LOWER CURVATURE BOUNDS, TOPONOGOV'S THEOREM, AND BOUNDED TOPOLOGY I / II

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LOWER CURVATURE BOUNDS, TOPONOGOV'S THEOREM, AND BOUNDED TOPOLOGY I

Introduction :

Classically the theory of non-compact Riemannian manifolds with negative sectional curvature is based on the visibility axiom (c.f. [EN]); heuristically speaking this axiom requires that the curvatures do not decay to zero to quickly. In contrast the theory of manifolds M^n with positive curvature does not require an additional hypothesis in the non-compact case. There are even nice results, if one supposes the curvature only to be non-negative : By the Toponogov splitting theorem such a manifold is isometric to a Riemannian product $N^k \times \mathbb{R}^{n-k}$, where the factor N^k does not contain a line. The soul theorem due to Cheeger and Gromoll claims that any non-negatively curved manifold M^n is diffeomorphic to the normal bundle of a compact, embedded submanifold. Moreover Gromov has shown that there is a universal upper bound $C(n)$ on the sum of the Betti numbers $\beta_i(M^n)$.

In this paper we are going to study a larger class of manifolds and include for instance "asymptotically flat manifolds".

Definition :

A complete Riemannian manifold (M^n,g) *with base point* o *is called asymptotically non-negatively curved , iff there exists a monotone function* $\lambda:[0,\infty)\longrightarrow[0,\infty)$ *such that*

i) $b_0(\lambda) := \int_0^\infty r\cdot\lambda(r)\,dr < \infty$

and ii) the sectional curvatures at any point $p\in M^n$ *are bounded from below by* $-\lambda(d(o,p))$.

The convergence of the integral $b_0(\lambda)$ implies a decay condition on the lower curvature bound λ. This condition is analyzed in more detail in chapter II. For instance it asserts that there is a unique non-negative solution of the Riccati equation $u'(r)=u(r)^2-\lambda(r)$ which decays to zero for $r\to\infty$. Thus one has another numerical invariant

$$b_1(\lambda) := \int_0^\infty u(r)\,dr$$

Both b_0 and b_1 depend on λ in a monotone way, and they can be regarded as invariants of the manifold M^n by taking the minimal monotone function λ which meets the conditions (i) and (ii) ; notice that the numbers do not change when the metric g on M^n is scaled with a global factor.

Main Results :

A) a generalized triangle comparison theorem of Toponogov type :

The model spaces will be arbitrary simply-connected surfaces of revolution with non-positive curvature, and the comparison triangle will have one vertex at the pole of the model space (c.f. I.3.1 and I.3.2).

We apply this theorem to triangles in asymptotically non-negatively curved manifolds which have one vertex at the base point o . Employing in addition the analysis done in chapter II , we obtain lower bounds on their angles which are uniform with respect to the size of the triangles (c.f. III.1) . Such uniform bounds can be derived from the standard Toponogov theorem only in the case of non-negative curvature, and in this setting they provide an important tool. Similarly our uniform estimates are the key to the following theorem.

B) Theorem :

For asymptotically non-negatively curved manifolds M^n there exist universal upper bounds on the number of ends and on the Betti numbers :

1.) $$\#\{\text{ends of } M^n\} \leq 2\cdot\pi^{n-1}\cdot\exp((n-1)\cdot b_1(M^n))$$

2.) $$\sum_i \beta_i(M^n) \leq C(n)\cdot\exp\left(\frac{15n-13}{4}\cdot b_1(M^n)\right) .$$

The function $C(n)$ can be effectively estimated by an expression which grows exponentially in n^3 .

The proof of B1 is carried out in chapter III , while B2 is deferred to a subsequent paper. Finally theorem B is optimal in the sense that the topology of a surface M^2 is not necessarily bounded, when its integral b_o diverges. We can prove even more :

C) Theorem :

Suppose that the integral $b_o(\lambda)$ of a function $\lambda : [0,\infty)\longrightarrow[0,\infty)$ diverges. Then every non-compact, connected surface M^2 with base point o carries a C^2-metric whose curvature κ obeys :

$$\kappa(p) = -\lambda(d(o,p)) \qquad , p\in M^2 .$$

I. Models and Toponogov's Theorem :

The standard Toponogov theorem compares triangles in a Riemannian manifold (M^n,g) to the corresponding Alexandrov triangles in suitable spaces of constant curvature (c.f. [CE] or [K]). It is worthwhile noticing that the models are essentially two-dimensional. We are going to extend the theorem and allow for any simply-connected surface of revolution with non-positive curvature as model space.

1. More precisely we consider all continuous functions $k:[0,\infty)\to[0,\infty)$. Each of them determines a unique surface of revolution $M^2(-k)$ with pole p_o such that the curvature equals the function $-k(d(.,p_o))$; here d denotes the Riemannian distance in $M^2(-k)$.

It is convenient to simultaneously consider the approximating functions $k_\varepsilon:[0,\infty)\to[0,\infty)$ which are defined by

1.1 $$k_\varepsilon(r) \;:=\; \sup\{\,k(r')\mid r'\geq 0 \text{ and } |r-r'|\leq\varepsilon\,\} \quad , \quad \varepsilon\geq 0$$

By notation $k_o=k$. In polar coordinates (r,ϕ) the metric of $M^2(-k_\varepsilon)$ looks like :

1.2 $$dr^2 + y_\varepsilon(r)^2\cdot d\phi^2$$

where the function y_ε is given by the Jacobi field equation :

1.3 $$y_\varepsilon'' = k_\varepsilon\cdot y_\varepsilon \quad , \quad y_\varepsilon(0)=0 \quad , \text{ and } \quad y_\varepsilon'(0)=1$$

2. We proceed to summarizethe elementary properties of our model spaces

2.1 Lemma :

The coordinate functions $r(s)$ *and* $\phi(s)$ *along a unit-speed geodesic* $s\mapsto\gamma(s)$ *in the model surface* $M^2(-k)$ *obey the equations :*

i) $$r'^2 + (y\circ r)^2\cdot\phi'^2 = 1$$

ii) $$(y\circ r)^2\cdot\phi' = \text{const} \qquad (\textit{ Clairaut })$$

iii) $$(y\circ r)^2\cdot(1-r'^2) = \text{const}^2$$

We skip the obvious proof and recall that by notation p_o always denotes the pole of the model space. When looking at a geodesic triangle $\Delta=(p_o,p_1,p_2)$ with edges of length $\ell_i=d(p_{i+1},p_{i+2})$ – indices taken modulo 3 – , formula 2.1iii becomes :

$$y(\ell_1)\cdot\sin(\measuredangle \text{ at } p_2) = y(\ell_2)\cdot\sin(\measuredangle \text{ at } p_1)$$

This generalizes the well-known Law of Sines in euclidean geometry ($k\equiv 0$, $y\equiv id$) and in hyperbolic geometry ($k\equiv -1$, $y\equiv\sinh$).

2.2 Lemma :

Given triangles $\Delta=(p_o,p_1,p_2)$ *and* $\Delta'=(p_o,p_1',p_2')$ *in a surface* $M^2(-k)$ *such that* $\ell_o=\ell_o'$ *and* $\ell_1=\ell_1'$ *, one has monotonicity :*

$$\measuredangle\text{at } p_2 < \measuredangle\text{at } p_2' \iff \ell_2 < \ell_2'$$

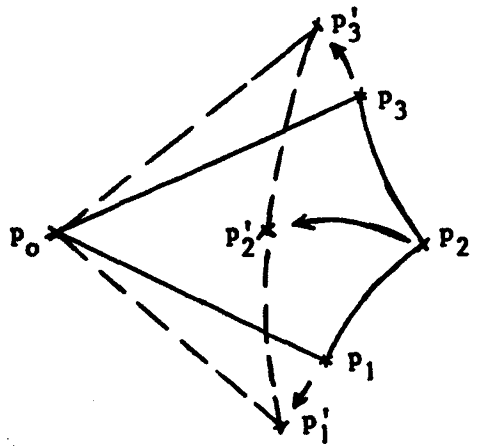
Proof :

We may rotate Δ' about p_o and without loss of generality may assume that $p_2'=p_2$. Then the claim becomes obvious, since $M^2(-k)$ is simply-connected and has non-positive curvature.

2.3 Lemma :

Let p_o , p_1 , p_2 , *and* p_3 *be the vertices of a quadrilateral in* $M^2(-k)$. *Moreover suppose that* $\measuredangle$*at* $p_2<\pi$ *and that*

$$d(p_1,p_2)+d(p_2,p_3) < d(p_3,p_o)+d(p_o,p_1)$$



Then there is a triangle $\Delta=(p_o,p_1',p_3')$, *unique up to rotation about* p_o *, such that :*

$$d(p_o,p_1') = d(p_o,p_1)$$

$$d(p_o,p_3') = d(p_o,p_3)$$

$$d(p_1',p_3') = d(p_1,p_2)+d(p_2,p_3)$$

$$\measuredangle\text{at } p_1' < \measuredangle\text{at } p_1$$

$$\measuredangle\text{at } p_3' < \measuredangle\text{at } p_3$$

Proof :

The idea is to bend in the corner at p_2 : we move p_2 towards the pole p_o . We keep the length of all edges fixed by moving the vertices p_1 and p_3 in an appropriate way. Obviously

$$\ell_{crit} := \max\{d(p_o,p_1)-d(p_1,p_2)\ ,\ d(p_o,p_3)-d(p_2,p_3)\} > 0 .$$

If $d(p_o,p_2)$ gets as small as ℓ_{crit} , one of the triangles (p_o,p_1,p_2) and (p_o,p_2,p_3) becomes degenerate, and the quadrilateral has $\measuredangle$at $p_2>\pi$. Now the claim is obvious, since the angle depends continuously on $d(p_o,p_2)$.

For later use we state another continuity property, which is due to the fact that the functions k_ε converge to k uniformly on compact subsets.

2.4 Lemma :

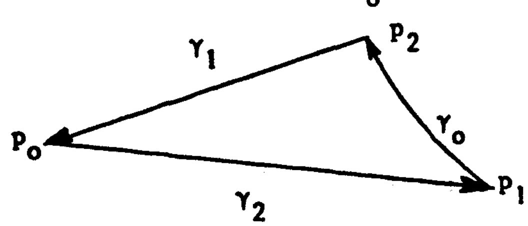
Given triangles $\Delta = (p_o, p_1, p_2)$ *in* $M^2(-k)$ *and* $\Delta^\varepsilon = (p_o^\varepsilon, p_1^\varepsilon, p_2^\varepsilon)$ *in* $M^2(-k_\varepsilon)$ *which have edges of equal length* $\ell_i^\varepsilon = \ell_i$, $i=1,2,3$, *then their angles depend continuously on* ε , *e.g.:*

$$\lim_{\varepsilon \to o} \sphericalangle \text{at } p_i^\varepsilon = \sphericalangle \text{at } p_i \quad .$$

3. In this section we are going to establish the generalized comparison theorem. Notation will be changed slightly : if there is a bar above a letter this symbol will refer to data in the model space, whereas unbarred letters will refer to data in the Riemannian manifold (M^n, g) .

3.1 Assumptions :

i) p_o , p_1 , *and* p_2 *are the vertices of a (generalized) geodesic triangle in a Riemannian manifold* M^n *; the edges* γ_1 *and* γ_2 *are supposed to be minimizing, whereas* γ_o *is only required to be a geodesic.*



We continue denoting the length of γ_i *by* ℓ_i , $i=1,2,3$.

ii) at any point $p \in M^n$ *the sectional curvatures shall be bounded from below by* $-k(d(p,p_o))$.

iii) the pole $\bar{p}_o$ *of* $M^2(-k)$ *shall be a vertex of the comparison triangle* $\bar{\Delta} = (\bar{p}_o, \bar{p}_1, \bar{p}_2)$.

3.2 Theorem :

Under the assumptions 3.1 the following conclusions hold:

a) if $\ell_i = \bar{\ell}_i$ *for all the edges,*
then $\sphericalangle$ *at* $p_1 \geq \sphericalangle$ *at* $\bar{p}_1$ *and* $\sphericalangle$ *at* $p_2 \geq \sphericalangle$ *at* $\bar{p}_2$.

b) if $\ell_o = \bar{\ell}_o$, $\ell_1 = \bar{\ell}_1$, *and* $\sphericalangle$ *at* $p_2 \leq \sphericalangle$ *at* $\bar{p}_2$,
then $\ell_2 \leq \bar{\ell}_2$.

3.3 Remarks :

i) Actually it is sufficient to require condition 3.1ii only for those points $p \in M^n$ which are ε-close to any minimizing geodesic from p_o to a point on γ_o. These points p obey the conditions :

$$d(p,p_o) + d(p,p_i) \leq \ell_o + \ell_i + \varepsilon \quad ; \; i = 1 , 2 ,$$

and

$$d(p,p_o) + \frac{1}{2}\cdot(d(p,p_1)+d(p,p_2)) \leq \ell_o + \frac{1}{2}\cdot(\ell_1 + \ell_2) + \varepsilon \quad .$$

ii) The Alexandrov triangle $\bar{\Delta} = (\bar{p}_o,\bar{p}_1,\bar{p}_2)$ which is required in part (a) of the theorem exists , if and only if $\ell_o \leq \ell_1 + \ell_2$.

iii) Even if one assumes in addition that the edge γ_o is minmizing , there is in contrast to the constant curvature case in general no easy way to restore the information on the angles at p_o and $\bar{p}_o$. The reason is that $\bar{p}_o$ plays the rather special role of the pole in $M^2(-k)$.

Proof : (c.f. [CE] , [K])

We give a straight forward extension of the classical argument.

a) $\Longrightarrow$ b) :

If $\ell_o \geq \ell_1 + \ell_2$, the claim is an obvious consequence of the triangle inequality $\bar{\ell}_o \leq \bar{\ell}_1 + \bar{\ell}_2$ which holds in $M^2(-k)$. Else there exists an Alexandrov triangle and the claim can be deduced from part (a) by means of the monotony principle 2.2 .

a) We pick $r_\Delta > 0$ and $K > 0$ such that the edge γ_o is contained in the ball $B(p_o,r_\Delta)$ and that the sectional curvatures in $B(p_o,r_\Delta+1)$ are bounded from above by K . Next we put the comparison triangles into the model spaces $M^2(-k_\varepsilon)$ instead of $M^2(-k)$. By a limiting argument based on lemma 2.4 it is sufficient to prove the result (a) for all surfaces $M^2(-k_\varepsilon)$ with $0 < \varepsilon < \min\{1 , \pi/\sqrt{K}\}$. We fix the value of ε , subdivide γ_o into pieces shorter than $\varepsilon/2$, and pick minimizing geodesics from p_o to all the partition points on γ_o . Provided that the claimed comparison result holds for all the small triangles, the deformation lemma 2.3 extends the inequalities to Δ and $\bar{\Delta}$. In view of the monotony principle 2.2 we have reduced the proof to showing :

$$(*) \qquad \ell_1 = \bar{\ell}_1 < r_\Delta \quad , \quad \ell_o = \bar{\ell}_o < \varepsilon/2 \quad ,\text{and} \quad \sphericalangle \text{at } p_2 = \sphericalangle \text{at } \bar{p}_2 \quad \Longrightarrow \quad \ell_2 \leq \bar{\ell}_2$$

In order to see this , we extend $-\gamma_o'(\ell_o)$ and $-\bar{\gamma}_o'(\ell_o)$ to parallel vector fields along the edges γ_1 and $\bar{\gamma}_1$ repectively. They give raise to ruled surfaces c and $\bar{c}$. For example $c: \mathbb{R} \times [0,\ell_1] \longrightarrow M^n$, $(s,t) \longmapsto c(s,t)$ is characterized by the formulae :

$$c(0,t) = \gamma_1(t) \qquad\qquad c'(0,0) = -\gamma_o'(\ell_o)$$
$$\nabla_{\dot{c}} c' \Big|_{(0,t)} = 0 \qquad\qquad \nabla_{c'} c' = 0 \qquad .$$

Here as usual a prime denotes a derivative with respect to s and a dot denotes a derivative with respect to t .

Observations :

i) $\bar{\gamma}_2$ is contained in the image of $[0,\infty) \times [0,\ell_1]$ under $\bar{c}$. By notation $\bar{\gamma}_2(\bar{\ell}_2) = \bar{p}_1$ and $\bar{\gamma}_2(0) = \bar{p}_o$. Because of the Gauß-Bonnet theorem $\sphericalangle(\bar{\gamma}_2', \bar{c}')$ is non-decreasing along $\bar{\gamma}_2$. Hence $\bar{\gamma}_2$ is contained in $\bar{V} := \bar{c}(U)$, where U stands for the cube $[0,\varepsilon/2) \times [0,\ell_1]$.

ii) $c(U)$ is contained in $B(p_o, r_\Delta + 1)$, and therefore our choices above imply that in U there are no focal points on the geodesics $s \longmapsto c(s,t)$

By construction the inequality

$$\begin{aligned} -k(d(p_o, c(s,t))) &\geq -k_{\varepsilon/2}(d(p_o, c(0,t))) \\ &= -k_{\varepsilon/2}(d(\bar{p}_o, \bar{c}(0,t))) \geq -k_\varepsilon(d(\bar{p}_o, \bar{c}(s,t))) \end{aligned}$$

holds for all $(s,t) \in U$. Hence Rauch's comparison theorem yields :

$$|\dot{c}^\perp| \leq |\dot{\bar{c}}^\perp| \qquad \text{on } U$$

Here $\perp$ denotes the component orthogonal to the unit vectors c' resp. $\bar{c}'$. We conclude that the map $c \circ \bar{c}^{-1}: \bar{V} \longrightarrow M^n$ is distance non-increasing.

It follows that $t \longmapsto c \circ \bar{c}^{-1} \circ \bar{\gamma}_2(t)$ defines a curve in M^n which joins p_o and p_1 and is not longer than $\bar{\gamma}_2$. This proves (*) .

II. Analyzing the Decay Condition

Throughout this chapter we assume $\lambda : [0,\infty) \longrightarrow [0,\infty)$ to be a monotone non-increasing function. Roughly speaking the integral $b_o(\lambda)$ converges, iff $\lambda(r)$ decays a little quicker than r^{-2} for $r \to \infty$. We start making this observation more precise.

1.1 Lemma :

Whenever $b_o(\lambda)$ converges, there exist monotone non-increasing functions :

$$\lambda_1 : r \longmapsto \int_r^\infty \lambda(\rho)\, d\rho$$

$$\lambda_2 : r \longmapsto \int_r^\infty \lambda_1(\rho)\, d\rho$$

$$\lambda_b : r \longmapsto \int_r^\infty \rho \cdot \lambda(\rho)\, d\rho = \lambda_2(r) + r \cdot \lambda_1(r)$$

Moreover the following estimates hold: $(r \geq 0)$

$$r^2 \cdot \lambda(r) \leq 2 \cdot b_o(\lambda)$$

$$r \cdot \lambda_1(r) \leq b_o(\lambda)$$

$$\tilde{b}_o(\lambda) := \int_0^\infty \inf\{\lambda_1(r), \sqrt{\lambda(r)}\}\, dr \leq \lambda_2(0) = b_o(\lambda) .$$

Proof :

The expressions $\lambda_1(r)$ and $\lambda_b(r)$ obviously converge. The existence of $\lambda_2(r)$ follows from the theorems by Fubini and Tonelli :

$$\lambda_2(r) = \int_r^\infty \int_t^\infty \lambda(\rho)\, d\rho\, dt = \int_r^\infty (\rho - r) \cdot \lambda(\rho)\, d\rho$$

$$= \lambda_b(r) - r \cdot \lambda_1(r)$$

The remaining estimates are due to the computations :

$$r^2 \cdot \lambda(r) = 2 \cdot \lambda(r) \cdot \int_0^r \rho\, d\rho \leq 2 \cdot \int_0^r \rho \cdot \lambda(\rho)\, d\rho \leq 2 \cdot b_o(\lambda)$$

and $$r \cdot \lambda_1(r) = \lambda_1(r) \cdot \int_0^r d\rho \leq \int_0^r \lambda_1(\rho)\, d\rho \leq b_o(\lambda) .$$

1.2 Remarks :

i) Almost the same computations give raise to the formulae :

$$\lim_{r\to\infty} r^2\cdot\lambda(r) = 0$$

and $$\lim_{r\to\infty} r\cdot\lambda_1(r) = 0$$

ii) Observe that for $T_c\lambda(r) := c^2\cdot\lambda(c\cdot r)$, $r \geq 0$, $c > 0$, one has :

$$(T_c\lambda)_1(r) = c\cdot\lambda_1(c\cdot r)$$

and $$(T_c\lambda)_2(r) = \lambda_2(c\cdot r)$$

Therefore the invariant $b_o(M^n)$ does not change, when the metric of the asymptotically non-negatively curved manifold is scaled with some global factor.

iii) We point out that it does not depend on the choice of the base point o in a Riemannian manifold (M^n,g) whether the integral b_o converges. However its numerical value is very sensitive with respect to the position of o. This is related to the fact that b_o does not detect certain curvature singularities at o. Such a task would require much refined numerical invariants.

Later on we shall need information about the models $M^2(-\lambda)$. This means basically that we have to study the Jacobi field equation :

$$(*) \qquad y''(r) = \lambda(r)\cdot y(r)$$

2.1 Lemma :

The following conditions are equivalent :

i) $b_o(\lambda) < \infty$

ii) for any solution y *of equation* $(*)$ *there exists* $y'(\infty) := \lim_{r\to\infty} y'(r)$.

Proof :

We assume that $r > r_1 > 0$ and compute :

$$|y'(r)-y'(r_1)| \leq \int_{r_1}^{r} \lambda(\rho)\cdot|y(\rho)|\,d\rho$$

$$\leq \lambda_1(r_1)\cdot|y(r_1)| + \lambda_2(r_1)\cdot|y'(r_1)|$$

$$+ \lambda_2(r_1)\cdot\max\{\,|y'(\rho)-y'(r_1)|\ \big|\ r_1 \leq \rho \leq r\,\}$$

Provided that r_1 is sufficiently large, we know that $\lambda_2(r_1) \leq \frac{1}{2}$, and hence we obtain for $r > r_1 \gg 0$ that :

$$|y'(r)-y'(r_1)| \leq 2\cdot\lambda_1(r_1)\cdot|y(r_1)| + 2\lambda_2(r_1)\cdot|y'(r_1)| =: C(r_1)$$

This already shows that y' remains bounded. We iterate the inequality and conclude that for $r > r_2 > r_1 >> 0$ we have :

$$|y'(r) - y'(r_2)| \leq 2 \cdot \lambda_1(r_2) \cdot |y(r_1)| + 2 \cdot \left(r_2 \cdot \lambda_1(r_2) + \lambda_2(r_2)\right) \cdot \left(|y'(r_1)| + C(r_1)\right) .$$

The right-hand side converges to zero for $r_2 \to \infty$.

2.2 Lemma :

$$(1 + b_0(\lambda)) \cdot z_\infty(\infty) \leq 1 .$$

This estimate is an obvious consequence of the following formulae :

$$z_\infty'(r) = \int_r^\infty \lambda(\rho) \cdot z_\infty(\rho)\, d\rho$$

$$z_\infty(r) = z_\infty(\infty) + \int_r^\infty (\rho - r) \cdot \lambda(\rho) \cdot z_\infty(\rho)\, d\rho \geq z_\infty(\infty) \cdot (1 + \lambda_2(r)) .$$

The Jacobi field z_∞ is closely related to the invariant b_1 ; observe that the function $-z_\infty(r)^{-1} \cdot z_\infty'(r)$ converges to zero for $r \to \infty$, and that it obeys the Riccati equation :

$$(**) \qquad u'(r) = u(r)^2 - \lambda(r)$$

2.3 Lemma :

Let $b_0(\lambda) < \infty$ *; then there is a unique* *non-negative* *solution* u *of* $(**)$ *such that* $u(r) \longrightarrow 0$ *for* $r \longrightarrow \infty$ *. Even more one has the estimates :*

i) $\quad 0 \leq u(r) \leq \min\{\lambda_1(r), \sqrt{\lambda(r)}\}$

ii) $\quad b_1(\lambda) := \int_0^\infty u(r)\, dr \leq \tilde{b}_0(\lambda) \leq b_0(\lambda)$

Proof :

Consider the continuous functions u_ℓ which vanish identically on $[\ell, \infty)$ and solve for $(**)$ on $[0, \ell]$. Since $0' = 0 \geq -\lambda$ and $\lambda_1' = -\lambda \leq \lambda_1^2 - \lambda$, standard monotony arguements yield the estimate

$$0 \leq u_\ell(r) \leq \lambda_1(r)$$

Therefore the limits

$$u(r) := \lim_{\ell \to \infty} u_\ell(r)$$

exist and the function u meets the desired conditions.

i) It is also easy to verify that the functions u_ℓ are monotone and that hence $u_\ell(r) = \sqrt{\lambda(r)}$.

ii) This inequality is clear from the definitions.

2.4 Remarks :

i) $z_\infty(r) = \exp(-\int_0^r u(\rho)\,d\rho) \geq z_\infty(\infty) > 0$

ii) By lemma 2.2 and lemma 2.3ii it is clear that all our invariants associated to a function λ are *equivalent* in some non-linear sense:

$$b_1(\lambda) \leq \tilde{b}_o(\lambda) \leq b_o(\lambda) \leq \exp(b_1(\lambda)) - 1$$

Moreover all the invariants depend on the function λ in a monotone way.

In chapter III there will be a situation where some uniform control on a family of model spaces $M^2(-\lambda)$ is required. This estimate can be done comparing the solutions z_ℓ of (*) to the function z_∞ . For the sake of brevity we shall use the notation :

$$\beta := z_\infty(\infty) = \exp(-b_1(\lambda))$$

2.5 Lemma :

i) $\beta \leq z_\infty(r) \leq 1 \qquad , 0 \leq r < \infty$

ii) $(1 - \frac{r}{\ell}) \cdot z_\infty(\ell) \leq z_\ell(r) \qquad , 0 \leq r \leq \ell$

iii) $\frac{\beta}{\ell} \leq -z_\ell'(\ell) \leq -z_\ell'(0) \leq -z_\infty'(0) + \frac{1}{\ell}\cdot z_\infty(\ell) \leq \frac{1}{\ell} + \lambda_1(0)$

Proof :

i) c.f. remark 2.4i .

ii) The difference $\Delta z := z_\infty - z_\ell$ also solves the differential equation (*) . As it is non-negative on $[0,\infty)$, it is a convex function :

$$\Delta z(r) \leq \frac{r}{\ell}\cdot\Delta z(\ell) + (1 - \frac{r}{\ell})\cdot\Delta z(0) = \frac{r}{\ell}\cdot z_\infty(\ell)$$

$$\Longrightarrow \qquad z_\ell(r) \geq z_\infty(r) - \frac{r}{\ell}\cdot z_\infty(\ell) \geq (1 - \frac{r}{\ell})\cdot z_\infty(\ell)$$

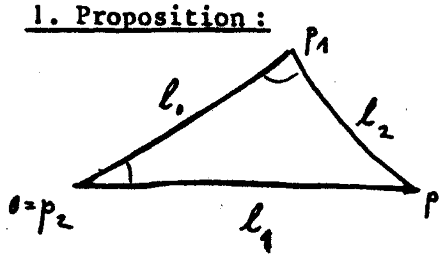
iii) Using part (ii) and monotony, the first, the second , and the last inequality are obvious. In order to obtain the third inequality , we use the convexity of Δz and compute :

$$z_\infty'(0) - z_\ell'(0) = \Delta z'(0) \leq \frac{1}{\ell}\cdot\Delta z(\ell) = \frac{1}{\ell}\cdot z_\infty(\ell)$$

III. Geodesic Triangles and the Number of Ends in asymptotically non-negatively curved Manifolds

The generalized triangles which have one of their vertices at the base point o of M form a rather distinguished class of objects. They might serve as tools to study the properties of M ; in view of packing arguements their angle at o deserves special interest. At a first look this very angle seems to cause difficulties : there might be conjugate points which prevent one from contoling the Jacobi fields along a family of geodesics emanating from o ; moreover there is no hypothesis on the cut-locus, and hence one does not know the lower curvature bound along these geodesics explicitely. Nevertheless the generalized Toponogov theorem allows for some rough estimates.

1. Proposition :



Let $a,\varepsilon \in (0,1)$ *and let* $\Delta = (p_0, p_1, p_2)$ *be a generalized geodesic triangle in an asymptotically non-negatively curved manifold* M^n . *Suppose moreover that* $\ell_0 \le (1-\varepsilon)\cdot\ell_1$ *and that* p_2 *is the base point* o *of* M^n . *Then the following estimates hold:*

i) $\cos(\sphericalangle \text{ at } o) \ge \sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2}$
$\Longrightarrow \ell_2 \le \ell_1 - \ell_0\cdot\sqrt{1-a^2}$

ii) $\cos(\sphericalangle \text{ at } p_1) \ge -\sqrt{1-a^2}$
$\Longrightarrow \ell_1 \le \ell_2 + \ell_0\cdot\sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2}$

iii) $\sphericalangle$ *at* o *acute,*
$\Longrightarrow |\sin(\sphericalangle \text{ at } o)| \ge \beta^2\cdot\varepsilon^2\cdot|\sin(\sphericalangle \text{ at } p_1)|$.

Proof :

We put $\ell := \ell_1 = d(p_0, o)$ and $k(r) := \lambda(|\ell - r|)$, $r > 0$. Making use of the triangle inequality and the monotonicity of λ , we see that for all p in M^n :

1.1 $$\text{curvatures at } p \ge -\lambda(d(p,o)) \ge -k(d(p,p_0))$$

i) We can apply the generalized comparison theorem (c.f. I.3.2.b) and reduce things to a problem in the model space $M^2(-k)$, where the radial

Jacobi field y is a multiple of the function $z_\ell(\ell-.)$ defined in section II.2 . We consider the function r along the edge $\bar{\gamma}_o$. This unit-speed geodesic joins $\bar{p}_1 = \bar{\gamma}_o(0)$ and $\bar{p}_2 = \bar{\gamma}_o(\ell_o)$. By monotonicity we may restrict to the case where

$$r'(\ell_o) = \cos(\sphericalangle \text{at } \bar{p}_2) = \sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2}$$

The conservation law I.2.1iii becomes :

1.2 $$z_\ell(\ell-r)^2\cdot(1-r'^2) = \text{const}$$

Observing that $r(\ell_o) = \bar{\ell}_1 = \ell_1$, we obtain :

$$\begin{aligned} a^2\cdot\beta^2\cdot\varepsilon^2 &= z_\ell(0)^2\cdot(1-\cos^2(\sphericalangle \text{at } \bar{p}_2)) \\ &= z_\ell(\ell-r(s))^2\cdot(1-r'(s)^2) \\ &\geq \varepsilon^2\cdot z_\infty(\ell)^2\cdot(1-r'(s)^2) \qquad , 0 \leq s \leq \ell_o \end{aligned}$$

here the inequality is due to lemma II.2.5ii and to the fact that $r(s) \geq \ell_1 - \ell_o \geq \varepsilon\cdot\ell$. We conclude that

$$a^2 \geq 1 - r'(s)^2 \qquad , 0 \leq s \leq \ell_o$$

Thereforethe continuous function r' does not vanish in the interval $[0,\ell_o]$, and there we get :

$$r' \geq \sqrt{1-a^2}$$

Hence $\bar{\ell}_2 = r(0) \leq r(\ell_o) - \ell_o\cdot\sqrt{1-a^2} = \ell_1 - \ell_o\cdot\sqrt{1-a^2}$.

ii) Here an indirect proof works : assume that $\ell_1 > \ell_2 + \ell_o\cdot\sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2}$. Again we make use of the generalized Toponogov theorem. Exchanging the roles of p_1 and p_2 , we obtain a triangle $\bar{\Delta} = (\bar{p}_o,\bar{p}_1,\bar{p}_2)$ in the model $M^2(-k)$ such that

$$\bar{\ell}_o = \ell_o \quad , \quad \bar{\ell}_1 \geq \ell_1 \quad , \quad \bar{\ell}_2 = \ell_2 \quad , \quad \cos(\sphericalangle \text{at } \bar{p}_1) = -\sqrt{1-a^2}.$$

The function r along $\bar{\gamma}_o$ obeys $r(0) = \ell_2$ and $r'(0) = \sqrt{1-a^2}$. Since $\ell - \bar{\ell}_2 \leq \ell_o \leq (1-\varepsilon)\cdot\ell$, we can deduce from formula 1.2 that

$$\begin{aligned} z_\ell(\ell-r)^2\cdot(1-r'^2) &= z_\ell(\ell-\bar{\ell}_2)^2\cdot(1-r'(0)^2) \\ &\geq a^2\cdot z_\ell((1-\varepsilon)\cdot\ell)^2 \geq a^2\cdot\beta^2\cdot\varepsilon^2 \end{aligned}$$

As long as $r(s) \leq \ell$,we have $z_\ell(\ell-r) \leq 1$, and hence :

$$r(s) \leq \ell_2 + s\cdot\sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2}$$

The standard continuity arguement now yields the contradiction

$$\bar{\ell}_1 = r(\ell_o) \leq \ell_2 + \ell_o\cdot\sqrt{1-a^2\cdot\beta^2\cdot\varepsilon^2} < \ell_1$$

iii) Put $a_1 := \sin(\measuredangle \text{ at } p_1)$; then we conclude with the aid of part (ii) that

$$\ell_1 \leq \ell_2 + \ell_0 \cdot \sqrt{1 - a_1^2 \cdot \beta^2 \cdot \varepsilon^2}$$

Reversing the implication in (i), we obtain :

$$\cos(\measuredangle \text{ at } o) \leq \sqrt{1 - a_1^2 \cdot \beta^4 \cdot \varepsilon^4}$$

The proof is then finished, as the angle at o is acute by hypothesis.

2. We recall that two curves $c_1, c_2 : [0,\infty) \longrightarrow M$ are said to be *cofinal*, if and only if for every compact set $K \subset M$ there is some $t > 0$ such that $c_1(t_1)$ and $c_2(t_2)$ lie in the same connected component of $M \setminus K$ for all $t_1, t_2 \geq t$. An equivalence class of cofinal curves is called an *end of* M.

Elementary Properties :

2.1. Any family of relatively compact open sets $(U_i)_{i \in \mathbb{N}}$ which exhaust M, i.e. which obey $U_i \subset\subset U_{i+1}$ and $\bigcup_i U_i = M$, defines a bijection :

$$\{\text{ends } E \text{ of } M\} \overset{\cong}{\longleftrightarrow} \{(E_i)_{i \in \mathbb{N}} \mid E_{i+1} \subset E_i \text{ and } E_i \text{ is a connected component of } M \setminus \overline{U_i}\}$$

Notice that for each of these inverse systems $E = (E_i)_{i \in \mathbb{N}}$ the sets E_i are non-empty and their closures $\overline{E_i}$ in M are non-compact. Moreover, if M has only finitely many ends, then there is some $i_o > 0$ such that all the inverse systems $(E_i)_{i \in \mathbb{N}}$ stabilize for $i \geq i_o$, i.e. $E_i = E_{i_o}$.

2.2. Given a point $p \in M$, then any end E of M contains a *ray* γ emanating from p ; recall that by definition γ is a geodesic $[0,\infty) \longrightarrow M$ such that each of its segments is shortest and that $\gamma(0) = p$.

2.3. Given any two distinct ends E^1 and E^2 of M, there is a *line* $\gamma : \mathbb{R} \longrightarrow M$ such that the rays $\gamma_\pm : [0,\infty) \longrightarrow M$, $t \longmapsto \gamma(\pm t)$ are contained in E^1 and E^2 respectively.

2.4. As is the case for the *ideal boundary* in the theory of non-compact surfaces, the set of ends carries a natural topology ; a basis for the open sets is parametrized by the non-compact closed subsets $C \subset M$:

$$U_C := \{\text{ends } (E_i)_{i \in \mathbb{N}} \mid E_i \subset C \text{ for } i \text{ sufficiently large}\}$$

In this way {ends E of M} becomes a compact, separable, totally disconnected space. (c.f. [AS], [R].)

3. Theorem :

Every asymptotically non-negatively curved manifold M^n *has at most finitely many ends. More precisely :*

$$\#\{\text{ends } E \text{ of } M\} \leq 2 \cdot \pi^{n-1} \cdot \exp((n-1) \cdot b_1)$$

Here b_1 is the invariant introduced in II.2.3 .

Proof : For each end E of M we pick a unit-speed ray γ_E which emanates from o and is contained in E . We consider the set of unit vectors $v_E := \gamma_E'(0)$ in T_oM . It is a consequence of proposition 1 that for any unit vector $v \in T_oM$ which is sufficiently close to some v_E the geodesic $\gamma : [0,\infty) \longrightarrow M$, $t \longmapsto \exp_o(t \cdot v)$ is contained in the end E ; in some more detail one obtains :

$$\langle v , v_E \rangle = \sqrt{1-a^2 \cdot \beta^2 \cdot \varepsilon^2} \quad , \quad 0 < a^2 , \varepsilon^2 < 1$$

$$\Longrightarrow \quad d(\gamma_E(t) , \gamma((1-\varepsilon)\cdot t)) \leq t - (1-\varepsilon)\cdot t \cdot \sqrt{1-a^2} =: q_{a,\varepsilon} \cdot t$$

where $0 < q_{a,\varepsilon} < 1$; therefore γ is contained in E provided $\langle v , v_E \rangle > \sqrt{1-\beta^2}$. Thus the balls B^S_E in the unit sphere $S^{n-1} \subset T_oM$ with centres v_E and radii $\frac{1}{2} \cdot \arcsin(\beta)$ are pairwise disjoint. Notice that $\arcsin(\beta) \geq \beta = \exp(-b_1)$; so the claimed bound on the number of ends is a direct consequence of the following well-known packing lemma.

3.1 Lemma :

Let $0 < \rho \leq \frac{\pi}{2}$ *; then the number of disjoint balls* $B^S(\rho) \subset S^{n-1}$ *with radius* ρ *does not exceed*

$$\frac{\text{vol } S^{n-1}}{\text{vol } B^S(\rho)} \leq 2 \cdot \left(\frac{\pi}{2 \cdot \rho}\right)^{n-1}$$

IV. Surfaces and other Examples

In this chapter we are going to discuss the hypothesis and conclusions of theorem B. A first set of examples shows that for surfaces the theorem is definitely wrong when the integral b_o diverges (c.f. IV.1). Moreover we shall see that the given bounds on the number of ends and on the Betti numbers are reasonable in a certain sense : in section IV.2 we construct surfaces with large invariants b_o and b_1 such that theorem B overestimates the number of ends by not more than a factor of 2π ; in section IV.3 we consider the higher-dimensional case and give a set of examples where the bounds actually grow exponentially in $n \cdot b_1(M^n)$.

1. Theorem :

Let $\lambda : [0,\infty) \longrightarrow [0,\infty)$ *be a continuous function such that the integral*

$$\int_0^\infty r \cdot \lambda(r)\, dr$$

diverges. Then every non-compact, connected surface M^2 *with base point* o *carries a* C^2*-metric* g *with curvature*

(*) $\kappa(p) = -\lambda(d(o,p))$ *at any point* $p \in M^2$

Remarks :

1.1. Obviously $\mathbb{R}^2$ becomes the surface of revolution $M^2(-\lambda)$, which has been described in chapter I.1 .

1.2. Suppose that the curvature of a surface (M^2,g) with base point o obeys condition (*) above. Then the complement of the cut-locus of o is isometric to a tree-like open subset in $M^2(-\lambda)$; the isometry is given by $\exp_o$ and the obvious identifications.

Moreover the generic cut points, i.e. those cut points which are joined to o by precisely two minimizing geodesics, lie on open geodesic segments in (M^2,g) .

1.3. In order to reverse the preceeding observation and construct some more examples, we look at two non-intersecting geodesics γ_1 and γ_2 in $M^2(-\lambda)$ which have equal distance to the base point. Notice that they can be mapped onto each other by an isometry ϕ of $M^2(-\lambda)$. We take that component of $M^2(-\lambda) \setminus (\gamma_1 \cup \gamma_2)$ which contains the pole. We take its closure and glue the boundary components γ_1 and γ_2 by means of ϕ. The differentiable structure of the quotient manifold

M^2 is conveniently described using normal exponential coordinates around the geodesics γ_1 and γ_2. The quotient metric g on the surface M^2 turns out to be of class C^2 ; the reason is that the curvature function of $M^2(-\lambda)$ is invariant under the clutching map ϕ.

1.4. This construction can be iterated as long as one can find an appropriate pair of geodesics γ_1, γ_2 in

$$(T_o M_j^2)^{int} := \{ x \in T_o M_j^2 \mid 0 \text{ and } x \text{ are joined by an arc which does not contain a cut point.} \}$$

It gives rise to a surface M_{j+1}^2, which differs from M_j^2 topologically and the metric g_{j+1} still obeys condition (*). Depending on the position of the geodesics and the orientation of ϕ there are four distinct cases :

i) If γ_1 and γ_2 lie in the same end E_j of (M_j^2, g_j), then either E_j is split into two ends E_{j+1}^1 and E_{j+1}^2 (Fig. a & Fig. b) or a cross cap is attached to E_j (Fig. c).

ii) If γ_1 and γ_2 lie in different ends E_j^1 and E_j^2, then these ends are glued ; a handle (Fig. d) resp. a Kleinian bottle (Fig. e) is attached.

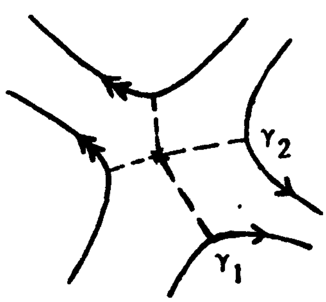


Fig. a

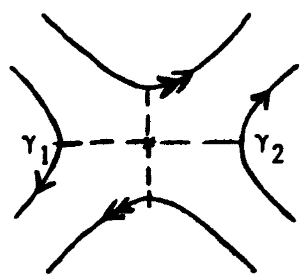


Fig. b

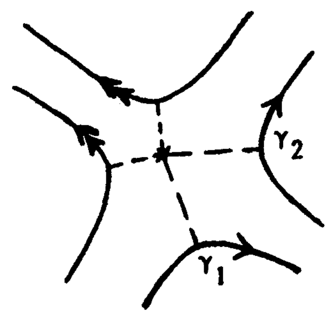


Fig. c

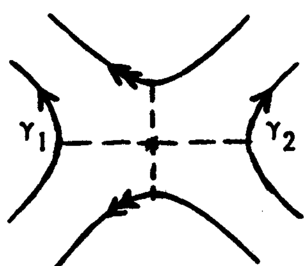


Fig. d

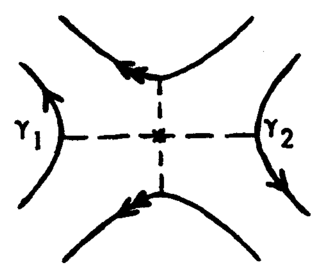


Fig. e

1.5. Basically it is the function λ which determines how often the constructions 1.4i and 1.4ii can be applied. Let us assume that

$$\int_0^\infty r \cdot \lambda(r)\, dr$$

diverges. Then the integral of the curvature over an arbitrary sector in $M^2(-\lambda)$ also diverges, and this surface turns out to be a visibility manifold (c.f. [EN]); the angle $\alpha(\gamma)$ of the sector in which a geodesic γ is seen from the pole o decreases to zero when $dist(o,\gamma) \to \infty$. Hence in any conical end of a surface M_j^2 one can go out far enough and find geodesics γ_1 and γ_2, suitable for the constructions 1.4i and 1.4ii. Moreover it is possible to pick these geodesics in such a way that the manifold M_{j+1}^2 has only conical ends, provided M_j^2 had.

Therefore - whenever the above integral of λ diverges - metrical considerations do not impose any conditions on the combinatorial patterns for iterating the constructions 1.4.

Proof of the Theorem:

Standard classification results imply that in remark 1.5 we have constructed all non-compact, orientable and non-orientable surfaces which have finite genus and finitely many ends. Next we consider a sequence of surfaces M_j^2 and perform all the surgery simultaneously. This yields a manifold M_∞^2 which carries a metric g_∞ obeying condition (*). Our goal is to finish the arguement using the classification of surfaces. This classification result is due to Richardson [R] and requires the following data:

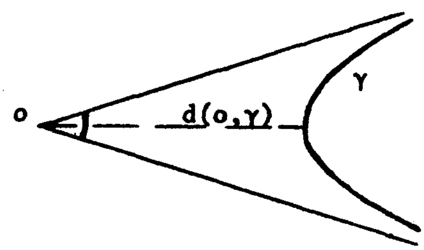
i) a triple of totally-disconnected, separable, compact sets $A \subset B \subset C$

ii) orientability (dispensible, if $A \neq \emptyset$.)

iii) genus (dispensible, if $B \neq \emptyset$.)

Here A describes the infinitely non-orientable ends, B the ends with infinite genus, and C simply contains all points of the ideal boundary. It is known that totally-disconnected, separable, compact sets like Cantor sets are related to equivalence classes of trees. These trees govern the combinatorial pattern according to which the constructions 1.4 have been iterated.

2. Next we consider a function λ such that the integral $b_o(\lambda)$ is finite. We are going to construct a surface M^2 which has as many ends as possible. For this purpose we look at the geodesic triangle Δ in $M^2(-\lambda)$ which is given by the pole o and an arbitrary geodesic γ and which has two vertices at infinity. The Gauß-Bonnet theorem yields :



$$(**) \qquad \measuredangle \text{ at } o = \pi - \int_\Delta \lambda(d(o,.))\, dvol$$

The differential equation $y'' = \lambda \cdot y$ and the initial data $y(0) = 0$, $y'(0) = 1$ as usual describe a radial Jacobi field and determine polar coordinates. Moreover it is possible to define another invariant

$$b_2(\lambda) \ := \ \lim_{r\to\infty} y'(r)$$

We can now proceed and estimate the right-hand side of (**) :

$$\measuredangle \text{ at } o \ \leq \ \pi - \int_0^{d(o,\gamma)} \lambda(r)\cdot y(r)\, dr \cdot \measuredangle \text{ at } o$$

Hence $(\measuredangle \text{ at } o)\cdot y'(d(o,\gamma)) \leq \pi$, and we can pick at least $2\cdot[y'(d)]$ non-intersecting geodesics γ in $M^2(-\lambda)$, each of them with distance d to the base point o . Applying the construction from 1.4i as often as we can, we obtain a surface with $[y'(d)]$ ends. Finally we pass to the limit $d\to\infty$:

2.1 Proposition :

Whenever the invariant integral $b_o(\lambda)$ *of some function* $\lambda:[0,\infty)\to[0,\infty)$ *is finite, then there exists a surface* (M^2,g) *which has at least* $\exp(b_2(\lambda)) - 1$ *ends and whose curvature obeys the condition*

$$\kappa(p) \ = \ -\lambda(\text{dist}(p, \textit{base point})) \qquad \textit{for all } p\in M^2$$

This proposition shows that for surfaces the previously given upper bound on the number of ends is sharp up to a factor of at most 2π ; we pick λ to be the characteristic function of $[0,d]$ and compute :

$$b_o(\lambda) = \frac{1}{2}\cdot d^2 \quad , \quad b_1(\lambda) = \ln\cosh(d) \quad , \quad b_2(\lambda) = \ln\sinh(d) \quad ;$$

asymptotically b_1 and b_2 coincide.

3. Our last examples shall demonstrate that for asymptotically non-negatively curved manifolds M^n the number of ends and the sum of the Betti numbers can grow exponentially in $n \cdot b_1(M^n)$ each. We point out that Riemannian products of the above surfaces are totally inadequate in either case. Partially this is due to the fact that the function λ changes when passing to products.

We are going to construct some tree-like looking objects. Roughly speaking the desired growth in $n \cdot b_1(M^n)$ is acchieved by using building blocks of the same type only. In order to describe these pieces it is convenient to think of a hypersurface in $\mathbb{R}^{n+1}$ which is obtained by glueing cylinders $\mathbb{R}^+ \times S^{n-1}$ perpendicular onto a hyperplane $\mathbb{R}^n$ where appropriate balls have been removed. The curvature is kept bounded by plugging in some intermediate tubes. Again we use the "same" tube everywhere, and a packing arguement assures that the number of ends of a single building block grows exponentially in n .

3.1. the intermediate tubes :

We fix some $t_o > 0$ and consider the warped products $Tb^n(t_o) = ([0,t_o] \times S^{n-1} , ds^2)$, where the metric is defined by :

$$ds^2 = dt^2 + \sinh^{-2}(t_o) \cdot \cosh^2(t_o - t) \cdot d\omega^2$$

Here $d\omega^2$ denotes the standard metric on S^{n-1} .

Properties :

i) $$\operatorname{diam} Tb^n(t_o) \leq \max_{o \leq t \leq t_o} t + \pi \cdot \sinh^{-1}(t_o) \cdot \cosh(t_o - t)$$
$$= \pi \cdot \coth(t_o) + \max\{ 0 , t_o - \pi \cdot \tanh(\tfrac{1}{2} \cdot t_o) \}$$

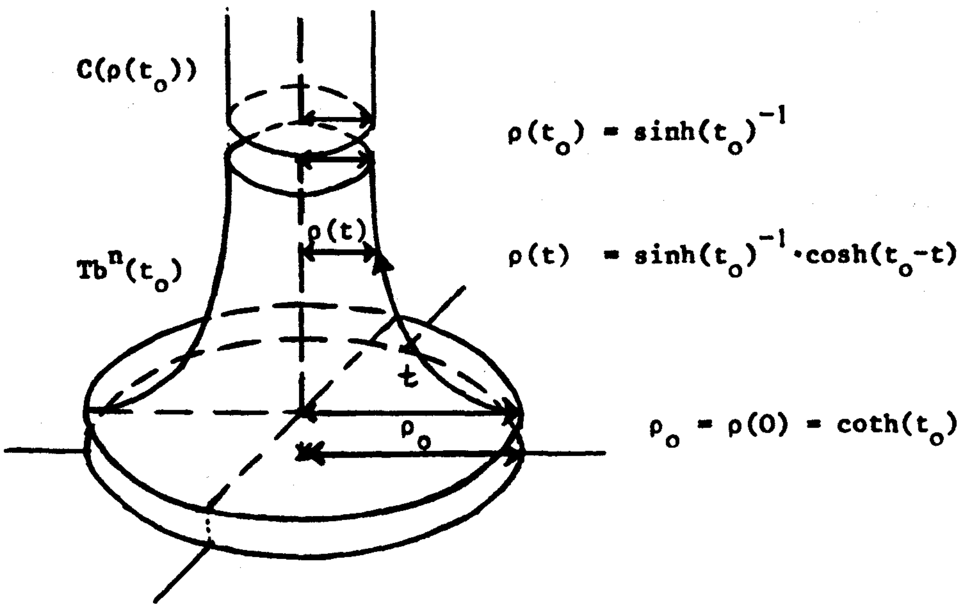
ii) the tubes $Tb^n(t_o)$ can be embedded as rotationally symmetrical hypersurfaces in $\mathbb{R}^{n+1}$.

iii) $Tb^2(t_o)$ has constant curvature equal to -1 , and for $n > 2$ the $Tb^n(t_o)$ have sectional curvatures ≥ -1 .

iv) the boundary components $\{0\} \times S^{n-1}$ and $\{t_o\} \times S^{n-1}$ are spheres with constant curvature $\tanh^2(t_o)$ and $\sinh^2(t_o)$ respectively. $(n > 2)$. As submanifolds in $Tb^n(t_o)$ they have principal curvatures $\tanh(t_o)$ and 0 respectively.

v) the tube $Tb^n(t_o)$ can be doubled in an analytical way along the boundary component $\{t_o\} \times S^{n-1}$. The same boundary component of the tube can be glued isometrically to the boundary of a cylinder $C(\sinh^{-1}(t_o)) := \sinh^{-1}(t_o) \cdot (\mathbb{R}^+ \times S^{n-1})$ with radius $\sinh^{-1}(t_o)$; this time curvature is only bounded, but non-continuous.

vi) at $\{0\} \times S^{n-1}$ the tube $Tb^n(t_o)$ can be glued with bounded, but non-continuous curvature to $\mathbb{R}^n \setminus B(x, \coth(t_o))$, $x \in \mathbb{R}^n$ arbitrary.



3.2 the building blocks A^n :

Let $t_o > 0$, $r_o \geq 2$, $\rho_o := \coth(t_o)$, and let B_o be the ball $B(0,\rho_o)$ in $\mathbb{R}^n$. We pick a maximal family of mutually disjoint open balls $B_1, \ldots, B_N$ with radius ρ_o in the subset $B(0,(r_o+1)\cdot\rho_o) \setminus B_o$. We remove all $N+1$ balls $B_o, \ldots, B_N$ and – as described in 3.1vi – attach tubes $Tb^n(t_o)$ to the boundaries $\partial B_o, \ldots, \partial B_N$. The boundary of the resulting manifold A^n consists of the spheres $\{t_o\} \times S^{n-1}$ in the attached tubes. For later use it is convenient to single out the boundary of the central tube which has been glued to ∂B_o ; we shall call it a^n .

Properties :

i) $\#\{\text{ends of } A^n\} = 1$

$\#\{\text{boundary components of } A^n\} = N+1$

ii) in $\mathbb{R}^n$ the enlarged balls $2 \cdot B_j$, $0 \leq j \leq N$, cover $B(0, r_o \cdot \rho_o)$; hence :

$$N + 1 \geq \left(\frac{1}{2} \cdot r_o\right)^n$$

iii) in $\mathbb{R}^n$: $\quad \mathrm{dist}(B_o, B_j) = (r_o - 2) \cdot \rho_o \qquad ; 1 \leq j \leq N$

Moreover for each j there exists a curve in $\mathbb{R}^n \setminus \bigcup_{\nu \neq o} B_\nu$ which joins ∂B_o and ∂B_j and which has length $\geq (r_o - 2) \cdot \rho_o$.

iv) for any point $p \in A^n$ which is non-flat and for any point $p \notin \partial A^n$ one has :

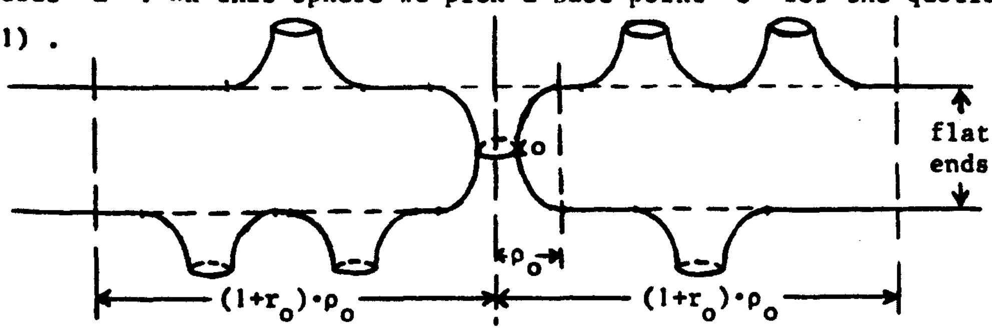
$$d(p, a^n) \leq \frac{\pi}{2} \cdot (r_o - 2) \cdot \rho_o + \mathrm{diam}\, Tb^n(t_o) + t_o$$

v) the sectional curvatures of A^n are ≥ -1 .

3.3 the trees $A^n(\mu)$:

We use the following inductive construction :

i) We glue two copies of the manifold A^n by identifying their boundary spheres a^n . On this sphere we pick a base point o for the quotient $A^n(1)$.



ii) We assume that $A^n(\mu)$ has been constructed and that all its $2 \cdot N^\mu$ boundary components are totally geodesic spheres with diametre equal to $\pi \cdot \sinh^{-1}(t_o)$. Then we can attach to each of these spheres the central boundary a^n of a new copy of A^n , and thus we can glue $2 \cdot N^\mu$ copies of A^n to $A^n(\mu)$. We define this larger manifold to be the $\mu+1^{st}$ generation object $A^n(\mu+1)$.

3.4 the manifolds $M^n(\mu)$:

We obtain non-compact, complete Riemannian manifolds $M^n(\mu)$ by glueing to each boundary sphere of $A^n(\mu)$ a cylinder $C(\sinh^{-1}(t_o))$.

Properties :

i) $$\#\{\text{ends of } M^n(\mu)\} = 2 \cdot \frac{N^{\mu+1} - 1}{N - 1}$$

$$\beta_{n-1}(M^n(\mu)) = \#\{\text{ends of } M^n(\mu)\} - 1$$

$$\geq 2 \cdot N^\mu \geq \left(\left(\frac{1}{2} \cdot r_o\right)^n - 1\right)^\mu$$

ii) the sectional curvatures of $M^n(\mu)$ at any point p are bounded from below by $-\lambda_{(\mu)}(d(p,o))$, where $\lambda_{(\mu)}$ is the characteristic function of the interval $[0\,,\,d_\mu]$, and d_μ is given by :

$$d_\mu \;:=\; \pi\cdot\sinh^{-1}(t_o) + \mu\cdot\left(\frac{\pi}{2}\cdot r_o\cdot\coth(t_o) + t_o + \max\{0\,,\; t_o - \pi\cdot\tanh(\tfrac{1}{2}\cdot t_o)\}\right)$$

iii) for any integer $\mu \geq 1$ the following inequalities hold :

$$b_1(M^n(\mu)) \;\leq\; d_\mu + \ln(1+e^{-2}) - \ln(2)$$

$$\ln \beta_{n-1}(M^n(\mu)) \;\geq\; \mu\cdot\left(n\cdot\ln\left(\frac{r_o}{2}\right) + \ln\left(1-\left(\frac{r_o}{2}\right)^{-n}\right)\right)$$

Specializing to the case $r_o = 7$ and $t_o = 2.5$, we obtain :

$$\ln \beta_{n-1}(M^n(\mu)) \;\geq\; 1.2\cdot n\cdot\mu \;\geq\; \frac{1}{12}\cdot n\cdot b_1(M^n(\mu))$$

Notice that we still have the freedom to pick μ large. Therefore , when working in terms of the invariant $b_1(M^n)$ and the dimension , any estimate on the number of ends or on the $n-1^{st}$ Betti number has to grow at least exponentially in $n\cdot b_1(M^n)$. Such a result has been acchieved in Theorem B .

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LOWER CURVATURE BOUNDS, TOPONOGOV'S THEOREM, AND BOUNDED TOPOLOGY II

An upper bound on the Betti numbers of asymptotically non-negatively curved manifolds

Introduction

In this paper we continue studying asymptotically non-negatively curved manifolds. Our goal is to estimate their Betti numbers from above in terms of curvature decay and dimension. In special cases bounds of this type are due to Gromov [G] ; he deals with non-negatively curved manifolds and with compact manifolds. Related is also the work of Berard and Gallot [BG] who have applied heat equation methods in order to get bounds for all topological invariants of compact manifolds.

We recall that a complete Riemannian manifold (M^n, g) with base point o is said to be *asymptotically non-negatively curved*, iff there exists a monotone function $\lambda : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\text{i)} \quad b_o(\lambda) := \int_0^\infty r \cdot \lambda(r)\, dr < \infty$$

and ii) the sectional curvatures at p $\geq -\lambda(d(o,p))$ for all $p \in M^n$. A detailed exposition of the analytical impact of the convergence of the integral $b_o(\lambda)$ has been given in chapter II of part one; for instance there exists a unique non-negative solution of the Riccati equation $u' = u^2 - \lambda$ with the property that $u(r) \to 0$ for $r \to \infty$. This gives rise to another numerical invariant

$$b_1(\lambda) := \int_0^\infty u(r)\, dr$$

Both b_o and b_1 depend on λ in a monotone way, and they can be regarded as invariants of the manifold M^n by taking the minimal monotone function λ which obeys the conditions (i) and (ii).

In principle b_o and b_1 can be regarded as equivalent invariants: $b_1 \leq b_o \leq \exp(b_1) - 1$. However , b_1 is better adapted to our problem. A natural family of weighted L^1-norms on the Betti numbers of a space X are induced by the Poincaré series

$$P_t(X) := \sum_i t^i \cdot \beta_i(X)$$

<u>Main Theorem :</u>

For any asymptotically non-negatively curved manifold (M^n, g, o) *the Betti numbers with respect to an arbitrary coefficient field can be bounded universally in terms of the dimension and the invariant* b_1 :

$$P_{t(n)^{-1}}(M^n) \leq C(n) \cdot \exp\left(\frac{15n-13}{4} \cdot b_1(M^n)\right)$$

where $\quad C(n) := \exp(5 \cdot n^3 + 8 \cdot n^2 + 5 \cdot n + 2)$

and $$t(n) := 5^{n^2} \cdot 8^n \cdot \exp(\frac{9}{4} \cdot \frac{1}{n+1})$$

Moreover these manifolds have finitely many ends and the Betti numbers at infinity are bounded as follows :

$$\sum_{\text{ends } E} P_{t(n)^{-1}}(E) \leq C(n) \cdot \exp((n-1) \cdot b_1(M^n))$$

Remarks :

i) By the examples given in chapter IV of part one it is reasonable that the bounds in both the estimates grow exponentially in $n \cdot b_1(M^n)$. However there is no geometric reason known so far , why the constants $C(n)$ and $t(n)^n$ should grow exponentially in n^3 .

ii) Notice that :

$$\#\{\text{ends of } M^n\} \leq \sum_{\text{ends}} P_1(E) \leq t(n)^{n-1} \cdot \sum_{\text{ends}} P_{t(n)^{-1}}(E)$$

Thus we have recovered a weaker version of Theorem III.3 in part one.

iii) Using the long exact homology sequence , one obtains an estimate on the relative Betti numbers :

$$P_{t(n)^{-1}}(M^n , \bigcup_{\text{ends}} E) \leq (1+t(n)) \cdot C(n) \cdot \exp(\frac{15n-13}{4} \cdot b_1(M^n))$$

iv) Because of Poincaré duality the inequality

$$\sum_i \beta_i(M^n) \leq \tilde{C}(n) \cdot \exp(\frac{15n-13}{4} \cdot b_1(M^n))$$

holds with $$\tilde{C}(n) := 3 \cdot t(n)^{(n+1)/2} \cdot C(n)$$
$$\leq \exp(6 \cdot n^3 + 10 \cdot n^2 + 6 \cdot n + 3)$$

Special Cases :

a) M^n has non-negative sectional curvature :

$$P_{t(n)^{-1}}(M^n) \leq C(n)$$

b) the sectional curvatures of M^n are bounded from below by $-k^2$ and even more are non-negative outside a ball with radius d around the base point o : (i.e.: M^n compact with diametre d)

$$P_{t(n)^{-1}}(M^n) \leq C(n) \cdot \exp(\frac{15n-13}{4} \cdot k \cdot d)$$

Method of Proof :

We use a modification of Gromov's direct geometric proof. The basic idea is to combine Morse theory arguements on the distance function and covering arguements. In a first step we do things locally and derive an estimate for small balls (sections 1 - 3). In a second step we reduce the theorem to these local bounds (sections 4 and 5).

In principal the local result is already contained in Gromov's paper (c.f. [G]) ; however, we shall rearrange the details in a more subtle way. Therefore our constants grow only exponentially in n^3 ; they do not depend doubly exponentially on n . The key to this improvement is a non-standard packing lemma (c.f. Appendix A).

The way in which we put together the local estmates is essentially new. We use metrical annuli as intermediate objects when extending the estimate from small balls to all of the manifold M^n .

1. A Topological Lemma :

In this section we are going to do the topological part of the arguement. There are two reasons for avoiding the Betti numbers in the intermediate steps in the proof :

a) Given a point $p \in M^n$ and any number $N > 0$, it is easy to put a bumpy on M^n such that $\dim H_1(B(p,1)) \geq N$. The idea is to produce a sufficiently complicated intersection of the distance sphere $S(p,1) \subset M^n$ with the cut-locus of p .

b) For arbitrary subsets $X_1 , X_2 \subset M^n$ it is impossible to estimate the dimension of $H_*(X_1 \cup X_2)$ in terms of $\dim H_*(X_1)$ and $\dim H_*(X_2)$ only. Some pieces of information about $X_1 \cap X_2$ are required in addition.

These obstructions towards an " obvious proof" are related, and they both can be circumvented looking at topological pairs (Y,X) where $X \subset Y \subset M^n$ are open subsets. We consider the numbers

<u>1.1</u>
$$\begin{aligned} rk_i(Y,X) &:= rk\,(H_i(X) \longrightarrow H_i(Y)) \\ rk_*^t(Y,X) &:= \sum_{i \geq 0} t^i \cdot rk_i(Y,X) \end{aligned}$$

It is worthwhile noticing that under the hypothesis above the numbers $rk_i(Y,X)$ vanish for $i > n$.

<u>1.2</u> We consider open subsets $B_j^o \subset B_j^1 \subset \ldots \subset B_j^{n+1}$, $1 \leq j \leq N$, such that

$$X \subset \bigcup_{j=1}^{N} B_j^o$$

and
$$Y \supset \bigcup_{j=1}^{N} B_j^{n+1}$$

<u>Lemma :</u>

Let $t > 0$ *and suppose that any* B_j^n *intersects at most* t *distinct sets* $B_{j'}^n$, $j' \neq j$; *then there holds the following inequality :*

$$rk_*^{t^{-1}}(Y,X) \leq rk_*^{t^{-1}}\Big(\bigcup_{j=1}^{N} B_j^{n+1} , \bigcup_{j=1}^{N} B_j^o\Big)$$

$$\leq (e-1)\cdot N \cdot \sup\{rk_*^{t^{-1}}(B_{j_o}^{\sigma+1} \cap \ldots \cap B_{j_{n-\sigma}}^{\sigma+1} , B_{j_o}^{\sigma} \cap \ldots \cap B_{j_{n-\sigma}}^{\sigma}) \mid$$

$$0 \leq \sigma \leq n \;,\; 1 \leq j_o < \ldots < j_{n-\sigma} \leq N \;\} .$$

Essentially this lemma is already contained in Gromov's paper (c.f. [G]). For the sake of completeness we include an elementary

<u>Proof:</u>

Consider open subsets $X_1 \subset X_2 \subset X_3$ and $Y_1 \subset Y_2 \subset Y_3$ in M^n. The Mayer-Vietoris sequence gives raise to a commutative diagrame with exact rows :

$$\begin{array}{ccccccc}
\longrightarrow H_\mu(X_1) \oplus H_\mu(Y_1) & \longrightarrow & H_\mu(X_1 \cup Y_1) & \longrightarrow & H_{\mu-1}(X_1 \cap Y_1) & \longrightarrow \\
\downarrow & & \downarrow i_\mu & & \downarrow i'_{\mu-1} & \\
\longrightarrow H_\mu(X_2) \oplus H_\mu(Y_2) & \longrightarrow & H_\mu(X_2 \cup Y_2) & \longrightarrow & H_{\mu-1}(X_2 \cap Y_2) & \longrightarrow \\
\downarrow j_{\mu,X} \oplus j_{\mu,Y} & & \downarrow j_\mu & & \downarrow & \\
\longrightarrow H_\mu(X_3) \oplus H_\mu(Y_3) & \longrightarrow & H_\mu(X_3 \cup Y_3) & \longrightarrow & H_{\mu-1}(X_3 \cap Y_3) & \longrightarrow
\end{array}$$

All the vertical homomorphisms are induced by inclusions. The standard diagrame chasing technique shows that :

1.3 $$\mathrm{rk}\,(j_\mu \cdot i_\mu) \leq \mathrm{rk}\,(j_{\mu,X} \oplus j_{\mu,Y}) + \mathrm{rk}\,(i'_{\mu-1})$$

or in different terminology :

$$\mathrm{rk}_\mu(X_3 \cup Y_3\, ,\, X_1 \cup Y_1)$$
$$= \mathrm{rk}_\mu(X_3\, ,\, X_2) + \mathrm{rk}_\mu(Y_3\, ,\, Y_2) + \mathrm{rk}_{\mu-1}(X_2 \cap Y_2\, ,\, X_1 \cap Y_1)$$

We apply this formula inductively to the family B^i_j and obtain :

1.4 $$\mathrm{rk}_i\Big(\bigcup_{j=1}^{N} B_j^{i+1+\nu}\, ,\, \bigcup_{j=1}^{N} B_j^{\nu}\Big)$$
$$\leq \sum_{\mu=o}^{i} \sum_{j_o<\dots<j_{i-\mu}} \mathrm{rk}_\mu\big(B_{j_o}^{\mu+\nu+1} \cap \dots \cap B_{j_{i-\mu}}^{\mu+\nu+1}\, ,\, B_{j_o}^{\mu+\nu} \cap \dots \cap B_{j_{i-\mu}}^{\mu+\nu}\big) \quad ;$$

here ν denotes some non-negative integer which does not exceed $n-i$.

We specialize to the case $\nu = n-i$ and compute :

$$\mathrm{rk}_*^{t^{-1}}\Big(\bigcup_{j=1}^{N} B_j^{n+1}\, ,\, \bigcup_{j=1}^{N} B_j^{o}\Big)$$
$$\leq \sum_{i=o}^{n} t^{-i} \cdot \mathrm{rk}_i\Big(\bigcup_{j=1}^{N} B_j^{n+1}\, ,\, \bigcup_{j=1}^{N} B_j^{n-i}\Big)$$
$$\leq \sum_{\nu=o}^{n} \sum_{\sigma=\nu}^{n} \sum_{j_o<\dots<j_{n-\sigma}} t^{\nu-n} \cdot \mathrm{rk}_{\sigma-\nu}\big(B_{j_o}^{\sigma+1} \cap \dots \cap B_{j_{n-\sigma}}^{\sigma+1}\, ,\, B_{j_o}^{\sigma} \cap \dots \cap B_{j_n}^{\sigma}$$

$$= \sum_{\sigma=o}^{n} \sum_{j_o<\dots<j_{n-\sigma}} t^{\sigma-n} \cdot \sum_{\nu=o}^{\sigma} t^{\nu-\sigma} \cdot rk_{\sigma-\nu}(\dots , \dots) \qquad ,$$

hence :

$$rk_*^{t^{-1}} \Big(\bigcup_{j=1}^{N} B_j^{n+1} , \bigcup_{j=1}^{N} B_j^{o} \Big)$$

1.5

$$\leq \sum_{\sigma=o}^{n} \sum_{j_o<\dots<j_{n-\sigma}} t^{\nu-n} \cdot rk_*^{t^{-1}} (B_{j_o}^{\sigma+1} \cap \dots \cap B_{j_{n-\sigma}}^{\sigma+1} , B_{j_o}^{\sigma} \cap \dots \cap B_{j_{n-\sigma}}^{\sigma}) \quad .$$

To complete the proof , we point out that the number of non-empty intersections

$$B_{j_o}^{\sigma} \cap \dots \cap B_{j_{n-\sigma}}^{\sigma}$$

does not exceed

$$\frac{N}{n-\sigma+1} \cdot \binom{t}{n-\sigma} \qquad , \; 0 \leq \sigma \leq n \qquad ;$$

therefore the number of non-vanishing terms on the right-hand side of 1.5 is bounded from above by

$$\sum_{\sigma=o}^{n} \frac{N}{n-\sigma+1} \cdot \binom{t}{n-\sigma} \cdot t^{\sigma-n} \leq N \cdot \sum_{\sigma=o}^{n} \frac{1}{(n-\sigma+1)!} \leq N \cdot (e-1) \quad .$$

In most of our applications the sets B_j^i will be open metrical balls. We shall use the notation $\rho \cdot B(p,r) := B(p, \rho \cdot r)$. It is convenient to draw the following

__1.6 Corollary :__

Let $\rho > 1$, $t \geq 1$ *and suppose that :*

i) X M^n *is covered by open metrical balls* B_j^o , $1 \leq j \leq N$,

ii) $i < n$ *and* $B_j^i \cap B_{j'}^i \neq \emptyset \implies \rho \cdot B_j^i \subset B_{j'}^{i+1}$,

iii) $\rho \cdot B_j^n \subset B_j^{n+1} \subset Y$, $1 \leq j \leq N$, *and*

iv) *each ball* B_j^n *intersects at most* t *other balls* $B_{j'}^n$.

Then the following estimate holds :

$$rk_*^{t^{-1}}(Y,X) \leq rk_*^{t^{-1}} \Big(\bigcup_{j=1}^{N} B_j^{n+1} , \bigcup_{j=1}^{N} B_j^{o} \Big)$$

$$\leq (e-1) \cdot N \cdot \sup \{ rk_*^{t^{-1}} (\rho \cdot B_j^i , B_j^i) \mid 0 \leq i \leq n , 1 \leq j \leq N \}$$

__Remark :__

Condition (ii) is obviously met , if all the balls B_j^o have equal radii and if $B_j^i = (2+\rho)^i \cdot B_j^o$, $0 \leq i \leq n$, $1 \leq j \leq N$.

2. The Morse Theory of the Distance Function

Any point $p \in M$ gives raise to a function $d_p : M \longrightarrow \mathbb{R}$ defined by $d_p(\tilde{p}) := d(p,\tilde{p})$. $\tilde{p} \in M$ is called a *critical point of* d_p , iff for any $v \in T_{\tilde{p}}M$ there is a minimizing geodesic segment γ which joins $\tilde{p} = \gamma(0)$ to p , and which obeys $\langle v , \gamma'(0) \rangle \geq 0$. $\mathcal{S}_p :=$ { critical points of d_p } is said to be the *singular set of* d_p . Notice that $\tilde{p}$ is *non-critical* for d_p , iff the initial vectors of all minimizing geodesics joining $\tilde{p}$ to p lie in an open half-space of $T_{\tilde{p}}M$. Hence for any non-critical point $\tilde{p}$, there is an open neighborhood U and a contiuous non-vanishing vector fiel v_U which is defined on U and has an acute angle with the initial vector of any minimizing geodesic joining a point in U to p . As $\mathcal{S}_p$ is closed one can reason in the standard way and obtain :

<u>2.1 Lemma :</u>

For any $p \in M$ *there exists a smooth vector field* $v_p : M \longrightarrow TM$ *, which obeys* $\langle v_p|_{\gamma(0)} , \gamma'(0) \rangle > 0$ *for all minimizing geodesics* γ *which join some point* $\gamma(0) \in M \setminus \mathcal{S}_p$ *to* p .

We shall use the vector field v_p in order to construct retraction maps; we point out that d_p is monotone decreasing along the integral curv of v_p . Under some suitable hypothesis this tool even gives raise to isotopies rather than homotopy equivalences. (c.f. [G] , [GS] .)

In order to establish a Morse theory on d_p , it is moreover necessa to determine how the topology of M changes at the critical points of d_p and to count the strata of the singular set $\mathcal{S}_p$ in a reasonable manner. A d_p is not differentiable at the cut-locus of p , both these problems canno be tackled in the usual way.

<u>2.2</u> However , it is possible to bound in some sense the number of critica points of d_p : let $L > 1$; we consider $p_1 , \dots , p_k \in M$ such that

i) $d_p(p_{i-1}) \geq L \cdot d_p(p_i)$

and ii) p_i is critical for d_p , $2 \leq i \leq k$

Such a sequence $(p_i)_{i=1}^{k}$ will be called a *metrical* $(k;L_1,L,L_k)$ - *frame of* p provided L_1 and L_k are positive real numbers which satisfy $\ell_1 := d_p(p_1) \leq$ and $\ell_k := d_p(p_k) \geq L_k$. We shall say that a subset $X \subset M$ is $(k;L_1,L,L_k)$-*framo* iff for each $p \in X$ there exists a metrical $(k;L_1,L,L_k)$ - frame.

2.3 Lemma :

i) In a manifold (M^n,g,o) *which has asymptotically non-negative curvature any metrical* $(k;L_1,L,L_k)$ *- frame of the base point* o *obeys :*

$$k \leq 2 \cdot \pi^{n-1} \cdot \left(\frac{L}{L-1}\right)^{2n-2} \cdot \exp\left((2n-2) \cdot b_1 \right)$$

ii) Let p *be any point in an arbitrary Riemannian manifold, and suppose that the sectional curvatures in the ball* $B(p\,,(1+L^{-1})\cdot L_1)$ *are bounded from below by* $-\eta^2$ *,* $\eta \geq 0$ *. If moreover*

$$3 \cdot (1+\sqrt{2})^{n-1} \cdot \eta \cdot L_1 \cdot \coth(\eta \cdot L_1) \leq L$$

then $k \leq 2n$ *holds for any metrical* $(k;L_1,L,L_k)$ *- frame of* p *.*

Remark :

The lemma relates the parameter k to the dimension n of M^n , and thus it justifies the terminology , although the word "frame" might be a little misleading. In a similar context Gromov heuristically speaks of the "number of essential directions in M^n ".

Proof :

We fix minimizing geodesics γ_i which join p with p_i ; their initial vectors in T_pM will be denoted by v_i . We head towards a lower bound on the angle between any two of these vectors and then make use of some packing arguements : let $1 \leq i < j \leq k$ and study the geodesic triangle $\Delta = (p_i,p,p_j)$ with edges γ_i , γ_j , and a minimizing geodesic γ_{ij} . We observe that p_j is critical for d_p , and thus γ_j can be replaced by another minmizing geodesic $\tilde{\gamma}_j$ such that the angle at p_j in the modified triangle $\tilde{\Delta}$ does not exceed $\frac{\pi}{2}$. The data on Δ and $\tilde{\Delta}$ are turned into inequalities by means of the (generalized) Toponogov theorem. We start with $\tilde{\Delta}$:

i) Proposition III.1ii applies with $\varepsilon = \frac{L-1}{L}$, and in the limit $a \longrightarrow 1$ it yields :

$$d_p(p_i) \leq d(p_i,p_j) + d_p(p_j)\cdot\sqrt{1-\left(\frac{L-1}{L}\right)^2 \cdot \exp(-2\cdot b_1)}$$

ii) Obviously a minimizing geodesic which joins p_i to any point on γ_j is not longer than $d_p(p_i) + d_p(p_j)$, and therefore it is contained in the ball $B(p\,, d_p(p_i) + d_p(p_j)) \subset B(p\,, (1+L^{-1})\cdot L_1)$. Hence the hyperbolic plane with curvature $-\eta^2$ is an admissible model. We deform the Alexandrov triangle such that $\bar{\ell}_i \geq d_p(p_i)$, $\bar{\ell}_j = d_p(p_j)$, $\bar{\ell} = d(p_i, p_j)$, and $\measuredangle$ at $\bar{p}_j = \frac{\pi}{2}$; then the Law of Cosines yields :

$$\cosh \eta \cdot \bar{\ell} = \frac{\cosh \eta \cdot \bar{\ell}_i}{\cosh \eta \cdot d_p(p_j)} \geq \frac{\cosh \eta \cdot d_p(p_i)}{\cosh \eta \cdot d_p(p_j)}$$

Using the above estimates we can treat the triangle Δ in a similar way :

i) Reversing the implication in Proposition III.1i , we obtain :

$$\cos \measuredangle(v_i , v_j) < \sqrt{1 - \left(\frac{L-1}{L}\right)^4 \cdot \exp(-4 \cdot b_\lambda)}$$

or :
$$|\measuredangle(v_i , v_j)| \geq |\sin \measuredangle(v_i , v_j)| > \left(\frac{L-1}{L}\right)^2 \cdot \exp(-2 \cdot b_\lambda)$$

In this case the claim immediately follows from the standard packing estimate, which has been stated in Lemma III.3.1 for instance.

ii) Here we apply the Law of Cosines directly to the Alexandrov triangle : $\bar{\ell}_i = d_p(p_i)$, $\bar{\ell}_j = d_p(p_j)$, $\bar{\ell} = d(p_i, p_j)$, and $\measuredangle$ at $\bar{p} \leq \measuredangle$ at $p = \measuredangle(v_i, v_j)$ we obtain the inequality :

$$\begin{aligned}
\cos \measuredangle(v_i , v_j) &\leq \eta^2 \cdot \frac{\cosh(\eta \cdot \bar{\ell}_i) \cdot \cosh(\eta \cdot \bar{\ell}_j) - \cosh(\eta \cdot \bar{\ell})}{\sinh(\eta \cdot \bar{\ell}_i) \cdot \sinh(\eta \cdot \bar{\ell}_j)} \\
&\leq \frac{\eta^2 \cdot \coth(\eta \cdot \bar{\ell}_i)}{\sinh(\eta \cdot \bar{\ell}_j)} \cdot \left(\coth(\eta \cdot \bar{\ell}_j) - \frac{1}{\coth(\eta \cdot \bar{\ell}_j)} \right) \\
&= \coth(\eta \cdot \bar{\ell}_i) \cdot \tanh(\eta \cdot \bar{\ell}_j) \\
&\leq \frac{\bar{\ell}_j}{\bar{\ell}_i} \cdot \eta \cdot \bar{\ell}_i \cdot \coth(\eta \cdot \bar{\ell}_i)
\end{aligned}$$

hence :
$$\cos \measuredangle(v_i , v_j) \leq L^{-1} \cdot \eta \cdot L_1 \cdot \coth(\eta \cdot L_1)$$

By assumption the packing arguement given in appendix A applies.

□

3. Morse Theory and Coverings

We have no idea how to control in which way the topology of M^n changes at the critical strata of d_p ; thus the gluing arguements have to be eliminated from the Morse theory. We are going to use covering arguements along the lines of section 1 instead. The idea of deformation will be applied to reduce to special covering situations which we know quite a lot about. For this purpose we introduce some more language :

3.1 Definition :

Given $\rho > 1$ *; a ball* $B = B(p,r)$ *in* M^n *is said to be* ρ *- compressible to* $\tilde{B} = B(\tilde{p},\tilde{r})$ *, if and only if :*

i) $\tilde{r} \leq (1-\rho^{-1}) \cdot r$

ii) $\rho \cdot \tilde{B} \subset \rho \cdot B$

and *iii)* $\tilde{B}$ *is a deformation retract of some subset* $X \subset \rho \cdot B$ *, which also contains* B *.*

B *is called* ρ *- incompressible, iff there does not exist a ball* $\tilde{B}$ *as above.*

The injectivity radius is a continuous positive function on M^n, which has a positive lower bound r_0 on $\rho \cdot B$. ρ − Compressing the given ball B repeatedly, one will therefore arrive at a ρ − incompressible ball or at a topological ball within finitely many steps. Thus it is natural to try and reduce to incompressible balls, when bounding the invariant $rk_*^{t^{-1}}(\rho \cdot B , B)$.

3.2 Lemma :

If $t > 0$ *, and if the ball* B *is* ρ *- compressible to* $\tilde{B}$ *, then :*

$$rk_*^{t^{-1}}(\rho \cdot B , B) \leq rk_*^{t^{-1}}(\rho \cdot \tilde{B} , \tilde{B})$$

Proof :

The claim is an immediate consequence of the following commutative diagrame, where the graded maps are induced by inclusion :

$$\begin{array}{ccccc} H_*(\tilde{B}) & \xrightarrow[\text{deformation retract}]{\cong} & H_*(X) & \longleftarrow & H_*(B) \\ \downarrow & & \downarrow & & \\ H_*(\rho \cdot \tilde{B}) & \longrightarrow & H_*(\rho \cdot B) & & \end{array}$$

Moreover incompressible balls allow for some statements about the critical points of several distance functions.

3.3 Lemma :

If $B = B(p,r)$ *is* ρ *- incompressible, then for any* $\tilde{p} \in \frac{\rho-1}{2} \cdot B$ *there is a critical point* p_c *of* $d_{\tilde{p}}$ *such that :*

$$\min\{(1-\rho^{-1}) \cdot r \,,\, r - \rho^{-1} \cdot d(p,\tilde{p})\} \le d(\tilde{p},p_c) \le r + d(p,\tilde{p}) \le \frac{1+\rho}{2} \cdot r \,.$$

Proof :

Conversely, let us assume that there is no critical point p_c of $d_{\tilde{p}}$ which obeys the above inequalities. We put :

$$\tilde{r} \;:=\; \min\{(1-\rho^{-1}) \cdot r \,,\, r - \rho^{-1} \cdot d(p,\tilde{p})\}$$

and consider the balls $\tilde{B} := B(\tilde{p},\tilde{r})$ and $X := B(\tilde{p}\,,\, r + d(p,\tilde{p}))$. We point out that $r + 2 \cdot d(p,\tilde{p}) \le \rho \cdot r$, hence $B \cup \tilde{B} \subset X \subset B(p\,,\, r + 2 \cdot d(p,\tilde{p})) \subset \rho \cdot B$ and $\rho \cdot \tilde{B} \subset \rho \cdot B$. Lemma 2.1 gives raise to a vector field $v_{\tilde{p}}$ which does not vanish on the closed annulus $\overline{X} \setminus \tilde{B}$. As $d_{\tilde{p}}$ is monotone decreasing along the integral curves of $v_{\tilde{p}}$, we obtain a retraction map , and in contrast to the hypothesis B turns out to be ρ - compressible to $\tilde{B}$.

We proceed and consider the covering situation in some more detail.

3.4 Assumptions :

Let ξ , $\tilde{\xi}$, L , and L_1 be some positive real numbers ; define functions ρ , q , t_o , and N_o by

$$\rho \;:=\; 3 + 2 \cdot L^{-1}$$

$$q \;:=\; (2 + 3 \cdot L)^{-1}$$

$$t_o \;:=\; \left(\frac{2L}{\tilde{\xi} \cdot L_1} \cdot \sinh \frac{\tilde{\xi} \cdot L_1}{2L}\right)^{n-1} \cdot \left(1 + 8 \cdot \left(5 + \frac{2}{L}\right)^n\right)^n$$

and $$N_o \;:=\; \left(\frac{L}{\tilde{\xi} \cdot L_1} \cdot \sinh \frac{\tilde{\xi} \cdot L_1}{L}\right)^{n-1} \cdot \left(1 + 4 \cdot (2 + 3L) \cdot \left(5 + \frac{2}{L}\right)^n\right)^n$$

We *suppose* that :

i) the ball $\rho \cdot B$ associated to $B = B(p,r)$ is $(k;L_1,L,(1+2L) \cdot r)$-framed.

ii) the curvatures in $\rho \cdot B$ are bounded from below by $-\tilde{\xi}^2$.

iii) the curvatures in $B(p\,,\,(1+L^{-1}) \cdot L_1)$ are bounded from below by $-\xi^2$.

Furthermore it is useful to introduce the notation :

$\mathrm{cont}_{*}^{t^{-1}}(L_1, L, \tilde{\xi}) := \sup\{\, \mathrm{rk}_{*}^{t^{-1}}(\rho\cdot B, B) \mid$ the ball B meets the conditions 3.4i and 3.4ii $\}$

3.5 Lemma :

i) Suppose that the assumptions 3.4 hold ; then for any $t \geq t_o$ *there is the estimate :*

$$\mathrm{rk}_{*}^{t^{-1}}(\rho\cdot B, B) \leq (e-1)\cdot N_o \cdot \sup\{\, \mathrm{rk}_{*}^{t^{-1}}(\rho\cdot\tilde{B}, \tilde{B}) \mid \tilde{B} = B(\tilde{p}, \tilde{r}) \text{ where } \tilde{r} \leq q\cdot r \text{ and } \tilde{p} \text{ lies in } B \,\}.$$

ii) If moreover B *is* ρ *- incompressible, then all the balls* $\rho\cdot\tilde{B}$ *on the right-hand side of the above estimate are* $(k+1\,;\, L_1, L, (1+2L)\cdot q\cdot r)$ *- framed.*

iii) If $t \geq t_o(L_1, L, \tilde{\xi})$ *, then :*

$$\mathrm{cont}_{k}^{t^{-1}}(L_1, L, \tilde{\xi}) \leq \max\{\, 1 \,,\, (e-1)\cdot N_o \cdot \mathrm{cont}_{k+1}^{t^{-1}}(L_1, L, \tilde{\xi}) \,\}$$

iv) If condition 3.4iii *holds and if* $L \geq \sqrt{1+\xi^2\cdot L_1^{\,2}} \cdot 3\cdot(1+\sqrt{2})^{n-1}$ *then :*

$$\mathrm{cont}_{2n}^{t^{-1}}(L_1, L, \tilde{\xi}) = 1$$

Proof :

i) We put $\rho_i := (2+\rho)^i$ for $0 \leq i \leq n$ and $\rho_{n+1} := \rho\cdot\rho_n$. We pick a maximal set of pairwise disjoint metrical balls B_j^{-1} , $1 \leq j \leq N$, whose centres lie in B and whose radii equal $r_{-1} := 0.5\cdot q\cdot\rho_n^{-1}\cdot r$. Obviously for $0 \leq i \leq n$ the families $B_j^i := 2\cdot\rho_i\cdot B_j^{-1}$, $1 \leq j \leq N$, cover B . Since $1+q\cdot\rho = 1+L^{-1} < \rho$, we conclude that the balls B_j^{n+1} are contained in $\rho\cdot B$. Therefore the estimate is a consequence of corollary 1.6 , provided that a) $N_o \geq N$ and that b) t_o bounds from above the number of balls $B_{j'}^n$, which intersect any fixed B_j^n . In order to verify both the conditions , we point out that the B_j^{-1} are disjoint, and that :

a) $$B_j^{-1} \subset \left(1+\frac{q}{2\cdot\rho_n}\right)\cdot B \subset (1+4\cdot\rho_n\cdot q^{-1})\cdot B_j^{-1} \subset \left(3+\frac{q}{2\cdot\rho_n}\right)\cdot B$$

b) $$B_{j'}^n \cap B_j^n \neq \emptyset \Longrightarrow B_{j'}^{-1} \subset \left(2+\frac{1}{2\cdot\rho_n}\right)\cdot B_j^n \subset \left(4+\frac{1}{2\cdot\rho_n}\right)\cdot B_{j'}^n \subset \left(1+4\cdot q+\frac{q}{2\cdot\rho_n}\right)\cdot B$$

These inclusions yield :

a) $$N \leq \sup_j \frac{\mathrm{vol}\ (1+0.5\cdot q\cdot \rho_n^{-1})\cdot B}{\mathrm{vol}\ B_j^{-1}} \leq \sup_j \frac{\mathrm{vol}\ (1+4\cdot\rho_n\cdot q^{-1})\cdot B_j^{-1}}{\mathrm{vol}\ B_j^{-1}}$$

b) $$\# \{ B_{j'}^n \mid B_{j'}^n \cap B_j^n \neq \emptyset \} \leq \sup_{j'} \frac{\mathrm{vol}\ (1+8\cdot\rho_n)\cdot B_{j'}^{-1}}{\mathrm{vol}\ B_{j'}^{-1}}$$

Since all the balls are contained in $\rho\cdot B$, the right-hand sides of these inequalities can be evaluated by means of the volume comparison theorem for concentric metrical balls (c.f. [BC]) ; we may use model spaces with constant curvature $-\tilde{\xi}^2$, and we compute :

a) $$N \leq \frac{\int_0^{1+4\rho_n/q} \sinh(\tilde{\xi}\cdot\sigma\cdot r_{-1})^{n-1}\, d\sigma}{\int_0^1 \sinh(\tilde{\xi}\cdot\sigma\cdot r_{-1})^{n-1}\, d\sigma}$$

$$\leq (1+\frac{4}{q}\cdot\rho_n)\cdot \sup\{\frac{\sinh \tilde{\xi}\cdot\sigma\cdot(1+\frac{4}{q}\cdot\rho_n)\cdot r_{-1}}{\sinh \tilde{\xi}\cdot\sigma\cdot r_{-1}} \mid 0\leq\sigma\leq 1\}^{n-1}$$

$$\leq (1+\frac{4}{q}\cdot\rho_n)^n \cdot (\frac{\sinh \tilde{\xi}\cdot(1+\frac{4}{q}\cdot\rho_n)\cdot r_{-1}}{\tilde{\xi}\cdot(1+\frac{4}{q}\cdot\rho_n)\cdot r_{-1}})^{n-1} \leq N_o$$

The last step is due to the fact that :

$$(1+\frac{4}{q}\cdot\rho_n)\cdot r_{-1} = (2+\frac{q}{2\cdot\rho_n})\cdot r \leq (2+L^{-1})\cdot r \leq {}^{L_1}/_{L}$$

and $\rho_n\cdot q^{-1} = (2+3L)\cdot(5+\frac{2}{L})^n$

b) $$\# \{ B_{j'}^n \mid B_{j'}^n \cap B_j^n \neq \emptyset \} \leq \frac{\int_0^{1+8\rho_n} \sinh(\tilde{\xi}\cdot\sigma\cdot r_{-1})^{n-1}\, d\sigma}{\int_0^1 \sinh(\tilde{\xi}\cdot\sigma\cdot r_{-1})^{n-1}\, d\sigma}$$

$$\leq (1+8\cdot\rho_n)^n \cdot (\frac{\sinh \tilde{\xi}\cdot(1+8\rho_n)\cdot r_{-1}}{\tilde{\xi}\cdot(1+8\rho_n)\cdot r_{-1}})^{n-1} \leq t_o$$

This time the last estimate is due to the fact that :

$$(1+8\cdot\rho_n)\cdot r_{-1} = (4+\frac{1}{2\cdot\rho_n})\cdot q\cdot r \leq \frac{9}{2}\cdot(2+3L)^{-1}\cdot(1+2L)^{-1}\cdot L_1$$

$$\leq \frac{L_1}{2\cdot L}$$

ii) It follows from lemma 3.3 that for any point $\tilde{p} \in (1+q\cdot\rho)\cdot B$ there exists a critical point p_c of $d_{\tilde{p}}$, which obeys :

$$L\cdot d(p_c,\tilde{p}) \leq L\cdot(2+q\cdot\rho)\cdot r = (1+2L)\cdot r$$

and

$$d(p_c,\tilde{p}) \geq (1-\rho^{-1}\cdot(1+q\cdot\rho))\cdot r = q\cdot(\rho\cdot L-L-1)\cdot r$$
$$= q\cdot(1+2L)\cdot r$$

therefore the set $(1+q\cdot\rho)\cdot B$ as well as the subballs $\rho\cdot\tilde{B}$ are $(k+1\,;\ L_1, L, (1+2L)\cdot q\cdot r)$ - framed.

iii) Obviously $rk_*^{t^{-1}}(\rho\cdot B, B) = 1$, if the metrical ball B is a topological ball as well. Therefore lemma 3.2 reduces the proof to the case where B is a ρ - incompressible ball. The estimate given in (i) holds, and the property (ii) allows to bound the right-hand side as desired.

iv) Suppose B were a ρ - incompressible ball which obeyed the conditions 3.4i, ii, and iii) with $k=2n$; then by means of (ii) there would exist $(2n+1\,;\ L_1, L, 0)$ - framed balls. As by hypothesis

$$L \geq \sqrt{1+\xi^2\cdot L_1^2}\cdot 3\cdot(1+\sqrt{2})^{n-1}$$
$$\geq \xi\cdot L_1\cdot\coth(\xi\cdot L_1)\cdot 3\cdot(1+\sqrt{2})^{n-1}$$

the above conclusion contradicts to lemma 2.3ii .

<u>3.6 Proposition :</u>

Suppose that the ball $B=B(p,r)$ *obeys the conditions* 3.4ii *and* iii *with* $L > \sqrt{1+\xi^2\cdot L_1^2}\cdot 3\cdot(1+\sqrt{2})^{n-1}$ *; moreover assume that the boundary of* $B(p,L_1)$ *in* M^n *is non-empty and that* $L_1 \geq 2r\cdot(L+2+L^{-1})$. *Then for any* $t \geq t_o(L_1, L, \tilde{\xi})$ *, one has :*

$$rk_*^{t^{-1}}(\rho\cdot B, B) \leq (e-1)^{2n-1}\cdot N_o(L_1, L, \tilde{\xi})^{2n-1}$$

Proof :

We fix a point p_1 on the boundary of $B(p,L_1)$; it is easy to verify that $\rho\cdot B = (3+2\cdot L^{-1})\cdot B$ is $(1\,;\ L_1, L, (1+2L)\cdot r)$ - framed. We apply lemma 3.5iii inductively and use 3.5iv in order to stop at $k=2n$.

Heuristically speaking, the proposition bounds the topology of small metrical balls B in a Riemannian manifold M^n. All the conditions can be formulated in terms of curvature, the diametre of M^n, and the radius of the ball B. No assumption on the injectivity radius is required.

3.7 Corollary : (c.f. Gromov)

If M^n *is non-negatively curved, non-compact, and connected, and if* $t \geq 2^{1/n} \cdot 8^n \cdot 5^{n^2}$, *then for any ball* $B = B(p,r)$ *in* M^n *the following estimate holds :*

$$\mathrm{rk}_*^{t^{-1}}(M^n, B) \leq \mathrm{rk}_*^{t^{-1}}(3.3\cdot B, B) \leq \exp(5\cdot n^3 + 3.5\cdot n^2) \quad .$$

Proof :

We put $\xi := \tilde{\xi} := 0$, $L := 3\cdot(1+\sqrt{2})^{n-1}$, and $L_1 := 2r\cdot L\cdot(1+L^{-1})^2$. Then the proposition applies ; to make things more explicit, we make use of the following computations :

$$\begin{aligned} t_o &= (1 + 8\cdot 5^n\cdot(1+\frac{2}{5L})^n)^n \leq 8^n\cdot 5^{n^2}\cdot\exp(\frac{2n^2}{5\cdot L} + \frac{n}{8}\cdot 5^{-n}) \\ &\leq 2^{1/n}\cdot 8^n\cdot 5^{n^2} \end{aligned}$$

and

$$\begin{aligned} N_o &= (1 + 12\cdot L\cdot 5^n\cdot(1+\frac{2}{5L})^n\cdot(1+\frac{2}{3L}))^n \\ &\leq (12\cdot L)^n\cdot 5^{n^2}\cdot\exp(\frac{2n^2}{5L} + \frac{2n}{3L} + \frac{n}{12\cdot L}\cdot 5^{-n}) \end{aligned}$$

i.e.

$$\begin{aligned} ((e-1)\cdot N_o)^{2n-1} &\leq (5\cdot(1+\sqrt{2}))^{n^2\cdot(2n-1)}\cdot(\frac{36\cdot\sqrt{e-1}}{1+\sqrt{2}})^{n\cdot(2n-1)}\cdot \\ &\quad \cdot\exp(\frac{n\cdot(2n-1)}{L}\cdot(\frac{2n}{5}+\frac{2}{3}+\frac{1}{12}\cdot 5^{-n})) \\ &\leq (5\cdot(1+\sqrt{2}))^{(n^2+1.2\cdot n)\cdot(2n-1)}\cdot 5 \\ &\leq 5\cdot\exp(5\cdot n^3 + 3.5\cdot n^2 - 3\cdot n) \end{aligned}$$

□

4. Metrical Annuli in Asymptotically Non-negatively Curved Manifolds

Most of the preceding results are valid for arbitrary Riemannian manifolds ; especially proposition 3.6 holds in general. In this section we are going to specialize to asymptotically non-negatively curved manifolds (M^n, g, o) . Our goal is to get rid of the assumption on the diametre of M^n . Towards this purpose it is natural to consider metrical annuli

$$A(R_1, R_2) \; := \; \overline{B(o,R_2)} \setminus B(o,R_1)$$

around the base point o of M^n . We want to bound from above

$$rk_*^{t^{-1}} \left(A((1-\varepsilon)\cdot R_1, (1+\varepsilon)\cdot R_2), \; A(R_1, R_2) \right)$$

provided t and ε are sufficiently large. The idea is to cover the annuli by balls of a very special type : a metrical ball $B = B(p,r)$ in M^n is said to be δ-*small* $(\delta > 0)$, iff $r = \delta \cdot d_o(p)$.

We recall that by lemma II.1.1 the curvatures at a point $p \in M^n$ are bounded from below by

$$-2 \cdot b_o \cdot f(d_o(p)) \cdot d_o(p)^{-2} \qquad ;$$

here $r \longmapsto f(r)$ denotes a monotone non-increasing error function which takes values in $[0,1]$ and converges to 0 for $r \to \infty$.

4.1 Assumptions :

Let $L_o := 3\cdot(1+\sqrt{2})^{n-1}$ and let $\eta < \left(1 + \sqrt{b_o \cdot f(\frac{R_1}{2L_o})}\right)^{-1}$ be some positive number ; we put :

$$L \; := \; L_o \cdot \sqrt{1 + 2\cdot b_o \cdot f(\frac{R_1}{2L_o}) \cdot (\frac{\eta}{1-\eta})^2}$$

$$\rho \; := \; 3 + 2\cdot L^{-1}$$

and $\qquad \varepsilon_\eta \; := \; \frac{1}{2} \cdot L^3 \cdot (1+L)^{-4} \cdot \eta$

4.2 Lemma :

i) $\quad L_o \le L \le \sqrt{3}\cdot L_o \le 2\cdot L_o - 1$

$$\varepsilon_\eta \le \frac{\eta}{2\cdot(L+4)} \le \frac{\eta}{22}$$

$$\rho \cdot \varepsilon_\eta \le \min\{\frac{\eta}{2\cdot(L+3)}, \frac{\eta}{7}\}$$

ii) If $0 < \delta \le \varepsilon_\eta$ *and* $B = B(p,r)$ *is any* δ*-small ball in* M^n *with centre* p *in* $A(R_1, R_2)$ *, then the estimate*

$$rk_*^{t^{-1}} (\rho \cdot B, B) \le (e-1)^{2n-1} \cdot \tilde{N}_o^{\,2n-1}$$

holds for

$$\tilde{N}_o := (12\cdot L)^n \cdot 5^{n^2} \cdot \exp\left(\frac{n}{L}\cdot\left(\frac{2n}{5}+\frac{2}{3}+\frac{1}{12}\cdot 5^{-n}\right)+\frac{\sqrt{2}\cdot(n-1)}{L+2}\right)$$

and for all

$$t \geq \tilde{t}_o(n) := 8^n \cdot 5^{n^2} \cdot \exp\left(\frac{2n^2}{5L}+\frac{n-1}{\sqrt{2}\cdot(L+2)}+\frac{n}{8}\cdot 5^{-n}\right)$$

Proof :

i) The estimates are obvious conequences of the definitions.

ii) We define $L_1 := 2L\cdot(1+L^{-1})^2\cdot\varepsilon_n\cdot d_o(p) = (\frac{L}{L+1})^2\cdot\eta\cdot d_o(p)$. It is easy to verify that :

a) the curvatures in $B(p\,,\,(1+L^{-1})\cdot L_1) \subset B(p\,,\frac{L\cdot\eta}{L+1}\cdot d_o(p))$ are bounded from below by $-\xi^2$, where

$$\xi := \sqrt{2\cdot b_o\cdot f(\frac{R_1}{L+1})}\cdot\left(1-\frac{L\cdot\eta}{L+1}\right)^{-1}\cdot d_o(p)^{-1}$$

b)
$$\xi\cdot L_1 \leq \sqrt{2\cdot b_o\cdot f(\frac{R_1}{L+1})}\cdot\frac{\eta}{1-\eta}$$

i.e.: $L \geq \sqrt{1+\xi^2\cdot L_1^2}\cdot L_o$

c) the curvatures in the ball $\rho\cdot B \subset B(p\,,\rho\cdot\varepsilon_n\cdot d_o(p)) \subset B(p\,,\frac{\eta}{7}\cdot d_o(p))$ are bounded from below by $-\tilde{\xi}^2$, where

$$\tilde{\xi} := \sqrt{2\cdot b_o\cdot f(\frac{R_1}{L+1})}\cdot\left(1-\frac{\eta}{7}\right)^{-1}\cdot d_o(p)^{-1}$$

Therefore proposition 3.6 applies ; it remains to compute $N_o(L_1,L,\tilde{\xi})$ and $t_o(L_1,L,\tilde{\xi})$. Since

$$\frac{2\cdot(1+L^{-1})^2\cdot\varepsilon_n}{1-\frac{\eta}{7}} = \frac{L}{(1+L)^2}\cdot\frac{\eta}{1-\frac{\eta}{7}} \leq \frac{1}{L+2}\cdot\frac{\eta}{1-\eta}$$

we see that :

$$\frac{\tilde{\xi}\cdot L_1}{L} = \frac{1}{L+2}\cdot\frac{\eta}{1-\eta}\cdot\sqrt{2\cdot b_o\cdot f(\frac{R_1}{L+1})} \leq \frac{\sqrt{2}}{L+2}$$

Because of the inequality

$$\frac{\sinh(x)}{x} \leq \cosh(x) \leq \exp(|x|)$$

we obtain that :

$$t_o \leq \exp\left(\frac{n-1}{\sqrt{2}\cdot(L+2)}\right)\cdot\left(1+8\cdot\left(5+\frac{2}{L}\right)^n\right)^n \leq \tilde{t}_o(n)$$

and
$$N_o \leq \exp\left(\frac{\sqrt{2}\cdot(n-1)}{L+2}\right)\cdot\left(1+12\cdot L\cdot\left(1+\frac{2}{3L}\right)\cdot\left(5+\frac{2}{L}\right)^n\right)^n \leq \tilde{N}_o .$$

4.3 Construction :

There is a sequence of numbers $0 < \varepsilon_{-1} < \varepsilon_o < \dots < \varepsilon_n < \varepsilon_{n+1} < 1$, uniquely determined by the following conditions :

$$\begin{array}{lll} \varepsilon_n & \text{is the number given in 4.1 ,} & \\ \varepsilon_{n+1} & := \rho\cdot\varepsilon_n & \\ \varepsilon_{i+1} & = (2+\rho\cdot(1+\varepsilon_i))\cdot(1-\varepsilon_i)^{-1}\cdot\varepsilon_i & , 0 \leq i < n , \end{array}$$

and $\varepsilon_o := 2\cdot\varepsilon_{-1}\cdot(1-\varepsilon_{-1})^{-1}$

(i.e. $\varepsilon_{-1} = \varepsilon_o\cdot(2+\varepsilon_o)^{-1}$.)

We put $\rho_i := \varepsilon_i\cdot\varepsilon_o^{-1}$ for $-1 \leq i \leq n+1$.

We pick a maximal family of disjoint ε_{-1} - small balls B_j^{-1} , $1 \leq j \leq N$, whose centres p_j lie in $A(R_1, R_2)$. Moreover we shall consider all balls

$$B_j^i := \rho_i\cdot(2+\varepsilon_o)\cdot B_j^{-1} \quad , \quad 1 \leq j \leq N \quad , \quad 0 \leq i \leq n+1$$

4.4 Immediate Consequences :

i) $p \in B_j^i \implies (1-\varepsilon_i)\cdot d_o(p_j) \leq d_o(p) \leq (1+\varepsilon_i)\cdot d_o(p_j)$

ii) $B_{j'}^i \cap B_j^i \neq \emptyset \implies p_{j'} \in 2\cdot(1-\varepsilon_i)^{-1}\cdot B_j^i$

and $\sigma\cdot B_{j'}^i \subset (2+\sigma\cdot(1+\varepsilon_i))\cdot(1-\varepsilon_i)^{-1}\cdot B_j^i$ for all $\sigma>0$.

iii) the balls B_j^o , $1 \leq j \leq N$, cover the annulus $A(R_1, R_2)$.

In order to prove (iii) , let us assume that there is some $p_{N+1} \in A(R_1, R_2) \setminus \bigcup\{B_j^o \mid 1 \leq j \leq N\}$; it is a consequence of (ii) that the ball $B_{N+1}^{-1} := B(p_{N+1}, \varepsilon_{-1}\cdot d_o(p_{N+1}))$ is disjoint from the other B_j^{-1} . This conclusion contradicts the maximality of the family $(B_j^{-1})_{j=1}^N$.

We point out that for sufficiently large t the corollary 1.6 gives raise to an upper bound on

$$rk_*^{t^{-1}}(A((1-\varepsilon_{n+1})\cdot R_1, (1+\varepsilon_{n+1})\cdot R_2) , A(R_1, R_2))$$

in terms of the quantities $rk_*^{t^{-1}}(\rho\cdot B_j^i, B_j^i)$, $0 \leq i \leq n$, $1 \leq j \leq N$, which in turn have already been controlled in lemma 4.3.

We continue listing some inequalities which will be used in the subsequent computations.

iv) $5^i \leq \rho_i \leq 5^i \cdot \exp(\frac{2i}{5L} + \frac{1}{4\cdot(L+4)})$, $0 \leq i \leq n$.

$$2\cdot 5^n \leq \varepsilon_n \cdot \varepsilon_{-1}^{-1} = \rho_n \cdot (2+\varepsilon_o) = 2\cdot\rho_n + \varepsilon_n$$

$$\leq 2\cdot 5^n \cdot \exp(\frac{2n}{5L} + \frac{1+5^{-n}}{4\cdot(L+4)})$$

v) $\varepsilon_{-1}^{-1} \leq 4 \cdot 5^n \cdot \frac{L}{\eta} \cdot (1+L^{-1})^4 \cdot \exp(\frac{2n}{5L} + \frac{1+5^{-n}}{4\cdot(L+4)})$

$$\leq 4 \cdot 5^n \cdot \frac{L}{\eta} \cdot \exp(\frac{2n}{5L} + \frac{17}{4L})$$

vi) $B_{j'}^n \cap B_j^n \neq \emptyset$

$$\Longrightarrow B_{j'}^{-1} \subset (2 + \varepsilon_{-1}\cdot\varepsilon_n^{-1}\cdot(1+\varepsilon_n))\cdot(1-\varepsilon_n)^{-1}\cdot B_j^n \subset \tau_n \cdot B_{j'}^n$$

where $\tau_n := \frac{2}{1-\varepsilon_n} \cdot (1 + \frac{1}{1-\varepsilon_n}) + \varepsilon_{-1}\cdot\varepsilon_n^{-1} \cdot (1 + \frac{2\varepsilon_n}{1-\varepsilon_n})^2$

vii) $\frac{2}{1-\varepsilon_n} \cdot (1 + \frac{1}{1-\varepsilon_n}) \leq 2\cdot(1 + \frac{1}{2L+7})\cdot(2 + \frac{1}{2L+7})$

$$\leq 4 \cdot (1 + \frac{3}{4L+13})$$

$$(1 + \frac{2\varepsilon_n}{1-\varepsilon_n})^2 \leq (1 + \frac{2}{2L+7})^2 \leq 1 + \frac{2}{L+3}$$

viii) $\tau_n \cdot \varepsilon_n \leq (\frac{8}{4L+13} + 5^{-n} \cdot \frac{L+5}{4\cdot(L+3)\cdot(L+4)}) \cdot \eta$

$$\leq \frac{2\eta}{L+3} \cdot (1 - \frac{1}{4L+13} + 5^{-n} \cdot \frac{L+5}{8\cdot(L+4)}) \leq \frac{2\eta}{L+3} .$$

ix) $\frac{\tau_n \cdot \varepsilon_n}{\varepsilon_{-1}} \leq 8 \cdot 5^n \cdot (1 + \frac{3}{4L+13}) \cdot \exp(\frac{2n}{5L} + \frac{1+5^{-n}}{4\cdot(L+4)}) + 1 + \frac{2}{L+3}$

$$\leq 8 \cdot 5^n \cdot \exp(\frac{2n}{5L} + \frac{5^{-n}}{8} + \frac{4}{4L+13} + \frac{5^{-n}}{4} \cdot (\frac{1}{L+4} + \frac{1}{L+3}))$$

$$\leq 8 \cdot 5^n \cdot \exp(\frac{2n}{5L} + \frac{4}{4L+13} + \frac{7}{8}\cdot 5^{-n-1})$$

__4.5 Lemma :__

The number of balls $B_{j'}^n$ *which intersect a given ball* B_j^n *is bounded from above by*

$$t_1(n) = 5^{n^2} \cdot 8^n \cdot \exp(\frac{9}{4\cdot(n+1)})$$

__Proof :__

Since by construction the balls $B_{j'}^{-1}$ are disjoint , we can deduce from

4.4vi that :

$$\# \{B^n_{j'} \mid B^n_{j'} \cap B^n_j \neq \emptyset\} \leq \sup_{j'} \frac{\mathrm{vol}\ \tau_n \cdot B^n_{j'}}{\mathrm{vol}\ B^{-1}_{j'}} \qquad (*)$$

The curvatures in the ball $\tau_n \cdot B^n_{j'}$ are bounded from below by $-\xi^2_{j'}$ where

$$\xi_{j'} := \sqrt{2 \cdot b_o \cdot f(\tfrac{R_1}{L+1})} \cdot (1 - \tau_n \cdot \varepsilon_n)^{-1} \cdot d_o(p_{j'})^{-1}$$

Since

$$\frac{\tau_n \cdot \varepsilon_n}{1 - \tau_n \cdot \varepsilon_n} \leq \frac{2}{L+3} \cdot \frac{\eta}{1-\eta}$$

and since $p_{j'} \in A(R_1, R_2)$, it follows that :

$$\xi_{j'} \cdot \tau_n \cdot \varepsilon_n \cdot d_o(p_{j'}) \leq \frac{2}{L+3}$$

Therefore the volume comparison for concentric metrical balls yields :

$$\frac{\mathrm{vol}\ \tau_n \cdot B^n_{j'}}{\mathrm{vol}\ B^{-1}_{j'}} \leq \frac{\int_0^{\tau_n \cdot \varepsilon_n \cdot d_o(p_{j'})} \sinh(\xi_{j'} \cdot \sigma)^{n-1}\, d\sigma}{\int_0^{\varepsilon_{-1} \cdot d_o(p_{j'})} \sinh(\xi_{j'} \cdot \sigma)^{n-1}\, d\sigma}$$

$$\leq \left(\frac{\tau_n \cdot \varepsilon_n}{\varepsilon_{-1}}\right)^n \cdot \left(\frac{\sinh(\xi_{j'} \cdot \tau_n \cdot \varepsilon_n \cdot d_o(p_{j'}))}{\xi_{j'} \cdot \tau_n \cdot \varepsilon_n \cdot d_o(p_{j'})}\right)^{n-1}$$

$$\leq \left(\frac{\tau_n \cdot \varepsilon_n}{\varepsilon_{-1}}\right)^n \cdot \exp\left(\frac{2n-2}{L+3}\right)$$

We plug this estimate into (*) and obtain by means of 4.4ix that :

$$\# \{B^n_{j'} \mid B^n_{j'} \cap B^n_j \neq \emptyset\} \leq 8^n \cdot 5^{n^2} \cdot \exp\left(\frac{2n^2}{5L} + \frac{3n-2}{L+3} + \frac{7n}{8} \cdot 5^{-n-1}\right)$$

$$\leq t_1(n)$$

□

In order to bound the number N of balls B^o_j in the covering, we look at the pulled back situation under $\exp_o : T_oM \longrightarrow M$. We pick a family of vectors $v_j \in T_oM$ such that

i) $\exp_o(v_j)$ is the centre p_j of B^o_j ;

ii) $\| v_j \| = d_o(p_j)$, i.e. the curve $t \longmapsto \exp_o(t \cdot v_j)$, $t \in [0,1]$, is minimizing.

Given numbers $0 < \zeta_1, \zeta_2, \zeta_3 < 1$ and a non-zero vector $w \in T_oM$, we consider the sets

$$AC(w;\zeta_1,\zeta_2,\zeta_3) := \{v \in T_oM \mid |\measuredangle(v,w)| \leq \zeta_1\cdot\zeta_3\cdot\exp(-b_1) \text{ and } (1-\zeta_1)\cdot(1-\zeta_2)\cdot|w| \leq |v| \leq (1-\zeta_1)\cdot(1+\zeta_2)\cdot|w| \}.$$

4.6 Lemma :

i) If $1-\varepsilon_{-1} \leq (1-\zeta_1)\cdot(\sqrt{1-\zeta_3^2}-\zeta_2)$,
then $\exp_o$ *maps the set* $AC(v_j;\zeta_1,\zeta_2,\zeta_3)$ *into* B_j^{-1} *for all* j .

ii) The number of disjoint sets $AC(w;\zeta_1,\zeta_2,\zeta_3)$ *in the annulus*

$$A := \{v \in T_oM \mid (1-\zeta_1)\cdot(1-\zeta_2)\cdot R_1 \leq |v| \leq (1-\zeta_1)\cdot(1+\zeta_2)\cdot R_2 \}$$

is bounded from above by

$$(2+\zeta_2^{-1}\cdot\ln\frac{R_2}{R_1})\cdot\pi^{n-1}\cdot(2\cdot\zeta_1\cdot\zeta_3)^{1-n}\cdot\exp((n-1)\cdot b_1)$$

iii) The number N *of distinct balls* B_j^{-1} *in* M^n *is bounded from above by*

$$N_1 := (\frac{5^{-n}}{8L+32}+\ln\frac{R_2}{R_1})\cdot 16\cdot(16\cdot\pi)^{n-1}\cdot\sqrt{\frac{5^n\cdot L}{n}}^{\,3n-1}\cdot$$
$$\cdot\exp(\frac{3n-1}{L}\cdot(\frac{n}{5}+\frac{17}{8}))\cdot\exp((n-1)\cdot b_1)$$

Proof :

i) It is sufficient to show that $\exp_o$ maps the sets $AC(v_j;\zeta_1,0,\zeta_3)$ into the $(\varepsilon_{-1}-\zeta_2\cdot(1-\zeta_1))$ - small balls $(1-\zeta_2\cdot(1-\zeta_1)\cdot\varepsilon_{-1}^{-1}\cdot B_j^{-1}$. This amounts to studying the generalized geodesic triangle $\Delta = (p_j,p,o)$, where p is the image under $\exp_o$ of any vector $v \in AC(v_j;\zeta_1,0,\zeta_3)$. Thus , in the notation of proposition III.1 , we have :

$$\ell_1 = d_o(p_j)$$
$$\ell_o = d_o(p) \leq (1-\zeta_1)\cdot\ell_1$$
and $$\cos(\measuredangle \text{at } o) \geq \sqrt{1-\zeta_1^2\cdot\zeta_3^2\cdot\exp(-2\cdot b_1)}$$

Hence, the proposition applies and yields the desired inequality :

$$\begin{aligned} d(p,p_j) &\leq \ell_1 - \ell_o\cdot\sqrt{1-\zeta_3^2} \\ &\leq (1-(1-\zeta_1)\cdot\sqrt{1-\zeta_3^2})\cdot d_o(p_j) \\ &\leq (\varepsilon_{-1}-\zeta_2\cdot(1-\zeta_1))\cdot d_o(p_j) \end{aligned}$$

ii) We make use of the diffeomorphism $\Phi: T_oM\setminus\{0\} \longrightarrow S^{n-1}\times\mathbb{R}$, $v \longmapsto (|v|^{-1}\cdot v, \ln|v|)$ and the canonical volume form on $S^{n-1}\times\mathbb{R}$, and we compute - $B^S(r)$ will denote a ball of radius r in S^{n+1} - :

$$\# \text{ disjoint sets } AC(w;\zeta_1,\zeta_2,\zeta_3) \text{ in } A$$

$$\leq \sup_w \frac{\text{vol } \Phi(A)}{\text{vol } \Phi(AC(w;\zeta_1,\zeta_2,\zeta_3))}$$

$$\leq \ln \frac{(1+\zeta_2)\cdot R_2}{(1-\zeta_2)\cdot R_1} \cdot \left(\ln \frac{1+\zeta_2}{1-\zeta_2}\right)^{-1} \cdot \frac{\text{vol } S^{n+1}}{\text{vol } B^S(\zeta_1\cdot\zeta_3\cdot\exp(-b_1))}$$

$$\leq (1 + \left(\ln \frac{1+\zeta_2}{1-\zeta_2}\right)^{-1} \cdot \ln \frac{R_2}{R_1}) \cdot 2 \cdot \pi^{n-1} \cdot (2\cdot\zeta_1\cdot\zeta_3\cdot\exp(-b_1))^{1-n}$$

This inequality immediately yields the claimed bound, as it is known that $\ln(1+\zeta_2) - \ln(1-\zeta_2) \geq 2\zeta_2$ for all $\zeta_2 \geq 0$.

iii) If $\zeta_1 + \zeta_2 + \zeta_3^2 = \varepsilon_{-1}$, then the hypothesis of (i) are meet, the sets $AC(v_j;\zeta_1,\zeta_2,\zeta_3)$ are disjoint, and part (ii) yields the required control on N . We put $\zeta_1 = 2\cdot\zeta_2 = 2\cdot\zeta_3^2 = 0.5\cdot\varepsilon_{-1}$ and obtain the estimate :

$$N \leq (2\cdot\varepsilon_{-1} + 4\cdot\ln \frac{R_2}{R_1}) \cdot (2\cdot\pi)^{n-1} \cdot \sqrt{\varepsilon_{-1}}^{-(3n-1)} \cdot \exp((n-1)\cdot b_1)$$

Using the estimates given in 4.4 , one easily verifies that the right-hand side is dominated by N_1 .

4.7 Proposition :

Assume like in 4.1 that $0 < R_1 \leq R_2$ *and that* $\eta \leq (1 + \sqrt{b_o\cdot f(\frac{R_1}{2L_o})})^{-1}$
If moreover

$$t \geq t_1(n) := 5^{n^2} \cdot 8^n \cdot \exp(\frac{9}{4\cdot(n+1)})$$

the following estimates hold:

i) $$rk_*^{t^{-1}}(A(\tfrac{6}{7}\cdot R_1 , \tfrac{8}{7}\cdot R_2) , A(R_1 , R_2))$$
$$\leq C_a(n) \cdot (\frac{1}{2000} + \ln \frac{R_2}{R_1}) \cdot \sqrt{\eta\cdot\sqrt{2}}^{-(3n-1)} \cdot \exp((n-1)\cdot b_1) ,$$

where $C_a(n) := (e-1)^{2n} \cdot (5\cdot(1+\sqrt{2}))^{\tilde{c}_a(n)}$

and $\tilde{c}_a(n) := 2\cdot n^3 + \frac{19}{6}\cdot n^2 + \frac{5}{12}\cdot n + \frac{3}{4}$

ii) $$\lim_{R_1\to\infty} rk_*^{t^{-1}}(A(\tfrac{6}{7}\cdot R_1 , \tfrac{8}{7}\cdot R_1) , A(R_1 , R_1)) \leq C_e(n) \cdot \exp((n-1)\cdot b_1) ,$$

where $C_e(n) := (5\cdot(1+\sqrt{2}))^{\tilde{c}_e(n)}$

and $\tilde{c}_e(n) := 2\cdot n^3 + \frac{19}{6}\cdot n^2 - \frac{1}{3}\cdot n - \frac{1}{3}$

Remark :

We point out that

$$\frac{1}{\sqrt{2}}\cdot(1+\sqrt{b_o\cdot f(\frac{R_2}{R_1})}) \leq \frac{1}{\sqrt{2}}\cdot(1+\sqrt{b_o}) \leq \sqrt{1+b_o}$$

Since by lemma II.2.1 the last term does not exceed $\exp(\frac{1}{2}\cdot b_3)$, it is *admissible* to pick :

$$\frac{1}{\sqrt{2}\cdot\eta} := \exp(\frac{1}{2}\cdot b_3)$$

Proof :

By continuity it is sufficient to treat the case $\eta < (1+\sqrt{b_o\cdot f(\frac{R_1}{2L_o})})^{-1}$. We are going to consider the covering constructed in 4.3 . One easily verifies that $t_1(n) \geq \tilde{t}_o(n)$; hence it is possible to apply corollary 1.6 and lemma 4.2ii :

$$\begin{aligned} &\mathrm{rk}_*^{t^{-1}}(A(\tfrac{6}{7}\cdot R_1, \tfrac{8}{7}\cdot R_2), A(R_1, R_2)) \\ &\leq (e-1)\cdot N_1\cdot \sup\{\mathrm{rk}_*^{t^{-1}}(\rho\cdot B, B) \mid B \text{ is } \delta\text{-small with } \delta\leq\varepsilon_n \\ &\qquad\qquad \text{and has centre in the set } A(R_1, R_2)\} \\ &\leq (e-1)^{2n}\cdot\tilde{N}_o^{\,2n-1}\cdot N_1 \\ &= (e-1)^{2n}\cdot c_1\cdot c_2\cdot(\frac{5^{-n}}{8L+32}+\ln\frac{R_2}{R_1})\cdot\sqrt{\eta}^{\,1-3n}\cdot\exp((n-1)\cdot b_1) \end{aligned}$$

Here we have used the abbreviations :

$$c_1 := ((12\cdot L)^n\cdot 5^{n^2})^{2n-1}\cdot\sqrt{5^n\cdot L}^{\,3n-1}\cdot(16\cdot\pi)^{n-1}\cdot 16$$

and $$c_2 := \exp(\frac{2n^2-n}{L}\cdot(\frac{2n}{5}+\frac{8+5^{-n}}{12}) + \frac{3n-1}{L}\cdot(\frac{n}{5}+\frac{17}{8}) + \frac{(2n-1)\cdot(n-1)\cdot\sqrt{2}}{L+2})$$

Since $L\geq 3\cdot(1+\sqrt{2})^{n-1}$, we have $c_2 \leq 36$ for all $n\geq 2$.

i) Observing that $\frac{5^{-n}}{8L+32} \leq \frac{1}{2000}$, it is sufficient to show that

$$\sqrt{2}^{\,3n-1}\cdot c_1\cdot c_2 \leq C_a(n)$$

Obviously

$$\begin{aligned} \sqrt{2}^{\,3n-1}\cdot c_1 &\leq (5\cdot(1+\sqrt{2}))^{n^2\cdot(2n-1)+n\cdot(3n-1)/2}\cdot(\frac{36\cdot\sqrt{3}}{1+\sqrt{2}})^{n\cdot(2n-1)} \\ &\quad\cdot(\frac{3\cdot\sqrt{6}}{1+\sqrt{2}})^{(3n-1)/2}\cdot(16\cdot\pi)^{n-1}\cdot 16 \end{aligned}$$

and the result is due to the inequalities :

$$\frac{36\cdot\sqrt{3}}{1+\sqrt{2}} \leq (5\cdot(1+\sqrt{2}))^{4/3} \quad , \quad \left(\frac{3\cdot\sqrt{6}}{1+\sqrt{2}}\right)^{3/2}\cdot 16\cdot\pi \leq (5\cdot(1+\sqrt{2}))^{9/4} \quad ,$$

and $\frac{3\cdot\sqrt{6}}{1+\sqrt{2}} \cdot 16\cdot c_2 \leq (5\cdot(1+\sqrt{2}))^{3}$.

ii) Since the error function f converges to zero for $R_1\to\infty$, it is possible to pick a function $\eta(R_1)$ which converges to 1 for $R_1\to\infty$. Thus one is reduced to checking that

$$(e-1)^{2n}\cdot c_1\cdot c_2\cdot \frac{5^{-n}}{8\cdot L} \leq C_e(n)$$

This can be done calculating in a similar way as above.

5. The Global Estimates

The special case of non-negative curvature has been treated in corollary 3.7 , and the Betti numbers of the ends of M^n have been bounded in proposition 4.7ii . It remains to consider the general case and piece together the estimates on metrical annuli in an arbitrary asymptotically non-negatively curved manifold.

We look at a sequence of critical points $p_1, \dots, p_k$ of the distance function d_o such that :

$$d_o(p_i) \geq e \cdot d_o(p_{i+1}) \qquad , 1 \leq i < k ,$$

and that its length k is maximal. In the terminology of 2.2 this is a metrical $(k;\infty,e,0)$-frame of the base point o . It is useful to consider the annuli

$$A_i \quad := \quad A(e^{-1} \cdot d_o(p_i) \ , \ e \cdot d_o(p_i)) \qquad , 1 \leq i \leq k .$$

5.1 Immediate Consequences :

i) $k \leq 2 \cdot \pi^{n-1} \cdot (\frac{e}{e-1})^{2n-2} \cdot \exp((2n-2) \cdot b_1)$ [lemma 2.3] .

ii) $M^n \setminus \bigcup_i A_i$ does not contain a critical point of d_o ;therefore lemma 2.1 gives raise to a vector field v_o which does not vanish on this set.

There are numbers $0 < x_{\tilde{k}} < y_{\tilde{k}} < \dots < x_2 < y_2 < x_1 < y_1 < x_o := \infty$ such that :

iii) $$\bigcup_{i=1}^{k} A_i \subset \bigcup_{i=1}^{k} A(\tfrac{6}{7} \cdot e^{-1} \cdot d_o(p_i) \ , \tfrac{8}{7} \cdot e \cdot d_o(p_i)) = \bigcup_{j=1}^{\tilde{k}} A(\tfrac{6}{7} \cdot x_j \ , \ \tfrac{8}{7} \cdot y_j)$$

iv) the annuli $A(\frac{6}{7} \cdot x_j \ , \ \frac{8}{7} \cdot y_j)$ are disjoint.

It is convenient to also introduce the annuli $\tilde{A}_j$ which are defined as follows :

$$\tilde{A}_j \quad := \quad A(\tfrac{3}{4} \cdot x_j \ , \ x_{j-1}) \qquad , 1 \leq j < \tilde{k} ,$$

$$\tilde{A}_{\tilde{k}} \quad := \quad \overline{B(o \ , x_{\tilde{k}-1})} \setminus \{o\}$$

5.2 Observations :

i) $\sum_{j=1}^{\tilde{k}} (\ln \frac{4}{3} + \ln \frac{y_j}{x_j}) \leq k \cdot (2 + \ln \frac{4}{3})$.

ii) $M^n \setminus \{o\} = \bigcup_{j=1}^{\tilde{k}} \tilde{A}_j$

iii) $\tilde{A}_j \cap \tilde{A}_{j+1} = A(\frac{3}{4} \cdot x_j , x_j)$, $1 \leq j < \tilde{k}$, and

$\tilde{A}_j \cap \tilde{A}_{j'} = \emptyset$,if $|j-j'| \geq 2$

iv) the Mayer - Vietoris sequence yields estimates on the values of the Poincaré series (for positive t):

$$P_t(M^n \setminus \{o\}) \leq \sum_{j=1}^{\tilde{k}} P_t(\tilde{A}_j) + \sum_{j=1}^{\tilde{k}-1} P_t(A(\frac{3}{4} \cdot x_j , x_j))$$

$$P_t(M^n) \leq \sum_{j=1}^{\tilde{k}} (P_t(\tilde{A}_j) + P_t(A(\frac{3}{4} \cdot x_j , x_j)))$$

v) the inclusions

$A(x_j , y_j) \hookrightarrow A(\frac{6}{7} \cdot x_j , \frac{8}{7} \cdot y_j) \hookrightarrow \tilde{A}_j$ and

$A(\frac{7}{8} \cdot x_j , \frac{7}{8} \cdot x_j) \hookrightarrow A(\frac{3}{4} \cdot x_j , x_j)$

allow for deformation retracts along the vector field v_o as has been mentioned above.

vi) $$P_{t^{-1}}(M^n) \leq \sum_{j=1}^{\tilde{k}} (rk_*^{t^{-1}} (A(\frac{6}{7} \cdot x_j , \frac{8}{7} \cdot y_j) , A(x_j , y_j)) + rk_*^{t^{-1}} (A(\frac{3}{4} \cdot x_j , x_j) , A(\frac{7}{8} \cdot x_j , \frac{7}{8} \cdot x_j)))$$

vii) if $t \geq t_2(n)$ the right hand side of the previous formula can be estimated by means of proposition 4.7 :

$$P_{t^{-1}}(M^n) \leq \sum_{j=1}^{\tilde{k}} (\frac{1}{1000} + \ln \frac{y_j}{x_j}) \cdot C_a(n) \cdot \exp(\frac{7n-5}{4} \cdot b_1) .$$

We use 5.1i and 5.2i in order to compute the bound. This gives us the following result :

5.3 Proposition :

Let M^n *be an asymptotically non-negatively curved manifold,* b_1 *be its curvature invariant (as above), and let* t *be some number* *greater than :*

$$t_1(n) \quad := \quad 5^{n^2} \cdot 8^n \cdot \exp(\frac{9}{4} \cdot \frac{1}{n+1})$$

then :

$$P_{t-1}(M^n) \quad \leq \quad (5\cdot(1+\sqrt{2}))^{c(n)} \cdot \exp(\frac{15n-13}{4} \cdot b_1(M^n))$$

where :

$$c(n) \quad := \quad 2\cdot n^3 + \frac{19}{6}\cdot n^2 + \frac{9}{5}\cdot n + \frac{3}{5}$$

Remarks :

i) Up to this point all the numerical estimates done in order to get a simple explicit bound have been chosen in such a way that they do not spoil the leading order terms of $c(n)$ and $t_1(n)$. As far as $c(n)$ is concerned both the factors in the basis directly stem from the geometric constructions : the Fibonacci number 5 reflects the geometry in the local covering arguement (c.f. corollary 1.6) , and the number $1+\sqrt{2}$ is due to the packing arguement in appendix A .

ii) The lower order terms have not been treated that carefully; they could even be improved easily by changing the geometric details in the arguement. For instance one could make use of the fact that the critical points of the distance function d_p cannot lie everywhere in an incompressible ball $B(p,r)$; they are contained in a rather small subset , and lemma 3.5 could be modified accordingly.

Appendix A

A Packing Problem in $S^{n-1} \subset \mathbb{R}^n$

We define sequences $(a_n)_{n\geq 1}$ and $(\alpha_n)_{n\geq 1}$ of real numbers in $(0,1]$ resp. $(0,\frac{\pi}{2}]$ by :

$$a_1 := 1$$

$$a_n = a_{n-1} \cdot (1 - a_n) \cdot \left(1 + \sqrt{\frac{2}{1+a_n}}\right)^{-1} \qquad , n \geq 2$$

and

$$\alpha_n := \arcsin(a_n) \qquad , n \geq 1$$

<u>Proposition :</u>

Let A_n *be the collection of all subsets* $A \subset S^{n-1}$ *which obey the condition*

(*) $\quad p, q \in A \;,\; p \neq q \quad \Longrightarrow \quad d(p,q) > \frac{\pi}{2} - \alpha_n$

Then :

$$\max\{\# A \mid A \in A_n\} = 2n$$

<u>Remarks :</u>

i) We point out that $a_2 = \sin(\frac{\pi}{10})$, and $\frac{\pi}{2} - \alpha_2 = \frac{2\pi}{5}$, which is the centriangle of the regular 5 - gon.

ii) As explained in example 1 in appendix B, there is the estimate :

$$\frac{1}{3} \cdot (1 + \sqrt{2})^{1-n} \leq a_n \leq (1 + \sqrt{2})^{1-n} \qquad , n \geq 2 \;.$$

iii) We may view α_n as a lower bound on the angle $\tilde{\alpha}_n$ defined by

$$\frac{\pi}{2} - \tilde{\alpha}_n = 2 \cdot \sup\{\rho \mid \text{there exist } 2n+1 \text{ disjoint balls of radius } \rho \text{ in } S^{n-1}\}$$

In principal such a bound could also have been obtained computing the packing densities of n balls of radius ρ mutually touching each other with respect to the simplex spanned by their centres. (c.f. [Bö].)

<u>Proof :</u>

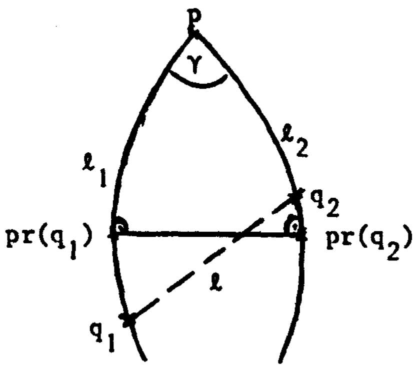
The vertices of the generalized octahedron in $\mathbb{R}^n$ define a set $A \in A_n$. This proves "$\geq$". The opposite inequality is shown by induction :

We take some $A \in A_n$, If A contains no more than 2 elements, we are done ; else we pick 2 points $p, \bar{p} \in A$ such that their distance $d(p,\bar{p})$ is maximal. To finish the proof, we construct a projection of $A' := A \setminus \{p, \bar{p}\}$ onto some set in A_{n-1} :

i) *The estimate* $\frac{\pi}{2} - \alpha_n < d(p,q) < \frac{3\pi}{4} + \frac{1}{2} \cdot \alpha_n$ *holds for all* $q \in A'$.

Assuming the converse, one concludes that q and $\bar{p}$ both lie in the ball of radius $\frac{\pi}{4}-\frac{1}{2}\cdot\alpha_n$ around the antipodal point of p . This observation immediately yields a contradiction to property (*) .

ii) *The projection* $pr: A' \longrightarrow S^{n-2} = \{ x\in S^{n-1} \mid d(p,x) = \frac{\pi}{2} \}$ *along the great circles through* p *can be controlled by means of spherical trigonometry :*



Suppose that q_1 , q_2 are two distinct points in A' ; put :

$$\gamma := d(pr(q_1), pr(q_2))$$

$$\ell := d(q_1, q_2)$$

$$\ell_i := d(p, q_i) \qquad , i = 1,2 .$$

The Law of Cosines may be written as follows:

$$(**) \quad \cos\gamma = \frac{\cos\ell - \cos\ell_1 \cdot \cos\ell_2}{\sin\ell_1 \cdot \sin\ell_2}$$

It is elementary analysis to verify that under the constraints

$$\frac{\pi}{2}-\alpha_n \leq \ell_2 \leq \ell_1 \leq \frac{3\pi}{4}+\frac{1}{2}\cdot\alpha_n \quad , \quad \frac{\pi}{2}-\alpha_n \leq \ell$$

the right-hand side of (**) has a unique maximum , which is acchieved at:

$$\ell = \ell_2 = \frac{\pi}{2}-\alpha_n \quad , \quad \ell_1 = \frac{3\pi}{4}+\frac{1}{2}\cdot\alpha_n$$

Therefore , if q_1 , $q_2 \in A'$, we conclude :

$$\cos\gamma > \cot(\frac{\pi}{2}-\alpha_n) \cdot \frac{1-\cos(\frac{3\pi}{4}+\frac{1}{2}\cdot\alpha_n)}{\sin(\frac{3\pi}{4}+\frac{1}{2}\cdot\alpha_n)}$$

$$= \tan\alpha_n \cdot \frac{1+\sqrt{\frac{1}{2}\cdot(1-\cos(\frac{\pi}{2}+\alpha_n))}}{\sqrt{\frac{1}{2}\cdot(1+\cos(\frac{\pi}{2}+\alpha_n))}}$$

$$= \frac{a_n}{\sqrt{1-a_n^2}} \cdot \frac{\sqrt{2}+\sqrt{1+a_n}}{\sqrt{1-a_n}}$$

$$= \frac{a_n}{1-a_n} \cdot \left(1+\sqrt{\frac{2}{1+a_n}}\right) = a_{n-1}$$

hence :

$$\cos\gamma > \cos(\frac{\pi}{2}-\alpha_{n-1})$$

□

Appendix B

A Lemma on Recursively Defined Sequences :

Let $\phi: [0,1) \longrightarrow [a,\infty)$ be a monotone function such that $\phi(0) = a > 1$.
Then $f: [0,1) \longrightarrow [0,\infty)$, $x \longmapsto x \cdot \phi(x)$ is invertible. For any $x_o \in (0,1]$ there is a unique sequence of real numbers $(x_n)_{n \in \mathbb{N}_0}$ defined by :

$$x_n = f(x_{n+1}) \qquad , \quad x_{n+1} \in (0,1]$$

Lemma :

i) $x_{n+1} \leq a^{-1} \cdot x_n \leq a^{-n} \cdot x_1$

If moreover $\phi(x) \leq a \cdot (1+x)/(1-x)$ *for all* $x \in [0,1)$ *, there are also lower bounds for the* x_n *:*

ii) $x_{n+1}/x_n = \phi(x_{n+1})^{-1} \geq \phi(a^{-n} \cdot x_1)^{-1}$

iii) $x_n/x_o \geq a^{-n} \cdot \left(\frac{1-x_1}{1+x_1}\right)^{\frac{a}{a-1}} \geq a^{-n} \cdot \left(\frac{a-x_o}{a+x_o}\right)^{\frac{a}{a-1}} \quad ; \; n \geq 1$

Proof : (i) and (ii) are obvious. In order to prove the last claim , notice that by induction :

$$(*) \qquad x_n/x_o \geq \prod_{j=o}^{n-1} \phi(a^{-j} \cdot x_1)^{-1} \geq a^{-n} \cdot \prod_{j=o}^{n-1} \frac{1-a^{-j} \cdot x_1}{1+a^{-j} \cdot x_1}$$

We compute :

$$\sum_{j=o}^{n-1} \ln\left(\frac{1-a^{-j} \cdot x_1}{1+a^{-j} \cdot x_1}\right) = -2 \cdot \sum_{j=o}^{n-1} \sum_{k=o}^{\infty} \frac{1}{2k+1} \cdot a^{-j \cdot (2k+1)} \cdot x_1^{2k+1}$$

$$\geq -2 \cdot \sum_{k=o}^{\infty} \frac{1}{2k+1} \cdot \frac{1}{1-a^{-2k-1}} \cdot x_1^{2k+1}$$

$$\geq \frac{a}{a-1} \cdot \ln\left(\frac{1-x_1}{1+x_1}\right)$$

Combining this inequality with (*) gives the required estimate.

Examples :

1.) $\phi(x) = \left(1+\sqrt{\frac{2}{1+x}}\right) / (1-x) \quad ; \quad x_o = 1 \quad :$

Clearly $a = 1+\sqrt{2}$ and $x_1 = \sin\left(\frac{\pi}{10}\right)$; therefore one has :

$$x_n \geq \frac{1}{3} \cdot (1+\sqrt{2})^{-n} \qquad , \text{provided } n \geq 1 \; .$$

2.) $\phi(x) = (2+\rho\cdot(1+x))\,/\,(1-x)$, where $2+\rho = 5+2\cdot L^{-1}$, $L>0$;

$x_o \leq \frac{1}{2\cdot(L+4)}$

Clearly $a = 2+\rho$, and one easily computes that :

$$\left(\frac{a+x_o}{a-x_o}\right)^{a/a-1} \leq \left(\frac{5+\frac{2}{L}+\frac{1}{2\cdot(L+4)}}{5+\frac{2}{L}-\frac{1}{2\cdot(L+4)}}\right)^{5/4}$$

$$\leq \left(1+\frac{1}{5\cdot(L+4)}\right)^{5/4} \leq \exp\left(\frac{1}{4\cdot(L+4)}\right)$$

Therefore

$$5^n \leq \frac{x_o}{x_n} \leq 5^n \cdot \exp\left(\frac{2n}{5L}+\frac{1}{4\cdot(L+4)}\right)$$

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