Exact triangles in Seiberg-Witten-Floer theory. Part IV: $\mathbb{Z}$-graded monopole homology

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1 Introduction

The content of this paper is part of our ongoing project of establishing the surgery formulae for Seiberg-Witten-Floer theory. As explained in the introductory part of [2], if K is a knot in a homology 3-sphere Y, we expect the Floer homologies of Y and of the manifolds Y_1 and Y_0 obtained by Dehn surgery on K with framing 1 and 0, respectively, to be related by an exact triangle

$$\begin{tikzcd}
HF_*^{SW}(Y_1, g_1) \arrow[rr, "I_*"] & & HF_*^{SW}(Y, g) \arrow[dl, "L_*"] \\
& \bigoplus_k HF_{(*)}^{SW}(Y_0, \mathfrak{s}_k) \arrow[ul, "\Delta_{(*)}"] &
\end{tikzcd}$$

There are several subtleties involved in the precise definition of the term

$$\bigoplus_k HF_{(*)}^{SW}(Y_0, \mathfrak{s}_k)$$

in the previous diagram. In this paper we explain the precise meaning of this term.

More specifically, we analyze some aspects of the construction of the Seiberg-Witten-Floer homology of the manifold Y_0, obtained by 0-surgery on the knot K in the homology sphere Y. The manifold Y_0 has the homology of $S^1 \times S^2$. For manifolds with $b_1(Y_0) > 0$ we know (cf. [3], [10], [11], [20]) that there is a well defined $\mathbb{Z}_\ell$ graded Seiberg-Witten-Floer homology $HF_*^{SW}(Y_0, \mathfrak{s})$, for every choice of a non-trivial Spin^c structure $\mathfrak{s} \in \mathcal{S}(Y_0)$, where the integer $\ell = \ell(\mathfrak{s})$ satisfies

$$c_1(L)(H_2(Y_0, \mathbb{Z})) = \ell\mathbb{Z},$$

and $\mathcal{S}(Y_0)$ denotes the set of Spin^c structures on Y_0. Here $L = \det W$ is the determinant line bundle of the spinor bundle W associated to the Spin^c structure $\mathfrak{s}$. We use the equivalent notation $L = \det \mathfrak{s}$. Here we are assuming that $c_1(L)$ is non-torsion, hence $\ell \neq 0$.

In this paper we shall discuss several issues regarding the Floer homology $HF_*^{SW}(Y_0, \mathfrak{s})$. The first is connected to the integer lift of the $\mathbb{Z}_\ell$ graded Floer homology. Analogous constructions of integer lifts of Floer homologies were derived by Fintushel and Stern [4], in the case of instanton homology, and by Weiping Li [7], in the case of symplectic Floer homology. We follow closely the construction of [4] and show that, in our case, there is a well defined integer lift of $HF_{*,(\omega)}^{SW}(Y_0, \mathfrak{s})$ of the $\mathbb{Z}_\ell$-graded Floer homology, here $\omega \in \mathbb{R}$ is a regular value of the Chern-Simons-Dirac functional on the infinite cyclic cover space of the gauge equivalence classes of connections and spinor sections. By studying the Chern-Simons-Dirac function on this infinite cyclic cover space, we will define an integer lift $\mathfrak{i}_{Y_0}^{(\omega)}$ of the indices of the critical points. We thus form a chain complex $C_*^{(\omega)}(Y_0, \mathfrak{s})$ depending on $\omega \in \mathbb{R}$. For any $n \in \mathbb{Z}_\ell$, the original $\mathbb{Z}_\ell$-graded Seiberg–Witten–Floer chain complex satisfies $C_n(Y_0, \mathfrak{s}) = \bigoplus_{k \in \mathbb{Z}} C_{j+k\ell}^{(\omega)}(Y_0, \mathfrak{s})$ where $j = n(mod \ell)$. In general, after defining a suitable boundary operator on $C_*^{(\omega)}(Y_0, \mathfrak{s})$, we observe that the resulting homology groups $HF_{*,(\omega)}^{SW}(Y_0, \mathfrak{s})$ do not satisfy the simple relation $\bigoplus_{k \in \mathbb{Z}} HF_{*+k\ell,(\omega)}^{SW}(Y_0, \mathfrak{s}) = HF_*^{SW}(Y_0, \mathfrak{s})$. However, the Floer homologies $HF_{*+k\ell,(\omega)}^{SW}(Y_0, \mathfrak{s})$ and $HF_*^{SW}(Y_0, \mathfrak{s})$ are related via a spectral sequence determined by a filtration of the chain complex $C_*(Y_0, \mathfrak{s})$. This spectral sequence converges to $HF_*^{SW}(Y_0, \mathfrak{s})$, and the E^1 term coincides with $HF_{*,(\omega)}^{SW}(Y_0, \mathfrak{s})$. For instance, in the particular case where all the higher differentials in the spectral sequence are zero, then we simply have, for $m \in \{0, 1, \cdots, \ell - 1\}$,

$$HF_m^{SW}(Y_0, \mathfrak{s}) = \bigoplus_{n \in \mathbb{Z}} HF_{m+n\ell,(\omega)}^{SW}(Y_0, \mathfrak{s}).$$

The arguments we develop in this section extend easily to the general case of any 3-manifold $(Y_0, \mathfrak{s})$ with non-trivial rational homology and with a Spin^c-structure satisfying $c_1(\mathfrak{s})(H_2(Y_0, \mathbb{Z})) = \ell\mathbb{Z} \neq 0$.

The second main issue discussed in this paper is a different construction of a $\mathbb{Z}$-graded Seiberg–Witten–Floer homology, where the integer lift of the original $\mathbb{Z}_\ell$-graded Seiberg–Witten–Floer homology is determined by the exact triangle, as derived in [2] [12] [13]. Under the surgery identifications of the Seiberg–Witten monopoles on Y and Y_0, proved in [2], there exists a one-to-one map

$$j: \qquad \bigsqcup_{\mathfrak{s}} \mathcal{M}_{Y_0}(\mathfrak{s}) \longrightarrow \mathcal{M}_{Y,\mu} \backslash \mathcal{M}_{Y_1}. \tag{1}$$

Then the $\mathbb{Z}$-valued indices on $\mathcal{M}_{Y,\mu}$ define a $\mathbb{Z}$-lifting $\mathrm{i}_{Y_0}^{(Y)}$ of the grading $\mathcal{M}_{Y_0}(\mathfrak{s})$. The corresponding chain complex $C_{(n)}(Y_0, \mathfrak{s})$ is defined to be $\mathbb{Z}\{a \in \mathcal{M}_{Y_0}(\mathfrak{s}) : \mathrm{i}_{Y_0}^{(Y)}(a) = n\}$. We will show that the restriction of the boundary operator on $C_*(Y_0, \mathfrak{s})$ to $C_{(n)}(Y_0, \mathfrak{s})$ is a well-defined boundary operator, and the resulting homology groups $HF_{(*)}(Y_0, \mathfrak{s})$ satisfy

$$HF_n^{SW}(Y_0, \mathfrak{s}) = \bigoplus_{k \in \mathbb{Z}} HF_{(n+k\ell)}^{SW}(Y_0, \mathfrak{s}). \tag{2}$$

The analysis of the splitting and gluing of the moduli spaces of flow lines in [12] play an essential role in the proof of the direct sum formula (2). It is precisely these $\mathbb{Z}$-graded monopole homology groups that we use in the exact triangles [13].

The third main issue that we discuss in this paper is the construction of the Seiberg-Witten-Floer homology on a 3-manifold Y_0 with $b_1(Y_0) > 0$, in the case of a Spin^c structure $\mathfrak{s}_0$ with torsion $c_1(\det \mathfrak{s}_0)$. This case was not included in our original paper [11]. We show that, in this case, the boundary operator of the Floer complex is well defined only after separating the components of uniform energy in the moduli space $\mathcal{M}(a, b)$. This is done by introducing a Novikov complex with coefficients in $\mathbb{Z}[[t]]$, which depends on the non-trivial cohomology class of the perturbation for the Chern-Simons-Dirac functional. The component of this Novikov-Floer homology $HF_*(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])$ that we are interested in, for the purpose of the exact triangle, is the evaluation

$$HF_{(*)}(Y_0, \mathfrak{s}_0) = HF_*(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])|_{t=0}.$$

We also generalize the construction of $HF_*(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])$ to a more natural setting over the coefficient field of formal Laurent series $\mathbb{Q}((t)) = \mathbb{Q}[[t]][t^{-1}]$. One reason for this definition is that, in this setting, we can associate a relative Seiberg–Witten invariant for any 4-manifold $(X, \mathfrak{s})$ with boundary $(Y_0, \mathfrak{s}_0)$, where $c_1(\mathfrak{s}|_{Y_0})$ is a torsion class. Here we introduce a non-trivial 1-cycle Γ in Y_0, which depends on the non-trivial cohomology class of the perturbation. The corresponding Seiberg-Witten-Floer homology is denoted by $HF_{\Gamma,*}^{SW}(Y_0, \mathfrak{s}_0, \mathbb{Q}((t)))$.

Suppose given 4-manifold $(X, \mathfrak{s})$ with a cylindrical end modeled on $(Y_0, \mathfrak{s}_0 = \mathfrak{s}|_{Y_0})$, where $c_1(\mathfrak{s}_0)$ is a torsion cohomology class in $H^2(Y_0.\mathbb{Z})$.

We study the perturbed 4-dimensional Seiberg-Witten equations on $(X, \mathfrak{s})$, the perturbation for $(X, \mathfrak{s})$ is compactible with the perturbation on $(Y_0, \mathfrak{s}_0)$ used to the construction of $HF^{SW}_{\Gamma,*}(Y_0, \mathfrak{s}_0, \mathbb{Q}((t)))$. For simplicity, we assume that Γ bounds a relative 2-cycle in X, then the relative Seiberg–Witten invariant $SW_X(\mathfrak{s})$ takes values in $HF^{SW}_{\Gamma,*}\big(Y_0, \mathfrak{s}_0, \mathbb{Q}((t))\big)$. We derive the gluing formulae for this case. When Γ does not bound any relative 2-cycle in X, the relative invariant takes values in the Seiberg-Witten-Floer homology $(Y_0, \mathfrak{s}_0)$ over the coefficient field of formal Laurent series on $H^1(Y_0, \mathbb{Z})$. The gluing formulae along the torsion Spin^c structure can be formulated similarly.

As an example, our gluing formula can be applied to show that, for certain particular choice of metric and perturbation on the three torus $S^1 \times T^2$ with the trivial Spin^c structure $\mathfrak{s}_0$, we have

$$HF_*\big(S^1 \times T^2, \mathfrak{s}_0, \mathbb{Q}((t))\big) = \mathbb{Q}((t)),$$

and for $D^2 \times T^2$ with the cylindrical end modeled on $S^1 \times T^2$, we get

$$SW_{D^2\times T^2}(\mathfrak{s}_0) = \frac{1}{t - t^{-1}}.$$

In the more general setting, these gluing formulae provide useful tools in the study of the structure of the Seiberg–Witten–Floer homology groups (cf. [17]). The relative invariant $SW_{D^2\times T^2}(\mathfrak{s}_0)$ can also be applied to knot surgery of 4-manifolds to give some of the gluing formulae obtained in [5] and [16].

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2 Seiberg-Witten-Floer homology

In the following two sections we present two different constructions of an integer lift of the Seiberg–Witten–Floer homology. The first case applies to the general setting where Y_0 is a compact 3-manifold with $b_1(Y_0) > 0$, and the Spin^c-structure is non-torsion. The second applies to the special case where Y_0 is obtained by zero-surgery on a knot in a homology 3-sphere

Y, again with a non-torsion Spin^c-structure. In this particular case, the integer lift of the Seiberg–Witten–Floer homology that we consider appears naturally in the context of the exact triangle formula.

We give some preliminary definitions.

Let $\mathcal{M}_{Y_0}(\mathfrak{s})$ be the moduli space of gauge classes of solutions of suitably perturbed Seiberg–Witten equations on a 3-manifold $(Y_0, \mathfrak{s})$, with $b_1(Y_0) > 0$ and $c_1(\mathfrak{s})$ a non-torsion element. A suitable class of perturbations that achieves transversality has been introduced in [2]. The perturbation can be chosen so that all the solutions in $\mathcal{M}_{Y_0}(\mathfrak{s})$ are irreducible critical points. Thus, under the choice of a generic perturbation, $\mathcal{M}_{Y_0}(\mathfrak{s})$ is a compact, oriented, 0-dimensional manifold, cut out transversely by the equations inside the configuration space $\mathcal{B} = \mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is the space of pairs formed by a $U(1)$-connection on $det(\mathfrak{s})$ and a spinor section of W, and $\mathcal{G}$ is the group of gauge transformations on $(Y_0, \mathfrak{s})$.

Recall that we have the Chern–Simons–Dirac functional on $\mathcal{A}$, defined as

$$
\begin{aligned}
CSD(A,\psi) &= -\frac{1}{2}\int_{Y_0} (A - A_0) \wedge (F_A + F_{A_0} - 2\sqrt{-1} * \rho) \\
&\quad + \int_{Y_0} \langle \psi, \partial_A \psi \rangle dvol_{Y_0},
\end{aligned}
$$

with ρ a co-closed 1-form on Y_0. In order to achieve transversality in the moduli spaces of flow lines, we consider an additional perturbation of CSD by functions $U(\tau_1, \dots, \tau_N) + V(\zeta_1, \dots, \zeta_K)$ as explained in [2]. Here τ_j and ζ_j are functions on $\mathcal{A}$ respectively associated to a complete basis $\{\mu_j\}_{j=1}^{\infty}$ for the co-closed L^2 one-forms on Y_0 and a complete basis $\{\nu_j\}_{j=1}^{\infty}$ for the imaginary-valued one-forms on Y_0,

$$
\begin{aligned}
\tau_j(A,\psi) = \tau_j(A - A_0) &= \int_{Y_0} (A - A_0) \wedge *\sqrt{-1}\mu_j \\
\zeta_j(A,\psi) = \zeta_j(\psi,\psi) &= \int_{Y_0} \langle \nu_j.\psi, \psi \rangle dvol_{Y_0}.
\end{aligned}
$$

The pair (U, V) is chosen from a subspace of functions in

$$
\bigcup_{N \geq b_1, K > 0} C^{\infty}(\mathbb{R}^N, \mathbb{R}) \times C^{\infty}(\mathbb{R}^K, \mathbb{R})
$$

completed to a Banach space in the Floer ϵ-norm (as specified in [2]). This perturbation term is gauge invariant by construction, and CSD changes by

$$
CSD(\lambda.(A,\psi)) - CSD(A,\psi) = \langle (8\pi^2 c_1(L) + 4\pi[*\rho]) \cup [\lambda], [Y_0] \rangle,
$$

where $[\lambda] \in H^1(Y_0,\mathbb{Z})$ determines the connected component of $\mathcal{G}$ that contains λ, and is represented by the closed 1-form $\frac{1}{2\pi\sqrt{-1}}\lambda^{-1}d\lambda$.

With this choice of perturbation, the Seiberg–Witten equations are of the form

$$\begin{cases} *F_A = \sigma(\psi,\psi) + \sum_{j=1}^{N} \dfrac{\partial U}{\partial \tau_j}\mu_j \\ \partial_A(\psi) + \sum_{j=1}^{K} \dfrac{\partial V}{\partial \zeta_j}\nu_j.\psi = 0, \end{cases}$$

for the critical points of the Chern–Simons–Dirac functional, and

$$\begin{cases} \dfrac{\partial A}{\partial t} &= - * F_A + \sigma(\psi,\psi) + \sum_{j=1}^{N} \dfrac{\partial U}{\partial \tau_j}\mu_j \\ \dfrac{\partial \psi}{\partial t} &= -\partial_A\psi - \sum_{j=1}^{K} \dfrac{\partial V}{\partial \zeta_j}\nu_j.\psi \end{cases}$$

for the corresponding downward gradient flow.

The Seiberg–Witten–Floer homology groups $HF_*^{SW}(Y_0,\mathfrak{s})$ (cf. [3], [10], [11]) are the homology groups of $(C_*(Y_0,\mathfrak{s}),\partial)$, where $C_*(Y_0,\mathfrak{s})$ is generated by the critical points of the perturbed CSD on $\mathcal{B}$. The entries of the boundary operator ∂ are defined by counting the points in the zero dimensional components of the moduli space of unparameterized flow lines of the perturbed functional CSD on $\mathcal{B}$. A considerable amount of technical work goes into checking that the moduli spaces of flow lines have all the desired properties that make this definition rigorous, and we refer the reader to [11] for a precise account. The resulting Floer homology $HF_*^{SW}(Y_0,\mathfrak{s})$ is $\mathbb{Z}_\ell$-graded with $\ell\mathbb{Z} = c_1(\mathfrak{s})(H_2(Y_0,\mathbb{Z}))$. This is due to an ambiguity in the index formula computed in terms of the spectral flow of the Hessian operator for CSD around a loop in $\mathcal{B}$. To understand this ambiguity, let (A,ψ) represent a critical point a of CSD on $\mathcal{B}$, let λ be a gauge transformation whose class $[\lambda] \in H^1(Y_0,\mathbb{Z})$ is non-trivial. Then the spectral flow of the Hessian operator H from $\lambda.(A,\psi)$ to (A,ψ) can be calculated from the index formula for the linearization of the 4-dimensional Seiberg–Witten equations on $Y_0 \times S^1$. This gives

$$SF(H)^{(A,\psi)}_{\lambda.(A,\psi)} = \langle [\lambda] \cup c_1(\mathfrak{s}), [Y_0] \rangle \in \ell\mathbb{Z}. \tag{3}$$

Remark 2.1. *Notice that the periodicity ℓ is an even number (cf. [2]). In fact, the map $\mathcal{S}(Y_0) \to H^2(Y_0,\mathbb{Z})$, given by $\mathfrak{s} \mapsto c_1(\det \mathfrak{s})$, is equivariant with respect to the action of $H^2(Y_0,\mathbb{Z})$, so that we have $\det(W \otimes H) = L \otimes H^2$, with $L = \det(W) = \det(\mathfrak{s})$ and $H \in H^2(Y_0,\mathbb{Z})$, cf. [9].*

There is a natural and non-trivial way to lift these $\mathbb{Z}_\ell$-graded homology groups to $\mathbb{Z}$-graded homology groups, so that the Euler characteristic number agrees with the original one. Following the idea of [4], we will discuss this integer lift and construct a spectral sequence which has the integer lift as E^1 term and converges to the original $\mathbb{Z}_\ell$-graded Floer homology. We shall then consider a different lifting of the $\mathbb{Z}_\ell$-graded homology groups which is relevant to the exact triangles considered in [2], [13], and [14].

3 $\mathbb{Z}$-graded homology groups and the spectral sequence

The construction we present in this section holds in the general case of a 3-manifold Y_0 with $b_1(Y_0) > 0$ and a Spin^c-structure with non-torsion class $c_1(L)$.

As explained in [11], there is a cyclic covering of $\mathcal{B}$, obtained by taking the quotient of $\mathcal{A}$, the space of $U(1)$-connections and spinors, with respect to the subgroup $\mathcal{G}_\ell$ of the gauge group $\mathcal{G}$ given by

$$\mathcal{G}_\ell = \{\lambda \in \mathcal{G} \mid \langle c_1(L) \cup [\lambda], [Y_0] \rangle = 0\},$$

where ℓ satisfies $\ell\mathbb{Z} = c_1(L)(H_2(Y_0, \mathbb{Z}))$. This subgroup depends on $c_1(L)$.

The space $\mathcal{B}$ has the homotopy type of $\mathbb{C}P^\infty \times K(H^1(Y_0, \mathbb{Z}), 1)$, hence it has a universal covering obtained by taking the quotient of $\mathcal{A}$ by the identity component of the gauge group. The resulting space $\tilde{\mathcal{B}}$ covers $\mathcal{B}$ with fibers $H^1(Y_0, \mathbb{Z})$. Define H_ℓ to be

$$H_\ell = \{h \in H^1(Y_0, \mathbb{Z}) | \langle c_1(L) \cup h, [Y_0] \rangle = 0\}.$$

The group H_ℓ also depends on $c_1(L)$. Then the space $\mathcal{B}_\ell = \mathcal{A}/\mathcal{G}_\ell$ is a covering of $\mathcal{B}$ with fiber $H^1(Y_0, \mathbb{Z})/H_\ell \cong \mathbb{Z}$. Hence $\mathcal{B}_\ell$ is an infinite cyclic covering space of $\mathcal{B}$.

The perturbed Chern–Simons–Dirac functional is a real valued functional $CSD : \mathcal{B}_\ell \to \mathbb{R}$ on the covering space $\mathcal{B}_\ell$. The critical manifold $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$ is a $\mathbb{Z}$-covering of $\mathcal{M}_{Y_0}(\mathfrak{s})$. The critical values form a discrete set in $\mathbb{R}$, which is a finite set mod $\mathbb{Z}$. Let $\Omega \subset \mathbb{R}$ denote the set of regular values. Let $\omega \in \Omega$ be a regular value. Given any point $a \in \mathcal{M}_{Y_0}(\mathfrak{s})$, there is a unique element a^ω in the fiber $\pi^{-1}(a)$ in $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$ that satisfies $CSD(a^\omega) \in (\omega, \omega + 8\pi^2\ell)$.

We have the following Lemma which shows that the relative indices on $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$, defined by the spectral flow of the Hessian operator, take values in $\mathbb{Z}$.

Lemma 3.1. *1. Consider the Hessian operator $H_{A(t),\psi(t)}$ defined in [2]. The spectral flow of the operator $H_{A(t),\psi(t)}$ around a loop in $\mathcal{B}_\ell$ is zero, hence the relative index of any two points $\tilde{a}$ and $\tilde{b}$ in $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$ is well defined in $\mathbb{Z}$.*

2. Let $\tilde{a}$ be a critical point in $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s}) \in \mathcal{B}_\ell$, and let $\lambda \in \mathcal{G}/\mathcal{G}_\ell$ be a gauge transformation. We have the following identities:

$$CSD(\lambda(\tilde{a})) - CSD(\tilde{a}) = 8\pi^2 \langle [\lambda] \cup c_1(\mathfrak{s}), [Y_0] \rangle;$$
$$SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})} = \langle [\lambda] \cup c_1(\mathfrak{s}), [Y_0] \rangle,$$

with $[\lambda] = [\frac{1}{2\pi\sqrt{-1}}\lambda^{-1}d\lambda]$. Here $SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})}$ denotes the spectral flow of the Hessian operator $H_{(A(t),\psi(t))}$ along a path from $\lambda(\tilde{a})$ to $\tilde{a}$ in $\mathcal{B}_\ell$.

Proof. By the Atiyah-Patodi-Singer index theorem, we have

$$SF(H_{(A(t),\psi(t))})|^{\tilde{a}}_{\lambda(\tilde{a})} = Index(\frac{\partial}{\partial t} + H_{(A(t),\psi(t))}).$$

This index calculates the virtual dimension of the 4-dimensional Seiberg–Witten monopole moduli space on $Y_0 \times S^1$, with the Spin^c structure obtained by gluing the Spin^c structure $\mathfrak{s}$ on $Y_0 \times \mathbb{R}$ along the two ends with the gauge transformation λ. By the index formula for the Seiberg–Witten monopoles, we obtain

$$\begin{aligned} & Index(\frac{\partial}{\partial t} + H_{(A(t),\psi(t))}) \\ = & \frac{1}{2\pi\sqrt{-1}} \int_{Y_0} c_1(L) \wedge \lambda^{-1} d\lambda \\ = & \langle [\lambda] \cup c_1(\mathfrak{s}), [Y_0] \rangle. \end{aligned}$$

The remaining claims are direct consequence of this index formula. □

Upon fixing a base point $\tilde{a}_0$ in $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$, we can define the following $\mathbb{Z}$-lifting of the $\mathbb{Z}_\ell$-grading of the elements of $\mathcal{M}_{Y_0}(\mathfrak{s})$.

Definition 3.2. *We define the grading of elements in $\mathcal{M}_{Y_0}(\mathfrak{s})$ as the relative index of the ω-lifting in $\tilde{\mathcal{M}}_{Y_0}(\mathfrak{s})$,*

$$\mathfrak{i}^{(\omega)}_{Y_0}(a) = \mathfrak{i}_{Y_0}(a^\omega) = SF(H_{(A(t),\psi(t))})|^{\tilde{a}_0}_{a^\omega}.$$

This definition of the grading depends on the choice of the base point $\tilde{a}_0$ and on the choice of a regular value ω. Notice that we can reduce the choice of $\tilde{a}_0$ to the choice of a base point a_0 in $\mathcal{M}_{Y_0}(\mathfrak{s})$. From now on, we fix a base point a_0 in $\mathcal{M}_{Y_0}(\mathfrak{s})$. Then $\tilde{a}_0$ is chosen to be the unique critical point in $\pi^{-1}(a_0)$ with $CSD(\tilde{a}_0) \in (\omega, \omega + 8\pi^2)$. It is easy to see, from the definition, that we have $\mathrm{i}_{Y_0}^{(\omega)} = \mathrm{i}_{Y_0}^{(\omega')}$ whenever ω and ω' are connected by a path in the set Ω of regular values. Moreover, we have $\mathrm{i}_{Y_0}^{(\omega+8\pi^2 k\ell)} = \mathrm{i}_{Y_0}^{(\omega)}$, for any $k \in \mathbb{Z}$.

Definition 3.3. *For any $q \in \mathbb{Z}$, we define*

$$\mathcal{M}_{Y_0,q}^{(\omega)}(\mathfrak{s}) = \{a \in \mathcal{M}_{Y_0}(\mathfrak{s}) \ |\mathrm{i}_{Y_0}^{(\omega)}(a) = q\}.$$

The q-chains in the $\mathbb{Z}$-graded Floer complex are the elements of the abelian group $C_q^{(\omega)}(Y_0, \mathfrak{s})$ generated by the monopoles in $\mathcal{M}_{Y_0,q}^{(\omega)}(\mathfrak{s})$. The boundary operator

$$\partial_q^{(\omega)} : C_q^{(\omega)}(Y_0, \mathfrak{s}) \to C_{q-1}^{(\omega)}(Y_0, \mathfrak{s})$$

is defined as

$$\partial_q^{(\omega)}(a) = \sum_{b \in \mathcal{M}_{Y_0,q-1}^{(\omega)}(\mathfrak{s})} \#\big(\hat{\mathcal{M}}^0(a,b)\big)\, b,$$

where $\hat{\mathcal{M}}^0(a,b)$ is the zero dimensional components of the moduli space of unparameterized flow lines in $\hat{\mathcal{M}}(a,b)$. The compactness theorem (cf. [11]) tells us that $\hat{\mathcal{M}}^0(a,b)$ is an oriented, compact, 0-dimensional manifold. Thus, the coefficient $\#(\hat{\mathcal{M}}^0(a,b))$ is well-defined as the algebraic sum of points in $\hat{\mathcal{M}}^0(a,b)$.

The following Lemma shows that we have $\partial_{q-1}^{(\omega)} \circ \partial_q^{(\omega)} = 0$. The resulting homology groups are denoted as $HF_{*,(\omega)}^{SW}(Y_0, \mathfrak{s})$, $* \in \mathbb{Z}$.

Lemma 3.4. *1. For $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q = n (mod\ \ell)$, we have*

$$C_n(Y_0, \mathfrak{s}) = \bigoplus_{k \in \mathbb{Z}} C_{q+k\ell}^{(\omega)}(Y_0, \mathfrak{s}).$$

2. Under the decomposition as above, the boundary operator ∂_n on

$C_n(Y_0,\mathfrak{s})$ can be expressed as

$$\partial_n = \begin{pmatrix} \partial^{(\omega)}_{*} & \partial^{(\omega)}_{*,1} & \partial^{(\omega)}_{*,2} & \cdots & \cdots & \partial^{(\omega)}_{*,*} \\ 0 & \partial^{(\omega)}_{*+\ell} & \partial^{(\omega)}_{*+\ell,1} & \cdots & \cdots & \partial^{(\omega)}_{*+\ell,*} \\ 0 & 0 & \partial^{(\omega)}_{*+2\ell} & \cdots & \cdots & \partial^{(\omega)}_{*+2\ell,*} \\ 0 & 0 & 0 & \ddots & & \partial^{(\omega)}_{*+3\ell,*} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \partial^{(\omega)}_{*+*\ell} \end{pmatrix}.$$

The meaning of this matrix notation can be explained more precisely as follows. Assume that a is a generator in $C^{(\omega)}_q(Y_0,\mathfrak{s})$. Upon regarding a as a generator of $C_n(Y_0,\mathfrak{s})$ for $n = q(mod\ \ \ell)$, we obtain

$$\partial_n(a) = \partial^{(\omega)}_q(a) + \sum_{k>0} \partial^{(\omega)}_{q,k}(a),$$

with $\partial^{(\omega)}_{q,k} : C^{(\omega)}_q(Y_0,\mathfrak{s}) \to C^{(\omega)}_{q-1+k\ell}$ for $k > 0$. In particular, the relation $\partial_{n-1}\circ\partial_n = 0$ implies that $\partial^{(\omega)}_{n-1}\circ\partial^{(\omega)}_n = 0$ is also satisfied.

Proof. The first statement about the decomposition of the chain complex is obvious from the definition. Now we study the boundary operator under this decomposition.

For any $k < 0$, $a \in \mathcal{M}^{(\omega)}_{Y_0,q}(\mathfrak{s})$, and $b \in \mathcal{M}^{(\omega)}_{Y_0,q-1+k\ell}(\mathfrak{s})$, we shall prove that the 0-dimensional components $\hat{\mathcal{M}}^0(a,b)$ in $\hat{\mathcal{M}}(a,b)$ is empty, hence the entry of the boundary operator is trivial, $\langle b, \partial_n(a)\rangle = 0$.

Notice that we have

$$\mathrm{i}^{(\omega)}_{Y_0}(a) = \mathrm{i}_{Y_0}(a^\omega) = q \quad \text{and} \quad \mathrm{i}^{(\omega)}_{Y_0}(b) = \mathrm{i}_{Y_0}(b^\omega) = q-1+k\ell.$$

Thus, the moduli space of flow lines on $\mathcal{B}_\ell$, $\hat{\mathcal{M}}(a^\omega, b^\omega)$ has virtual dimension $-kl > 0$. There exists a unique element $[\lambda] \in \mathcal{G}/\mathcal{G}_\ell$, such that

$$\langle [\lambda] \cup c_1(L), [Y_0]\rangle = -kl.$$

This implies that we have $\mathrm{i}_{Y_0}(\lambda(b^\omega)) = SF(H_{(A(t),\psi(t))})^{\tilde{a}_0}_{\lambda(b^\omega)} = q-1$, and $CSD(\lambda(b^\omega)) = CSD(b^\omega) - 8\pi^2 k\ell$, see Lemma 3.1.

When non-empty, $\hat{\mathcal{M}}^0(a,b)$ is isomorphic to $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$. We can prove that $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is empty, since the CSD functional is non-increasing along the gradient flow lines and the difference of CSD between a^ω and $\lambda(b^\omega)$ is negative:

$$\begin{aligned}
& CSD(a^\omega) - CSD(\lambda(b^\omega)) \\
= \;& CSD(a^\omega) - CSD(b^\omega) - (CSD(\lambda(b^\omega)) - CSD(b^\omega)) \\
= \;& CSD(a^\omega) - CSD(b^\omega) + 8\pi^2 k\ell < 0,
\end{aligned}$$

as $k < 0$ and $|CSD(a^\omega) - CSD(b^\omega)| < 8\pi^2\ell$. This proves that the entries below the diagonal are always zero.

For $a \in \mathcal{M}^{(\omega)}_{Y_0,q}(\mathfrak{s})$ and $b \in \mathcal{M}^{(\omega)}_{Y_0,q-1}(\mathfrak{s})$, it is easy to see that we have

$$\langle \partial_n(a), b\rangle = \langle \partial^{(\omega)}_q(a), b\rangle.$$

For $a \in \mathcal{M}^{(\omega)}_{Y_0,q}(\mathfrak{s})$ and $b \in \mathcal{M}^{(\omega)}_{Y_0,q-1+k\ell}(\mathfrak{s})$ with $(k > 0)$, we can define

$$\langle \partial^{(\omega)}_{q,k}(a), b\rangle = \langle \partial_n(a), b\rangle = \#\big(\hat{\mathcal{M}}^0(a,b)\big).$$

This counts the points in the zero dimensional components $\hat{\mathcal{M}}^0(a,b)$ in the moduli space of trajectories on $\mathcal{B}$ from a to b. Equivalently, we have

$$\langle \partial^{(\omega)}_{q,k}(a), b\rangle = \#\big(\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))\big),$$

where $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is the moduli space of trajectories on $\mathcal{B}_\ell$ from a^ω to $\lambda(b^\omega)$ and λ represents the unique element $[\lambda]$ in $\mathcal{G}/\mathcal{G}_\ell$ such that

$$\langle [\lambda] \cup c_1(L), [Y_0]\rangle = -kl.$$

Thus we have $\mathrm{i}_{Y_0}(\lambda(b^\omega)) = q - 1$. Therefore, if non-empty, the moduli space $\hat{\mathcal{M}}(a^\omega, \lambda(b^\omega))$ is an oriented, compact 0-dimensional manifold with energy given by

$$8\pi^2 k\ell + CSD(a^\omega) - CSD(b^\omega) > 8\pi^2(k-1)\ell > 0.$$

This completes the proof of the Lemma. □

The expression of ∂_n and the appearance of $\partial^{(\omega)}_{q,k}$ in Lemma 3.4 lead us naturally to introduce a filtration of of the $\mathbb{Z}_\ell$ graded complex $C_*(Y_0,\mathfrak{s})$. The filtration is given by

$$F^{(\omega)}_q C_n = \bigoplus_{k\geq 0} C^{(\omega)}_{q+k\ell}(Y_0,\mathfrak{s}).$$

for $n \in \mathbb{Z}_\ell$, and $q \in \mathbb{Z}$ with $q \equiv n (mod\ \ell)$. Thus, we have

$$\cdots \subset F^{(\omega)}_{q+\ell} C_n \subset F^{(\omega)}_q C_n \subset F^{(\omega)}_{q-\ell} C_n \subset \cdots \subset C_n(Y_0, \mathfrak{s}), \tag{4}$$

a finite length decreasing filtration of the $\mathbb{Z}$-graded Seiberg–Witten–Floer chain complex. From Lemma 3.4, we see that the boundary operator

$$\partial_n : \qquad F^{(\omega)}_q C_n \longrightarrow F^{(\omega)}_{q-1} C_{n-1}$$

preserves the filtration. Let $F^{(\omega)}_q H_n$ denote the homology of the complex

$$\cdots \xrightarrow{\partial} F^{(\omega)}_q C_n \xrightarrow{\partial} F^{(\omega)}_{q-1} C_{n-1} \xrightarrow{\partial} \cdots .$$

Define

$$F^{(\omega)}_q HF^{SW}_n(Y_0, \mathfrak{s}) = Im\big(F^{(\omega)}_q H_n \to HF^{SW}_n(Y_0, \mathfrak{s})\big).$$

We thus obtain a bounded filtration on $HF^{SW}_n(Y_0, \mathfrak{s})$,

$$\begin{aligned} \cdots \subset F^{(\omega)}_{q+\ell} HF^{SW}_n(Y_0, \mathfrak{s}) &\subset F^{(\omega)}_q HF^{SW}_n(Y_0, \mathfrak{s}) \\ \subset F^{(\omega)}_{q-\ell} HF^{SW}_n(Y_0, \mathfrak{s}) &\subset \cdots \subset HF^{SW}_n(Y_0, \mathfrak{s}). \end{aligned}$$

The standard procedure of constructing the spectral sequence for a filtration [18] gives the following theorem on the relation between the $\mathbb{Z}$-graded and the $\mathbb{Z}_\ell$-graded homology groups.

Theorem 3.5. *There exists a spectral sequence $(E^k_{q,n}(Y_0, \mathfrak{s}), d^k)$ with*

$$E^1_{q,n}(Y_0, \mathfrak{s}) \cong HF^{SW}_{q,(\omega)}(Y_0, \mathfrak{s})$$

for $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q \equiv n(mod\ \ell)$. The higher differentials

$$d^k : \qquad E^k_{q,n}(Y_0, \mathfrak{s}) \longrightarrow E^k_{q-1+k\ell, n-1}(Y_0, \mathfrak{s})$$

are induced by the maps $\partial^{(\omega)}_{q,k}$ defined in Lemma 3.4. Furthermore, the spectral sequence $(E^k_{q,n}(Y_0, \mathfrak{s}), d^k)$ converges to the $\mathbb{Z}_\ell$-graded homology groups $HF^{SW}_n(Y_0, \mathfrak{s})$.

Proof. By construction, the $\mathbb{Z}_\ell$-graded chain complex $C_n(Y_0, \mathfrak{s})$ has a bounded filtration (4) with the associated graded complex given by

$$F^{(\omega)}_q C_n / F^{(\omega)}_{q+\ell} C_n = C^{(\omega)}_q(Y_0, \mathfrak{s}).$$

Then by the standard technique of [18], we derive the existence of a spectral sequence

$$(E^k_{q,n}(Y_0,\mathfrak{s}), d^k)$$

with $E^1_{q,n}(Y_0,\mathfrak{s}) \cong HF^{SW}_{q,(\omega)}(Y_0,\mathfrak{s})$, and

$$\begin{aligned} Z^k_{q,n}(Y_0,\mathfrak{s}) &= \{a \in F^{(\omega)}_q C_n |\ \partial_n(a) \in F^{(\omega)}_{q-1+k\ell} C_{n-1}\} \\ E^k_{q,n}(Y_0,\mathfrak{s}) &= Z^k_{q,n}(Y_0,\mathfrak{s}) / \big(Z^{k-1}_{q+\ell,n}(Y_0,\mathfrak{s}) + \partial_{n+1} Z^{k-1}_{q+1-(k-1)\ell,n+1}(Y_0,\mathfrak{s})\big). \end{aligned}$$

The higher differentials are induced by ∂_n. The expression of ∂_n, as discussed in Lemma 3.4, tells us that the higher differentials acting on $E^k_{q,n}(Y_0,\mathfrak{s})$ are defined by $\partial^{(\omega)}_{q,k}$. $\square$

4 $\mathbb{Z}$-graded homology for exact triangles

In this section, we present a different construction which applies to the special case where Y_0 is obtained by Dehn surgery with framing 0 on a knot K in a $\mathbb{Z}$-homology 3-sphere Y. Again we assume that the Spin^c-structure is non-torsion. The torsion case will be analyzed in the next Section.

Under our assumptions, Y_0 has the homology of $S^1 \times S^2$. We denote by Y_1 the $\mathbb{Z}$-homology 3-sphere obtained by Dehn surgery on K with framing 1. With minor modifications, the arguments in this subsection can be extended to the case of a knot K representing a zero homology class in $H_1(Y,\mathbb{Z})$ in a general 3-manifold Y.

In [2], we have identified the (perturbed) irreducible Seiberg–Witten monopoles (as critical points of the perturbed CSD functional) on (Y,g), with monopoles on (Y_1,g_1) and on Y_0, with

$$\mathcal{M}^*_{Y,g,\mu} \cong \mathcal{M}^*_{Y_1,g_1} \cup \bigcup_{\mathfrak{s}} \mathcal{M}_{Y_0}(\mathfrak{s}). \tag{5}$$

The metric and perturbation on Y have been carefully chosen to met the above identification. In particular, the perturbation μ is constructed as in [2] so as to "simulate the effect of surgery". Namely, it is obtained as a deformation of the circle of flat connections on the tubular neighborhood of the knot $\nu(K)$ in Y, so that it approximates the union of the lines of flat connections on $\nu(K)$ inside Y_1 and Y_0.

For each Spin^c structure $\mathfrak{s}$ on Y_0, with non-empty moduli space $\mathcal{M}_{Y_0}(\mathfrak{s})$, we have an injective map

$$j_{\mathfrak{s}}: \qquad \mathcal{M}_{Y_0}(\mathfrak{s}) \longrightarrow \mathcal{M}_{Y,g,\mu} \tag{6}$$

defined by the identification (5).

Definition 4.1. *Assume that $c_1(\mathfrak{s})$ is a non-torsion class in $H^2(Y_0, \mathbb{Z})$. We define the $\mathbb{Z}$-grading of elements in $\mathcal{M}_{Y_0}(\mathfrak{s})$ as induced from the grading on $\mathcal{M}_{Y,g,\mu}$,*

$$\mathrm{i}^{(Y)}_{Y_0}(a) = \mathrm{i}_Y(j_{\mathfrak{s}}(a))$$

for any $a \in \mathcal{M}_{Y_0}(\mathfrak{s})$.

Recall that the grading on $\mathcal{M}_{Y,g,\mu}$ is defined by the spectral flow of the Hessian operator of CSD along a path from the critical point in $\mathcal{M}_{Y,g,\mu}$ to the unique reducible point ϑ in $\mathcal{M}_{Y,g,\mu}$. The relative index $\mathrm{i}_{Y_0}(a,b)$ of two points a and b in $\mathcal{M}_{Y_0}(\mathfrak{s})$, defined as the spectral flow of the Hessian operator of CSD function for $(Y_0, \mathfrak{s})$, agrees with the relative index $\mathrm{i}_Y(j_{\mathfrak{s}}(a), j_{\mathfrak{s}}(b))$, cf. [2]. Thus, we have

$$\mathrm{i}^{(Y)}_{Y_0}(a) - \mathrm{i}^{(Y)}_{Y_0}(b) = \mathrm{i}_{Y_0}(a,b) \ (mod\ \ell)$$

where $\ell \in \mathbb{Z}$ satisfies $c_1(\mathfrak{s})(H_2(Y_0,\mathbb{Z})) = \ell\mathbb{Z}$. Thus, $\mathrm{i}^{(Y)}_{Y_0}$ is a $\mathbb{Z}$-lifting of the $\mathbb{Z}_\ell$-grading on $\mathcal{M}_{Y_0}(\mathfrak{s})$.

Definition 4.2. *For any $q \in \mathbb{Z}$, we define*

$$\mathcal{M}^{(q)}_{Y_0}(\mathfrak{s}) = \{a \in \mathcal{M}_{Y_0}(\mathfrak{s}) \mid \mathrm{i}^{(Y)}_{Y_0}(a) = q\}.$$

The q-chains in the corresponding $\mathbb{Z}$-graded Floer complex are the elements of the abelian group $C_{(q)}(Y_0, \mathfrak{s})$ generated by the points of $\mathcal{M}^{(q)}_{Y_0}(\mathfrak{s})$. The boundary operator

$$\partial_{(q)} : C_{(q)}(Y_0, \mathfrak{s}) \to C_{(q-1)}(Y_0, \mathfrak{s})$$

is defined as

$$\partial_{(q)}(a) = \sum_{b \in \mathcal{M}^{(q-1)}_{Y_0}(\mathfrak{s})} \#\big(\hat{\mathcal{M}}^0(a,b)\big) b,$$

where $\hat{\mathcal{M}}^0(a,b)$ is the zero dimensional components of the moduli space of unparameterized flow lines in $\hat{\mathcal{M}}(a,b)$. The compactness theorem tells us that $\hat{\mathcal{M}}^0(a,b)$ is an oriented, compact 0-dimensional manifold. Thus, the coefficient $\#(\hat{\mathcal{M}}^0(a,b))$ is well-defined as the algebraic sum of points in $\hat{\mathcal{M}}^0(a,b)$.

In order to prove that we have $\partial_{(q)} \circ \partial_{(q-1)} = 0$, we need the following Lemma to know when the zero-dimensional moduli space of flow lines between a and b over $(Y_0(r), \mathfrak{s})$ is non-empty. Here we follow the notation of [2], where $Y_0(r)$ and $Y(r)$ denote the 3-manifolds Y_0 and Y endowed with long cylinders $T^2 \times [-r, r]$ along the boundary of the tubular neighborhood $\nu(K)$ around the knot K.

Lemma 4.3. *Let $\mathfrak{s}$ be a* Spin^c *structure on Y_0 with non-empty $\mathcal{M}_{Y_0}(\mathfrak{s})$, and $c_1(\mathfrak{s})$ is a non-trivial class in $H^2(Y_0, \mathbb{Q})$. Then the moduli spaces of flow lines can be described as follows. Assume that a, b are two critical points in $\mathcal{M}_{Y_0}(\mathfrak{s})$ of relative index 1 modulo ℓ, here $\ell \in \mathbb{Z}$ satisfies $c_1(\mathfrak{s})(H_2(Y_0, \mathbb{Z})) = \ell\mathbb{Z}$. If for sufficiently large r, the 1-dimensional component $\mathcal{M}^1_{Y_0(r)\times\mathbb{R}}(a, b)$ of $\mathcal{M}_{Y_0(r)\times\mathbb{R}}(a, b)$, is non-empty, then we have*

$$\mathfrak{i}^{(Y)}_{Y_0}(a) - \mathfrak{i}^{(Y)}_{Y_0}(b) = 1.$$

Proof. The result is the consequence of the results on the geometric limits in [12] [13], together with the dimension formulae for various moduli spaces.

Suppose given a solution $[A_r(t), \Psi_r(t)]$ in a one-dimensional component of $\mathcal{M}_{Y_0(r)\times\mathbb{R}}(a, b)$, with asymptotic values

$$[A_r(\pm\infty), \Psi_r(\pm\infty)] = [A'_{\pm\infty}, \Psi'_{\pm\infty}]\#^r_{a\pm}[a^{\pm}, 0],$$

for $t \to \pm\infty$. The geometric limits as $r \to \infty$ are determined in [12]. Among these geometric limits, there are two parameterized oriented paths $a_V(\tau), a_0(\tau)$ on $\partial_\infty(\mathcal{M}^*_V)$ and $\mathcal{M}_{\nu(K)\subset Y_0} = L_{Y_0}$, respectively, where both paths connect $[a^{\pm}, 0]$, consistently with the orientations. Here $\mathcal{M}_{\nu(K)\in Y_0}$ is the circle of flat $U(1)$-connections on $\nu(K) \subset Y_0$ modulo gauge equivalence. There is also a holomorphic map a_D from the unit half-disc to $\chi(T^2, Y_0)$ with boundary along the paths $a_V(\tau)$ and $a_0(\tau)$.

First we show that, if the moduli space $\mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), j_\mathfrak{s}(b))$ is non-empty, then we have

$$\mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), j_\mathfrak{s}(b)) \cong \mathcal{M}^1_{Y_0(r)\times\mathbb{R}}(a, b).$$

Under this assumption, the geometric limits of a family of solutions $[A_r(t), \Psi_r(t)]$ in $\mathcal{M}_{Y_0(r)\times\mathbb{R}}(a, b)$, for $r \to \infty$, can be deformed by a homotopy in $H^1(T^2, i\mathbb{R})$ to geometric limits for a monopole in $\mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), j_\mathfrak{s}(b))$. In particular, this can happen if and only if the path $a_0(\tau) = [A(\tau), 0]$ along

L_{Y_0} can be homotopically deformed to a path along the curve $\partial_\infty \mathcal{M}_{\nu(K),\mu}$, by a homotopy that moves the endpoints $a^\pm$ to the corresponding points $a^\pm(\epsilon)$, along $a_V(\tau)$. (We use here the same notation as in [2], [12], [13].) Both paths $a_V(\tau)$ and $a_0(\tau)$ must be contractible in $\chi(T^2, Y_0) = S^1 \times \mathbb{R}$. In fact, the geometric limits of any solution in $\mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), j_\mathfrak{s}(b))$ define an approximate solution to the monopole equations on $Y_0(r) \times \mathbb{R}$, which can be deformed to a solution in the minimal energy component of $\mathcal{M}_{Y_0(r)\times\mathbb{R}}(a, b)$. Conversely, in this case, the geometric limits of $[A_r(t), \Psi_r(t)]$ can be spliced together to form a solution to the monopole equations on $(Y(r) \times \mathbb{R}, \mu, g)$ between the critical points $j_\mathfrak{s}(a)$ and $j_\mathfrak{s}(b)$. Thus the identification of the moduli spaces implies that

$$\mathrm{i}^{(Y)}_{Y_0}(a) - \mathrm{i}^{(Y)}_{Y_0}(b) = 1.$$

Now assume that the moduli space $\mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), j_\mathfrak{s}(b))$ is empty. Notice that oriented paths $a_V(\tau), a_0(\tau)$ which are non-contractible in the cylinder $\chi_0(T^2, \nu(K)) \cong S^1 \times \mathbb{R}$ correspond to flowlines in the components with higher energy. Here

$$\chi(T^2, Y_0) = \chi_0(T^2, \nu(K))$$

is a covering of the torus $\chi(T^2)$ of flat $U(1)$-connections on T^2 modulo gauge equivalence, obtained by taking the quotient only by those gauge transformations on T^2 that extend to $Y_0(r)$, cf. [2]. Since we are considering the flowlines in the minimal energy components $\mathcal{M}^1_{Y_0(r)\times\mathbb{R}}(a, b)$, the paths $a_V(\tau)$ and $a_0(\tau)$ are also contractible in $\chi(T^2, Y_0) = S^1 \times \mathbb{R}$. The analysis of the moduli space on the cobordant manifold W_0 with cylindrical ends $Y_0(r) \times (-\infty, 0]$ and $Y(r) \times [0, \infty)$ in [13] shows that the holomorphic half-disc a_D, from the geometric limits of $[A_r(t), \Psi_r(t)]$ in the one-dimensional components of $\mathcal{M}_{Y_0(r)\times\mathbb{R}}(a, b)$, is the degenerate limit of the assembled holomorphic triangles Δ^ϵ as $\epsilon \to 0$, where ϵ is the parameter in the perturbation on $Y(r)$ that "simulates the effect of surgery". In particular, we have the following identification

$$\mathcal{M}^1_{Y_0(r)\times\mathbb{R}}(a, b) \cong \bigcup_{c \in \mathcal{M}^*_{Y_1}} \mathcal{M}_{Y(r)\times\mathbb{R}}(j_\mathfrak{s}(a), i(c)) \times \mathcal{M}^{W_0,(0)}(i(c), b).$$

Here $i(c)$ is the corresponding monopole in $\mathcal{M}^*_{Y,\mu,g}$, under (5), satisfying $\mathrm{i}_Y(j_\mathfrak{s}(a)) - \mathrm{i}_Y(i(c)) = 1$, and $\mathcal{M}^{W_0,(0)}(i(c), b)$ is given by the non-empty 0-dimensional components of $\mathcal{M}^{W_0,(0)}(i(c), b)$. Here we follow the notation of [13]. Then the dimension formulae developed in [13] can be applied to show that

$$\mathrm{i}_Y(i(c)) - \mathrm{i}_{Y_0}^{(Y)}(b) = 0.$$

Thus we have $\mathrm{i}_{Y_0}^{(Y)}(a) - \mathrm{i}_{Y_0}^{Y}(b) = 1$. This completes the proof of the statement. □

Using the result of Lemma 4.3, we now show that $\partial_{(q-1)} \circ \partial_{(q)} = 0$, and we relate the resulting homology groups to the original $\mathbb{Z}_\ell$-graded Floer homology groups for $(Y_0, \mathfrak{s})$.

Proposition 4.4. *1. For $n \in \mathbb{Z}_\ell$ and $q \in \mathbb{Z}$ with $q = n(mod\ \ell)$, then*

$$C_n(Y_0, \mathfrak{s}) = \bigoplus_{k \in \mathbb{Z}} C_{(q+k\ell)}(Y_0, \mathfrak{s}).$$

2. Under the decomposition as above, the boundary operator ∂_n on $C_n(Y_0, \mathfrak{s})$ can be expressed as $\oplus_{k \in \mathbb{Z}} \partial_{(q+k\ell)}$. Thus, in particular, we have $\partial_{(q-1)} \circ \partial_{(q)} = 0$ and

$$HF_n^{SW}(Y_0, \mathfrak{s}) = \bigoplus_{k \in \mathbb{Z}} HF_{(q+k\ell)}^{SW}(Y_0, \mathfrak{s}),$$

where $HF_{(q)}^{SW}(Y_0, \mathfrak{s})$ is the q-th homology group of the chain complex $(C_{(q)}(Y_0, \mathfrak{s}), \partial_{(q)})$.

Proof. We only need to prove the second statement. For any generator $a \in C_{(q)}(Y_0, \mathfrak{s})$, as an element in $C_n(Y_0, \mathfrak{s})$, we shall verify that $\partial_n(a) = \partial_{(q)}(a)$. From the definition, we know that

$$\partial_n(a) = \sum_{b \in \mathcal{M}_{Y_0}(\mathfrak{s}) : \mathrm{i}_{Y_0}(a,b) = 1 (mod\ \ell)} \#\big(\hat{\mathcal{M}}^0_{Y_0 \times \mathbb{R}}(a, b)\big) b,$$

where $\hat{\mathcal{M}}^0_{Y_0 \times \mathbb{R}}(a, b)$ is the zero dimensional component of the moduli space of the unparameterized flow lines in $\hat{\mathcal{M}}_{Y_0 \times \mathbb{R}}(a, b)$. By the result of Lemma 4.3, we know that the critical points $b \in \mathcal{M}_{Y_0}(\mathfrak{s})$ with non-empty zero dimensional component in $\hat{\mathcal{M}}_{Y_0 \times \mathbb{R}}(a, b)$ are contained in the set

$$\mathcal{M}_{Y_0}^{(q-1)}(\mathfrak{s}) = \{b \in \mathcal{M}_{Y_0}(\mathfrak{s}) \mid \mathrm{i}_{Y_0}^{(Y)}(b) = q - 1\}.$$

This implies that $\partial_n(a) = \partial_{(q)}(a)$ for $a \in C_{(q)}(Y_0, \mathfrak{s})$, hence $\partial_n = \oplus_{k \in \mathbb{Z}} \partial_{(q+k\ell)}$. Then the statement follows from standard arguments in homological algebra. □

In the exact triangle for the Seiberg–Witten–Floer homologies relating $HF_*^{SW}(Y,g)$, $HF_*^{SW}(Y_1,g_1)$, and $HF_{(*)}^{SW}(Y_0,\mathfrak{s}_k)$, as studied in [12][13], the $\mathbb{Z}$-graded Seiberg–Witten–Floer homology for $(Y_0.\mathfrak{s})$ with $\mathfrak{s}$ a non-trivial Spin^c structure, is precisely the $\mathbb{Z}$-graded $HF_{(*)}^{SW}(Y_0,\mathfrak{s})$ described in this subsection.

Remark 4.5. *In the case where Y_0 is obtained as 0-surgery on a knot K in a homology 3-sphere, both the constructions of $\mathbb{Z}$-graded Floer homology are possible.*

5 Monopole homology for trivial Spin^c-structures

In this section we consider again the general case where Y_0 is a compact 3-manifold with $b_1(Y_0)>0$. We analyze the Seiberg–Witten Floer theory in the case of a Spin^c-structure $\mathfrak{s}_0$ with $c_1(\det(\mathfrak{s}_0))$ torsion. The results we obtain specialize to the case of Y_0 a 0-surgery on a knot in a homology 3-sphere, with the trivial Spin^c-structure with $c_1(\det(\mathfrak{s}_0))=0$.

A cohomologically trivial perturbation would introduce a torus of reducible monopoles among the critical points of the Chern–Simons–Dirac functional given by the flat $U(1)$-connections $H^1(Y_0,\mathbb{R})/H^1(Y_0,\mathbb{Z})$. In order to avoid the torus of reducibles, we need to introduce a small cohomologically non-trivial perturbation of the Seiberg–Witten equations on Y_0 given by a 1-form ρ satisfying $[*\rho]\in H^2(Y_0,\mathbb{R})$ non-trivial. With this perturbation, $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ consists of a finite set of points in $\mathcal{B}$, cut out transversely by the 3-dimensional (perturbed) Seiberg–Witten monopole equations.

The different nature of the case $c_1(\mathfrak{s}_0)$ torsion can be regarded as a result of the following phenomenon. In the case analyzed previously, with $c_1(\mathfrak{s})$ non-torsion, we studied the critical points of CSD on the infinite cyclic covering space $\mathcal{B}_\ell$ of $\mathcal{B}$. The main point of our argument has been the following: the non-trivial covering $\mathcal{B}_\ell$ is the essential tool in separating the various components of $\hat{\mathcal{M}}(a,b)$ of different dimensions, with each of them separately admitting a nice compactification to a smooth manifold with corners as in [11], [15].

In the case with $c_1(\mathfrak{s}_0)$ torsion, however, the relative index $\mathfrak{i}(a)-\mathfrak{i}(b)$ is already well defined as an integer in $\mathcal{B}$. Thus, the moduli space $\hat{\mathcal{M}}(a,b)$ does not break into components of different dimensions. In fact, all components in $\hat{\mathcal{M}}(a,b)$ have constant dimension given by the relative index $\mathfrak{i}(a)-\mathfrak{i}(b)\in\mathbb{Z}$. In order to define the Floer homology, we still need a nice compactification of the moduli spaces $\hat{\mathcal{M}}(a,b)$. However, we lack the uniform energy bound on $\hat{\mathcal{M}}(a,b)$, since the CSD functional is only circle-valued, due to the presence of the non-trivial cohomology class of the perturbation $[*\rho]$. This lack of a uniform energy estimate implies that we cannot define the usual boundary operator

$$\langle \partial a, b\rangle = n_{ab}$$

of the Floer complex by setting

$$n_{ab} = \#\hat{\mathcal{M}}(a,b),$$

since $\hat{\mathcal{M}}(a,b)$ may be non-compact in this case.

Thus, we need to separate $\hat{\mathcal{M}}(a,b)$ into components of uniform energy and define a boundary operator in terms of components of a fixed energy. We identify $\hat{\mathcal{M}}(a,b)$ with a subset of $\mathcal{M}(a,b)$ consisting of those trajectories with equal energy distributions on $(-\infty,0]$ and $[0,\infty)$, that is,

$$\begin{aligned} &\int_{-\infty}^{0}(\|\partial_t A(t)\|^2_{L^2(Y_0)} + \|\partial_t\psi(t)\|^2_{L^2(Y_0)})dt \\ = &\int_{0}^{\infty}(\|\partial_t A(t)\|^2_{L^2(Y_0)} + \|\partial_t\psi(t)\|^2_{L^2(Y_0)})dt. \end{aligned} \tag{7}$$

Every non-constant trajectory $[A(t),\psi(t)]$ in a non-empty connected component of $\hat{\mathcal{M}}(a,b)$ has positive energy

$$\int_{\mathbb{R}}(\|\partial_t A(t)\|^2_{L^2(Y)} + \|\partial_t\psi(t)\|^2_{L^2(Y)})dt > 0.$$

The energy is constant, independent of the trajectory, in each connected component of $\hat{\mathcal{M}}(a,b)$. This energy agrees with the variation of the perturbed CSD functional along $[A(t),\psi(t)]$. In fact, we have

$$-\|\nabla CSD_{U,V}\|^2_{L^2} = \langle \partial_t(A,\psi), \nabla CSD_{U,V}\rangle,$$

if $CSD_{U,V} = CSD + U + V$ denotes the perturbed CSD functional, and (A,ψ) is a solution of the flow equation

$$\partial_t(A,\psi) = -\nabla CSD_{U,V}.$$

This gives

$$\begin{aligned} &\int_t^{\infty}\|\nabla CSD_{U,V}(A(\tau),\psi(\tau))\|^2_{L^2(Y)}d\tau = \\ &CSD_{U,V}(A(t),\psi(t)) - CSD_{U,V}(A_b,\psi_b). \end{aligned}$$

Thus, we can define the energy function over $\hat{\mathcal{M}}(a,b)$ as

$$\begin{aligned} \mathcal{E}: \qquad & \hat{\mathcal{M}}(a,b) \longrightarrow \mathbb{R} \\ \mathcal{E}([A(t),\psi(t)]) = & \int_{\mathbb{R}} (\|\partial_t A(t)\|^2_{L^2(Y)} + \|\partial_t \psi(t)\|^2_{L^2(Y)}) dt. \end{aligned} \tag{8}$$

The image of $\mathcal{E}$ is positive and discrete in $\mathbb{R}$. The energies of any two connected components may differ by some multiple of a positive number

$$e_\rho = min\{|[*\rho]([\Sigma])| : [\Sigma] \in H_2(Y_0,\mathbb{Z})\}.$$

Denote $e_{min}(a,b)$ the minimal energy on $\hat{\mathcal{M}}(a,b)$. Then, for any $n \geq 0$ in $\mathbb{Z}$, we can define

$$\hat{\mathcal{M}}^{(n)}(a,b) = \mathcal{E}^{-1}(e_{min}(a,b) + ne_\rho).$$

We have the following compactness theorem for the moduli space $\hat{\mathcal{M}}^{(n)}(a,b)$ with the fixed energy $e_{min}(a,b) + ne_\rho$, for some $n \geq 0$.

Proposition 5.1. *Let $(Y_0,\mathfrak{s}_0)$ be a 3-manifold with $b_1(Y) > 0$ and with $c_1(\mathfrak{s}_0)$ torsion. Let $\hat{\mathcal{M}}^{(n)}(a,b)$ be the component of $\hat{\mathcal{M}}(a,b)$ defined as above, for two critical points a and b of relative index $q+1$. Then $\hat{\mathcal{M}}^{(n)}(a,b)$ can be compactified to be a smooth manifold of dimension q with corner structure, by adding lower dimensional boundary strata. The codimension k boundary strata are given by*

$$\bigcup_{a_1,\cdots,a_k} \hat{\mathcal{M}}^{(n_0)}(a,a_1) \times \hat{\mathcal{M}}^{(n_1)}(a_1,a_2) \times \cdots \times \hat{\mathcal{M}}^{(n_k)}(a_k,b).$$

Here the union is over all possible sequences $a_0 = a, a_1, \cdots a_k, a_{k+1} = b$ with strictly decreasing indices and with $\sum_{j=0}^k n_j = n$. In particular, we have the following results.

1. *The component of fixed energy in $\hat{\mathcal{M}}(a,b)$ with relative index 1 is compact.*

2. *If a and c are two critical points of relative index $\mathfrak{i}(a) - \mathfrak{i}(c) = 2$, then the compactified moduli space $\hat{\mathcal{M}}^{(n)}(a,c)$ is an oriented, smooth manifolds with boundary points given by*

$$\bigcup_{b,i+j=n} \hat{\mathcal{M}}^{(i)}(a,b) \times \hat{\mathcal{M}}^{(j)}(b,c),$$

where b runs over all critical points with relative index $\mathfrak{i}(a) - \mathfrak{i}(b) = 1$.

Proof. The convergence and gluing theorems developed in Section 4 of [11] can be applied to this case to give $\hat{\mathcal{M}}^{(n)}(a,b)$ the structure of a smooth manifold structure with corners. In particular, the local diffeomorphism provided by the convergence and gluing theorem in [11] is energy preserving. Then the above claims follow from the arguments of Section 4 of [11]. Notice that we have both inequalities

$$e_{min}(a,b) \leq e_{min}(a,a_1) + \cdots + e_{min}(a_k,b)$$

and

$$e_{min}(a,b) \geq e_{min}(a,a_1) + \cdots + e_{min}(a_k,b),$$

where we use essentially the identification between energy and variation of the perturbed CSD functional. $\square$

We now introduce a chain complex over the ring $\mathbb{Z}[[t]]$ of formal power series as follows. We define the chain complex $C_*(Y_0,\mathfrak{s}_0,\mathbb{Z}[[t]])$ to be the $\mathbb{Z}[[t]]$-module generated by the critical points in $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$, with the grading given by the relative grading $\mathfrak{i}(a)-\mathfrak{i}(a_0)$, where a_0 is a fixed critical point in $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$. The boundary operator ∂ is defined by

$$\partial_{[t]}(a) = \sum_{b:\mathfrak{i}(a)-\mathfrak{i}(b)=1} \sum_{k\geq 0} \#\big(\hat{\mathcal{M}}^{(n)}(a,b)\big)t^n b. \tag{9}$$

By the result of Proposition 5.1, we know that the boundary operator $\partial_{[t]}$ in (9) is well-defined and satisfies $\partial^2=0$. The corresponding homology groups are the homology groups of the chain complex $(C_*(Y_0,\mathfrak{s}_0,\mathbb{Z}[[t]]),\partial_{[t]})$,

$$HF_*(Y_0,\mathfrak{s}_0,\mathbb{Z}[[t]]) = H_*(C_*(Y_0,\mathfrak{s}_0,\mathbb{Z}[[t]]),\partial_{[t]}). \tag{10}$$

In the construction of the exact triangle in [2], [12], [13], we set

$$HF_{(*)}(Y_0,\mathfrak{s}_0) = HF_*(Y_0,\mathfrak{s}_0,\mathbb{Z}[[t]])|_{t=0}.$$

Here the grading on $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ is induced from the injective map

$$j_{\mathfrak{s}_0}: \qquad \mathcal{M}_{Y_0}(\mathfrak{s}_0) \longrightarrow \mathcal{M}_{Y,g,\mu},$$

that is, for any $a\in\mathcal{M}_{Y_0}(\mathfrak{s}_0)$, we define the induced grading from $j_{\mathfrak{s}_0}$ as

$$\mathfrak{i}^{(Y)}_{Y_0}(a) = \mathfrak{i}_Y(j_{\mathfrak{s}_0}(a)).$$

This induced grading on $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ satisfies (cf. [2])

$$\mathfrak{i}^{(Y)}_{Y_0}(a) - \mathfrak{i}^{(Y)}_{Y_0}(b) = \mathfrak{i}_{Y_0}(a) - \mathfrak{i}_{Y_0}(b).$$

After a suitable choice of a base point a_0 in $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$, we can assume that

$$\mathrm{i}^{(Y)}_{Y_0}(a) = \mathrm{i}_{Y_0}(a) = \mathrm{i}_Y(j_{\mathfrak{s}_0}(a)). \tag{11}$$

Notice that, in the case of the 0-surgery Y_0 that has $b_1(Y_0) = 1$, the Floer homology $HF_*(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])$ defined as in (10) depends on the cohomology class $[*\rho] \in H^2(Y_0, \mathbb{R})$. This dependence can be seen from the fact that, when the value $[*\rho] \in \mathbb{R}$ crosses zero, some irreducible solutions in $\mathcal{M}_{Y_0,\rho}(\mathfrak{s}_0)$ may hit the circle of reducibles, cf. [8].

We have the analog of Lemma 4.3, in the case of a 3-manifold Y_0 that is obtained as a 0-surgery on a knot in a homology 3-sphere Y.

Lemma 5.2. *Let $\mathfrak{s}_0$ be a trivial* Spin^c *structure on Y_0, where Y_0 is obtained as a 0-surgery on a knot in a homology 3-sphere Y. Let a, b be two critical points in $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ with relative index 1. Then if $\hat{\mathcal{M}}^{(0)}_{Y_0(r)\times\mathbb{R}}(a,b)$ is non-empty, for sufficiently large r the oriented paths $a_0(\tau)$, $a_V(\tau)$ in the geometric limits of any flow line in the component of minimal energy are contractible in $\chi_0(T^2, Y_0)$. Here the moduli spaces $\mathcal{M}_{Y_0}(\mathfrak{s}_0)$ and $\hat{\mathcal{M}}^{(0)}_{Y_0(r)\times\mathbb{R}}(a,b)$ are considered with an additional perturbation ρ with $[*\rho] = \eta > 0$ in $H^2(Y_0, \mathbb{R}) = \mathbb{R}$, as in [2], cf. Remark 5.3.*

Proof. This is again a direct applications of the geometric limit results in [12] and [13]. The argument proceeds as in the proof of Lemma 4.3.

□

The homology groups for $(Y_0, \mathfrak{s}_0)$, which appear in the exact triangle, are precisely given by the homology groups

$$HF_{(*)}(Y_0, \mathfrak{s}_0) = HF_*(Y_0, \mathfrak{s}_0, \mathbb{Z}[[t]])|_{t=0},$$

after a possible global index shifting.

Remark 5.3. *The Seiberg–Witten–Floer homology for $(Y_0, \mathfrak{s}_0)$ depends on the perturbation $[*\rho]$, as the energy function $\mathcal{E}$ and the Novikov-type chain complex all depend on the choice of $[*\rho]$. In the exact triangle, we use a particular choice of $[*\rho]$, satisfying*

$$[*\rho] = \eta PD_{Y_0}(m) \in H^2(Y_0, \mathbb{R}),$$

with m the meridian of the knot K.

6 Gluing formulae along the trivial Spin^c structure

Motivated by the general ideas of topological field theory, one can associate to any 4-manifold $(X_i, \mathfrak{s}_i)$ with a boundary $(Y_0, \mathfrak{t})$ Seiberg–Witten invariants with values in the monopole homology groups $HF_*^{SW}(Y_0, \mathfrak{t})$,

$$SW_{X_i}(\mathfrak{s}_i, \cdot): \qquad \mathbb{A}(X_i) \longrightarrow HF_*^{SW}(Y_0, \mathfrak{t}),$$

where $\mathbb{A}(X_i) = Sym^*(H_0(X_i)) \otimes \Lambda^*(H_1(X_i))$ is the free graded algebra generated by elements in $H_0(X_i)$ of degree 2 and cycles in $H_1(X_i)$ of degree 1. Moreover, if a closed 4-manifold $X = X_1 \cup_{Y_0} X_2$ splits in two components X_i, $i = 1, 2$, along a 3-manifold Y_0, then the natural pairing between $HF_*^{SW}(Y_0, \mathfrak{t})$ and $HF_*^{SW}(-Y_0, -\mathfrak{t})$ can be applied to give gluing formulae for the Seiberg–Witten invariants of X in terms of the SW_{X_i}. For any 3-manifold $(Y_0, \mathfrak{t})$ with $b_1(Y) > 0$ and $c_1(\mathfrak{t})$ non-torsion, these gluing formulae were developed in [3]. For a manifold Y with a torsion Spin^c structure $\mathfrak{t}$, we discuss the corresponding gluing formulae in this section.

To the purpose of gluing relative invariants, it is natural to construct the monopole homology groups over the field $\mathbb{Q}((t))$ of formal Laurent series and use another definition of the energy in the construction of monopole homology groups.

We assume that the perturbation term $[*\rho]$ is Poincaré dual to $\epsilon[\Gamma]$ for some 1-cycle Γ in Y_0 and a small $\epsilon \in \mathbb{R}^+$. Here Γ is an integer cycle, that is, it defines a (non-torsion) integer homology class. The moduli space of critical points with this perturbation is denoted by $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$. Let a, b be two critical points in $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$ of relative index $k+1 > 0$. We know that $\hat{\mathcal{M}}(a, b)$ has infinite many components of dimension k. Note that, if we identify elements in $\hat{\mathcal{M}}(a, b)$ with trajectories in $\mathcal{M}(a, b)$ satisfying the equal energy condition (7), we can now define the following map on $\hat{\mathcal{M}}(a, b)$:

$$\begin{aligned} \mathcal{E}_\rho: \qquad & \hat{\mathcal{M}}(a, b) \longrightarrow \mathbb{Z} \\ \mathcal{E}_\rho([A(t), \psi(t)]) &= \langle [\frac{\sqrt{-1}}{2\pi} F_{A(t)}], [\Gamma \times \mathbb{R}] \rangle. \end{aligned} \tag{12}$$

Here $F_{A(t)}$ is the 4-dimensional curvature of $A(t)$. $\mathcal{E}_\rho$ essentially measures the variation of the Chern-Simons-Dirac functional on $Y_0 \times \mathbb{R}$ up to a positive scalar. We denote by

$$\hat{\mathcal{M}}^{[n]}(a, b) = \mathcal{E}_\rho^{-1}(n)$$

the preimage of n under the energy map $\mathcal{E}_\rho$.

Lemma 6.1. *Let Y be a 3-manifold with $b_1(Y) > 0$, endowed with a* Spin^c *structure $\mathfrak{t}$ with $c_1(\mathfrak{t}) = 0$. Let a, b be two critical points in $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$ of relative index $k+1 > 0$. We have the following results.*

1. *$\hat{\mathcal{M}}^{[n]}(a,b)$ is empty for $n << 0$;*
2. *For any non-empty $\hat{\mathcal{M}}^{[n]}(a,b)$, there exists a compactification of $\hat{\mathcal{M}}^{[n]}(a,b)$ to a smooth manifold with corners, obtained by adding lower dimensional strata of the form*

$$\bigcup_{a_1,\cdots,a_k} \hat{\mathcal{M}}^{[n_0]}(a,a_1) \times \hat{\mathcal{M}}^{[n_1]}(a_1,a_2) \times \cdots \times \hat{\mathcal{M}}^{[n_k]}(a_k,b).$$

Here the union is over all possible sequences $a_0 = a, a_1, \cdots a_k, a_{k+1} = b$ in $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$ with strictly decreasing indices and with $\sum_{j=0}^{k} n_j = n$.

In particular, given any two critical points a and b of relative index 1, if the moduli space $\hat{\mathcal{M}}^{[n]}(a,b)$ is non-empty, then it is compact. For any two critical points a and c of relative index 2, the compactified moduli space $\hat{\mathcal{M}}^{[n]}(a,c)$ is an oriented, smooth manifold with boundary points given by

$$\bigcup_{b, i+j=n} \hat{\mathcal{M}}^{[i]}(a,b) \times \hat{\mathcal{M}}^{[j]}(b,c).$$

Here b runs over all critical points with relative index $\mathfrak{i}(a) - \mathfrak{i}(b) = 1$.

Proof. Any non-empty $\hat{\mathcal{M}}^{[n]}(a,b)$ has positive and constant energy defined by $\mathcal{E}$ (8). This implies that $\hat{\mathcal{M}}^{[n]}(a,b)$ must be empty for $n << 0$. The argument for the compactification is analogous to Proposition 5.1. □

Definition 6.2. *Define the chain complex $C_{\Gamma,*}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$ as the vector space generated by the critical points in $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$, over the coefficient field of formal Laurent series $\mathbb{Q}((t)) = \mathbb{Q}[[t]][t^{-1}]$, with the grading given by the relative index. The boundary operator is given by*

$$\partial(a) = \sum_{b: \mathfrak{i}_{Y_0}(a) - \mathfrak{i}_{Y_0}(b) = 1} \sum_{n \in Z} \#\big(\hat{\mathcal{M}}^{[n]}(a,b)\big) t^n b.$$

The fact that $\partial^2 = 0$ follows from Lemma 6.1. The homology groups of the chain complex $C_*(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$ are denoted by $HF^{SW}_{\Gamma,*}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$.

By the construction of $HF^{SW}_{\Gamma,*}(Y_0,\mathfrak{t},\mathbb{Q}((t)))$, we know that these homology groups depend on the choice of the non-trivial cohomology class $[*\rho]$, which in turn depends on the choice of the one-cycle Γ. There exists a natural isomorphism $HF^{SW}_{\Gamma,*}(Y_0,\mathfrak{t},\mathbb{Q}((t))) \cong HF^{SW,*}_{-\Gamma}(-Y_0,-\mathfrak{t},\mathbb{Q}((t)))$, where $-Y_0$ is the manifold Y_0 with the reversed orientation. For the definition of Seiberg–Witten–Floer cohomology, in the equivariant setting, the reader can refer to Section 5 of [11]. This yields a natural pairing

$$\langle , \rangle : HF^{SW}_{\Gamma,*}(Y_0,\mathfrak{t},\mathbb{Q}((t))) \times HF^{SW}_{-\Gamma,*}(-Y_0,-\mathfrak{t},\mathbb{Q}((t))) \longrightarrow \mathbb{Q}((t)). \tag{13}$$

6.1 Relative Seiberg–Witten invariants and gluing formulae

Let X_1 be an oriented Riemannian 4-manifold with a cylindrical end modeled on $Y_0 \times [0,\infty)$. Let $\mathfrak{s}_1$ be a Spin^c structure on X_1, such that the induced Spin^c structure $\mathfrak{t}$ on Y_0 satisfies $c_1(\mathfrak{t}) = 0$ in $H_2(Y_0,\mathbb{Q})$. Denote by $\mathcal{M}_{Y_0}(\mathfrak{t},\Gamma)$ the moduli space of Seiberg–Witten monopoles on $(Y_0,\mathfrak{t})$, with respect to a perturbation ρ satisfying $[*\rho] = \epsilon PD([\Gamma])$, for a non-trivial 1-cycle Γ in Y_0. Here again we assume that Γ defines a non-torsion integer homology class. Moreover, for simplicity, we assume that Γ lies in the image of the map $H_2(X_1,Y_0;\mathbb{Z}) \to H_1(Y,\mathbb{Z})$. As we discuss in this section, the choice of Σ with $\partial\Sigma = \Gamma$ will provide a convenient way of separating the moduli spaces on $(X_1,\mathfrak{s}_1)$ into components of uniform energy, which admit a nice compactification by adding lower dimensional strata.

We assume also that we are working with an additional generic perturbation (U,V) as in section 2, in order to achieve transversality for the moduli space of trajectories. We write $\eta_0 = \rho + \sum_j \frac{\partial U}{\partial \tau_j}\mu_j$ and $\nu_0 = \sum_j \frac{\partial V}{\partial \zeta_j}\nu_j$. Choose $c : X_1 \to \mathbb{R}^+$ to be a cut-off function supported away from a compact set and equaling 1 on the end. Choose $\mu \in \Lambda^{2,+}(X_1, i\mathbb{R})$, satisfying $\mu - c\sqrt{-1}\eta_0^+ \in C^k_\delta$. Here C^k_δ denotes the space of δ-decaying C^k-forms for some large integer k and η_0^+ is the self-dual 2-form obtained by η_0. Next we choose an imaginary valued 1-form ν on X_1, with $\nu - c\nu_0 \in C^k_\delta$. Over the cylindrical end, we write (A,ψ) in temporal gauge. When restricted to the cylindrical end, μ and ν can be thought of as functions of (A,ψ) on the cylindrical end. The relative Seiberg–Witten invariant for X_1 is obtained from the perturbed Seiberg–Witten moduli space on X_1 with asymptotic limit representing a critical point in $\mathcal{M}_{Y_0}(\mathfrak{t},\Gamma)$.

For each a in $\mathcal{M}_{Y_0}(\mathfrak{t},\Gamma)$, define $\mathcal{M}_{X_1}(\mathfrak{s},a)$ to be the moduli space of the Seiberg–Witten equations on X_1 with asymptotic limit in the class of a, modulo the group of gauge transformations. The function spaces for the monopole equations modeled on a are L^2_1-Sobolev spaces with 'δ-decay' where $\delta > 0$ is determined by the least absolute eigenvalue of the Hessian of the CSD at a. Let (A_a, ψ_a) be a Spin^c connection and a spinor on X_1 which agrees with the pull-back of some gauge representatives of a on the cylindrical end. Note that the gauge class of (A_a, ψ_a) can be identified with the set $H^1(Y_0, \mathbb{Z})/Im(H^1(X_1, \mathbb{Z}) \to H^1(Y_0, \mathbb{Z}))$.

Let $\mathcal{M}_{X_1}(\mathfrak{s}_1, a)$ be the moduli space of solutions to the perturbed Seiberg–Witten equations on $(X_1, \mathfrak{s})$:

$$\begin{cases} F_A^+ = q(\psi) + \mu \\ D_A\psi + \nu.\psi = 0, \end{cases} \tag{14}$$

for (A, ψ) in the weighted Sobolev space

$$\{(A, \psi) | A - A_a \in L^2_{k,\delta}, \psi - \psi_a \in L^2_{l,\delta,A_a}\},$$

and with gauge group given by $\{u : X \to \mathbb{C} | |u| = 1, 1 - u \in L^2_{l+1,\delta}\}$. Here we fix a $U(1)$-connection A_a on the spinor bundle to define the covariant derivatives on the spinor sections. For generic (μ, ν), the moduli space $\mathcal{M}_{X_1}(\mathfrak{s}_1, a)$, if non-empty, is a smooth, oriented manifold whose dimension is given by the index of the deformation complex for (14). Denote this index by $\mathfrak{i}_{X_1}(a)$. We then have

$$\mathfrak{i}_{X_1}(a) - \mathfrak{i}_{X_1}(b) = \mathfrak{i}_{Y_0}(a) - \mathfrak{i}_{Y_0}(b).$$

In general, we cannot compactify $\mathcal{M}_{X_1}(\mathfrak{s}_1, a)$ naturally by adding lower dimensional strata. Instead, we define some energy map on $\mathcal{M}_{X_1}(\mathfrak{s}_1, a)$. Assume that the 1-cycle Γ is bounded by a 2-cycle Σ in X_1 with $\partial\Sigma = \Gamma$. Then the energy map is defined as the relative Chern class of $[A, \psi]$ restricted to (Σ, Γ):

$$\begin{aligned} \mathcal{E}_\Sigma : \qquad \mathcal{M}_{X_1}(\mathfrak{s}_1, a) &\longrightarrow H^2(\Sigma, \Gamma; \mathbb{Z}) \\ \mathcal{E}_\Sigma([A, \psi]) &= \langle [\frac{\sqrt{-1}}{2\pi} F_A], [\Sigma] \rangle. \end{aligned} \tag{15}$$

Denote by $\mathcal{M}^{[n]}_{X_1}(\mathfrak{s}_1, a)$ the preimage of $n \in \mathbb{Z}$ under the energy map $\mathcal{E}_\Sigma$. Then we can compactify $\mathcal{M}^{[n]}_{X_1}(\mathfrak{s}_1, a)$ to a smooth manifold with corners. The following proposition can be proved by adapting the convergence and gluing theorems of [11] to the present setting.

Proposition 6.3. *Assume that the moduli space $\mathcal{M}_{X_1}(\mathfrak{s}_1, a)$ is non-empty. Then $\mathcal{M}^{[n]}_{X_1}(\mathfrak{s}_1, a)$ is empty for $n << 0$. Moreover, if non-empty, $\mathcal{M}^{[n]}_{X_1}(\mathfrak{s}_1, a)$*

has a compactification to a smooth manifold with corners, with lower dimensional strata of the form

$$\bigcup_{a_1,\cdots,a_k} \mathcal{M}^{[n_0]}_{X_1}(\mathfrak{s}_1, a_1) \times \hat{\mathcal{M}}^{[n_1]}(a_1, a_2) \times \cdots \times \hat{\mathcal{M}}^{[n_k]}(a_k, a).$$

Here the union is over all possible sequences $a_1, \cdots a_k, a_{k+1} = a$ in $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$ with strictly decreasing indices and with $\sum_{j=0}^{k} n_j = n$.

Proof. From the Weitzenböck formula and the Seiberg–Witten equations, we know that $\mathcal{E}_\Sigma$ is bounded from below. Thus, $\mathcal{M}^{[n]}_{X_1}(\mathfrak{s}_1, a)$ is empty for $n << 0$. The remaining statements follow from the convergence and gluing theorems of [11]. □

Denote by $\mathbb{A}(X_1)$ the free graded algebra

$$Sym^*(H_0(X_1)) \otimes \Lambda^*(H_1(X_1))$$

where the degree of the elements in $H_0(X_1)$ is 2 and the degree of the elements in $H_1(X_1)$ is 1. For any monomial $z = [x]^n \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_k$ in $\mathbb{A}(X_1)$, and for each $m \in \mathbb{Z}$, we need to consider all the components of dimension $2n + k$ in $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1) = \cup_a \mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a)$. Suppose that $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$ is the union of components of dimension $2n+k$ and has energy m under the energy map (15). Here the $\{a_{(2n+k)}\}$ denote all those 3-dimensional monopoles (with perturbations) on $(Y_0, \mathfrak{t})$ which satisfy

$$\mathfrak{i}_{X_1}(a_{(2n+k)}) = 2n + k.$$

Thus, any non-empty component of $\mathcal{M}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$ has dimension $2n+k$. Choose smooth loops which represent 1-cycles $\gamma_1, \cdots, \gamma_k$. The holonomy map of the $[A]$-part in $[A, \psi]$ around these k loops defines maps

$$hol: \qquad \mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \mapsto U(1)^k.$$

These maps can be extended naturally to the compactification of the moduli spaces $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$, as described by Proposition 6.3. The generic fiber of *hol* defines a smooth submanifold $V_{\gamma_1,\cdots,\gamma_k}$ of dimension $2n$ in $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$. This submanifold $V_{\gamma_1,\cdots,\gamma_k}$ can be compactified by the corresponding fiber of *hol* on the boundary strata of $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$.

On $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$, there is a canonical $U(1)$ bundle, defined by the based monopole moduli space $\tilde{\mathcal{M}}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$, which consists of those solutions representing points in $\mathcal{M}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)})$ but considered modulo only those gauge transformations which fix the fiber of the spinor bundle over x. A different choice of base point provides an isomorphic $U(1)$ bundle. Therefore, we have an associated complex rank n vector bundle over $\mathcal{M}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)})$

$$E_n(a_{(2n+k)}) = \tilde{\mathcal{M}}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)}) \times_{U(1)} \mathbb{C}^n.$$

This vector bundle $E_n(a_{(2n+k)})$ is compatible with the boundary strata of $\mathcal{M}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)})$ in the sense that $E_n(a_{(2n+k)})$ has a natural extension to the boundary strata of $\mathcal{M}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)})$, to a vector bundle in the category of manifolds with corners (cf. [15]). Note that $E_n(a_{(2n+k)})$ is isomorphic to $\oplus_{i=1}^{n}\mathcal{L}(x_i)$, where $\mathcal{L}(x_i)$ is complex line bundle associated with the $U(1)$-bundle $\tilde{\mathcal{M}}_{X_1}^{[m]}(\mathfrak{s}_1, a_{(2n+k)})$ with the base point at $x_i \in X_1$. By construction, the line bundle $\mathcal{L}(x_i)$ extends to a line bundle in the category of manifolds with corners over the compactification. Since $\oplus$ is well defined in the category of manifolds with corners, so does $E_n(a_{(2n+k)})$.

Now consider the restriction of $E_n(a_{(2n+k)})$ to the $2n$-dimensional submanifold $V_{\gamma_1,\cdots,\gamma_k}$, which is also compatible with its compactification.

Choose a generic transversal section σ_0, which is also transversal along the boundary strata, i.e. a strata transverse section of a bundle over a manifold with corners. By the transversality along the boundary strata, we know that all the zeroes of this section in $V_{\gamma_1,\cdots,\gamma_k}$ lie within a compact set and consist of finitely many points with an orientation. Counting these points gives a number which is denoted by $SW_{X_1}^{[m]}(\mathfrak{s}_1, z; a_{(2n+k)})$. The relative Seiberg–Witten invariant is defined to be

$$SW_{X_1}(\mathfrak{s}_1, z) = \sum_{m\in\mathbb{Z}, a_{(deg(z))}} SW_{X_1}^{[m]}(\mathfrak{s}_1, z; a_{(deg(z))}) < a_{(deg(z))} > t^m,$$

as an element in the Seiberg–Witten–Floer complex. It is a routine to prove that this element is in fact a cycle, thus defining an element in the Seiberg–Witten–Floer homology groups $HF_{\Gamma,*}^{SW}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$. We also have the analog of Proposition 3.3 in [3], stating that $SW_{X_1}(\mathfrak{s}_1, z)$, as an element in $HF_{\Gamma,*}^{SW}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$, is independent of the choice of $V_{\gamma_1,\cdots,\gamma_k}$ and the choice of the strata transversal section of E_n. Note that, if we choose another relative 2-cycle Σ in the definition of the energy map (15), the relative invariant $SW_{X_1}(\mathfrak{s}_1, z)$ only changes by multiplication by a certain power of t.

Proposition 6.4. *$SW_{X_1}(\mathfrak{s}_1, z) \in HF_{\Gamma,*}^{SW}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$ is well-defined, that* *is, it is independent of the choices of $V_{\gamma_1,\cdots,\gamma_k}$ and the choice of the strata transversal section σ_0 of E_n.*

Proof. For any $m \in \mathbb{Z}$, let $V'_{\gamma_1,\cdots,\gamma_k}$ be another $2n$-dimensional submanifold in $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$ obtained from the construction of the holonomy map and let σ_1 be a strata transversal section of E_n over $V'_{\gamma_1,\cdots,\gamma_k}$. We will prove that the difference between $\#\sigma_0^{-1}(0)$ and $\#\sigma_1^{-1}(0)$ defines an element in $C_{\Gamma,*}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$ which is homologous to zero.

Over $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times [0,1]$, we can choose a strata transverse section σ of E_0, with $\sigma(\cdot, 0) = \sigma_0$ and $\sigma(\cdot, 1) = \sigma_1$. We also choose a codimension k submanifold V in $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times [0,1]$, whose intersection with $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times \{0\}$ is $V_{\gamma_1,\cdots,\gamma_k}$, and whose intersection with $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times \{1\}$ is $V'_{\gamma_1,\cdots,\gamma_k}$, respectively. V is also compatible with the boundary strata of $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times [0,1]$.

From this construction, the zero set of σ over V is a 1-manifold with boundary which consists of three parts: (1) the zero set of σ_0 in $V_{\gamma_1,\cdots,\gamma_k}$; (2) the zero set of σ_1 in $V'_{\gamma_1,\cdots,\gamma_k}$; (3) the zero set of σ in the intersection of V with $\partial^1\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)}) \times [0,1]$, where $\partial^1\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$ is the codimension one boundary of $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$.

By choosing the base points x in X_1 away from the cylindrical end for the $U(1)$-fibration $\tilde{\mathcal{M}}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$, and noticing that $[x]$ is the generator of $H_0(X_1, \mathbb{Z})$, we know that the contribution of (3) from the codimension one boundary of $\mathcal{M}^{[m]}_{X_1}(\mathfrak{s}_1, a_{(2n+k)})$ times $[0,1]$ only comes from

$$\bigcup_{a_{(2n+k-1)}} \bigcup_{m_1 \in \mathbb{Z}} \mathcal{M}^{[m_1]}_{X_1}(\mathfrak{s}_1, a_{(2n+k-1)}) \times \hat{\mathcal{M}}^{[m-m_1]}(a_{(2n+k-1)}, a_{(2n+k)})$$

where $a_{(2n+k-1)}$ runs over all the possible critical points with index $\mathrm{i}_{X_1}(a_{(2n+k-1)}) = 2n + k - 1$. Now the transversality condition over the boundary strata over the intersection of V with $\mathcal{M}^{[m_1]}_{X_1}(\mathfrak{s}_1, a_{(2n+k-1)}) \times [0,1]$ provides a set of finitely many oriented points. Counting these points gives a number, denoted by $H_{a_{(2n+k-1)};\, m_1}$. Then the difference of the two choices defines a number

$$\sum_{a_{(2n+k-1)}} H_{a_{(2n+k-1)};\, m_1} \#(\mathcal{M}^{[m-m_1]}(a_{(2n+k-1)}, a_{(2n+k)})),$$

thus,

$$\begin{aligned}\textstyle\sum_{m\in\mathbb{Z};m_1\in\mathbb{Z}}^{a_{(2n+k-1)}} & H_{a_{(2n+k-1)};\ m_1}\#(\mathcal{M}^{[m-m_1]}(a_{(2n+k-1)},a_{(2n+k)}))t^m a_{(2n+k)}\\ &= \partial\big(\textstyle\sum_{a_{(2n+k-1)};m_1\in\mathbb{Z}} H_{a_{(2n+k-1)};\ m_1}t^{m_1}a_{(2n+k-1)}\big),\end{aligned}$$

is homologous to zero. This proves that the two choices define the same homology class in $HF^{SW}_{\Gamma,*}(Y_0,\mathfrak{t},\mathbb{Q}((t)))$. $\square$

Consider a closed 4-manifold $(X,\mathfrak{s})$ with $b_2^+\geq 1$ and a Spin^c structure $\mathfrak{s}$. Suppose that X has a decomposition $X_1\cup_{Y_0}X_2$ along a 3-dimensional submanifold Y_0 with $b_1(Y_0)>0$, and with a Spin^c structure $\mathfrak{s}|_{Y_0}=\mathfrak{t}$. We assume that the class $c_1(\mathfrak{s}|_{Y_0})=c_1(\mathfrak{t})$ is a torsion element in $H^2(Y_0,\mathbb{Z})$. Here we need to choose a metric g_T on X such that X contains a very long 'neck' $Y_0\times[-T,T]$. For $b_2^+=1$, when the Seiberg–Witten invariant has a chamber structure, our choice of the perturbation compatible with Γ gives rise to a fixed chamber in the space of metrics and perturbations. Notice that, since we are dealing with the case of a torsion element $c_1(\mathfrak{t})$, in the case where $b_1(Y_0)=1$ the chamber also depends on the class $[*\rho]$ in $H^2(Y_0,\mathbb{R})$ Now we can state the gluing formulae of the Seiberg–Witten invariants on X as follows.

Theorem 6.5. *Let X be a closed manifold with $b^+\geq 1$ which is written as $X=X_1\cup_{Y_0}X_2$, with X_1 and X_2 two 4-manifolds with boundary, and with $\partial X_1=-\partial X_2=Y_0$. Suppose that we have* Spin^c *structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$, on X_1 and X_2 respectively, such that $\mathfrak{s}_1|_{Y_0}\cong\mathfrak{s}_2|_{Y_0}\cong\mathfrak{t}$ is a torsion* Spin^c *structure on Y_0. Suppose given 1-cycle Γ in Y_0 bounds two relative 2-cycles Σ_i in X_i respectively. Then the relative Seiberg-Witten invariants $SW_{X_1}(\mathfrak{s}_1)$ and $SW_{X_2}(\mathfrak{s}_1)$ take values in $HF^{SW}_{\Gamma,*}(Y_0,\mathfrak{t},\mathbb{Q}((t)))$ and $HF^{SW}_{-\Gamma,*}(-Y,-\mathfrak{t},\mathbb{Q}((t)))$ respectively. Let $\Sigma=\Sigma_1+\Sigma_2\in H_2(X,\mathbb{Z})$, under the natural pairing (13), we have the following gluing formula for $z_i\in\mathcal{A}(X_i)$, $i=1,2$,*

$$\sum_{\{\mathfrak{s}:\mathfrak{s}|_{X_i}=\mathfrak{s}_i\}}SW_{X,\mathfrak{s}}(z_1z_2)t^{c_1(\mathfrak{s})\cdot\Sigma}=\langle SW_{X_1}(\mathfrak{s}_1,z_1),SW_{X_2}(\mathfrak{s}_2,z_2)\rangle.$$

When $b^+=1$, the Seiberg–Witten invariants correspond to a fixed chamber determined by the perturbation which is compatible with Γ.

Proof. Let i_k denote the boundary embedding maps of Y_0 into X_k. Then the set of Spin^c-structures on X which agree with $\mathfrak{s}_i$ over X_i, denoted by Spin$^c(X;\mathfrak{s}_1,\mathfrak{s}_2)$, form an affine space over

$$H^1(Y_0,\mathbb{Z})/\big(Imi_1^*+Imi_2^*\big),$$

with $i_1^* : H^1(X_1, \mathbb{Z}) \to H^1(Y_0, \mathbb{Z})$ and $i_2^* : H^1(X_2, \mathbb{Z}) \to H^1(Y_0, \mathbb{Z})$. For any $\mathfrak{s} \in \mathrm{Spin}^c(X; \mathfrak{s}_1, \mathfrak{s}_2)$, after a generic perturbation, the moduli space $\mathcal{M}_X(\mathfrak{s})$ is an oriented, compact, smooth manifold with dimension given by

$$\frac{1}{4}(c_1(\mathfrak{s})^2 - (2\chi + 3\sigma)) = d_X(\mathfrak{s}) \geq 0.$$

The Seiberg–Witten invariant is a linear functional

$$SW_X(\mathfrak{s}, \cdot) : \qquad \mathbb{A}(X) \longrightarrow \mathbb{Z}$$

where $\mathbb{A}(X) = Sym^*(H_0(X)) \otimes \Lambda^*(H_1(X))$. We summarize this definition briefly. We shall only consider monomials $z = [x]^n \gamma_1 \wedge \cdots \wedge \gamma_k \in \mathbb{A}(X)$ with $2n + k = d_X(\mathfrak{s})$. For any other degree of z, we assign $SW_X(\mathfrak{s}, z)$ to be zero. For any monomial as above, we can choose smooth 1-dimensional submanifolds representing $\gamma_1, \cdots, \gamma_k$. Then the holonomy along these loops defines a map: $\mathcal{M}_X(\mathfrak{s}) \mapsto U(1)^k$, whose generic fiber is a closed codimension k submanifold $V_{\gamma_1, \cdots, \gamma_k}$ in $\mathcal{M}_X(\mathfrak{s})$. The based monopoles define a $U(1)$-fiber bundle $\tilde{\mathcal{M}}_X(\mathfrak{s})$ over $\mathcal{M}_X(\mathfrak{s})$. Then $SW_X(\mathfrak{s}, z)$ is defined to be the result of integration of the n^{th} power of the first Chern class of this $U(1)$ bundle over $V_{\gamma_1, \cdots, \gamma_k}$. Equivalently, we can consider the associated rank n complex vector bundle E over the 2n-dimensional manifold $V_{\gamma_1, \cdots, \gamma_k}$. Then $SW_X(\mathfrak{s}, z)$ is obtained by counting points with the orientation in the zero set of a generic strata transverse section of E over $V_{\gamma_1, \cdots, \gamma_k}$.

We can adapt the gluing theory for Seiberg–Witten monopoles developed in [11] to the setting of [6] and [19]. Thus, when T is sufficiently large, we obtain a gluing theorem for Seiberg–Witten monopoles, which identifies the 4-dimensional monopoles on X with the following product:

$$\bigcup_{\mathfrak{s} \in \mathrm{Spin}^c(X, \mathfrak{s}_1, \mathfrak{s}_2)} \mathcal{M}_X(\mathfrak{s}) \cong \bigcup_a \mathcal{M}_{X_1}(\mathfrak{s}_1, a) \times \mathcal{M}_{X_2}(\mathfrak{s}_2, a). \tag{16}$$

Notice that, for $[A, \Psi] \in \mathcal{M}_X(\mathfrak{s})$, we write $[A, \Psi]$ as $[A_1, \Psi_1] \#^T [A_2, \Psi_2]$ under the gluing map (16). We have

$$\mathcal{E}_{\Sigma_1}([A_1, \Psi_1]) + \mathcal{E}_{\Sigma_2}([A_2, \Psi_2]) = c_1(\mathfrak{s}) \cdot \Sigma.$$

Now the gluing formulae follow from the definition of the relative invariants $SW_{X_1}(\mathfrak{s}_1, z_1)$ and $SW_{X_2}(\mathfrak{s}_2, z_2)$ for $z_i \in \mathbb{A}(X_i)$, with values in $HF^{SW}_{\Gamma, *}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$ and $HF^{SW}_{-\Gamma, *}(-Y_0, -\mathfrak{t})$ respectively.

Note that, in the case of $b_2^+(X) = 1$, the choice of the 1-cycle Γ fixes the choice of the chamber. In fact, the moduli space $\mathcal{M}_{Y_0}(\mathfrak{t}, \Gamma)$, with the perturbation $[*\rho] = \eta PD(\Gamma)$, contains no reducibles. Thus, in particular, since the relative invariants take values in $HF^{SW}_{\Gamma,*}(Y_0, \mathfrak{t}, \mathbb{Q}((t)))$, once Γ is fixed, there are no wall crossing terms for $SW_{X,\mathfrak{s}}$ when we change the perturbation and the metric. $\square$

We conclude with the following example.

Example 6.6. *([17]) Let $(Y_0, \mathfrak{s}_0)$ be the 3-manifold $\Sigma_g \times S^1$ with the trivial* Spin^c *structure $\mathfrak{s}_0$, where Σ_g is a closed Riemann surface of genus g. Then the gluing formulae in Theorem 6.5 for 4-manifold $\Sigma_g \times S^2$ can be applied to study $HF^{SW}_{\Gamma,*}(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{Q}((t)))$ and its ring structure. Here we have $\Gamma = [pt \times S^1]$, with the surface Σ of Theorem 6.5 corresponding to $pt \times S^2$. It turns out that, as a $Sp(2g, \mathbb{Z})$-equivariant vector space, we have*

$$HF^{SW}_{\Gamma,*}(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{Q}((t))) \cong H_*(Sym^{g-1}(\Sigma), \mathbb{Q}) \otimes \mathbb{Q}((t)).$$

In particular, when Σ_g is a torus, then

$$HF^{SW}_{\Gamma,*}(T^2 \times S^1, \mathfrak{s}_0; \mathbb{Q}((t))) \cong \mathbb{Q}((t)).$$

Upon choosing the metric on $T^2 \times S^2$ such that it has a long neck around $T^2 \times S^1$, the Seiberg-Witten invariant is computed as the wall-crossing term from the chamber corresponding to the unperturbed equations for the positive scalar curvature which has trivial Seiberg-Witten invariant, and the other chamber determined by the choice of Γ, then we obtain the following result from the gluing formulae:

$$\begin{aligned} & \langle SW_{T^2 \times D^2}(\mathfrak{s}_0, 1), SW_{T^2 \times D^2}(\mathfrak{s}_0, 1) \rangle \\ = & \textstyle\sum_{n \geq 1} SW_{T^2 \times S^2, 2nPD([T^2])}(1) t^{2n} \\ = & \textstyle\sum_{n \geq 1} n t^{2n}. \end{aligned}$$

Thus, up to sign, we can express the relative Seiberg–Witten invariant of $T^2 \times D^2$ with the trivial Spin^c *structure as*

$$SW_{T^2 \times D^2}(\mathfrak{s}_0, 1) = \frac{1}{t - t^{-1}}.$$

For $z \in \mathbb{A}(T^2 \times D^2)$ of non-zero degree, we have $SW_{T^2 \times D^2}(\mathfrak{s}_0, z) = 0$.

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