mathbb{Z}_{K_m}\diagdown\{0_{K_m}\}$ *and any* $\varepsilon>0$ *there exists* $C^*_{\Delta,m}(q,\varepsilon)>0$ *such that*

$$b^{\beta(\Delta,m)+\varepsilon}\max_{\sigma\in Gal(K/\mathbb{Q})}(|q^{\sigma}b\log((2+\exp(2\pi i/m))^{\sigma})-p^{\sigma}|)\geq \tag{9}$$

$$C^*_{\Delta,m}(q,\varepsilon),$$

where $p\in\mathbb{Z}_{K_m}$, $b\in\mathbb{N}$.

For the proof I use the same method, as in [37] – [67]. I work on the Riemann surface $\mathfrak{F}$ of the function $Log(z)$ and identify it with the direct product of the multiplicative group $\mathbb{R}^*_+=\{r\in\mathbb{R}\colon r>0\}$ of all the positive real numbers with the operation $\times$, not to be written down explicitly as usual, and the additive group $\mathbb{R}$ of all the real numbers, so that

$$z_1z_2=(r_1r_2,\phi_1+\phi_2)$$

for any two points $z_1 = (r_1, \phi_1)$ and $z_2 = (r_2, \phi_2)$ on $\mathfrak{F}$. I will illustrate the appearing situations on the half plain (ϕ, r), where $r > 0$.

For each $z = (r, \phi) \in \mathfrak{F}$, let

$$\theta_0(z) = r \exp i\phi,\ Log(z) = \ln(r) + i\phi,\ \eta^*_\alpha(z) = (r, \phi - \alpha),$$

where $\alpha \in \mathbb{R}$. Clearly, $Log(z_1 z_2) = Log(z_1) + Log(z_2)$ for any $z_1 \in \mathfrak{F}$ $z_2 \in \mathfrak{F}$. Let $\rho(z_1, z_2) = |Log(z_1) - Log(z_2)|$, where $z_1 \in \mathfrak{F}$ and $z_2 \in \mathfrak{F}$; clearly, $(\mathfrak{F}, \rho)$ is a metric space. Clearly, $\rho(zz_1, zz_2) = \rho(z_1, z_2)$ for any z_1, z_2 and z in $\mathfrak{F}$. Clearly, $\theta_0(z) = \exp(Log(z))$ for any $z \in \mathfrak{F}$. Clearly, for any $\alpha \in \mathbb{R}$ the map $z \to \eta^*_\alpha(z)$ is the bijection of $\mathfrak{F}$ onto $\mathfrak{F}$ and

$$\theta_0((\eta^*_\alpha)^m(z)) = \exp(-im\alpha)\theta_0(z)$$

for each $z = (r, \phi) \in \mathfrak{F}$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$. Clearly, the group $\mathfrak{F}$ may be considered as $\mathbb{C}$-linear space, if for any $z \in \mathfrak{F}$ and any $s \in \mathbb{C}$ we let

$$z^s = (|\exp(sLog(z))|, \Im(sLog(z)).$$

Let us fix a domain D in $\mathfrak{F}$. Let $f(z) = f^\wedge(r, \phi)$ for a complex-valued function $f(z)$ on D, It is well known that $f(z)$ is holomorphic in D if the complex-valued function $f^\wedge(r, \phi)$ of two real variables r and ϕ has continuous partial derivatives in D, and the Cauchy-Riemann conditions

$$r(((\partial/\partial r)f^\wedge)(r, \phi)) = -i((\partial/\partial\phi)f^\wedge)(r, \phi)) := \tag{10}$$

$$(\delta f)(z) := \theta_0(z)((\partial/\partial z)f)(z))$$

are satisfied for every point $z = (r, \phi) \in D$. The equalities (10) determine a differentiations $\frac{\partial}{\partial z}$ and $\delta = \theta_0(z)\frac{\partial}{\partial z}$ on the ring of all the holomorphic in the domain D functions. In particular, the function $Log(z)$ is holomorphic on $\mathfrak{F}$ and we have the equalities

$$((\partial/\partial z)Log)\,(z) = \theta_0(z^{-1}),\ (\delta Log)(z) = 1.$$

For the proof I use the functions of C.S.Mejer. Let $\Delta \in \mathbb{N} + 1$, $\delta_0 = 1/\Delta$,

$$\gamma_1 = (1 - \delta_0)/(1 + \delta_0), \quad d_l = \Delta + (-1)^l, \quad l = 1, 2.$$

To introduce the first of my auxiliary function $f_1(z, \nu)$, I use the auxiliary set

$$\Omega_0 = \{z \in \mathfrak{F} :\ |z| \le 1\}.$$

I prove that, for each $\nu \in \mathbb{N}$, the function $f_1(z, \nu)$ belongs to the ring $\mathbb{Q}[\theta_0(z)]$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$. For $\nu \in \mathbb{N}$, let

$$f_1(z, \nu) = -(-1)^{\nu(\Delta+1)} G^{(1,1)}_{2,2}\left(z \middle| \begin{matrix} -\nu d_1, & 1 + \nu d_2 \\ 0, & \nu \end{matrix}\right) \tag{11}$$

$$= -(-1)^{\nu(\Delta+1)} \frac{1}{2\pi i} \int\limits_{L_1} g^{(1,1)}_{2,2}(s)ds,$$

where

$$g^{(1,1)}_{2,2}(s) = \theta_0(z^s)\Gamma(-s)\Gamma(1 + d_1\nu + s)/(\Gamma(1 - \nu + s)\Gamma(1 + d_2\nu - s))$$

and the curve L_1 passes from $+\infty$ to $+\infty$ encircling the set $\mathbb{N} - 1$ in the negative direction, but not including any point of the set $-\mathbb{N}$. So, for the parameters of the Meyer's functions we have

$$p = q = 2,\ m = n = 1,\ a_1 = -\nu d_1,\ a_2 = 1 + \nu d_2,\ b_1 = 0,\ b_2 = \nu,$$

$$\Delta^* = \left(\sum_{k=1}^{q} b_k\right) - \sum_{j=1}^{p} a_j = -\nu - 1 < -1,$$

and, since we take $|z| \le 1$, convergence conditions of the integral in (11) hold. To compute the function $f_1(z, \nu)$, we use the following formula

$$G = (-1)^k \sum_{s\in S_k} Res(g; s), \tag{12}$$

where $k = 1$, G denotes the integral (11) with $L = L_k$, g denotes the integrand of the integral (11), S_k denotes the set of all the unremovable singularities of g encircled by L_k, and $Res(g; s)$ denotes the residue of the function g at the point s. Then we obtain the equlity

$$f_1(z,\ \nu) =$$

$$(\nu d_1)!/(\nu\Delta)!z^\nu(-1)^{\nu\Delta}\sum_{k=0}^{\nu\Delta}(-\theta_0(z))^k\binom{\nu\Delta}{k}\binom{\nu\Delta + k}{\nu d_1}.$$

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$. Let

$$\Omega_1 = \{z \in \mathfrak{F} :\ |z| \ge 1\}.$$

Now, let me introduce my second auxiliary function defined for $z \in \Omega_1$. For $\nu \in \mathbb{N}$, let

$$f_2(z, \nu) = -(-1)^{\nu(\Delta+1)} G^{(2,1)}_{2,2}\left(z \middle| \begin{matrix} -\nu d_1, & 1 + \nu d_2 \\ 0, & \nu \end{matrix}\right) = \tag{13}$$

$$-(-1)^{\nu(\Delta+1)}\frac{1}{2\pi i}\int_{L_2} g^{(2,1)}_{2,2}(s)ds,$$

where

$$g^{(2,1)}_{2,2}(s) = \theta_0((\eta_\pi(z))^s)\Gamma(-s)\Gamma(\nu - s)\Gamma(1 + d_1\nu + s)/\Gamma(1 + d_2\nu - s).$$

and the curve L_2 passes from $-\infty$ to $-\infty$ encircling the set $-\mathbb{N}$ in the positive direction, but not including any point of the set $\mathbb{N}-1$. So, for the parameters of the Meyer's functions we have

$$p = q = m = 2,\ n = 1,\ a_1 = -\nu d_1,\ a_2 = 1 + \nu d_2,\ b_1 = 0,\ b_2 = \nu,$$

$$\Delta^* = \left(\sum_{k=1}^{q} b_k\right) - \sum_{j=1}^{p} a_j = -nu - 1 < -1,$$

and, since we take $|z| \geq 1$, convergence conditions of the integral in (13) hold. To compute the function $f_2(z,\nu)$, we use the formula (12) where $k = 2$, G denotes the integral in (13) with $L = L_k$, g denotes the integrand of the integral in (13), S_k denotes the set of all the unremovable singularities of g encircled by L_k, and $Res(g;s)$ denotes the residue of the function g at the point s. Then we obtain the equality

$$f_2(z,\nu)(\nu\Delta)!/(\nu d_1)! = (-1)^{\nu} \sum_{t=\nu+1}^{\infty} R_0(t;\nu)\theta_0(z^{-t+\nu}), \tag{14}$$

where

$$R_0(t;\nu) = (\nu\Delta)!/(\nu d_1)! \left(\prod_{\kappa=\nu+1}^{\nu\Delta} (t-\kappa)\right) \prod_{\kappa=0}^{\nu\Delta} (t+\kappa)^{-1}.$$

Let further

$$f_k^*(z,\,\nu) = f_k(z,\,\nu)(\nu\Delta)!/(\nu d_1)!, \tag{15}$$

where $k = 1, 2$. Expanding the function $R_0(t;\nu)$ into partial fractions, we obtain the equality

$$R_0(t;\nu) = \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}/(t+k)$$

with

$$\alpha^*_{\nu,k} = (-1)^{\nu+\nu\Delta+k} \binom{\nu\Delta}{k} \binom{\nu\Delta+k}{\nu\Delta-\nu}, \tag{16}$$

where $k = 0, \ldots, \nu\Delta$. It follows from (13), (14), (15) and (16) that

$$f_2^*(z,\nu) = (-\theta_0(z))^{\nu} \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t} R_0(t;\nu) = \tag{17}$$

$$= (-\theta_0(z))^{\nu} \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t-k+k} \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}/(t+k)$$

$$= (-\theta_0(z))^{\nu} \sum_{t=1+\nu}^{+\infty} ((\theta_0(z))^{-t-k}/(t+k)) \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k =$$

$$(-\theta_0(z))^{\nu} \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu+k}^{+\infty} ((\theta_0(z))^{-\tau}/\tau)) =$$

$$= \alpha^*(z;\nu)(-\log(1-1/\theta_0(z))) - \phi^*(z;\nu),$$

where $\log(\zeta)$ is a branch of $Log(\zeta)$ with $|\arg(\zeta)| < \pi$,

$$\alpha^*(z;\nu) = (-(\theta_0(z))^{\nu} \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,\,k}(\theta_0(z))^k = f_1^*(z;\nu), \tag{18}$$

(19) $$\phi^*(z;\nu) = (-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) =$$

$$(-\theta_0(z))^\nu \sum_{\tau=1}^{\nu}((\theta_0(z))^{-\tau}\alpha^*(z;\nu)/\tau+$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)).$$

The change of order of summation by passage to (17) is possible, because the series in the second sum in (17) is convergent, if $|z| \geq 1$ and $\theta_0(z) \neq 1$. Since

$$\deg_t\left(\prod_{\kappa=\nu+1}^{\nu\Delta}(t-\kappa)\right) - \deg_t\left(\prod_{\kappa=0}^{\nu\Delta}(t+\kappa)\right) = -\nu-1,$$

it follows that

$$\alpha^*(1;\nu) = Res(R_0(t;\nu); t=\infty) = 0$$

So in the domain $D_0 = \{z \in \mathfrak{F}\colon |z| > 1$ the funcion $f^*_2(z,\nu)$ coincides with the functin

(20) $$f^*_0(z,\nu) = \alpha^*(z;\nu)(-\log(1-1/\theta_0(z))) - \phi^*(z;\nu),$$

The form (20) may be used for various applikations. Espeshially it is pleasant, when both $1/\theta_0(z)$ and $\alpha^*(z;\nu)$ for some z is integer algebraic number. The following Lemma corresponds to this remark.

Lemma 1. *Let $m \in \mathbb{N}$, $m > 2$ $m \neq 2p^\alpha$, where p run over the all the prime numbers and α run over $\mathbb{N}$. Then $1+\exp(2\pi i/m)$ belongs to the group of the units of the field K_m. If $m = 2p^\alpha$, where p is a prime number and $\alpha \in \mathbb{N}$, then the ideal $\mathfrak{l} = (1+exp(2\pi i/m))$ is a prime ideal in the field K_m, and $\mathfrak{l}^{\phi(m)} = (p)$.*

Proof. Let polynomial $\Phi_m(z)$ is irreducible over $\mathbb{Q}$, has the leading coefficient equal to one and $\Phi_m(\exp(2\pi i/m)) = 0$. Let $\Lambda(n)$, as usual, denotes the Mangold's function. Since (see, for example, [27], end of the chapter 3)

$$\Phi_m(z) = \prod_{d|m}(z^{m/d}-1)^{\mu(d)},$$

it follows that

$$\Phi_m(-1) = (-2)^{\left(\sum\limits_{d|m}\mu(d)\right)} = 1,$$

if $m \in 1+2\mathbb{N}$,

$$\Phi_m(z) = \prod_{d|(m/2)}(((z)^{m/(2d)}-1)^{\mu(2)}((-z)^{m/d}-1)/((-z)-1))^{\mu(d)},$$

$$\Phi_m(-1) = \lim_{z\to-1}\prod_{d|(m/2)}(((-z)^{m/d}-1)/((-z)^{-}1))^{\mu(d)}\times$$

$$(-2)^{\mu(2)\left(\sum\limits_{d|(m/20}\mu(d)\right)} =$$

$$\exp\left(\sum_{d|(m/2)} \ln(m/(2d))\mu(2d)\right) = \exp(\Lambda(m/2)),$$

if $m \in 2(1+2\mathbb{N})$,

$$\Phi_m(z) = \prod_{d|(m/2)} (((-z)^{m/d} - 1)/((-z) - 1))^{\mu(d)},$$

and

$$\Phi_m(-1) = \lim_{z\to -1} \prod_{d|(m/2)} (((-z)^{m/d} - 1)/((-z) - 1))^{\mu(d)} =$$

$$\exp\left(\sum_{d|m/2} \ln(m/(2d))\mu(d)\right) = \exp(\Lambda(m/2)),$$

if $m \in 4\mathbb{N}$. If $m = 2p^\alpha$ with $\alpha \in \mathbb{N}$, then $\Phi_m(-1) = \exp(\Lambda(m/2)) = p$, and ideals $\mathfrak{l}_k = (1 + exp(2\pi ik/m))$, where $(k,m) = 1$, divide each other and in the standard equality $efg = n$ (see, [27], chapter 3, section 10) we have

$$e = n = \phi(m),\ f = g = 1. \blacksquare.$$

In connection with the above remark and with the Lemma 1, the following case is interesting for us:

(21)
$$\theta_0(z) = (-\rho)(1 + \exp(-i\beta)) = -(\rho\exp(i\beta/2))/(2cos(\beta/2)) =$$

$$-(\rho\exp(i\psi))(2cos(\psi)) = -(1 + i\tan(\psi))/2$$

with $\rho > 2/3$, $|\beta| < \pi$ and $-\pi/2 < \psi = \beta/2 < \pi/2$; then

$$\Re(1 - 1/\theta_0(z)) = \Re(2 + \exp(i\beta)/\rho) > 1/2,$$

and we have no problems with $\log(1 - 1/\theta_0(z))$. Of course, according to the Lemma 1, the case $\rho = 1$ is interesting especially. So, we will take further

(22)
$$z = (\rho/(2\cos(\psi)), \psi - \pi) = (\rho/(-2cos(\theta), \theta),$$

where $\rho > 2/3$, $|\psi| < \pi/2$ and $-3\pi/2 < \theta = \psi - \pi < -\pi/2$; clearly, the function (20) is analytic in the domain

$$D_1 = \{z = (\rho(2\cos(\psi))^{-1}, \psi - \pi))\colon \rho > 2/3,\ -\pi/2 < \psi < \pi/2\} =$$

$$\{z = ((-2\rho\cos(\theta))^{-1}, \theta))\colon \rho > 2/3,\ -3\pi/2 < \theta < -\pi/2\}.$$

Let

(23)
$$D_2(\delta_0) = \{z \in \mathfrak{F}\colon |z| > 1 + \delta_0/2\},\ D_3 = D_2(\delta_0) \cup D_1.$$

So, the funcion $f_2^*(z,\nu)$ coincides with the function (20) in $D_2(\delta_0) \subset D_0$. Since $D_2(\delta_0) \cap D_1 \neq \emptyset$, it follows that the join $D_3 = D_2(\delta_0) \cup D_1$ of the domains $D_2(\delta_0)$ and D_1 is a domain in $\mathfrak{F}$ and the function (20) is analytic in this domain.

The conditions, which imply the equality

$$(24)\qquad (-1)^{m+p-n}\exp(-i\alpha)\theta_0(z)\times$$
$$\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)(G\circ\eta^*_\alpha)\right)(z)=\left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)(G\circ\eta^*_\alpha)\right)(z)$$

hold in our case for the Mejer's function

$$G=G^{(m,n)}_{p,q}\left(z\middle|\begin{matrix}a_1, & \dots, & a_p\\ b_1, & \dots, & b_q\end{matrix}\right).$$

We have $p=q=2$, $m=n=1$, $\alpha=0$ for the function $f_1(z,\nu)$ and the equation (24) takes the form

$$\theta_0(z)((\delta+1+d_1\nu)(\delta-d_2\nu)f_1)(z,\nu)=(\delta(\delta-\nu)f_1)(z,\nu)$$

We have $p=q=m=2$, $n=1$, $\alpha=\pi$ for the function $f_2(z,\nu)$ and the equation (24) takes the form

$$\theta_0(z)((\delta+1+d_1\nu)(\delta-d_2\nu)f_2)(z,\nu)=(\delta(\delta-\nu)f_2)(z,\nu).$$

We see that both the functions $f(z,\nu)=f^*_k(z,\nu)$, where $k=1,2$ satisfy to the same differential equation

$$(25)\qquad \theta_0(z)(\delta+1+d_1\nu)(\delta-d_2\nu)f(z,\nu)=(\delta(\delta-\nu)f)(z,\nu).$$

in the domain D_0. According to the general properties of the Mejer's functions we have the equality

$$(26)\qquad \left(\prod_{\kappa=1}^{\Delta-1}(\nu(\Delta-1)+\kappa)\right)\prod_{\kappa=1}^{d_2}(\delta-d_2\nu-\kappa)f^*_k(z,\,\nu+1)=$$
$$\left(\prod_{\kappa=1}^{\Delta}(\nu\Delta+\kappa)\right)(\delta-\nu)\prod_{\kappa=1}^{d_1}(\delta+d_1\nu+\kappa)f^*_k(z,\,\nu),$$

where $k=1,2$ and $z\in D_0$. Since $f^*_0(z,\nu)$ and polynomial $f^*_1(z,\nu)$ are analytic in the domain $D_0\cup D_1$, and $f^*_0(z,\nu)$ coincides with $f^*_2(z,\nu)$, it follows that the equations (25) and (26) hold in $D_0\cup D_1$ for $k=0,1$.

Let

$$(27)\qquad D^\vee(w,\eta)=(\eta+1)(\eta+\gamma_1)-2(1+\gamma_1)w\eta,$$

$$(28)\qquad D^\wedge(z,\eta)=D^\vee(\theta_0(z),\eta),$$

where, in view of (21),

$$(29)\qquad w=\theta_0(z)=-r\exp(i\psi),\ r=1/(2cos(\psi)),\ |\psi|<\pi/2.$$

In view of (29), the polynomial (27) coincides with the polynomial (1) in [59]. Let

$$(30)\qquad h^\sim(\eta)=(\eta-1)(1-\delta_0)^{-d_1}(\eta+1)2^{-2}\,\eta^{d_1}.$$

As in [51], we consider ν^{-1} as an independent variable taking its values in the field $\mathbb{C}$ including 0. Let F be a bounded closed subset of $\mathfrak{F}$ (in particular, this compact F may be an one-point set). Let $\mathfrak{H}_0(F)$ be the subring of all those functions in $\mathbb{Q}(w)$, which are well defined for every $w \in \theta_0(F)$. For $\varepsilon \in (0,1)$, let $\mathfrak{H}(F,\varepsilon)$ be the subring of all those functions in $\mathbb{Q}(w,\nu^{-1})$, which are well defined for every (w,ν^{-1}) with $w \in \theta_0(F)$, $|\nu^{-1}| \leq \varepsilon_0$.

Lemma 2. *Let F be a closed bounded subset of $D_0 \cup D_1$ (in particular, F may be an one-point set). Let further for any $z \in F$ the polynomial (28) has only simple roots and on the set of all the roots η of the polynomial $D^\wedge(z,\eta)$ the map*

$$\eta \to h^\sim(\eta) \tag{31}$$

is injective. Then there is $\varepsilon \in (0,1)$ such that, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon]$, the functions $f_0^*(z,\nu)$, $f_1^*(z,\nu) = \alpha^*(z;\nu)$ and $\phi^*(z;\nu)$ are solutions of the difference equation

$$x(z,\nu+2) + \sum_{j=0}^{1} q_j^*(z,\nu^{-1})x(z,\nu+j) = 0, \tag{32}$$

moreover,

$$q_j^*(z,\, \nu^{-1}) \in \mathfrak{H}(F,\, \varepsilon) \tag{33}$$

for $j = 0,\, 1,$ and trinomial

$$w^2 + \sum_{j=0}^{1} q_j^*(z,0)w^j \tag{34}$$

coincides with

$$\prod_{k=0}^{1}(w - h(\eta_k)), \tag{35}$$

if

$$\prod_{k=0}^{1}(w - \eta_k),$$

coincides with $D^\vee(w,\eta)$ from (27).

Proof. Proof may be found in [51]. ∎

This Lemma shows the importance of the properties of the roots of the polynomial (27). In correspondence with (22) and with notations in [59], let

$$\rho > 2/3,\ r = \rho/(2\cos(\psi)),\ t = \cos(\psi),\ |\psi| < \pi/2. \tag{36}$$

Let $u = r^2, \delta_0 \leq 1/2 < 2/3 < \rho$. Then

$$2\delta_0 \leq 2/5 < 2/3 < \rho < 2\sqrt{u} = 2r. \tag{37}$$

Clearly,

$$(\partial/\partial\psi)r = (\rho\sin(\psi))/(2\cos^2(\psi)) = -2\rho(\sin(\psi) - 1) - 2\rho/(\sin(\psi) + 1),$$

$$(\partial/\partial\psi)^2 r = (2\rho\cos(\psi))/(\sin(\psi)-1)^2 + (2\rho\cos(\psi))/(\sin(\psi)+1)^2 > 0,$$

if $|\psi| < \pi/2$ In view of (3.1.10) in [52],

$$|D_0(r,\psi,\delta_0)|^2 = r^4 + r^2 + (\delta_0/2)^4 + \tag{38}$$

$$2r^2(\delta_0/2)^2(2t^2-1) + 2r(r^2+(\delta_0/2)^2)t =$$

$$u^2 + u + (\delta_0/2)^4 + (\delta_0/2)^2(\rho^2 - 2u) + \rho(u + (\delta_0/2)^2) =$$

$$u^2 + u(\rho + 1 - (\delta_0)^2/2) + (\delta_0/2)^2(\rho^2 + \rho + (\delta_0/2)^2),$$

$$|R_0(r,\psi,\delta_0)|^2 = |D_0(r,\psi,\delta_0)| = \tag{39}$$

$$\sqrt{u^2 + u(\rho+1-(\delta_0)^2/2) + (\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)}.$$

In view of (3.1.41) - (3.1.43) in [52] and (39),

$$p_1 = 8(|R_0^*(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2 = \tag{40}$$

$$8(r^2 + rt + 1/4 + |D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2 = 8(1+\delta_0)^{-2}\times$$

$$\left(u + \rho/2 + 1/4 + \sqrt{u^2 + u(\rho+1-(\delta_0)^2/2) + (\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)}\right),$$

$$p_2 = (8(|R_1^*(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2))/(1+\delta_0)^2 = \tag{41}$$

$$8(r^2 - r\delta_0 t + (\delta_0)^2/4 + |D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2 =$$

$$8(u - \delta_0\rho/2 + (\delta_0)^2/4)/(1+\delta_0)^2 +$$

$$8(1+\delta_0)^{-2}\sqrt{u^2 + u(\rho+1-(\delta_0)^2/2) + (\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)} =$$

$$8(1+\delta_0)^{-2}u(2 + (\rho+1-\delta_0\rho)/(2u) + O(1/u^2)),$$

$$q_1(r,\psi,\delta_0) = ((1-\delta_0)/(1+\delta_0))^2,\ q_2(r,\psi,\delta_0) = \tag{42}$$

$$(4r/(1+\delta_0))^2 = (16u)/(1+\delta_0)^2.$$

In view of (91) in [59], (36) and (37),

$$s = s_0(r,\psi) = |r\exp(i\psi)+1|/2 = \sqrt{(r^2+1+2rcos(\psi))/4} = \tag{43}$$

$$\sqrt{(u+1+\rho)/4} \in (\max(|r-1|/2,\ \delta_0/4),\ (r+1)/2]$$

and

$$t = cos(\psi) = (4s^2 - r^2 - 1)/(2r).$$

In view of (3.1.68) in [52], (3.1.70) – (3.1.71) in [52] and (39),

$$|R_{-1}^*(r,\psi,\delta_0)|^2 = r^2 + (2+\delta_0)^2/4 + r(2+\delta_0)\cos(\psi) =$$

$$u + (2+\delta_0)^2/4 + \rho(2+\delta_0)/2,$$

$$p_0 = 8(|R_{-1}^*(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2 = \tag{44}$$

$$8(u+(2+\delta_0)^2/4+\rho(2+\delta_0)/2)/(1+\delta_0)^2+$$
$$8(1+\delta_0)^{-2}\sqrt{u^2+u(\rho+1-(\delta_0)^2/2)+(\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)},$$

(45) $$q_0(r,\psi,\delta_0)(1+\delta_0)^2/16=(r^2+1+2rcos(\psi))=(u+1+\rho).$$

According to Lemma 4.4 in [59], (23) and (37),

(46) $$|\eta_1^\wedge(r,\psi,\delta_0)+\epsilon|<|\eta_0^\wedge(r,\psi,\delta_0)+\epsilon|,$$

if $\epsilon^2=\epsilon$ and $z\in D_3$. Therefore, according to (40), (42) and (46),

(47) $$(-1)^k(\partial/\partial u)|\eta_k^\wedge(r,\psi,\delta_0)|>0,$$

where $\frac{1}{3}<\rho/2<\sqrt{u}=r$, $k^2=k$. According to a) and c) of the Lemma 4.6 in [59], and in view of (23) and (43),

(48) $$|\eta_1^\wedge(r,\psi,\delta_0)-1|<|\eta_0^\wedge(r,\psi,\delta_0)-1|,$$

if $z\in D_3$. In view of (38),

(49) $$|D_0(r,\psi,\delta_0)|^2=$$
$$u^2+u(\rho+1-(\delta_0)^2/2)+(\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)=$$
$$(u+(\rho+1)/2-(\delta_0)^2/4)^2+(\delta_0/2)^2(\rho^2+\rho+(\delta_0/2)^2)-$$
$$(((\rho+1)/2)^2-(\rho+1)(\delta_0)^2/4+(\delta_0/2)^4)=$$
$$(u+(\rho+1)/2-(\delta_0)^2/4)^2+(\delta_0/2)^2(\rho^2+2\rho+1)-(\rho+1)^2/4=$$
$$(u+(\rho+1)/2-(\delta_0)^2/4)^2-(\rho+1)^2(1-(\delta_0)^2)/4.$$

Consequently,

(50) $$|D_0(r,\psi,\delta_0)|=u+\frac{\rho+1}{2}-\frac{(\delta_0)^2}{4}+O(1/u),$$

where $u\geq 1/4$. Since $u\geq 1/4>(\delta_0)^2/4$, it follows that

$$u+(\rho+1)/2-(\delta_0)^2/4>\sqrt{1-(\delta_0)^2}(\rho+1)/2.$$

If $\rho=1, u=1/4$ then in view of (49),

$$|D_0(r,\psi,\delta_0)|^2=(5/4-(\delta_0)^2/4)^2-(1-(\delta_0)^2)=$$
$$\left(\tau-5/4)^2\right)^2+4\tau-1,$$

where $0<\tau=\frac{(\delta_0)^2}{4}<\frac{1}{100}$; moreover, in this case

$$(\partial/\partial\tau)|D_0(r,\psi,\delta_0)|^2=2\tau-5/2+4>0;$$

therefore if $\delta_0\leq 1/5$, then

$$|D_0(r,\psi,\delta_0)|^2\Big|_{u=1/4,\rho=1}\leq(1,24)^2-0,96=0,5776$$

and
$$|D_0(r,\psi,\delta_0)|^2\Big|_{u=1/4,\rho=1}\le 0,76.$$
In view of (49),
$$1<(\partial/\partial u)|D_0(r,\psi,\delta_0)|=$$
$$\sqrt{\frac{(u+(\rho+1)/2-(\delta_0)^2/4)^2}{(u+(\rho+1)/2-\frac{(\delta_0)^2}{4})^2-(\rho+1)^2(1-(\delta_0)^2)/4}}=1+O(1/u^2),$$
in view of (40), (41) and (44),

(51) $$(\partial/\partial u)p_\epsilon=8(2+O(1/u^2))/(1+\delta_0)^2,$$

where $\epsilon^3=\epsilon$, and $(\partial/\partial u)|D_0(r,\psi,\delta_0)|$ decreases with increasing u; consequently,
$$(\partial/\partial u)^2|D_0(r,\psi,\delta_0)|<0,$$
if $u\ge 1/4$. In view of (40), (41) and (44),

(52) $$(\partial/\partial u)^2p_\epsilon=(\partial/\partial u)^2|D_0(r,\psi,\delta_0)|<0,$$

where $u\ge 1/4$, $0<\delta_0<2/3<\rho$, $\epsilon^3=\epsilon$. In view of (41), (42), (49) and (50), if $\rho=1$, $u>1/4$, $0<\delta_0\le 1/5$, then

(53) $$q_2((\partial/\partial u)p_2)/(\partial/\partial u)q_2-p_2/2=$$
$$8u(1+(u+1-(\delta_0)^2/4))/|D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2-$$
$$4(u-\delta_0/2+(\delta_0)^2/4+|D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2=$$
$$4(u+\delta_0/2-(\delta_0)^2/4)/(1+\delta_0)^2+$$
$$4((1+\delta_0)^2|D_0(r,\psi,\delta_0)|)^{-1}(2u^2+u(2-(\delta_0)^2/2)-$$
$$4((1+\delta_0)^2|D_0(r,\psi,\delta_0)|)^{-1}(u^2+u(2-(\delta_0)^2/2)+(\delta_0/2)^2(2+(\delta_0/2)^2))=$$
$$4(u+\delta_0/2-(\delta_0)^2/4)/(1+\delta_0)^2+$$
$$4(1+\delta_0)^2|D_0(r,\psi,\delta_0)|)^{-1}(u^2-(\delta_0/2)^2(2+(\delta_0/2)^2))>0,$$
$$q_2((\partial/\partial u)p_2)/(\partial/\partial u)q_2-p_2=\frac{8}{u}(1+(u+1-(\delta_0)^2/4))/|D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2-$$
$$8(u-\delta_0/2+(\delta_0)^2/4+|D_0(r,\psi,\delta_0)|)/(1+\delta_0)^2=$$
$$8u(2+O(1/u^2))/(1+\delta_0)^2-$$
$$8(u-\frac{\delta_0}{2}+\frac{(\delta_0)^2}{4}+u+1-\frac{(\delta_0)^2}{4}+O(1/u))/(1+\delta_0)^2=$$
$$-8(1-\frac{\delta_0}{2}+O(1/u))/(1+\delta_0)^2.$$
In view of (44), (45), (53), (49), (51), (50), if $\rho=1$, $u>1/4$, $0<\delta_0\le 1/5$, then
$$(u+1)(\partial/\partial u)p_0-p_0/2>8(2u+2)/(1+\delta_0)^2-$$
$$4(u+(2+\delta_0)^2/4+(2+\delta_0)/2+u+1-(\delta_0)^2/4)/(1+\delta_0)^2=$$

$$\frac{8}{(}1/2+u-(3\delta_0)/4)/(1+\delta_0)^2>0,$$

(54) $$q_0((\partial/\partial u)p_0)/(\partial/\partial u)q_0-p_0/2=(u+2)(\partial/\partial u)p_0-p_0/2>$$

$$(u+1)(\partial/\partial u)p_0-p_0/2>0,$$

(55)
$$q_0(\partial/\partial u)p_0)/(\partial/\partial u)q_0-p_0=8(u+2)(2+O(1/u^2))/(1+\delta_0)^2-$$

$$8(u+(2+\delta_0)^2/4+(2+\delta_0)/2+u+1-(\delta_0)^2/4)/(1+\delta_0)^2=$$

$$8(4+O(1/u))/(1+\delta_0)^2-(2+\delta_0)^2/4-(2+\delta_0)/2-1+(\delta_0)^2/4+O(\frac{1}{/}u)=$$

$$8(1-(3/2)\delta_0+O(1/u))/(1+\delta_0)^2,$$

where $u>1/4$. In view of (45), (54) and (52),

$$(\partial/\partial u)((q_0(\partial/\partial u)p_0)/(\partial/\partial u)q_0-p_0)(\partial/(\partial u)p_0+(\partial/\partial u)q_0)=$$

$$(\partial/\partial u)(((u+2)(\partial/\partial u)p_0-p_0)(\partial/\partial u)p_0)=$$

$$((\partial/\partial u)p_0)^2+((u+2)(\partial/\partial u)^2p_0-(\partial/\partial u)p_0)(\partial/\partial u)p_0+$$

$$((u+2)(\partial/\partial u)p_0-p_0)(\partial/\partial u)^2p_0=$$

$$((u+2)(\partial/\partial u)^2p_0)(\partial/\partial u)p_0+((u+2)(\partial/(\partial u)p_0-p_0)(\partial/\partial u)^2p_0=$$

$$(2(u+2)(\partial/\partial u)p_0-p_0)(\partial/\partial u)^2p_0<0.$$

Therefore, according to (55), (51) and (45),

(56)
$$\inf\{((u+2)(\partial/\partial u)p_0-p_0)(\partial/\partial u)p_0+(\partial/\partial u)q_0\colon u\geq 1/4\}=$$

$$\lim_{u\to+\infty}((u(\partial/\partial u)p_0-p_0)(\partial/\partial u)p_0+(\partial/\partial u)q_0)=$$

$$128(1-(3/2)\delta_0)/(1+\delta_0)^4+16/(1+\delta_0)^2)>0.$$

According to the Lemma 4.17 in [59] and in view of (53), (54), (56),

(57) $$(\partial/\partial u)|\eta_0(r,\psi,\delta_0)+\epsilon|^2>0,$$

where $\epsilon^2=1$, $u>1/4$,

(58) $$(\partial/\partial u)|\eta_1(r,\psi,\delta_0)-1|^2<0,$$

where $u>1/4$. The following Lemma describes the behavior of the value $h^{\sim}(\eta_k(r,\psi,\delta_0))$ with $k^2=k$ and h^sim in (30).

Lemma 3. *If* $\Delta\geq 5$, *then*

(59) $$(\partial/\partial u)(|h^{\sim}(\eta_0(r,\psi,\delta_0))|)>0,$$

$$(\partial/\partial u)(|h^{\sim}(\eta_1(r,\psi,\delta_0))|)<0,$$

where $u\in(1/4,+\infty)$.

Proof. The inequality (59) directly follows from (46), (57) and (30). So, we must prove the inequality (30) Clearly, if $\beta < 1$, $u > 1/4$ then

$$(\partial/\partial u)(u^{3/4} + (3/4)\beta u^{-1/4}) > 0,$$

We take

$$\beta = (4/3)(\delta_0/2)^2(2 + (\delta_0)^2)/4)/(2 - (\delta_0)^2/2).$$

Then, clearly, $\beta < (\delta_0)^2 = 1/(\Delta)^2 < 1$. Therefore, in view of (40) and (49), if $\rho = 1$, then

$$p_1 u^{-1/4} = 8(1+\delta_0)^{-2}\times$$

$$(u^{3/4} + (3/4)u^{-1/4} + \sqrt{u^{3/2} + u^{1/4}(\rho + 1 - (\delta_0)^2/2)(u^{3/4} + (3/4)\beta u^{-1/4})})$$

increases together with increasing $u \in (1/4, +\infty)$, and, in view of (42),

$$|\eta_0(r,\psi,\delta_0)|^2 u^{-1/4}| = p_1 u^{-1/4}/2 + \sqrt{(p_1 u^{-1/4}/2)^2 - q_1 u^{-1/2}} \tag{60}$$

increases together with increasing $u \in (1/4, +\infty)$.

In view of (47), (42), (60), (57) and (58), if $\Delta \geq 5$, then

$$|\eta_1(r,\psi,\delta_0)|^{2(\Delta-1)}|(\eta_1(r,\psi,\delta_0))^2 - 1|^2 =$$

$$|\eta_1(r,\psi,\delta_0)|^{2(\Delta-5)}\frac{(q_1)^4}{(|\eta_0(r,\psi,\delta_0)|^2 u^{-1/4})^4}\times$$

$$\frac{16}{(1+\delta_0)^2}|\eta_0(r,\psi,\delta_0)+1|^{-2}|\eta_1(r,\psi,\delta_0)-1|^2$$

decreases together with increasing $u \in (1/4, +\infty)$. ■

Let D is bounded domain in $\mathbb{C}$ or $\mathfrak{F}$. and D^* is closure of D. Let

$$a_0^{\sim}(z), \dots, a_n^{\sim}(z) \tag{61}$$

are the functions continuous on D^* and analytic in D. Let $a_n^{\sim}(z) = 1$ for any $z \in D^*$. Let

$$T(z,\lambda) = \sum_{i=0}^{n} a_i^{\sim}(z)\lambda^k. \tag{62}$$

Let $s \in \mathbb{N}$, $n_i \in \mathbb{N} - 1$, where $i = 1, \dots, s$ and $\sum\limits_{i=1}^{s} n_i = n$. We say that polynomial $T(z,\lambda)$ has $(n_1, \dots, n_s)$-disjoint system of roots on D^*, if for any $z \in D^*$ the set of all the roots λ of the polynomial $T(z,\lambda)$ splits in s klasses $\mathfrak{K}_1(z), \dots, \mathfrak{K}_s(z)$ with following properties:

a) the sum of the multiplicities of the roots of the klass $\mathfrak{K}_i$ is equal to n_i for $i = 1, \dots, s$;

b) if $i \in [1,s] \cap \mathbb{N}$, $j \in (i,s] \cap \mathbb{N}$ and $n_i n_j \neq 0$, then the absolute value of each roots of the klass $\mathfrak{K}_i(z)$ is greater than absolute value of the each roots of the klass $\mathfrak{K}_j(z)$.

If the polynomial (62) has $(n_1, \dots, n_s)$-disjoint system of roots on D^*, then for each $i = 1, \dots, s$ we denote by $\rho^*_{i,0}(z)$ and $\rho^*_{i,1}(z)$ respectively the maximal and minimal absolute value of the roots of the klass $\mathfrak{K}_i(z)$.

Let D is bounded domain in $\mathfrak{F}$ such that $D^* \in D_3$. Let

$$F^\wedge(z,\eta) = \prod_{i=1}^{2}(\theta_0(z) - h(\eta_{i-1}(r,\psi,\delta_0))), \tag{63}$$

$$n = s = 2,\ n_1 = n_2 = 1,\ \mathfrak{K}_i(z) = \{h(\eta_{i-1}(r,\psi,\delta_0))\},$$
$$\rho_{i,0} = \rho_{i,1} = |h(\eta_{i-1}(r,\psi,\delta_0))|,$$

where $i = 1, 2$.

Lemma 4. *The polynomial $F^\wedge(z,\eta)$ in (63) has* (1, 1)*-disjoint system of roots on D^*.*

Proof. The assertion of the Lemma follows from (46) and (48). ■

Corollary. *The map* (31) *is injective for every $z \in D^*$; all the conditions of the Lemma 2 are fulfilled for the functions $f_0^*(z,\nu)$ from (20), $\alpha^*(z,\nu)$ from (18) and $\phi^*(z,\nu)$ from (19) in every $z \in D^*$; therefore for every $z \in D^*$ these functions are solutions of the difference equation of Poincaré type (32), and the polynomial (35) coincides with characteristical polynomial of this equation.* ■

Let for each $\nu \in \mathbb{N} - 1$ are given continuous on D^* functions

$$a_0(z;\nu),\ \ldots\ ,\ a_n(z,\nu), \tag{64}$$

which are analytic in D.

Let $a_n(z:\nu) = 1$ for any $z \in D^*$ and any $\nu \in \mathbb{N} - 1$. Let for any $i = 1, \ldots, n-1$ the sequence of functions $a_i(z;\nu)$ converges to $a_i^\sim(z)$ uniformly on D^*, when $\nu \to \infty$. Let us consider now the difference equation

$$a_0(z;\nu)y(\nu+0)\ + \ldots + a_n(z;\nu)y(\nu+n) = 0, \tag{65}$$

i.e. we consider a difference equation of the Poincaré type, coefficients (64) of this equation are continuous on D^* and analytic in D, and they uniformly converge to limit functions (61), when $\nu \to \infty$.

Lemma 5. *Let polynomial (62) has $(n_1, \ldots, n_s)$-disjoint system of roots on D^*. Let $y(z,\nu)$ is a solution of the equation (65), and this solution is continuous on D^* and analytic on D. Let further $i \in [1,s]\cap\mathbb{Z}$. Let us consider the set of all the $z \in D$, for which the following inequality holds*

$$\limsup_{\nu\in\mathbb{N},\,\nu\to\infty} |y(z,\nu)|^{1/\nu}) < \rho_{i,1}(z); \tag{66}$$

if this set has a limit point in D, then the inequality (66) holds in D^*.

Proof. The proof may be found in [31] (Theorem 1 and its Corollary). ■

Lemma 6. *Let D is bounded domain in $\mathfrak{F}$ such that $D^* \in D_3$. Then*

$$\limsup_{\nu\in\mathbb{N},\,\nu\to\infty} |f_0^*(z,\nu)|^{1/\nu}) < \rho_{1,1}(z) = |h^\sim(\eta_0(r,\psi,\delta_0))| \tag{67}$$

for any $z \in D^*$.

Proof. In view of (23), expanding the domain D, if necessary, we can suppose that $\{(r,\phi)\colon\ r \in [2, 3], \phi = 0\} \in D$. Making use the same arguments, as in [55], Lemma 4.2.1, we see that the inequality (67) holds for any point $z = (r, \phi) \in \{r \in [2, 3], \phi = 0\}$. According to the Lemma 5, the inequality (67) holds for any $z \in D^*$. ■

For each prime $p \in \mathbb{N}$ let v_p denotes the p-adic valuation on $\mathbb{Q}$.

Lemma 7. *Let $p \in \mathbb{N} + 2$ is a prime number,*

$$d \in \mathbb{N} - 1,\ r \in \mathbb{N} - 1,\ r < p.$$

Then

$$v_p((dp + r)!/((-p)^d d!\, r!) - 1) \geq 1.$$

Lemma 8. *Let $p \in \mathbb{N} + 2$ is a prime number, $d \in \mathbb{N} - 1$, $d_1 \in \mathbb{N} - 1$,*

$$r \in [0, p - 1] \cap \mathbb{N},\ r_1 \in [0, p - 1] \cap \mathbb{N},\ d_1 p + r_1 \leq dp + r. \tag{68}$$

Then

$$v_p\left(\binom{dp + r}{d_1 p + r_1}\right) = v_p\left(\binom{d}{d_1}\right), \tag{69}$$

if $r_1 \leq r$,

$$v_p\left(\binom{dp + r}{d_1 p + r_1}\left(\binom{d}{d_1}\binom{r}{r_1}\right)^{-1} - 1\right) \geq 1, \tag{70}$$

if $r_1 \leq r$,

$$v_p\left(\binom{dp + r}{d_1 p + r_1}\right) = 1 + v_p\left((d - d_1)\binom{d}{d_1}\right), \tag{71}$$

if $r < r_1$,

(72)

$$v_p\left((-1)^{r_1 - r - 1}\binom{dp + r}{d_1 p + r_1}\binom{r_1}{r}(r_1 - r)\left(p\binom{d}{d_1}(d - d_1)\right)^{-1} - 1\right) \geq 1,$$

Proof. Clearly, $d_1 \leq d$. If $r_1 \leq r$, then let $r_2 = r - r_1$, $d_2 = d - d_1$. On the other hand, if $r_1 > r$, then, in view of (68), $d \geq d_1 + 1$; therefore in this case we let

$$r_2 = p + r - r_1,\ d_2 = d - d - 1. \tag{73}$$

Then $d = d_1 + d_2$, $r = r_1 + r_2$,

$$\binom{dp + r}{d_1 p + r_1} = (dp + r)!((d_1 p + r_1)!(d_2 p + r_2)!)^{-1}.$$

Accordindg to the Lemma 7,

$$v_p\left(\binom{dp + r}{d_1 p + r_1}(-p)^{-d + d_1 + d_2} d_1!\, r_1!\, d_2!\, r_2!/(d!\, r!) - 1\right) \geq 1, \tag{74}$$

$$v_p\left(\binom{dp + r}{d_1 p + r_1}\right) = d - d_1 - d_2 + \tag{75}$$

$$v_p(d!\, r!/(d_1!\, r_1!\, d_2!\, r_2!)).$$

The equality (69) and the inequality (13) directly follow from (74) and (75). If the inequality $r < r_1$ holds, then in view of (73) – (75),

$$r_2!\prod_{j=1} r_1-r-1(p+r-r_1+j)=(p-1)!,\ v_p(r_2!(r_1-r-1)!(-1)^{r_1-r}-1)\geq 1,$$

and (72) holds.

Corollary 1. *Let $p\in\mathbb{N}$ is a prime number,*

$$d\in\mathbb{N}-1, r\in\mathbb{N}-1, d_1\in\mathbb{N}-1, d_2\in\mathbb{N}-1, r_1\in\mathbb{N}-1, r_2\in\mathbb{N}-1,$$

$$max(r_1,r_2)<p.$$

Then

$$p^{-d}(dp+r)!\in(-1)^d d!r!+p\mathbb{Z},$$
$$\binom{(d_1+d_2)p+r_1+r_2}{d_1p+r_1}\in\binom{d_1+d_2}{d_1}\binom{r_1+r_2}{r_1}+p\mathbb{Z}.$$

Proof. This is direct corollary of the Lemma 7 and Lemma 8. See also Lemma 9 in [54]. ■

Corolary 2. *Let $p\in\mathbb{N}+2$ is a prime number,*

$$d\in\mathbb{N},\ r_1\in\mathbb{N},\ r_1<p,\ d_1\in\mathbb{N}-1,\ d_1<d.$$

Then

$$v_p\left(\binom{dp}{d_1p+r_1}\left(d\binom{d-1}{d_1}\binom{p}{r_1}\right)^{-1}+1\right)\geq 1 \tag{76}$$

Proof. Since,

$$d\binom{d-1}{d_1}=(d-d_1)\binom{d}{d_1},\ v_p\left(\binom{p}{r_1}r_1/p-(-1)^{r_1}\right)\geq 1,$$

the equality (76) directly follows from (72). ■

Corolary 3. *Let $p\in\mathbb{N}+2$ is a prime number,*

$$d\in\mathbb{N},\ r_1\in\mathbb{N},\ r_1<p,\ d^{\sim}\in\mathbb{N}-1,\ d^{\sim}<d.$$

Then

$$\binom{dp}{d_1p+r_1}\in d\binom{d-1}{d^{\sim}}\binom{p}{r_1}+p^2\mathbb{Z}.$$

Proof. This is a corollary of the Corrolary 2. See also Lemma 10 in [54] .■

Let let p be prime in $(2,+\infty)$, let K be a finite extension of $\mathbb{Q}$ let $\mathfrak{p}$ be a prime ideal in $\mathbb{Z}_K$ and $p\in\mathfrak{p}$, let f be the degree of $\mathfrak{p}$, let $(p)=\mathfrak{p}^e\mathfrak{b}$, with entire ideal $\mathfrak{b}$ not contained in $\mathfrak{p}$, let $v_{\mathfrak{p}}$ be additive $\mathfrak{p}$-valuation, which prolongs v_p; so, if π is a $\mathfrak{p}$-prime number, then $v_{\mathfrak{p}}(\pi)=1/e$. If f is the degree of the ideal $\mathfrak{p}$ then

$$v_{\mathfrak{p}}\left(w^{p^\beta}-w\right)\geq 1, \tag{77}$$

where $\beta\in\mathbb{N}f$, $w\in K$ and

$$v_{\mathfrak{p}}(w)\geq 0.$$

In viw of (77), (18), and (16),

$$v_{\mathfrak{p}}(\alpha^*(z;\, p^\beta l) - \alpha^*(z;\, l)) > 1/e,$$

if $\beta \in \mathbb{N}f$, $\theta_0(z) \in K$ and $v_{\mathfrak{p}}(\theta_0(z)) \geq 0$. In view of (19),

(78) $$\phi^*(z;\nu) = (-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) =$$

$$(-\theta_0(z))^\nu \sum_{\tau=1}^{\nu}((\theta_0(z))^{-\tau}\alpha^*(z;\nu)/\tau +$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) =$$

$$(-1)^\nu \sum_{\tau=1}^{\nu(\Delta+1)} \frac{1}{\tau} \sum_{k=\max(0,\,\tau-\nu)}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^{\nu-\tau+k};$$

therefore, if $\nu = p^\beta l$, $f = 1$, $\beta \in \mathbb{N}f$, $p > l(\Delta+1)$, $\theta_0(z) \in K$ and $v_{\mathfrak{p}}(\theta_0(z)) \geq 0$, then, according to the Lemma 2,

(79) $$1 - \beta \leq$$

$$v_{\mathfrak{p}}\left(\phi^*(z;\nu) - \sum_{\substack{\eta\in[1,\,\Delta+1]\cap\mathbb{Z}\\ k\in[p^\beta(\eta-l),\,p^\beta l\Delta]\cap\mathbb{Z}\\ k\geq 0,\,v_{\mathfrak{p}}(k)>0}} \frac{(-1)^{pl}}{p^\beta\eta}(\theta_0(z))^{p^\beta(l-\eta)+k}\alpha^*_{\nu,k}\right),$$

(80) $$1/e - \beta \leq$$

$$v_{\mathfrak{p}}\left(\phi^*(z;\nu) - \sum_{\substack{\eta\in[1,\,\Delta+1]\cap\mathbb{Z}\\ k\in[p^{\beta-1}(\eta-l),\,p^{\beta-1}l\Delta]\cap\mathbb{Z}\\ k\geq 0}} \frac{(-1)^{pl}p}{p^\beta\eta}(\theta_0(z))^{p^{\beta-1}(l-\eta)+k}\alpha^*_{\nu/p,k}\right).$$

We make the pass (79) $\to$ (80) β times and obtain the inequality

(81) $$1/e - \beta \leq$$

$$v_{\mathfrak{p}}\left(\phi^*(z;\, p^\beta l) - p^{-\beta}\phi^*(z;\, l)\right),$$

where

$$\{l,\, \beta\} \subset \mathbb{N},\, p > l(\Delta+1),\, p \in \mathfrak{p}$$

and $\mathfrak{p}$ is ideal of the first degree.

Lemma 9. *If* $m \in \mathbb{N}+1$, $K = \mathbb{Q}[\exp(2pi/m)]$,

$$\alpha^*(z;\, l_1)\phi^*(z;\, l_2)) \neq 0$$

for some $z \in K \setminus \{0\}$, $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}$, *then for any* $l \in \mathbb{N}$ *the sequenses*

$$\alpha^*(z;\,\nu),\ \phi^*(z;\,\nu), \tag{82}$$

where $\nu \in l + \mathbb{N}$ *form a linear independent system over* K.

Proof. There exists $d^* \in \mathbb{N}$ such that

$$d^*z \in \mathbb{Z}_K,\ d^*z\alpha^*(z;\,l_1) \in \mathbb{Z}_K,\ d^*z\phi^*(z;\,l_2) \in \mathbb{Z}_K.$$

Let a prime $p \in \mathbb{N}m + 1$ satisfies to the inequality

$$p > |Nm_{K/\mathbb{Q}}(d^*z\alpha^*(z;\,\nu))| + |Nm_{K/\mathbb{Q}}(d^*z\phi^*(z;\,\nu))+$$

$$|Nm_{K/\mathbb{Q}}(d^*z)| + |Nm_{K/\mathbb{Q}}(d^*) + (\Delta+1)(l_1+l_2).$$

Let $\mathfrak{p}$ is a prime ideal containing p. Then

$$v_{\mathfrak{p}}\left(\alpha^*(z;\,l_1)\right) = v_{\mathfrak{p}}\left(\phi^*(z;\,l_2)\right) = 0,$$

and, in view of (81),

$$v_{\mathfrak{p}}\left(\phi^*(z;\,p^{\beta}l_2)\right) = -\beta,$$

but

$$v_{\mathfrak{p}}\left(\alpha^*(z;\,p^{\beta}l_1)\right) = 0.$$

with $\beta \in \mathbb{N}$ ■.

Let $m \in \mathbb{N}$, $k \in \mathbb{Z}$, $2 \le 2|k| < m$, and let m and k have no common divisor with exeption ± 1. Let further $K_m = \mathbb{Q}[\exp(2\pi i/m)]$ is a cyclotomic field, $\mathbb{Z}_{K_m}$ is the ring of all the integers of the field K_m.

Lemma 10. *Let* $\Delta \in \{5,\,7\}$. *In correspondece with (21), (22) and (23), let* $z = (1/(2\cos(k\pi i/m), k\pi i/m - \pi)$, *where* $|k| < m/2$, $(|k|, m) = 1$.

Then for each $l \in \mathbb{N}$ *the two sequences (82) form a linear independent system over* $\mathbb{C}$.

Proof. We check the fulfilment of the conditions of the Lemma 9.

Let $\mathfrak{M} = \mathbb{N}\setminus\{1,\,2,\,6\}$ and $\mathfrak{M}_0 = \{m \in \mathfrak{M}\colon \Lambda_0(m) = 0\}$. According to the condition of the Lemma, $\theta_0(z) = -1/(1+\exp(2i\pi/m)$ with $m \in \mathfrak{M}$. If $m \in \mathfrak{M}$ and $\phi(m) > \Delta$, then, in view of (18) and (16), $\alpha^*(z;1) \neq 0$, because the numbers $(1+\exp(2i\pi/m)^k$, where $k = 0,\,\ldots\,,\,\phi(m)-1$, form a basis of the field K_m. Let $\Delta = p \in 2\mathbb{N}+1$, where p is a prime, $\mathfrak{p}$ is a prime ideal containing p, and, as before, let $(p) = \mathfrak{b}\mathfrak{p}^e$, $1_{K_m} \in \mathfrak{b} + \mathfrak{p}$. Then

$$\binom{2p-1}{p}\binom{p}{p-1} \equiv p \mod p^2,\ v_{\mathfrak{p}}\left(\binom{p+k}{1+k}\binom{p}{k}\right) = 2, \tag{83}$$

where $k = 1,\,\ldots\,,\,p-2$,

$$\binom{p}{1}\binom{p}{0} = p,\ \binom{2p}{p+1}\binom{p}{p} \equiv 2p \mod p^2. \tag{84}$$

If $m \in \mathfrak{M}$ and $(m,p) = 1$, or, if $m \in \mathfrak{M}_0$, then, according to the Lemma 1,

$$(1+\exp(2i\pi/m), p) = (1) \tag{85}$$

and, according to the Lemmata 7 and 8,

(86) $$\alpha^*(z;1)/(p\theta_0(z)) \equiv 1 + (\theta_0(z))^{p-1} - 2(\theta_0(z))^p \equiv$$
$$1 + (\exp(2i\pi/m) + 3)/(1 + \exp(2pi\pi/m)) \equiv$$
$$(\exp(2ip\pi/m) + \exp(2i\pi/m) + 4)/(1 + \exp(2pi\pi/m)) \mod p.$$

If $m = q^\alpha$ with $\alpha \in \mathbb{N}$ and prime q and there exists l in $\{0, \ldots, \phi(m) - 1\}$ such that $p \equiv l \mod (m)$, then

(87) $$\exp(2ip\pi/m) + \exp(2i\pi/m) + 4 \not\equiv 0 \mod p.$$

If $m = 2q^\alpha$ with odd prime q and $\alpha \in \mathbb{N}$, and there exist l in $\{0, \ldots, \phi(m/2)-1\}$ such that $p \equiv 2l \mod (m/2)$, then (87) holds.

If $p = 5$, then $\{3, 4, 5\ 8, 10, 12\} = \{m \in \mathfrak{M}\colon \phi(\mathfrak{m}) \leq \mathfrak{p}\}$.

If $m = 3, 4, 5, 8, 10$ then, clearly, (87) holds.

If $m = 12$, then $1, \exp(i\pi/2), \exp(2i\pi/3), \exp(i\pi/6),$ form a entire basis of K_{12}, $\exp(5i\pi/6) = \exp(i\pi/2) - \exp(i\pi/6),$ and (87) holds.

If $p = 7$ then $\{3, 4, 5 7, 8, 9, 10, 12, 14, 18\} = \{m \in \mathfrak{M}\colon \phi(\mathfrak{m}) \leq \mathfrak{p}$.

If $m = 3, 4, 5, 7, 9, 14$, then, clearly, (87) holds.

If $m = 8$, then $\exp(7i\pi/4) = -\exp(3i\pi/4)$ and (87) holds.

If $m = 12$, then $1, \exp(i\pi/2), \exp(2i\pi/3), \exp(i\pi/6),$ form a entire basis of K_{12}, $\exp(7i\pi/6) = -\exp(i\pi/6),$ and (87) holds.

If $m = 18$, then

$$\exp(7i\pi/9) = -\exp(-2i\pi/9) = \exp(4i\pi/9) + \exp(10i\pi/9),$$

and (87) holds.

The coefficient at $(\theta_0(z))^0$ in the expression (19) of $\phi^*(z;\nu)$ is equal to

$$\sum_{k=0}^{\nu\Delta}(-1)^\nu \alpha^*_{\nu,k}/(\nu + k)$$

and, if $\Delta = p$, $\nu = 1$, then in view (83) – (84), the value of v_p on this coefficient is equal to 0. Therefore, if $m \in \mathfrak{M}$ and $\phi(m) > p = \Delta$, then $\phi^*(z;1) \neq 0$.

If $m \in \mathfrak{M} \diagdown \mathfrak{M}_0$, and $m \equiv 0 \mod p$ then $m = 2p^\alpha$, where $\alpha \in \mathbb{N}$. According to the Lemma 1, $\mathfrak{p} = (1 + \exp(2i\pi/m)$ is a prime ideal in K_m, and, furthermore, $\mathfrak{p}^{\phi(m)} = (p)$. Let $v_\mathfrak{p}$. is the $\mathfrak{p}$-adic valuation, which prolongs the valuation v_p. Clearly, $v_\mathfrak{p}(1 + \exp(2i\pi/m) = 1/\phi(m)$, $v_\mathfrak{p}(\theta_0(z)) = -1/\phi(m)$ In view of (19) with $\nu = 1$, for the summands of the sum

$$\sum_{k=1}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^{1+k} \sum_{\tau=2}^{1+k}((\theta_0(z))^{-\tau}/\tau))$$

we have the inequality

$$v_\mathfrak{p}((\theta_0(z))^{\Delta+k-y\alpha^*_{\nu,k}/y} \geq -(k-1)/\phi(m) + 2 - v_\mathfrak{p}(\tau) \geq -(p-3)/\phi(m) + 2,$$

if $k = 1, \ldots, p-2$, because in this case $\tau \in [2, p-1]$,

$$v_p((\theta_0(z))^{\Delta+k-y\alpha^*_{\nu,k}/y} \geq -(k)/\phi(m) + 1 - v_p(\tau) \geq -(p-1)/\phi(m),$$

where $k \in \{q-1, q\}$, and the equality reaches only for $k = \tau = p$; on the other hand, $v_{\mathfrak{p}}(\alpha^*(z;1)) \geq 1 - (p+1)/\phi(m) \geq -2/(p-1) \geq -2/\phi(m)$. So, if $p \geq 5$, then $v_{\mathfrak{p}}(\phi^*(z;1)) = -(p-1)/\phi(m)$. If $m \in \mathfrak{M} \setminus \mathfrak{M}_{\mathfrak{o}}$, then $m = 2q^\alpha$, with prime q, according to the Lemma 1, $\mathfrak{l} = (1 + \exp(2i\pi/m))$ is a prime ideal in K_m, and $\mathfrak{l}^{\phi(m)} = (q)$. Therefore in this case $v_{\mathfrak{p}}(\theta_0(z) = 0$ If $m \in \mathfrak{M}_0$, then, according to the Lemma 1, $v_{\mathfrak{p}}(\theta_0(z) = 0$. According to (19), in both last cases,

$$v_{\mathfrak{p}}(\phi^*(z;1)) + \alpha_{\nu,p-1}/p + \theta_0(z)\alpha_{\nu,p}/p) \geq 1.$$

In view of (83), (84),

$$v_{\mathfrak{p}}(\alpha_{\nu,p-1}/p + \theta_0(z)\alpha_{\nu,p}/p) =$$

$$v_{\mathfrak{p}}(exp(2i\pi/m) - 1)/(exp(2i\pi/m) + 1)).$$

If $p = 5$ and $m \in \{3, 4, 5, 7, 8, 9, 10\}$ then, clearly,

$$v_{\mathfrak{p}}((exp(2i\pi/m) - 1)) \leq 1/4. \tag{88}$$

If $p = 5$ and $m = 12$, then $Nm_{K_{12}}((exp(i\pi/6) - 1)) = 3$ and (88) holds.

If $p = 7$, and $m \in \{3, 4, 5, 7, 8, 9, 10, 12, 14, 18\}$, then

$$v_{\mathfrak{p}}((exp(2i\pi/m) - 1)) \leq 1/6.$$

■

Lemma 11. *Let are fulfilled all the conditions of the Lemma 10. Then*

$$\limsup_{\nu\in\mathbb{N},\,\nu\to\infty}\left(|f_0^*(z,\,\nu)|^{1/\nu} = \rho_{2,1}(z)\Big|_{\theta_0(z)=-1/(1+\exp(2ik\pi/m))}\right) = \tag{89}$$

$$|h^\sim(\eta_1(1/(2\cos(k\pi i/m)),\, k\pi i/m, \delta_0))|,$$

where $h^\sim(\eta)$ is defined in (30).

Proof. According to the Lemma 2, (20) and Lemma 10, $f_0^*(z,\,\nu)$ is a nonzero solution of the Poincaré type difference equation (32). According to the Perron's theorem and Lemma 5, the equality (89) holds. ■

Let $K/\mathbb{Q}$ be the finite extension of the field $\mathbb{Q}$,

$$[K:\mathbb{Q}] = d.$$

Let the field K has r_1 real places and r_2 complex places. Each such place is the monomorphism of the field K in the field $\mathbb{R}$, if a place is real, or in the field $\mathbb{C}$, if a place is not real; we will denote these monomorphisms respectively by $\sigma_1, \ldots \sigma_{r_1+r_2}$. Then $d = r_1 + 2r_2$. Let $\mathfrak{B}$ be the fixed integer basis

$$\omega_1, \ldots, \omega_d$$

of the field K over $\mathbb{Q}$. Clearly,K is an algebra over $\mathbb{Q}$. With extension of the ground field from $\mathbb{Q}$ to $\mathbb{R}$ appears an isomorphism of the algebra $\mathfrak{K} = K \otimes \mathbb{R}$ onto direct sum

$$\underbrace{\mathbb{R}\oplus\ldots\oplus\mathbb{R}}_{r_1 \text{ times}}\oplus\underbrace{\mathbb{C}\oplus\ldots\oplus\mathbb{C}}_{r_2 \text{ times}}$$

of r_1 copies of the field $\mathbb{R}$ and r_2 copies of the field $\mathbb{C}$. We identify by means of this isomorphism the algebra $\mathfrak{K}$ with the specified direct sum. We denote below by π_j, where $j = 1, \dots, r_1 + r_2$, the projection of $\mathfrak{K}$ onto its $j-$th direct summand and also the extension of this projection onto all kinds of matrices which have all the elements in $\mathfrak{K}$. So, $\pi_j(\mathfrak{K}) = \mathbb{R}$ for $j = 1, \dots, r_1$ and $\pi_j(\mathfrak{K}) = \mathbb{C}$ for $j = r_1 + 1, \dots, r_1 + r_2$. Further by $\mathfrak{i}_{\mathfrak{K}}$ we denote the embedding of $\mathbb{R}$ in $\mathfrak{K}$ in diagonal way and also the extension of this embedding onto all kinds of the real matrices. So, $\mathbb{R}$ is imbedded by means of $\mathfrak{i}_{\mathfrak{K}}$ in $\mathfrak{K}$ in diagonal way. Each element $Z \in \mathfrak{K}$ has a unique representation in the form:

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_{r_1+r_2} \\ \overline{z_{r_1+1}} \\ \vdots \\ \overline{z_{r_1+r_2}} \end{pmatrix},$$

with $z_j = \pi_j(Z) \in \mathbb{R}$ for any $j = 1, \dots, r_1$ and with $z_j = \pi_j(Z) \in \mathbb{C}$ for any $j = r_1 + 1, \dots, r_1 + r_2$. Further by $Tr_{\mathfrak{K}}(Z)$ we denote the sum

$$\sum_{j=1}^{r_1} z_j + \sum_{j=r_1+1}^{r_1+r_2} 2\Re(z_j) =$$

$$\sum_{j=1}^{r_1} \pi_j(Z) + \sum_{j=r_1+1}^{r_1+r_2} 2\Re(\pi_j(Z)),$$

and by $q_\infty^{(\mathfrak{K})}(Z)$ we denote the value

$$\max(|z_1|, \dots, |z_{r_1+r_2}|) =$$

$$\max(|\pi_1(Z)|, \dots, |\pi_{r_1+r_2}(Z)|).$$

Clearly,

$$q_\infty^{(\mathfrak{K})}(Z_1 Z_2) \le q_\infty^{(\mathfrak{K})}(Z_1) q_\infty^{(\mathfrak{K})}(Z_2),$$
$$q_\infty^{(\mathfrak{K})}(Z_1 + Z_2) \le q_\infty^{(\mathfrak{K})}(Z_1) + q_\infty^{(\mathfrak{K})}(Z_2),$$
$$q_\infty^{(\mathfrak{K})}(\mathfrak{i}_{\mathfrak{K}}(\lambda) Z) = |\lambda| q_\infty^{(\mathfrak{K})}(Z)$$

for any $Z_1 \in \mathfrak{K}$, $Z_2 \in \mathfrak{K}$, $Z \in \mathfrak{K}$ and $\lambda \in \mathbb{R}$. The natural extension of the norm $q_\infty^{(\mathfrak{K})}$ on the set of all the matrices, which have all the elements in $\mathfrak{K}$ (i.e. the maximum of the norm $q_\infty^{(\mathfrak{K})}$ of all the elements of the matrix) also will be denoted by $q_\infty^{(\mathfrak{K})}$. If

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} \in K,$$

then

$$z_j = \sigma_j(Z),$$

where $j = 1\,, \dots\,, r_1 + r_2$,

$$z_{r_1+r_2+j} = \overline{\sigma_{r_1+j}(Z)},$$

where $j = 1\,, \dots\,, r_2$. In particular,

$$\omega_k = \begin{pmatrix} \sigma_1(\omega_k) \\ \vdots \\ \sigma_{r_1+r_2}(\omega_k) \\ \overline{\sigma_{r_1+1}(\omega_k)} \\ \vdots \\ \overline{\sigma_{r_1+r_2}(\omega_k)} \end{pmatrix},$$

As usually, the ring of all the integer elements of the field K will be denoted by $\mathbb{Z}_K$. The ring $\mathbb{Z}_K$ is embedded in the ring $\mathfrak{K}$ as discrete lattice. Moreover, if $Z \in \mathbb{Z}_K \backslash \{0\}$, then

$$\left(\prod_{i=1}^{r_1} |\sigma_j(Z)|\right) \prod_{i=1}^{r_2} |\sigma_{r_1+i}(Z)|^2 = |Nm_{K/\mathbb{Q}}(Z)| \in \mathbb{N}$$

and therefore $q_\infty^{(\mathfrak{K})}(Z) \geq 1$. for any $Z \in \mathbb{Z}_K \backslash \{0\}$. The elements of $\mathbb{Z}_K$ we name below by K-integers. For each $Z \in \mathfrak{K}$ let

$$\|\mathbb{Z}\|_K = \inf_{W \in \mathbb{Z}_K} \{q_\infty^{(\mathfrak{K})}(Z - W)\}.$$

Let $\{m, n\} \subset \mathbb{N}$,

$$a_{i,k} \in \mathfrak{K}$$

for $i = 1\,, \dots\,, m,\; k = 1\,, \dots\,, n$,

$$\alpha_j^\wedge(\nu) \in \mathbb{Z}_K,$$

where $j = 1\,, \dots\,, m + n$ and $\nu \in \mathbb{N}$. Let there are $\gamma_0, r_1^\wedge \geq 1, \dots, r_m^\wedge \geq 1$ such that

$$q_\infty^{(\mathfrak{K})}(\alpha_i(\nu)) < \gamma_0 (r_i^\wedge)^\nu$$

where $i = 1\,, \dots\,, m$ and $\nu \in \mathbb{N}$. Let

$$y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum_{i=1}^{m} a_{i,k} \alpha_i^\wedge(\nu)$$

where $k = 1\,, \dots\,, n$ and $\nu \in \mathbb{N}$. If $X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathfrak{K}^n$, then let

$$y^\wedge(X) = y^\wedge(X, \nu) = \sum_{k=1}^{n} y_k^\wedge(\nu) Z_k$$

for $\nu \in \mathbb{N}$, let

$$\phi_i(X) = \sum_{k=1}^{n} a_{i,k} Z_k$$

for $i = 1, \dots, m$, and let

$$\alpha_0^{\wedge}(X, \nu) = \sum_{k=1}^{n} \alpha_{m+k}^{\wedge}(\nu) Z_k$$

for $\nu \in \mathbb{N}$. Clearly,

$$y^{\wedge}(X, \nu) = -\alpha_0^{\wedge}(X, \nu) + \sum_{i=1}^{m} \alpha_i^{\wedge}(\nu)\phi_i(X)$$

for $X \in \mathfrak{K}^n$ and $\nu \in \mathbb{N}$,

$$\alpha_0^{\wedge}(X, \nu) \in \mathbb{Z}_{\mathbb{K}}$$

for $X \in (\mathbb{Z}_K)^n$ and $\nu \in \mathbb{N}$.

Lemma 12. *Let* $\{l, n\} \subset \mathbb{N}$, $\gamma_1 > 0$, $\gamma_2 > \frac{1}{2}$, $R_1 \geq R_2 > 1$,

$$\alpha_i = (\log(r_i^{\wedge} R_1/R_2))/\log(R_2),$$

where $i = 1, \dots, m$, *let* $X \in (\mathbb{Z}_K)^n \backslash \{(0)\}$,

$$\gamma_3 = \gamma_1(R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \gamma_4 = \gamma_3 \left(\sum_{i=1}^{m} \gamma_0 (r_i^{\wedge})^{(log(2\gamma_2))/\log(R_2)+l} \right)^{-1}$$

and let for each $\nu \in \mathbb{N} - 1$ *hold the inequalities*

$$\gamma_1(R_1)^{-\nu} q_\infty^{(\mathfrak{K})}(X) \leq \sup\{q_\infty^{(\mathfrak{K})}(y^{\wedge}(X, \kappa)) \colon \kappa = \nu, \dots, \nu + l - 1\},$$

$$q_\infty^{(\mathfrak{K})}(y^{\wedge}(X, \nu)) \leq \gamma_2(R_2)^{-\nu} q_\infty^{(\mathfrak{K})}(X)$$

Then

$$\sup\{\|\phi_i(X)\|_K (q_\infty^{(\mathfrak{K})}(X))^{\alpha_i} \colon i = 1, \dots, m\} \geq \gamma_4.$$

Proof. Proof may be found in [56], Theorem 2.3.1. ■

Corollary. *Let* $a \in \mathfrak{K}$,

$$\alpha_1^{\wedge}(\nu) \in \mathbb{Z}_K,\ \alpha_2^{\wedge}(\nu) \in \mathbb{Z}_K, y(\nu) = -\alpha_2^{\wedge}(\nu) + a\alpha_1^{\wedge}(\nu) \tag{90}$$

where $\nu \in \mathbb{N}$. *Let there are* $\gamma_0, r_1^{\wedge} \geq 1$ *such that*

$$q_\infty^{(\mathfrak{K})}(\alpha_1(\nu)) < \gamma_0 (r_1^{\wedge})^{\nu},$$

where $\nu \in \mathbb{N}$. *Let* $l \in \mathbb{N}$, $\gamma_1 > 0$, $\gamma_2 > \frac{1}{2}$, $R_1 \geq R_2 > 1$,

$$\alpha_1 = (\log(r_1^{\wedge} R_1/R_2))/\log(R_2),\ \gamma_3 = \gamma_1(R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)},$$

$$\gamma_4 = \gamma_3 \left(\gamma_0 (r_1^{\wedge})^{(log(2\gamma_2))/\log(R_2)+l}\right)^{-1},$$

$X \in \mathbb{Z}_K$ *and let for each* $\nu \in \mathbb{N} - 1$ *hold the inequalities*

$$\gamma_1(R_1)^{-\nu} q_\infty^{(\mathfrak{K})}(X) \leq \sup\{q_\infty^{(\mathfrak{K})}(y_1(\kappa)X) \colon \kappa = \nu, \dots, \nu + l - 1))\},$$

$$q_\infty^{(\mathfrak{K})}(y(\nu)X) \le \gamma_2(R_2)^{-\nu} q_\infty^{(\mathfrak{K})}(X)$$

Then

$$\|aX\|_K (q_\infty^{(\mathfrak{K})}(X))^\alpha \ge \gamma_4. \tag{91}$$

Proof. This Corrolary is the Lemma 12 for $m = n = 1$. ■

Let $B \in \mathbb{N}$, $D^*(B) = \inf\{q \in \mathbb{N}\colon d/\kappa \in \mathbb{N}, \kappa \in \mathbb{N}, \kappa \le B\}$. It is known that

$$D^*(B) = \exp(B + O(B/\log(B)).$$

Let $d_0^*(\Delta,\nu) = D^*(\nu(\Delta+1))$. Then

$$d_0^*(\Delta,\nu) = \exp(\nu(\Delta+1) + O(\nu/\log(\nu))), \tag{92}$$

when $\nu \to \infty$.

Probably G.V. Chudnovsky was the first man, who discovered, that the numbers (16) have a great common divisor; Hata ([17]) in details studied this effect. Therefore I name the mentioned common divisor by Chudnovsky-Hata's multiplier and denote it by $d_1^*(\Delta,\nu)$. According to the Hata's results,

$$\log(d_1^*(\Delta,\nu)) = (1+o(1))\nu \times \tag{93}$$

$$\sum_{\mu=0}^{1}\left(\frac{\Delta+(-1)^\mu}{2}\log\left(\frac{\Delta}{\Delta+(-1)^\mu}\right) + (-1)^\mu\frac{\pi}{2}\sum_{\kappa=1}^{\left[\frac{\Delta+(-1)^\mu}{2}\right]}\cot\left(\frac{\pi\kappa}{\Delta+(-1)^\mu}\right)\right).$$

In view of (92),

$$d_0^*(5,\nu) = \exp(6\nu(\Delta+1) + O(\nu/\log(\nu))), d_0^*(7,\nu) = \tag{94}$$

$$\exp(8\nu(8) + O(\nu/\log(\nu))).$$

In view of (94)

$$\log(d_1^*(5,\nu)) = (1+o(1))\nu \times \tag{95}$$

$$(-3\log(1.2) + 2\log(0.8) + (\pi/2)(\cot(\pi/6) + \cot(\pi/3) + \cot(\pi/4))) =$$

$$(1+o(1))\nu \times 1.956124...,$$

$$\log(d_1^*(7,\nu)) = (1+o(1))\nu \times \tag{96}$$

$$(4\log(7/8) + 3\log(7/6)) + (1+o(1))(\pi/2)\nu\times$$

$$(-\cot(\pi/6) - \cot(\pi/3) + \cot(\pi/8) + \cot(3\pi/8) + \cot(\pi/4)) =$$

$$(1+o(1))\nu(4\log(7/8) + 3\log(7/6) + \pi(-2/\sqrt{3} + 2/\sqrt{2} + 1/2) =$$

$$(1+o(1))\nu \times 2.314407\ldots,$$

when $\nu \to \infty$.

In view of (18) and (19),

$$\alpha^*(z;\nu)d_0^*(\nu)/d_1^*(\nu) \in \mathbb{Z}[z],$$

$$\phi^*(z;\nu)d_0^*(\nu)/d_1^*(\nu) \in \mathbb{Z}[z].$$

Let

(97) $$U_\Delta(m,\nu)=d_0^*(\nu)/d_1^*(\nu),\ \Lambda_0(m)=0,$$

if $m\neq 2p^\alpha$, where p run over the all the prime numbers and α run over $\mathbb{N}$ and let

(98) $$U_\Delta(m,\nu)=\frac{d_0^*(\nu)}{d_1^*(\nu)}p^{[(\Delta+1)\nu/\phi(m)]+1},\ \Lambda_0(m)=\Lambda(m/2),$$

if $m=2p^\alpha$, where p is a prime number and $\alpha\in\mathbb{N}$. In view of the (18), (19) and Lemma 1,

(99) $$\alpha^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k\pi i}{m})},\,\frac{k\pi i}{m}-\pi\right)}U_\Delta(m,\nu)\in\mathbb{Z}_{\mathbb{Q}[\exp(2i\pi/m)]},$$

(100) $$\phi^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k\pi i}{m})},\,\frac{k\pi i}{m}-\pi\right)}U_\Delta(m,\nu)\in\mathbb{Z}_{\mathbb{Q}[\exp(2i\pi/m)]},$$

where $(k,m)=1$. In view of (98), (97), (93), (92), (2) and (3)

(101) $$\frac{d_0^*(\nu)}{d_1^*(\nu)}=$$

$$\nu(1+o(1))V_\Delta^*\log(U_\Delta(m,\nu))=\nu(1+o(1))V_\Delta(m),$$

when $\nu\to\infty$.

The polynomial (28) take the form

$$D^\wedge(z,\eta)=(\eta+1)\left(\eta+\frac{\Delta-1}{\Delta+1}\right)+\frac{2\Delta\exp(i\psi)\eta}{(\Delta+1)\cos(\psi)}=$$

$$((\Delta+1)\eta^2+2\Delta(2+iT)\eta+(\Delta-1))/(\Delta+1),$$

where $\psi\in(-pi/2,\,\pi/2)$ and $T=\tan(\psi)$; its roots are equal to

(102) $$-(2\Delta+\Delta iT+R)/(\Delta+1),$$

where $R^2=\Delta^2(3-T^2)+1+4\Delta^2 iT$. In view of (1), Then

$$R\in\left\{\pm\left(w_\Delta(T)+i2\Delta^2 iT/w_\Delta(T)\right\}\right).$$

In view of (102) and (46),

$$\eta_j^\wedge(r,\psi,\delta_0)=$$

$$-\frac{2\Delta+\Delta iT+(-1)^j\left(w_\Delta(T)+i2\Delta^2 iT/w_\Delta(T)\right)}{\Delta+1}=$$

$$-\frac{2\Delta+(-1)^j w_\Delta(T)+iT\Delta\left(1+(-1)^j 2\Delta/w_\Delta(T)\right)}{\Delta+1},$$

where $j=0,1$,

$$|\eta_j^\wedge(r,\psi,\delta_0)+k|^2=$$

$$\frac{\left(2\Delta+(-1)^jw_\Delta(T)-k(\Delta+1)\right)^2+T^2\Delta^2\left(1+(-1)^j2\Delta/w_\Delta(T)\right)^2}{(\Delta+1)^2},$$

where $j=0,1$; $k=0,\ 1,\ -1$. Therefore, in view of (30) and (4)

(103) $$\ln|h^{\sim}(\eta_j^{\wedge}(r,\psi,\delta_0))|=$$

$$(\eta_j(r,\psi,\delta_0)-1)(1-\delta_0)^{-d_1}(\eta_j(r,\psi,\delta_0)+1)2^{-2}\eta_j(r,\psi,\delta_0)^{d_1}=$$

$$-log\left(4(\Delta+1)^{\Delta+1}(1-1/\Delta)^(\Delta-1)\right)+$$

$$\frac{1}{2}\log\left(\left(2\Delta+(-1)^jw_\Delta(T)+(\Delta+1)\right)^2+T^2\Delta^2\left(1+\frac{(-1)^j2\Delta}{w_\Delta(T)}\right)^2\right)+$$

$$\frac{1}{2}\log\left(\left(2\Delta+(-1)^jw_\Delta(T)-(\Delta+1)\right)^2+T^2\Delta^2\left(1+\frac{(-1)^j2\Delta}{w_\Delta(T)}\right)^2\right)+$$

$$\frac{(\Delta-1)}{2}\log\left(\left(2\Delta+(-1)^jw_\Delta(T)\right)^2+T^2\Delta^2\left(1+\frac{(-1)^j2\Delta}{w_\Delta(T)}\right)^2\right)=$$

$$l_\Delta(j,T),$$

where $j=0,1$. Clearly,

$$w_\Delta(0)=\sqrt{3\Delta^2+1},$$

$$\eta_j^{\wedge}(1/2,0,\delta_0)=-\frac{2\Delta+(-1)^j\sqrt{3\Delta^2+1}}{\Delta+1},$$

where $j=0,1$,

$$\left|\eta_j^{\wedge}(1/2,0,\delta_0)+k\right|=\left|\frac{2\Delta+(-1)^j\sqrt{3\Delta^2+1}-k(\Delta+1)}{\Delta+1}\right|,$$

where $j=0,1$; $k=0,\ 1,\ -1$. Therefore

(104) $$l_\Delta(\epsilon,0)=(\log|h^{\sim}(\eta_\epsilon^{\wedge}(1/2,0,\delta_0))|)=$$

$$\log\left(|(\eta_\epsilon(1/2,0,\delta_0)-1)(1-\delta_0)^{-d_1}(\eta_\epsilon(1/2,0,\delta_0)+1)2^{-2}\eta_\epsilon(1/2,0,\delta_0)^{d_1}|\right)=$$

$$-\log\left(4(\Delta+1)^{\Delta+1}(1-1/\Delta)^(\Delta-1)\right)+$$

$$\log\left(|2\Delta+(-1)^\epsilon\sqrt{3\Delta^2+1}-(\Delta+1)|\right)+$$

$$\log\left(|2\Delta+(-1)^\epsilon\sqrt{3\Delta^2+1}+(\Delta+1)|\right)+$$

$$(\Delta-1)\log\left(|2\Delta+(-1)^\epsilon\sqrt{3\Delta^2+1}|\right).$$

Consequently

$$l_5(1,0)=-\|og(4)-6\log 6-4\log(0.8)+$$

$$\log(\sqrt{76}-4)+\log(16-\sqrt{76})+4\log(10-\sqrt{76})$$

I made computations below "by hands" using calculator of the firm "CASIO."

$$\log 4=1,386294361...;\ 6\log(6)=10,7505682...;$$

$$4\log(0.8) = -0,892574205...\,;$$
$$\sqrt{76} = 8,717797887...\,;\ \sqrt{76} - 4 = 4,717797887...\,;$$
$$16 - \sqrt{76} = 7,282202113...\,;\ 10 - \sqrt{76} = 1,282202113...\,;$$
$$\log\left(\sqrt{76} - 4\right) = 1.551342141...\,;\ \log\left(16 - \sqrt{76}\right) = 1.985433305...\,;$$
$$\log\left(10 - \sqrt{76}\right) = 0.248579...\,;\ 4\log\left(10 - \sqrt{76}\right) = 0,994316001...\,;$$

$$l_5(1,0) = -6.713196909...; \tag{105}$$

$$l_7(1,0) = -\log(4) - 8\log(8) - 6\log(6) + 6\log(7) +$$
$$\log\left(\sqrt{148} - 6\right) + \log\left(22 - \sqrt{148}\right) + 6\log\left(14 - \sqrt{148}\right);$$
$$8\log 8 = 16,63553233...\,;\ 6\log 6 = 10,75055682...;\ 6\log 7 = 11,67546089...;$$
$$\sqrt{148} = 12,16552506...\,;\ \sqrt{148} - 6 = 6,16552506...$$
$$22 - \sqrt{148} = 9,83474939...\,;\ 14 - \sqrt{148} = 1,83474939...\,;$$
$$\log(\sqrt{148} - 6) = 1,818973301;\ \log(22 - \sqrt{148}) = 2,285894063...;$$
$$\log(14 - \sqrt{148}) = 0,606758304...\,;\ 6\log(14 - \sqrt{148}) = 3,640549824...\,;$$

$$l_7(1,0) = -9,35150543.... \tag{106}$$

In view of $(2), (92), (93), (95),\ (96)$ and (101),

$$V_5^* = 6 - 1.956124... = 4,04387...;\ V_7^* = 8 - 2.314407 = 5,685593. \tag{107}$$

In view (105) – (107),

$$-V_5^* - l_5(1,0) > 0,\ -V_7^* - l_7(1,0) > 0. \tag{108}$$

So, the key inequalities (108) are checked "by hands". I view of (103), (108) and Lemma 3,

$$-V_5^* - l_5(1, \tan(\pi/m)) > 0,\ -V_7^* - l_7(1, \tan(\pi/m)) > 0,$$

where $m > 2$. Since $(\log(p))/(p^{\alpha-1}(p-1))$ decreases together with increasing of $p \in (3, +\infty)$ with fixed $\alpha \geq 1$, or icreasing of $\alpha \in (1, +\infty$ with fixed $p \geq 2$ (or, of course, increasing both $\alpha \in (1, +\infty$ and $p \in (3, +\infty)$), and

$$\lim_{p\to\infty}((\log(p))/(p^{\alpha-1}(p-1))) = 0,$$

where $\alpha \geq 1$,

$$\lim_{\alpha\to\infty}((\log(p))/(p^{\alpha-1}(p-1))) = 0,$$

where $p \geq 2$, it follows that the inequality (7) holds for all the sufficient big integers m. Computations on computer of class "Pentium" show that the inequality (7) holds for $m = 3$, $m = 4$, $m = 5$ and $m = 2 \times 5$; therefore inequality (7) holds for all the $m > 2\times 3$. Let $\varepsilon_0 = h_\Delta(m)/2$, with $h_\Delta(m)$ defined in (6). In view of (7), $\varepsilon_0 > 0$. We take now $K = K_m = \mathbb{Q}[\exp(2\pi i/m)]$. Let further $\{\sigma_1, \dots, \sigma_{\phi(m)}\} = Gal(K/\mathbb{Q})$. For each $j = 1, \dots, \phi(m)$ there exists $k_j \in (-m/2, m/2) \cap \mathbb{Z}$ such that

$$(|k_j|, m) = 1,\ \sigma_j\left(\exp\left(\frac{2\pi i}{m}\right)\right) = \exp\left(\frac{2\pi i k_j}{m}\right).$$

Let a be the element of $\mathfrak{K}$, such that

$$\pi_j(a) = \log(2 + \sigma_j(\exp(2\pi i/m))) = \log(2 + \exp(2\pi i k_j/m)),$$

where $j = 1, \dots, \phi(m)$; we suppose that $k_1 = 1$. In view of (99) and (100), let $\alpha_1^\vee(\nu)$, $\alpha_1^\wedge(\nu)$, $\alpha_2^\vee(\nu)$, $\alpha_2^\wedge(\nu)$, are elements in $\mathfrak{K}$ such that

$$\pi_j(\alpha_1^\vee(\nu)) = \alpha^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k_j\pi i}{m})}, \frac{k_j\pi i}{m} - \pi\right)},$$

$$\pi_j(\alpha_2^\vee(\nu)) = \phi^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k_j\pi i}{m})}, \frac{k_j\pi i}{m} - \pi\right)},$$

$$\pi_j(\alpha_1^\wedge(\nu)) = \alpha^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k_j\pi i}{m})}, \frac{k_j\pi i}{m} - \pi\right)} U_\Delta(m,\nu), \tag{109}$$

$$\pi_j(\alpha_2^\wedge(\nu)) = \phi^*(z;\nu)\Big|_{z=\left(\frac{1}{2\cos(\frac{k_j\pi i}{m})}, \frac{k_j\pi i}{m} - \pi\right)} U_\Delta(m,\nu), \tag{110}$$

where $j = 1, \dots, \phi(m)$. Then $\alpha_k^\wedge(\nu) \in \mathbb{Z}_K$ for $k = 1, 2$.

$$y^\vee(\nu) = -\alpha_2^\vee(\nu) + a\alpha_1^\vee(\nu), \tag{111}$$

and let $y(\nu)$ is defined by means the equality (90). According to the Corrollary of the Lemma 4, to the Theorem 4 in [58] (or Theorem 7 in [66]), to the Lemma 8, to (103), there exist $m_1^* \in \mathbb{N}$ having the following property:

for any $\varepsilon \in (0, \varepsilon_0)$ there exist $\gamma_0(\varepsilon) > 0$, $\gamma_1(\varepsilon) > 0$, and $\gamma_2(\varepsilon) > 0$ such that

$$|\pi_j(\alpha_k^\vee(\nu))| \le \tag{112}$$

$$\gamma_0(\varepsilon)\exp((l_\Delta(\tan((k_j\pi i)/m), 0) + \varepsilon/3)\nu),$$

where $k = 1, 2$, $j = 1, \dots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$,

$$\gamma_1(\varepsilon)\exp((l_\Delta(\tan((k_j\pi i)/m), 1) - \varepsilon/3)\nu) \le \tag{113}$$

$$\max(|\pi_j(y^\vee(\nu))|, |\pi_j(y^\vee(\nu+1))| \le$$

$$\gamma_2(\varepsilon)\exp((l_\Delta(\tan((k_j\pi i)/m), 1) + \varepsilon/3)\nu),$$

where $j = 1, \dots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$.

Let $\omega_1(m) = (m-1)/2$, if m is odd, $\omega_1(m) = m/2-2$, if $m \equiv 2(\mod 4)$ and $\omega(m) = m/2-1$, if $m \equiv 0(\mod 4)$. Then

$$\omega_1(m) = \sup\{k \in \mathbb{N}\colon k_j < m/2, (k,m) = 1\}.$$

According to the Lemma 3 and (103),

$$l_\Delta(\tan((k_j\pi i)/m), 0) \le l_\Delta(\tan((\omega_1(m)\pi i)/m), 0), \tag{114}$$

$$l_\Delta(\tan((\omega_1(m)\pi i)/m), 1) \le \tag{115}$$

$$l_\Delta(\tan((k_j\pi i)/m), 1) \le l_\Delta(\tan((\pi i)/m), 1)$$

where $j = 1, \ldots, \phi(m)$. In view of (112) – (115),

(116)
$$|\pi_j(\alpha_k^\vee(\nu))| \le \gamma_0(\varepsilon)\exp((l_\Delta(\tan((\omega_1(\nu)\pi i)/m), 0) + \varepsilon/3)\nu),$$

where $k = 1, 2$, $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$,

$$\gamma_1(\varepsilon)\exp((l_\Delta(\tan((\omega_1(m)\pi i)/m), 1) - \varepsilon/3)\nu) \le \tag{117}$$

$$\max(|\pi_j(y^\vee(\nu))|, |\pi_j(y^\vee(\nu+1))| \le$$

$$\gamma_2(\varepsilon)\exp((l_\Delta(\tan((\pi i)/m), 1) + \varepsilon/3)\nu),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$. In view of (101), there exists $m_2^* \in \mathbb{N} - 1 + m_1^*$, such that

$$\exp(V_\Delta(m) - \varepsilon/3)\nu \le U_\Delta(m,\nu) \le \exp(V_\Delta(m) - \varepsilon/3)\nu \tag{118}$$

where $\nu \in \mathbb{N} - 1 + m_2^*$.

In view of (115) – (118), (109) – (111), (6), (5),

$$|\pi_j(\alpha_k(\nu))| \le \gamma_0(\varepsilon)\exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu), \tag{119}$$

where $k = 1, 2$, $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$,

$$\gamma_1(\varepsilon)\exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu) \le \tag{120}$$

$$\max(|\pi_j(y^\vee(\nu))|, |\pi_j(y^\vee(\nu+1))| \le$$

$$\gamma_2(\varepsilon)\exp((-h_\Delta(m) + 2\varepsilon/3)\nu),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$.

Let $X \in \mathbb{Z}_{K_m} \diagdown \{0\}$. Then, in view of (119) and (120),

(121)
$$|\pi_j(X\alpha_k(\nu))|| \le \gamma_0(\varepsilon)\exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)|\pi_j(X)| \le$$

$$\gamma_0(\varepsilon)\exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)q_\infty^{(\mathfrak{K})}(X),$$

where $k = 1, 2$, $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$,

$$\gamma_1(\varepsilon)\exp((-g_{\Delta,1}(m) - 2\varepsilon/3)\nu)|\pi_j(X)| \le \tag{122}$$

$$\max(|\pi_j(Xy^\vee(\nu))|, |\pi_j(Xy^\vee(\nu+1))| \leq$$
$$\max(q^{(\mathfrak{K})}_\infty(Xy^\vee(\nu)), q^{(\mathfrak{K})}_\infty(Xy^\vee(\nu+1)),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$,

(123) $$\max(|\pi_j(Xy^\vee(\nu))|, |\pi_j(Xy^\vee(\nu+1))| \leq$$

$$\gamma_2(\varepsilon)\exp((-h_\Delta(m) + 2\varepsilon/3)\nu)|\pi_j(X)| \leq$$
$$\gamma_2(\varepsilon)\exp((-h_\Delta(m) + 2\varepsilon/3)\nu)q^{\mathfrak{K}}_\infty(X),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_2^*$.

In view of (121)

(124) $$q^{(\mathfrak{K})}_\infty(X\alpha_k(\nu)) \leq$$

$$\gamma_0(\varepsilon)\exp((g_{\Delta,0}(m) + 2\varepsilon/3)\nu)q^{(\mathfrak{K})}_\infty(X),$$

where $k = 1, 2,$ and $\nu \in \mathbb{N} - 1 + m_2^*$. In view of (123),

(125) $$\max(q^{(\mathfrak{K})}_\infty(Xy^\vee(\nu)), q^{(\mathfrak{K})}_\infty((Xy^\vee(\nu+1)) =$$

$$\sup(\{|\pi_j(Xy^\vee(\nu+\epsilon))|, : \epsilon \in \{0, 1\}, j = 1, \ldots, \phi(m)\}) \leq$$
$$\gamma_2(\varepsilon)\exp((-h_\Delta(m) + 2\varepsilon/3)\nu)q^{(\mathfrak{K})}_\infty(X),$$

where $\nu \in \mathbb{N} - 1 + m_2^*$.

Taking in acount (124), (125) and (122), we see that all the conditions of the Corollary of the Lemma 12 are fulfilled for

$$\varepsilon \in (0, \varepsilon_0), \gamma_0(\varepsilon), \gamma_1(\varepsilon), \gamma_2(\varepsilon), y = y(\nu), \alpha_1(\nu), \alpha_2(\nu),$$

$$r_1 = r_1(\varepsilon) = \exp(g_{\Delta,0}(m) + 2\varepsilon/3,$$
$$R_1 = R_1(\varepsilon) = \exp(g_{\Delta,1}(m) + 2\varepsilon/3),$$
$$R_2 = R_2(\varepsilon)\exp(h_\Delta(m) - 2\varepsilon/3),$$

and this proves the part of our Theorem connected with the inequality (8).

Let again $X \in \mathbb{Z}_{K_m} \diagdown \{0\}$ and let

$$q^{(\mathfrak{K})}_{min}(X) = \inf(|\{\pi_j(X)|: j = 1, \ldots, \phi(m)\})$$

Clearly, $q^{(\mathfrak{K})}_{min}(X) > 0$ According to the Theorem 4 in [58], or to the Theorem 7 in [66], there exist $m_1^* \in \mathbb{N}$ having the following property: for any $\varepsilon \in (0, \varepsilon_0)$ there exist $\gamma_0^*(X, \varepsilon) > 0$, $\gamma_1^*(X, \varepsilon) > 0$, and $\gamma_2^*(X, \varepsilon) > 0$ such that

$$|\pi_j(\alpha_k^\vee(\nu))| \leq \gamma_0^*(\varepsilon)\exp((l_\Delta(\tan((\omega_m\pi i)/m), 0) + \varepsilon/3)\nu),$$

where $k = 1, 2, j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$,

$$\gamma_1^*(X\varepsilon)\exp((l_\Delta(\tan((\pi i)/m), 1) - \varepsilon/3)\nu) \leq$$

$$\max(|\pi_j(y^\vee(\nu))|, |\pi_j(y^\vee(\nu+1))| \leq$$
$$\gamma_2(\varepsilon)\exp((l_\Delta(\tan((\pi i)/m), 1) + \varepsilon/3)\nu),$$

where $j = 1, \ldots, \phi(m)$ and $\nu \in \mathbb{N} - 1 + m_1^*$. Repeating the previous considerations, we see that all the conditions of the Corollary of the Lemma 12 are fulfilled for $\varepsilon \in (0, \varepsilon_0)$,

$$\gamma_0 = \gamma_0^*(X, \varepsilon),\ \gamma_1 = \gamma_1^*(X, \varepsilon),\ \gamma_2 = \gamma_2^*(X, \varepsilon),$$

$$y = y(\nu),\ \alpha_1(\nu), \alpha_2(\nu),\ r_1 = r_1(\varepsilon) = \exp(g_{\Delta,0}(m) + 2\varepsilon/3,$$

and

$$R_1 = R_2 = R_2(\varepsilon)\exp(h_\Delta(m) - 2\varepsilon/3),$$

and this proves the part of our Theorem connected with the inequality (9). ■

Below are values of β and α computed for $\Delta \in \{5, 7\}$ and some $m \in \mathbb{N}$.

$$(m;\ \Delta;\ \beta;\ \alpha) = (3;\ 5;\ 3,111228...\,;\ 3,111228...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (3;\ 7;\ 3,073525...\,;\ 3,073525...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (4;\ 5;\ 11,458947...\,;\ 11,458947...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (4;\ 7;\ 10,551730...\,;\ 10,551730...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (5;\ 5;\ 4,826751...\,;\ 5,607961...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (5;\ 7,\ 4,837858...\,;\ 5,684622...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (7;\ 5;\ 5,701485...\,;\ 6,977258...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (7;\ 7;\ 5,724804...\,;\ 7,114963...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (8;\ 5;\ 8,337857...\,;\ 9,436901...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (8;\ 7;\ 8,253047...\,;\ 9,433260...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (9;\ 5;\ 6,312056...\,;\ 7,960502...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (9;\ 7;\ 6,335274...\,;\ 8,134962...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (10;\ 5;\ 43,546644...\,;\ 46,230614...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (10;\ 7;\ 35,648681...\,;\ 38,043440...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (11;\ 5;\ 6,786990...\,;\ 8,735234...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (11;\ 7,\ 6,806087...\,;\ 8,934922...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (12;\ 5;\ 5,638541...\,;\ 6,813222...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (12;\ 7;\ 5,696732...\,;\ 6,983870...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (13;\ 5;\ 7,177155...\,;\ 9,376030...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (13;\ 7;\ 7,190814...\,;\ 9,594580...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (14;\ 5;\ 19,659885...\,;\ 21,835056...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (14;\ 7;\ 18,447228...\,;\ 20,668254...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (15\,;5;\ 7,508714...\,;\ 9,922761...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (15;\ 7;\ 7,516606...\,;\ 10,156245...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (16;\ 5,\ 7,951153...\,;\ 9,876454...),$$

$$(m;\ \Delta;\ \beta;\ \alpha) = (16;\ 7,\ 7,945763...\,;\ 10,039605...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (17;\ 5;\ 7,797153...\,;\ 10,399610...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (17;\ 7,\ 7,799343...\,;\ 10,645404...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (18;\ 5,\ 9,486110...\,;\ 10,955534...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (18;\ 7,\ 9,406368...\,;\ 10,989150...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (19;\ 5;\ 8,052478...\,;\ 10,822446...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (19;\ 7;\ 8,049182...\,;\ 11,078690...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (20;\ 5;\ 6,696241...\,;\ 8,559091...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (20;\ 7;\ 6,733979...\,;\ 8,774063...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (21;\ 5;\ 8,281548...\,;\ 11,202268...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (21;\ 7;\ 8,273039...\,;\ 11,467583...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (22;\ 5;\ 13,134623...\,;\ 15,504916...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (22;\ 7;\ 12,815391...\,;\ 15,331975...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (23;\ 5;\ 8,489281...\,;\ 11,547024...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (23;\ 7;\ 8,475843...\,;\ 11,820351...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (24;\ 5;\ 7,088338...\,;\ 9,210037...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (24;\ 7;\ 7,116679...\,;\ 8,439782...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (25;\ 5;\ 8,679328...\,;\ 11,862643...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (25;\ 7;\ 8,661235...\,;\ 12,143143...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (26;\ 5;\ 12,172520...\,;\ 14,674949...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (26;\ 7;\ 11,944943...\,;\ 14,618461...),$$
$$\dots$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (32;\ 5;\ 8,654733...\,;\ 11,466214...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (32;\ 7;\ 8,637697...\,;\ 11,705492...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (33;\ 5;\ 9,310125...\,;\ 12,911341...),$$
$$(m;\ \Delta;\ \beta;\ \alpha) = (33;\ 5;\ 9,275806...\,;\ 13,214792...),$$

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