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SUPERCONFORMAL INDICES OF
$\mathcal{N}=4$ SYM FIELD THEORIES

V. P. SPIRIDONOV AND G. S. VARTANOV

Abstract. Superconformal indices of $\mathcal{N}=4$ sypersymmetric Yang-Mills field theories with simple gauge groups $SU(N), SO(N), SP(2N), G_2, F_4, E_6, E_7, E_8$ are described in terms of elliptic hypergeometric integrals. For the latter four exceptional groups this yields first examples of integrals of such type. S-duality transformation for G_2 and F_4 theories does not change their superconformal indices being equivalent to a change of variables in the corresponding integrals. Some mathematical arguments are given in favor of the equality of indices for dual $SP(2N)$ and $SO(2N+1)$ theories conjectured by Gadde et al [19].

Contents

1. Introduction

The question of strong-weak duality of $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory in four dimensional space-time is a quite old area of research [1, 2, 3]. This duality (called also S-duality) states the equivalence of the theory with an "electric" gauge group G_c to a similar theory with a "magnetic" gauge group $G_c^\vee$ and the inverse coupling constant. If one introduces the coupling constant as

$$\tau \;=\; \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g^2}, \tag{1}$$

then the S-duality transformation of the theory maps τ for a simply-laced gauge group[1] to the coupling constant $-1/\tau$,

$$S: \ \tau \to -\frac{1}{\tau}. \qquad (2)$$

Together with the symmetry transformation

$$T: \ \tau \to \tau + 1,$$

the strong-weak duality becomes equivalent to the $SL(2,\mathbb{Z})$ group of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \ a, b, c, d \in \mathbb{Z}. \qquad (3)$$

For $\mathcal{N} = 4$ SYM theories with the non-simply laced gauge groups one has the following realization of S-duality

$$\widetilde{S}: \ \tau \to -\frac{1}{m\tau}, \qquad (4)$$

where m is the ratio of the lengths-squared of long and short roots of the corresponding root system ($m = 2$ for $SO(2N+1), SP(2N), F_4$ and $m = 3$ for G_2). In [4], $\mathcal{N} = 4$ theories with G_2 and F_4 gauge groups were analyzed from the algebraic point of view and the S-duality transformation of the moduli space was described. Here we would like to discuss another approach for testing validity of these and other conjectural dualities for $\mathcal{N} = 4$ SYM field theories.

For this purpose we use the technique based on the calculation of the superconformal indices for $\mathcal{N} = 4$ theories suggested by Kinney et al in [5] (for the definition of indices in $\mathcal{N} = 1$ theories, see [6, 7]). $\mathcal{N} = 4$ SYM theory has the $PSU(2,2|4)$ space-time symmetry group generated by $J_a, \overline{J}_a$, $a = 1, 2, 3$, representing $SU(2)$ subgroups (Lorentz rotations), $P_\mu, Q_{i,\alpha}, \overline{Q}_{i,\dot\alpha}$ (supertranslations) with $i = 1, 2, 3, 4$ and $\alpha, \dot\alpha = 1, 2$; $K_\mu, S^{i,\alpha}, \overline{S}^{i,\dot\alpha}$ (special superconformal transformations), and H (dilations) whose state eigenvalues are given by conformal dimensions [8]. As to the $SU(4)_R$ R-symmetry subgroup, we mention only its commuting maximal torus generators R_1, R_2, R_3. For a distinguished pair of supercharges, say, $Q \equiv Q_{1,1}$ and $Q^\dagger \equiv S^{1,1}$, in appropriate normalization one has

$$\{Q, Q^\dagger\} = H - 2J_3 - 2\sum_{k=1}^{3}\left(1 - \frac{k}{4}\right) R_k \equiv \Delta, \qquad (5)$$

and the superconformal index is defined by the matrix integral [5]

$$I(t, y, v, w) = \int_{G_c} [dU] \, \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m} f(t^m, y^m, v^m, w^m) \mathrm{Tr}(U^\dagger)^m \mathrm{Tr}\, U^m \right\}, \qquad (6)$$

where $[dU]$ is the $G_c = U(N)$ invariant measure and $f(t, y, v, w)\mathrm{Tr}\, U^\dagger \mathrm{Tr}\, U$ is the so-called single-particle states index with

$$f(t, y, v, w) \ = \ \frac{t^2(v + \frac{1}{w} + \frac{w}{v}) - t^3(y + \frac{1}{y}) - t^4(w + \frac{1}{v} + \frac{v}{w}) + 2t^6}{(1 - t^3 y)(1 - \frac{t^3}{y})}. \qquad (7)$$

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[1] A simply laced group is a Lie group whose Dynkin diagram contains only simple links, and therefore all roots of the corresponding Lie algebra have the same length. These groups are $SU(N), SO(2N), E_6, E_7$, and E_8.

As shown in [9] (see there the discussion following formula (5.33)) this expression can be obtained from the superconformal group character or partition function for $\mathcal{N}=4$ theories by imposing the shortening condition for the multiplets. The integrand in (6) is given by the following expression

$$\mathrm{Tr}\left((-1)^{\mathrm{F}} t^{2(H+J_3)} y^{2\overline{J}_3} v^{R_2} w^{R_3} e^{\sum_a g_a G^a} e^{-\beta\Delta}\right), \tag{8}$$

where F is the fermion number operator, G^a are gauge group generators, and t, y, v, w, g_a, β are the group parameters (chemical potentials). The trace is taken over the states corresponding to zero modes of the operator Δ because relation (5) is preserved by operators in (8) (the contributions from other states cancel together with the dependence on β). All the fields in $\mathcal{N}=4$ supermultiplet lie in the same representation of the gauge group G_c. It means that, in comparison with the superconformal indices in $\mathcal{N}=1,2$ SYM theories, the contribution from the fields will be given by the adjoint representation only. The problem of counting various BPS states in $\mathcal{N}=4$ theories and computation of the related characters was discussed in [9, 10].

The superconformal indices technique has already found many applications in supersymmetric field theories. In [7] the Seiberg duality for $\mathcal{N}=1$ SYM theories was conjectured to lead to the equality of indices of dual theories. Later on Dolan and Osborn explicitly confirmed this conjecture for a number of examples [11]. It appears that superconformal indices are expressed in terms of elliptic hypergeometric integrals whose theory was developed earlier in [12, 13] (see also [14] for a general survey). Moreover, equality of indices in dual theories happened to be equivalent either to exact computability of elliptic beta integrals discovered in [12] or to nontrivial Weyl group symmetry transformations for higher order elliptic hypergeometric functions [13, 15]. In a series of papers [16, 17, 18] we applied this technique for analyzing all previously found Seiberg dualities. We suggested also many new such dualities on the basis of known identities for elliptic hypergeometric integrals and showed that known nontrivial duality checks are satisfied for them. As a payback to mathematics, it happened that many old dualities lead to new, still unproven highly nontrivial relations for integrals.

This line of thoughts was further developed in beautiful papers by Gadde et al [19, 20]. In [19], a fresh identity from [21] describing $W(F_4)$ Weyl group transformation for a particular one dimensional elliptic hypergeometric integral was used for confirming S-duality for $\mathcal{N}=2$ SYM theory with $SU(2)$ gauge group and four hypermultiplets [22, 23] and for ensuring associativity of the operator algebra of 2D theories behind that duality. Using the inversion of the simplest elliptic hypergeometric integral transform of [24], the superconformal index for a E_6 SCFT theory was constructed in [20] from the index of $\mathcal{N}=2$ SYM theory with $G_c=SU(3)$ and six hypermultiplets and a new test of the Argyres-Seiberg duality [25] was suggested.

One of the purposes of our paper consists in the consideration of S-duality for $\mathcal{N}=4$ SYM theories with G_2 and F_4 gauge groups [1, 4] from the superconformal indices point of view. Similar consideration was performed already by Gadde et al in [19] in the case of $G_c=SP(2N)$ and $G_c^\vee=SO(2N+1)$ groups. We give here new sufficiently strong mathematical arguments in favor of the equality of indices for the latter dual theories. For completeness, we describe also the indices for $G_c=SU(N), SO(2N), E_6, E_7$, and E_8 theories. As a complementary result, we give two more examples: the identity coming from $\mathcal{N}=1$ SYM theories based on exceptional gauge group E_6 with 6 flavors [26, 27] and a relation between superconformal indices for a particular pair of $\mathcal{N}=2$ quiver theories. In the end we discuss briefly indices in relation to the exactly marginal deformations of $\mathcal{N}=4$ SYM theory.

2. Duality of $SO(2N+1)$ and $SP(2N)$ $\mathcal{N}=4$ SYM theories

Superconformal indices for $\mathcal{N}=4$ SYM theories with $SP(2N)$ and $SO(2N+1)$ gauge groups were described by Gadde et al in [19] and discussed briefly in the simplest case in [17]. Here we give some essential mathematical arguments supporting the conjecture that these two superconformal indices coincide.

The full single-particle index is

$$\frac{\sum_{k=1}^3 s_k - t^6\sum_{k=1}^3 s_k^{-1} - t^3(y+\frac{1}{y}) + 2t^6}{(1-t^3y)(1-\frac{t^3}{y})}\chi_{adj}(z), \tag{9}$$

where $\chi_{adj}(z)$ is the character of the adjoint representation of the corresponding gauge group (see the Appendix). For convenience, we have replaced the parameters v and w by s_1, s_2, s_3 using the notation

$$s_1 \;=\; t^2 v, \qquad s_2 \;=\; t^2\frac{1}{w}, \qquad s_3 \;=\; t^2\frac{w}{v}. \tag{10}$$

We stress that the single-particle state indices of all our theories discussed below differ only by the characters $\chi_{adj}(z)$. It is convenient to denote also

$$p \;=\; t^3 y, \qquad q \;=\; \frac{t^3}{y}.$$

Using the explicit form of the group invariant measures in terms of the maximal torus variables, the superconformal indices can be written as particular elliptic hypergeometric integrals [14]. The $SP(2N)$-electric theory index gets the following shape

$$I_E = \chi_N \int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm1} z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)} \prod_{j=1}^N \frac{\prod_{k=1}^3\Gamma(s_k z_j^{\pm2};p,q)}{\Gamma(z_j^{\pm2};p,q)} \prod_{j=1}^N \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{11}$$

and for $SO(2N+1)$-magnetic theory one has

$$I_M = \chi_N \int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\prod_{k=1}^3 \Gamma(s_k y_i^{\pm1} y_j^{\pm1};p,q)}{\Gamma(y_i^{\pm1}y_j^{\pm1};p,q)} \prod_{j=1}^N \frac{\prod_{k=1}^3\Gamma(s_k y_j^{\pm1};p,q)}{\Gamma(y_j^{\pm1};p,q)} \prod_{j=1}^N \frac{dy_j}{2\pi \mathrm{i} y_j}, \tag{12}$$

where $|s_k|<1$, $k=1,2,3$. For $|s_k|\ge 1$ the indices are defined as analytical continuations of the expressions (11) and (12).

Here $\mathbb{T}$ denotes the unit circle with positive orientation and we use conventions $\Gamma(a,b;p,q) := \Gamma(a;p,q)\Gamma(b;p,q)$, $\Gamma(az^{\pm1};p,q) := \Gamma(az;p,q)\Gamma(az^{-1};p,q)$, where

$$\Gamma(z;p,q) = \prod_{i,j=0}^\infty \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \quad |p|,|q|<1,$$

is the elliptic gamma function. The coefficient in front of the integrals is

$$\chi_N \;=\; \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \prod_{k=1}^3 \Gamma^N(s_k;p,q),$$

with $(a;q)_\infty=\prod_{k=0}^\infty(1-aq^k)$. The constraint

$$\prod_{k=1}^{3} s_k \;=\; pq \tag{13}$$

plays the role of the balancing condition for the integrals.

S-duality for these theories leads thus to a nice conjecture on the equality of elliptic hypergeometric integrals for $SO(2N+1)$ and $SP(2N)$ groups:

$$I_E \;=\; I_M \tag{14}$$

in the indicated domain of values of parameters.

We rewrite this equality as

$$\int_{\mathbb{T}^N}\Delta_E(\underline{z},\underline{s})\prod_{j=1}^N\frac{dz_j}{2\pi \mathrm{i} z_j}=\int_{\mathbb{T}^N}\Delta_M(\underline{y},\underline{s})\prod_{j=1}^N\frac{dy_j}{2\pi \mathrm{i} y_j}, \tag{15}$$

where the kernels $\Delta_E(\underline{z},\underline{s})$ and $\Delta_M(\underline{y},\underline{s})$ are read from the integrals (11) and (12). Then we compose the function

$$\rho(\underline{z},\underline{y},\underline{s}) \;=\; \frac{\Delta_E(\underline{z},\underline{s})}{\Delta_M(\underline{y},\underline{s})}. \tag{16}$$

We have verified that this function represents the so-called totally elliptic hypergeometric term [28, 17]. This is a rather rich mathematical statement giving a strong evidence on the validity of the stated equality of integrals. It means that all the functions

$$h_i^{(z)}=\frac{\rho(\ldots qz_i\ldots,\underline{y},\underline{s})}{\rho(\underline{z},\underline{y},\underline{s})},\quad h_i^{(y)}=\frac{\rho(\underline{z},\ldots qy_i\ldots,\underline{s})}{\rho(\underline{z},\underline{y},\underline{s})},\quad i=1,\ldots,N,$$
$$h_{kl}^{(s)}=\frac{\rho(\underline{z},\underline{y},\ldots qs_k,\ldots,q^{-1}s_l\ldots)}{\rho(\underline{z},\underline{y},\underline{s})},\quad k,l=1,2,3;\ k\neq l,$$

are elliptic functions of all their arguments z_i, y_i, s_k, and q. For instance,

$$\begin{aligned}&h_i^{(z)}(\ldots pz_j\ldots,\underline{y},\underline{s};q;p)=h_i^{(z)}(\underline{z},\ldots py_j\ldots,\underline{s};q;p)\\&=h_i^{(z)}(\underline{z},\underline{y},\ldots ps_k\ldots p^{-1}s_l;q;p)=h_i^{(z)}(\underline{z},\underline{y},\ldots ps_l\ldots;pq;p)\\&=h_i^{(z)}(\underline{z},\underline{y},\underline{s};q;p),\qquad k,l=1,2,3.\end{aligned}$$

This test is passed by all known integral identities; however, it is not sufficient for their validity. For further consequences of the total ellipticity and various technical details of such computations, we refer to papers [14, 17, 28].

3. Some direct checks

Proofs for $N=1,2$. For low ranks of the gauge group the equality of the indices follows from the change of variables associated with the rotation of the corresponding root system [19].

For $N=1$ the electric superconformal index is

$$I_E=\frac{(p;p)_\infty(q;q)_\infty}{2}\int_{\mathbb{T}}\frac{\prod_{k=1}^3\Gamma(s_kz^{\pm2};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{2\pi \mathrm{i} z}.$$

To obtain the magnetic index from this expression one has to substitute $z=\sqrt{y}$, take into account the factor $1/2$ coming from the Jacobian and the factor 2 coming after shrinking the integration contour from double $\mathbb{T}$ to $\mathbb{T}$ (which is easy to follow in terms of the $\theta \to \theta/2$ angle variables change with $z = e^{i\theta}$). Note that these electric and magnetic indices can be derived as particular reductions of the univariate elliptic beta integral of higher order with 24 and 10 parameters, respectively.

For $N = 2$ the electric superconformal index has the form

$$I_E = \chi_2 \int_{\mathbb{T}^2} \frac{\prod_{k=1}^3 \Gamma(s_k z_1^{\pm 1} z_2^{\pm 1}; p, q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1}; p, q)} \prod_{j=1}^{2} \frac{\prod_{k=1}^3 \Gamma(s_k z_j^{\pm 2}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{2} \frac{dz_j}{2\pi \mathrm{i} z_j}. \quad (17)$$

Corresponding magnetic index (12) is obtained from (17) after the substitutions

$$z_1 = \sqrt{y_1 y_2}, \qquad z_2 = \sqrt{y_1 y_2^{-1}}.$$

Indeed, for $|s_k| < 1$ the integral kernel can be represented as a convergent N-fold Laurent series in z_j-variables and the integration picks up its constant term. The change of the variables reshapes this Laurent series, but the constant term remains the same and it can be found by computing the integrals over the contours $y_j \in \mathbb{T}$.

The limit $s_k \to 1$. Suppose one of the parameters approaches 1, say, $s_1 \to 1$. Then a number of poles of the integral kernels in (11) and (12) approach the unit circle, but, because of the zeros already lying at the appropriate points, their residues vanish and no singularities appear on the integration contour. However, the factor χ_N is divergent in this limit. Because the product of two other parameters s_2 and s_3 becomes equal to pq, and $\Gamma(a, b; p, q) = 1$ for $ab = pq$, the integrands do not depend on all parameters s_j and are actually equal to 1. As a result, we have $\lim_{s_1 \to 1} I_E/I_M = 1$. For $N = 1$, from the physical point of view this limit can be associated with $\mathcal{N} = 2$ SYM theory with $SU(2)$ gauge group discussed in [22].

Reduction to $p = q = 0$. One can consider the integrals (11) and (12) in the limit $p, q \to 0$. Because of the balancing condition (13), some of the parameters should be rescaled by appropriate powers of p and q which can be done in many different ways. One simple possibility consists in fixing $s_{1,2}$ and setting

$$s_3 = \frac{pq}{s_1 s_2}.$$

For fixed z, the limit $p = 0$ and further limit $q = 0$ simplifies the elliptic gamma function to

$$\Gamma(z; p, q) \underset{p \to 0}{=} \frac{1}{(z; q)_\infty} \underset{q \to 0}{=} \frac{1}{1 - z},$$

so that integral (11) reduces first to q–integral

$$I_E^{p=0}(s_1, s_2 \text{ fixed}) = \frac{(q; q)_\infty}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \le i < j \le N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty (s_1 s_2 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty (s_2 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}$$
$$\times \prod_{j=1}^{N} \frac{(z_j^{\pm 2}; q)_\infty (s_1 s_2 z_j^{\pm 2}; q)_\infty}{(s_1 z_j^{\pm 2}; q)_\infty (s_2 z_j^{\pm 2}; q)_\infty} \prod_{j=1}^{N} \frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (18)$$

and then to the rational integral

$$I_E^{p=q=0}(s_1,s_2 \text{ fixed}) = \frac{1}{2^N N!}\int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{(1-z_i^{\pm1}z_j^{\pm1})(1-s_1s_2z_i^{\pm1}z_j^{\pm1})}{(1-s_1z_i^{\pm1}z_j^{\pm1})(1-s_2z_i^{\pm1}z_j^{\pm1})} \times \prod_{j=1}^N \frac{(1-z_j^{\pm2})(1-s_1s_2z_j^{\pm2})}{(1-s_1z_j^{\pm2})(1-s_2z_j^{\pm2})} \prod_{j=1}^N \frac{dz_j}{2\pi \mathrm{i} z_j}. \quad (19)$$

Integral (12) first reduces to

$$I_M^{p=0}(s_1,s_2 \text{ fixed}) = \frac{1}{2^N N!}\int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{(y_i^{\pm1}y_j^{\pm1};q)_\infty (s_1s_2y_i^{\pm1}y_j^{\pm1};q)_\infty}{(s_1y_i^{\pm1}y_j^{\pm1};q)_\infty(s_2y_i^{\pm1}y_j^{\pm1};q)_\infty} \times \prod_{j=1}^N \frac{(y_j^{\pm1};q)_\infty(s_1s_2y_j^{\pm1};q)_\infty}{(s_1y_j^{\pm1};q)_\infty(s_2y_j^{\pm1};q)_\infty} \prod_{j=1}^N \frac{dy_j}{2\pi \mathrm{i} y_j}, \quad (20)$$

and then becomes

$$I_M^{p=q=0}(s_1,s_2 \text{ fixed}) = \frac{1}{2^N N!}\int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{(1-y_i^{\pm1}y_j^{\pm1})(1-s_1s_2y_i^{\pm1}y_j^{\pm1})}{(1-s_1y_i^{\pm1}y_j^{\pm1})(1-s_2y_i^{\pm1}y_j^{\pm1})} \times \prod_{j=1}^N \frac{(1-y_j^{\pm1})(1-s_1s_2y_j^{\pm1})}{(1-s_1y_j^{\pm1})(1-s_2y_j^{\pm1})} \prod_{j=1}^N \frac{dy_j}{2\pi \mathrm{i} y_j}. \quad (21)$$

One can evaluate integrals (19) and (21) by computing the residues, since the integrands have now a finite number of poles. However, we did not find a simple way of performing these computations for arbitrary N.

We did such a residue calculus only for $N=3$. We shall not give details of the computation since the procedure is straightforward and cumbersome. We indicate only the poles relevant for this calculation. For $|s_k|<1$ the reduced integral for electric theory has the following poles

$$z_i = (s_k z_j^{\pm1})^{\pm1}, \qquad z_i = \pm s_k^{\pm\frac{1}{2}}, \quad (22)$$

where $k=1,2$ and $i,j=1,2,3,\ i\neq j$. Only the poles lying inside the three dimensional domain $\mathbb{T}^3$ give contributions and their residues can be computed successively in the integration variables. The poles for z_1 lying inside $\mathbb{T}$ are $s_k z_j^{\pm1}$ and $\pm s_k^{\frac{1}{2}}$, where $k=1,2,\ j=2,3$. Computing their residues we proceed further and calculate the residues of the poles in z_2 and, finally, we find the residues for z_3-poles. For the magnetic integral we have the following set of poles

$$y_i = (s_k y_j^{\pm1})^{\pm1}, \qquad y_i = s_k^{\pm1}, \quad (23)$$

where $k=1,2$ and $i,j=1,2,3,\ i\neq j$. In comparison with the electric case here we have much less residues to be taken into account. After lengthy Mathematica calculations we confirmed that the sums of residues for both integrals coincide.

Another way of taking the limit $p=q=0$ corresponds to a very natural choice of the chemical potentials v,w associated with R_2 and R_3 charges in the definition (8) equal to 1. This yields the simplest possibility of exact evaluation of the superconformal indices. Indeed, after fixing all $s_k=(pq)^{\frac{1}{3}}, k=1,2,3$, the limit $p,q\to 0$ strongly simplifies integral kernels leading to particular cases of the Selberg integral. The numerators of integrands in (11) and (12) become equal to 1, since $\Gamma((pq)^{\frac{1}{3}}a;p,q)\to 1$ for $p\to 0$. As a result, we obtain the electric index

$$I_E^{p=q=0}(s_k=(pq)^{\frac{1}{3}})=\frac{1}{2^N N!}\int_{\mathbb{T}^N}\prod_{1\leq i<j\leq N}(1-z_i^{\pm1}z_j^{\pm1})\prod_{j=1}^N(1-z_j^{\pm2})\frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (24)$$

and the magnetic one

$$I_M^{p=q=0}(s_k=(pq)^{\frac{1}{3}})=\frac{1}{2^N N!}\int_{\mathbb{T}^N}\prod_{1\leq i<j\leq N}(1-y_i^{\pm1}y_j^{\pm1})\prod_{j=1}^N(1-y_j^{\pm1})\frac{dy_j}{2\pi \mathrm{i} y_j}. \quad (25)$$

Then we denote $z_j=e^{\mathrm{i}\theta_j}$ and $y_j=e^{\mathrm{i}\phi_j}$, $j=1,\ldots,N$, and pass to the integration over θ_j and ϕ_j variables. In terms of new variables

$$x_j \;=\; \frac{1+\cos\theta_j}{2}, \qquad x_j' \;=\; \frac{1+\cos\phi_j}{2}, \quad j=1,\ldots,N, \quad (26)$$

the integrals reduce to special cases of the Selberg integral

$$I_S(\alpha,\beta,\gamma)=\int_0^1\cdots\int_0^1\prod_{j=1}^N x_j^{\alpha-1}(1-x_j)^{\beta-1}\prod_{1\leq i<j\leq N}(x_i-x_j)^{2\gamma}dx_1\ldots dx_N$$
$$=\prod_{j=1}^N\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(N+j-2)\gamma)\Gamma(1+\gamma)}, \quad (27)$$

where $\Gamma(x)$ is the usual gamma function and

$$\Re\alpha,\Re\beta>0, \qquad \Re\gamma>-\min\left(\frac{1}{N};\frac{\Re\alpha}{N-1};\frac{\Re\beta}{N-1}\right).$$

Explicit computations show that integral (24) is equal to

$$\frac{2^{2N^2+N}}{N!\pi^N}I_S(3/2,3/2,1)=1.$$

Integral (25) yields

$$\frac{2^{2N^2-N}}{N!\pi^N}I_S(1/2,3/2,1)=1.$$

Indeed, the electric integral differs from the magnetic one only in the integrand numerator term $\prod_{j=1}^N(1+z_j^{\pm1})$ which becomes $4^N\prod_{j=1}^N x_j$ after the change of variables (26). This results in the shift of the α parameter value and appearance of the relative coefficient 4^N in front of the corresponding Selberg integral. Inserting the indicated values of parameters into the right-hand side of (27) and taking into account the factor 4^N we see that the equality $I_E=I_M$ reduces to the identity $\prod_{j=1}^N(4j-2)/(j+N)=1$, which is easily proved using the formula for doubling the gamma function argument

$$\Gamma(x)\Gamma(x+1/2)=2^{1-2x}\sqrt{\pi}\Gamma(2x).$$

A $p=0$, $q\to1^-$ **limit.** Let us take the limit $p\to0$ with fixed parameters s_1 and s_2, as in the first case. In the resulting indices we set $s_1=q^\alpha$, $s_2=q^\beta$ and consider the limit $q\to1^-$ for fixed α and β. To compute this limit for the factor χ_N we use the well known formula

$$\lim_{q\to1^-}\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}=\Gamma(x)$$

and find the diverging expression

$$\chi_N = \frac{1}{2^N N!}\left(\frac{\Gamma(\alpha)\Gamma(\beta)}{(1-q)\Gamma(\alpha+\beta)}\right)^N (1+o(1)).$$

As to the integrals, we apply another known asymptotic formula

$$\lim_{q\to 1^-}\frac{(q^\alpha A;q)_\infty}{(q^\beta A;q)_\infty} = (1-A)^{\beta-\alpha}$$

and find that both integrands are equal to 1 giving the same value for the integrals. So, the leading diverging asymptotics of the indices coincide

$$I_{E,M}^{p=0,q\to 1^-}(s_1=q^\alpha, s_2=q^\beta) = \frac{1}{2^N N!}\left(\frac{\Gamma(\alpha)\Gamma(\beta)}{(1-q)\Gamma(\alpha+\beta)}\right)^N (1+o(1)). \quad (28)$$

A $p=0$, $s_2=0$ limit. Let us look now for another reduction of the integrals appearing after the $p\to 0$ limit. We are taking now the following sequential limit

$$p\to 0\,(\text{fixed } s_1,s_2),\quad s_2 \ \to\ 0. \quad (29)$$

It reduces integral (11) to the form

$$I_{SP(2N)}^{p=s_2=0} \ = \ \frac{1}{2^N N!}\frac{(q;q)_\infty^N}{(s_1;q)_\infty^N}\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(s_1 z_i^{\pm 1}z_j^{\pm 1};q)_\infty}\prod_{j=1}^N\frac{(z_j^{\pm 2};q)_\infty}{(s_1 z_j^{\pm 2};q)_\infty}\frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (30)$$

which can be evaluated exactly using the multivariable extension of the Askey-Wilson integral (or particular q-Selberg integral) found in [30]. Indeed, it is sufficient to set in the corresponding integral evaluation formula

$$b=s_1,\qquad a_{1,2}=\pm\sqrt{s_1},\qquad a_{3,4}=\pm\sqrt{qs_1}$$

and find

$$I_{SP(2N)}^{p=s_2=0} \ = \ \prod_{j=0}^{N-1}\frac{(qs_1^{2j+1};q)_\infty}{(s_1^{2j+2};q)_\infty}. \quad (31)$$

The limit (29) applied to (12) leads to the integral

$$I_{SO(2N+1)}^{p=s_2=0} \ = \ \frac{1}{2^N N!}\frac{(q;q)_\infty^N}{(s_1;q)_\infty^N}\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(s_1 z_i^{\pm 1}z_j^{\pm 1};q)_\infty}\prod_{j=1}^N\frac{(z_j^{\pm 1};q)_\infty}{(s_1 z_j^{\pm 1};q)_\infty}\frac{dz_j}{2\pi \mathrm{i} z_j}. \quad (32)$$

The BC_N root system q-beta integral of [30] can be reduced to this integral on the B_N root system (see, e.g., [29]). In order to obtain (32), it is necessary to choose the parameters as

$$b=s_1,\qquad a_1=s_1,\qquad a_2=-1,\qquad a_{3,4}=\pm\sqrt{q},$$

which leads to the same result (31) after some explicit computations. Therefore, we find that

$$I_{SP(2N)}^{p=s_2=0} \ = \ I_{SO(2N+1)}^{p=s_2=0} \ = \ \prod_{j=0}^{N-1}\frac{(qs_1^{2j+1};q)_\infty}{(s_1^{2j+2};q)_\infty}. \quad (33)$$

This is the most powerful check of the equality of superconformal indices (11) and (12) which we have found. Equality of indices in the limit $s_k=(pq)^{\frac{1}{3}}\to 0$, $k=1,2,3$, established above is a special case of relation (33) obtained after the choice $s_1=q=0$.

The integrals in (33) were computed under the assumption that $|s_1| < 1$, but for finite N we can analytically continue superconformal indices to arbitrary values of s_1 as meromorphic functions using the right-hand side expression. For $|s_1| < 1$ one can consider the limit $N \to \infty$ which yields a ratio of double infinite products resembling "halves" of the elliptic gamma function with the base $p = s_1^2$.

4. G_2 gauge group

We consider now the S-duality conjecture for $\mathcal{N} = 4$ SYM theory with G_2 gauge group following from the Goddard-Nyuts-Olive construction [1], which was made more explicit in [2, 3] and discussed in detail in [4].

The G_2-group has two dimensional maximal torus parametrized by z_1 and z_2, but it is convenient to introduce the third group variable $z_3 = z_1^{-1} z_2^{-1}$ as described in the Appendix. Then the electric superconformal index takes the form

$$I_E = \kappa_2 \int_{\mathbb{T}^2} \prod_{1 \le i < j \le 3} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^2 \frac{dz_j}{2\pi \mathrm{i} z_j}, \qquad (34)$$

where $|s_k| < 1$, $k = 1, 2, 3$, and

$$\kappa_2 = \frac{(p;p)_\infty^2 (q;q)_\infty^2}{2^2 3} \prod_{k=1}^3 \Gamma^2(s_k; p, q).$$

In the magnetic theory one has[2]

$$I_M = \kappa_2 \int_{\mathbb{T}^2} \prod_{1 \le i < j \le 3} \frac{\prod_{k=1}^3 \Gamma(s_k (y_i y_j)^{\pm 3}, s_k (y_i y_j^{-1})^{\pm 1}; p, q)}{\Gamma((y_i y_j)^{\pm 3}, (y_i y_j^{-1})^{\pm 1}; p, q)} \prod_{j=1}^2 \frac{dy_j}{2\pi \mathrm{i} y_j}, \qquad (35)$$

where $y_1 y_2 y_3 = 1$.

Validity of S-duality would suggest the equality of these elliptic hypergeometric integrals, $I_E = I_M$, in the indicated domain of values of parameters. Remarkably, this identity can be easily established by the following change of the integration variables

$$y_1 = (z_2 z_3^2)^{\frac{1}{3}}, \qquad y_2 = (z_3 z_1^2)^{\frac{1}{3}}, \qquad y_3 = (z_1 z_2^2)^{\frac{1}{3}}. \qquad (36)$$

This reparametrization is associated with the rotation of the G_2 root system [4]. The superconformal indices test confirms thus the S-duality in this case.

Application of the limit (29) from the previous section reduces integral (34) to the form

$$I_{G_2}^{p=s_2=0} = \frac{1}{2^2 3} \frac{(q;q)_\infty^2}{(s_1;q)_\infty^2} \int_{\mathbb{T}^2} \prod_{1 \le i < j \le 3} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^2 \frac{dz_j}{2\pi \mathrm{i} z_j}, \qquad (37)$$

where $z_1 z_2 z_3 = 1$. This integral admits the exact evaluation [29]

$$I_{G_2}^{p=s_2=0} = \frac{(q s_1, q s_1^5; q)_\infty}{(s_1^2, s_1^6; q)_\infty}, \qquad (38)$$

where we use the convention $(a, b; q)_\infty = (a; q)_\infty (b; q)_\infty$.

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[2] We are deeply indebted to S. Razamat for pointing to a misprint in our initial expression for this integral.

5. F_4 GAUGE GROUP

Now we consider S-duality for $\mathcal{N}=4$ SYM theory with F_4 gauge group [1, 2, 3, 4]. The electric superconformal index has the following form

$$I_E = \kappa_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2} z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2};p,q)} \prod_{j=1}^4 \frac{\prod_{k=1}^3 \Gamma(s_k z_j^{\pm 2};p,q)}{\Gamma(z_i^{\pm 2};p,q)}$$
$$\times \frac{\prod_{k=1}^3 \Gamma(s_k z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)}{\Gamma(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (39)$$

where $|s_k|<1$, $k=1,2,3$, and

$$\kappa_4 = \frac{(p;p)_\infty^4 (q;q)_\infty^4}{2^7 3^2} \prod_{k=1}^3 \Gamma^4(s_k;p,q).$$

In the derivation of this expression we have used the adjoint representation character given in the Appendix and made there the change of variables $z_i \to z_i^2$ corresponding to stretching all root system vectors.

Using similar prescription for the magnetic theory, we find

$$I_M = \kappa_4 \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{\prod_{k=1}^3 \Gamma(s_k y_i^{\pm 1} y_j^{\pm 1};p,q)}{\Gamma(y_i^{\pm 1} y_j^{\pm 1};p,q)} \prod_{j=1}^4 \frac{\prod_{k=1}^3 \Gamma(s_k y_j^{\pm 2};p,q)}{\Gamma(y_i^{\pm 2};p,q)}$$
$$\times \frac{\prod_{k=1}^3 \Gamma(s_k y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)}{\Gamma(y_1^{\pm 1} y_2^{\pm 1} y_3^{\pm 1} y_4^{\pm 1};p,q)} \prod_{j=1}^4 \frac{dy_j}{2\pi \mathrm{i} y_j}. \quad (40)$$

Note that these integrals are the first examples of multiple elliptic hypergeometric integrals defined for the F_4 root system (in [21] the integrals were defined on the $SU(2)$ group and the Weyl group $W(F_4)$ was emerging as a transformation symmetry in the parameter space).

The S-duality conjecture suggests the transformation formula $I_E = I_M$ in the indicated domain of values of parameters. We have checked that the ratio of the kernels of integrals I_E and I_M defines a totally elliptic hypergeometric term, as required. And again, in a remarkable way, this identity is easily established by the change of variables[3]

$$z_1 = \sqrt{y_1 y_2}, \qquad z_2 = \sqrt{\frac{y_1}{y_2}}, \qquad z_3 = \sqrt{y_3 y_4}, \qquad z_4 = \sqrt{\frac{y_3}{y_4}}. \quad (41)$$

This reparametrization is associated with the rotation of the F_4 root system [4]. We see thus validity of the superconformal indices test for this S–duality.

The limit (29) reduces integral (39) to the expression

$$I_{F_4}^{p=s_2=0} = \frac{1}{2^7 3^2} \frac{(q;q)_\infty^4}{(s_1;q)_\infty^4} \int_{\mathbb{T}^4} \prod_{1\le i<j\le 4} \frac{(z_i^{\pm 2} z_j^{\pm 2};q)_\infty}{(s_1 z_i^{\pm 2} z_j^{\pm 2};q)_\infty} \prod_{j=1}^4 \frac{(z_j^{\pm 2};q)_\infty}{(s_1 z_j^{\pm 2};q)_\infty}$$
$$\times \frac{(z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};q)_\infty}{(s_1 z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} z_4^{\pm 1};q)_\infty} \prod_{j=1}^4 \frac{dz_j}{2\pi \mathrm{i} z_j}, \quad (42)$$

[3]This change was suggested to us by S. Razamat.

which admits exact evaluation [29]

$$I_{F_4}^{p=s_2=0} = \frac{(qs_1, qs_1^5, qs_1^7, qs_1^{11}; q)_\infty}{(s_1^2, s_1^6, s_1^8, s_1^{12}; q)_\infty}. \tag{43}$$

6. $SU(N)$ and $SO(2N)$ gauge groups

Consider for completeness superconformal indices for $\mathcal{N}=4$ SYM theories with $SU(N)$ and $SO(2N)$ gauge groups, which are S–self-dual [1].

The superconformal index for the $SU(N)$ theory is

$$I_{SU(N)} = \chi_N \int_{\mathbb{T}^{N-1}} \prod_{1\leq i<j\leq N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{-1} z_j, s_k z_i z_j^{-1}; p, q)}{\Gamma(z_i^{-1} z_j, z_i z_j^{-1}; p, q)} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{44}$$

where $\prod_{j=1}^N z_j = 1$, parameters s_k satisfy the constraints $|s_k| < 1$, $k = 1, 2, 3$, and

$$\chi_N = \frac{(p;p)_\infty^{N-1} (q;q)_\infty^{N-1}}{N!} \prod_{k=1}^3 \Gamma^{N-1}(s_k; p, q).$$

Taking the ratio of the kernel of this integral to itself with different integration variables, one can get the totally elliptic hypergeometric term. However, consequences of this statement are much less informative than in the cases with nontrivial symmetry transformations for integrals.

The limit (29) reduces integral (44) to the expression

$$I_{SU(N)}^{p=s_2=0} = \frac{1}{N!} \frac{(q;q)_\infty^{N-1}}{(s_1;q)_\infty^{N-1}} \int_{\mathbb{T}^{N-1}} \prod_{1\leq i<j\leq N} \frac{(z_i^{-1} z_j, z_i z_j^{-1}; q)_\infty}{(s_1 z_i^{-1} z_j, s_1 z_i z_j^{-1}; q)_\infty} \prod_{j=1}^{N-1} \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{45}$$

which admits exact evaluation [29]

$$I_{SU(N)}^{p=s_2=0} = \prod_{j=1}^{N-1} \frac{(qs_1^j; q)_\infty}{(s_1^{j+1}; q)_\infty}. \tag{46}$$

For $N \to \infty$ this index equals to $(s_1;q)_\infty/(s_1;s_1)_\infty$, which coincides with the reduced form of the $N \to \infty$ asymptotics (after passing from $U(N)$ to $SU(N)$ gauge group) found in [5] from the AdS/CFT correspondence.

The superconformal index for the $SO(2N)$ theory is

$$I_{SO(2N)} = \chi_N \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^{N} \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{47}$$

where $|s_k| < 1$, $k = 1, 2, 3$, and

$$\chi_N = \frac{(p;p)_\infty^{N} (q;q)_\infty^{N}}{2^{N-1} N!} \prod_{k=1}^3 \Gamma^{N}(s_k; p, q).$$

The situation with the total ellipticity condition is similar to the one for (44).

The limit (29) reduces (47) to the integral

$$I_{SO(2N)}^{p=s_2=0} = \frac{1}{2^{N-1} N!} \frac{(q;q)_\infty^{N}}{(s_1;q)_\infty^{N}} \int_{\mathbb{T}^N} \prod_{1\leq i<j\leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(s_1 z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^{N} \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{48}$$

with the exact evaluation formula [29]

$$I_{SO(2N)}^{p=s_2=0} = \frac{(qs_1^{N-1};q)_\infty}{(s_1^N;q)_\infty}\prod_{j=0}^{N-2}\frac{(qs_1^{2j+1};q)_\infty}{(s_1^{2j+2};q)_\infty}. \tag{49}$$

In the same way as for $SP(2N)$ and $SO(2N+1)$ groups, this integral can be obtained from the q-Selberg integral of [30] using special parameter values:

$$b=s_1, \qquad a_{1,2}=\pm 1, \qquad a_{3,4}=\pm\sqrt{q}.$$

7. Exceptional gauge groups E_6, E_7, and E_8

Also for generality, we describe superconformal indices for $\mathcal{N}=4$ SYM theories with the exceptional E_6, E_7, and E_8 gauge groups.

E_6 **gauge group.** For the first representative of these theories we have the superconformal index of the form

$$I_{E_6}=\kappa_6\int_{\mathbb{T}^6}\prod_{j=1}^{6}\frac{dz_j}{2\pi \mathrm{i} z_j}\prod_{1\le i<j\le 5}\frac{\prod_{k=1}^3\Gamma(s_kz_i^{\pm2}z_j^{\pm2};p,q)}{\Gamma(z_i^{\pm2}z_j^{\pm2};p,q)}\frac{\prod_{k=1}^3\Gamma(s_k(z_6^3Z)^{\pm1};p,q)}{\Gamma((z_6^3Z)^{\pm1};p,q)}$$
$$\times\prod_{1\le i<j\le 5}\frac{\prod_{k=1}^3\Gamma(s_k(z_6^3z_i^2z_j^2Z)^{\pm1};p,q)}{\Gamma((z_6^3z_i^2z_j^2Z)^{\pm1};p,q)}\prod_{i=1}^{5}\frac{\prod_{k=1}^3\Gamma(s_k(z_6^{-3}z_i^2Z)^{\pm1};p,q)}{\Gamma((z_6^{-3}z_i^2Z)^{\pm1};p,q)}, \tag{50}$$

where for convenience we denoted $Z=(z_1z_2z_3z_4z_5)^{-1}$ and

$$\kappa_6 = \frac{(p;p)_\infty^6(q;q)_\infty^6}{2^73^45}\prod_{k=1}^{3}\Gamma^6(s_k;p,q).$$

The combinatorial factors appearing here are the same as, for example, those given in [29]. Similar to the F_4-group case, we took the adjoint representation character given in the Appendix and replaced in it $z_j\to z_j^2$ (the same was done for the E_7 and E_8 group cases considered below).

The limit (29) reduces (50) to the integral

$$I_{E_6}^{p=s_2=0}=\frac{1}{2^73^45}\frac{(q;q)_\infty^6}{(s_1;q)_\infty^6}\int_{\mathbb{T}^6}\prod_{j=1}^{6}\frac{dz_j}{2\pi \mathrm{i} z_j}\prod_{1\le i<j\le 5}\frac{(z_i^{\pm2}z_j^{\pm2};q)_\infty}{(s_1z_i^{\pm2}z_j^{\pm2};q)_\infty}$$
$$\times\frac{((z_6^3Z)^{\pm1};q)_\infty}{(s_1(z_6^3Z)^{\pm1};q)_\infty}\prod_{1\le i<j\le 5}\frac{((z_6^3z_i^2z_j^2Z)^{\pm1};q)_\infty}{(s_1(z_6^3z_i^2z_j^2Z)^{\pm1};q)_\infty}\prod_{i=1}^{5}\frac{((z_6^{-3}z_i^2Z)^{\pm1};q)_\infty}{(s_1(z_6^{-3}z_i^2Z)^{\pm1};q)_\infty}, \tag{51}$$

which can be computed explicitly using a q-hypergeometric constant term evaluation formula valid for arbitrary reduced root system (see, e.g., [29])

$$I_{E_6}^{p=s_2=0} = \frac{(qs_1,qs_1^4,qs_1^5,qs_1^7,qs_1^8,qs_1^{11};q)_\infty}{(s_1^2,s_1^5,s_1^6,s_1^8,s_1^9,s_1^{12};q)_\infty}. \tag{52}$$

E_7 **gauge group.** For $\mathcal{N}=4$ SYM theory with the E_7 gauge group the superconformal index is

$$I_{E_7}=\kappa_7\int_{\mathbb{T}^7}\prod_{j=1}^{6}\frac{\prod_{k=1}^3\Gamma(s_kz_7^{\pm2}(z_j^2Z)^{\pm1};p,q)}{\Gamma(z_7^{\pm2}(z_j^2Z)^{\pm1};p,q)}\prod_{1\le i<j\le 6}\frac{\prod_{k=1}^3\Gamma(s_kz_i^{\pm2}z_j^{\pm2};p,q)}{\Gamma(z_i^{\pm2}z_j^{\pm2};p,q)}$$
$$\times\frac{\prod_{k=1}^3\Gamma(s_kz_7^{\pm4};p,q)}{\Gamma(z_7^{\pm4};p,q)}\prod_{1\le i<j<l\le 6}\frac{\prod_{k=1}^3\Gamma(s_kz_7^{\pm2}z_i^2z_j^2z_l^2Z;p,q)}{\Gamma(z_7^{\pm2}z_i^2z_j^2z_l^2Z;p,q)}\prod_{j=1}^{7}\frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{53}$$

where we denoted $Z=(z_1z_2z_3z_4z_5z_6)^{-1}$ and

$$\kappa_7 \;=\; \frac{(p;p)_\infty^7(q;q)_\infty^7}{2^{10}3^4 5\, 7}\prod_{k=1}^3\Gamma^7(s_k;p,q).$$

The limit (29) reduces (53) to the integral

$$I_{E_7}^{p=s_2=0}=\frac{1}{2^{10}3^4 5\cdot 7}\frac{(q;q)_\infty^7}{(s_1;q)_\infty^7}\int_{\mathbb{T}^7}\prod_{j=1}^6\frac{(z_7^{\pm2}(z_j^2Z)^{\pm1};q)_\infty}{(s_1z_7^{\pm2}(z_j^2Z)^{\pm1};q)_\infty}\prod_{1\leq i<j\leq 6}\frac{(z_i^{\pm2}z_j^{\pm2};q)_\infty}{(s_1z_i^{\pm2}z_j^{\pm2};q)_\infty}$$
$$\times\frac{(z_7^{\pm4};q)_\infty}{(s_1z_7^{\pm4};q)_\infty}\prod_{1\leq i<j<l\leq 6}\frac{(z_7^{\pm2}z_i^2z_j^2z_l^2Z;q)_\infty}{(s_1z_7^{\pm2}z_i^2z_j^2z_l^2Z;q)_\infty}\prod_{j=1}^7\frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{54}$$

which can be evaluated explicitly [29]

$$I_{E_7}^{p=s_2=0} \;=\; \frac{(qs_1,qs_1^5,qs_1^7,qs_1^9,qs_1^{11},qs_1^{13},qs_1^{17};q)_\infty}{(s_1^2,s_1^6,s_1^8,s_1^{10},s_1^{12},s_1^{14},s_1^{18};q)_\infty}. \tag{55}$$

E_8 **gauge group.** Finally, for the largest exceptional gauge group E_8 theory the superconformal index is

$$I_{E_8}=\kappa_8\int_{\mathbb{T}^8}\prod_{j=1}^8\frac{dz_j}{2\pi \mathrm{i} z_j}\prod_{1\leq i<j\leq 8}\frac{\prod_{k=1}^3\Gamma(s_k(z_i^2z_j^2Z)^{\pm1};p,q)}{\Gamma((z_i^2z_j^2Z)^{\pm1};p,q)}\frac{\prod_{k=1}^3\Gamma(s_kZ^{\pm1};p,q)}{\Gamma(Z^{\pm1};p,q)} \tag{56}$$
$$\times\prod_{1\leq i<j\leq 8}\frac{\prod_{k=1}^3\Gamma(s_kz_i^{\pm2}z_j^{\pm2};p,q)}{\Gamma(z_i^{\pm2}z_j^{\pm2};p,q)}\prod_{1\leq i<j<l<m\leq 8}\frac{\prod_{k=1}^3\Gamma(s_k(z_i^2z_j^2z_l^2z_m^2Z)^{\pm1};p,q)}{\Gamma((z_i^2z_j^2z_l^2z_m^2Z)^{\pm1};p,q)},$$

where $Z=(z_1z_2z_3z_4z_5z_6z_7z_8)^{-1}$ and

$$\kappa_8 \;=\; \frac{(p;p)_\infty^8(q;q)_\infty^8}{2^{14}3^5 5^2 7}\prod_{k=1}^3\Gamma^8(s_k;p,q).$$

Again, the limit (29) reduces (56) to the integral

$$I_{E_8}^{p=s_2=0}=\frac{1}{2^{14}3^5 5^2 7}\frac{(q;q)_\infty^8}{(s_1;q)_\infty^8}\int_{\mathbb{T}^8}\prod_{j=1}^8\frac{dz_j}{2\pi \mathrm{i} z_j}\prod_{1\leq i<j\leq 8}\frac{((z_i^2z_j^2Z)^{\pm1};q)_\infty}{(s_1(z_i^2z_j^2Z)^{\pm1};q)_\infty} \tag{57}$$
$$\times\frac{(Z^{\pm1};q)_\infty}{(s_1Z^{\pm1};q)_\infty}\prod_{1\leq i<j\leq 8}\frac{(z_i^{\pm2}z_j^{\pm2};q)_\infty}{(s_1z_i^{\pm2}z_j^{\pm2};q)_\infty}\prod_{1\leq i<j<l<m\leq 8}\frac{((z_i^2z_j^2z_l^2z_m^2Z)^{\pm1};q)_\infty}{(s_1(z_i^2z_j^2z_l^2z_m^2Z)^{\pm1};q)_\infty},$$

which can be evaluated exactly [29]

$$I_{E_8}^{p=s_2=0} \;=\; \frac{(qs_1,qs_1^7,qs_1^{11},qs_1^{13},qs_1^{17},qs_1^{19},qs_1^{23},qs_1^{29};q)_\infty}{(s_1^2,s_1^8,s_1^{12},s_1^{14},s_1^{18},s_1^{20},s_1^{24},s_1^{30};q)_\infty}. \tag{58}$$

In all three integrals (50), (53), and (56) we assumed the restrictions $|s_k|<1$, $k=1,2,3$. As expected, ratios of their kernels to themselves with different integration variables yield totally elliptic hypergeometric terms. These integrals represent first known examples of elliptic hypergeometric integrals based on the exceptional root systems of E–type.

8. Some special $\mathcal{N}=1$ and $\mathcal{N}=2$ dualities

Much attention is paid in this paper to supersymmetric theories with the exceptional gauge groups. Therefore we would like to describe one more duality example for such theories known to us. We take $\mathcal{N}=1$ SYM theory with E_6 gauge group and matter fields given by 6 flavors in the fundamental representation of $SU(6)$ flavor group and in 27-dimensional representation of the gauge group E_6.

This electric theory and its particular magnetic dual were suggested in [26, 27] and validity of this duality was discussed further in [31]. The superconformal index for the electric theory has the form

$$I_E = \kappa_6 \int_{\mathbb{T}^6} \prod_{1\le i<j\le 5} \frac{\prod_{k=1}^6 \Gamma(s_k z_6^{-1} Z z_i^2 z_j^2; p,q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p,q)} \frac{\prod_{k=1}^6 \Gamma(s_k z_6^{-4}, s_k z_6^{-1} Z; p,q)}{\Gamma((z_6^3 Z)^{\pm 1}; p,q)} \tag{59}$$

$$\times \prod_{1\le i<j\le 5} \frac{1}{\Gamma((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p,q)} \prod_{i=1}^5 \frac{\prod_{k=1}^6 \Gamma(s_k z_6^2 z_i^{\pm 2}, s_k z_6^{-1} Z^{-1} z_i^{-2}; p,q)}{\Gamma((z_6^{-3} z_i^2 Z)^{\pm 1}; p,q)} \prod_{j=1}^6 \frac{dz_j}{2\pi \mathrm{i} z_j},$$

where $|s_k|<1$, $k=1,\ldots,6$, we denoted $Z=(z_1z_2z_3z_4z_5)^{-1}$ and

$$\kappa_6 \;=\; \frac{(p;p)_\infty^6 (q;q)_\infty^6}{2^7 3^4 5}.$$

The magnetic theory is described by 6 antifundamentals of the flavor group lying in 27-dimensional representation of the gauge group. There are also the singlet mesons given by the absolute symmetric representation of the third rank of the flavor group. The magnetic superconformal index is

$$I_M = \kappa_6 \prod_{j=1}^6 \Gamma(s_j^3; p,q) \prod_{i,j=1;\, i\neq j}^6 \Gamma(s_i s_j^2; p,q) \int_{\mathbb{T}^6} \prod_{1\le i<j\le 5} \frac{1}{\Gamma((z_6^3 z_i^2 z_j^2 Z)^{\pm 1}; p,q)}$$

$$\times \prod_{1\le i<j\le 5} \frac{\prod_{k=1}^6 \Gamma(S^{\frac13} s_k^{-1} z_6^{-1} Z z_i^2 z_j^2; p,q)}{\Gamma(z_i^{\pm 2} z_j^{\pm 2}; p,q)} \frac{\prod_{k=1}^6 \Gamma(S^{\frac13} s_k^{-1} z_6^{-4}, S^{\frac13} s_k^{-1} z_6^{-1} Z; p,q)}{\Gamma((z_6^3 Z)^{\pm 1}; p,q)}$$

$$\times \prod_{i=1}^5 \frac{\prod_{k=1}^6 \Gamma(S^{\frac13} s_k^{-1} z_6^2 z_i^{\pm 2}, S^{\frac13} s_k^{-1} z_6^{-1} Z^{-1} z_i^{-2}; p,q)}{\Gamma((z_6^{-3} z_i^2 Z)^{\pm 1}; p,q)} \prod_{j=1}^6 \frac{dz_j}{2\pi \mathrm{i} z_j}, \tag{60}$$

where $|s_k|<1$, $k=1,\ldots,6,$. The balancing condition for both elliptic hypergeometric integrals has the form $S=\prod_{i=1}^6 s_i = pq$.

We have checked that the ratio of these integral kernels yields a totally elliptic hypergeometric term, which is an important test suggesting that these dualities and the equality $I_E = I_M$ might be valid. It is interesting to note that the limit $s_6 \to 1$ reduces the integrals in such a way, that one obtains superconformal indices of peculiar E_6 and F_4 SYM theories dual to each other [27].

As the last but not least remark and an additional advertisement of the applications of the elliptic hypergeometric integrals techniques, we would like to present the superconformal index of a particular $\mathcal{N}=2$ quiver SYM theory described in

[32]. Define

$$
\begin{aligned}
I_E \;=\;& \frac{(p;p)_\infty^6 (q;q)_\infty^6}{8} \int_{\mathbb{T}} \frac{dx}{2\pi \mathrm{i} x} \int_{\mathbb{T}} \frac{dy}{2\pi \mathrm{i} y} \int_{\mathbb{T}^2} \prod_{j=1}^{2} \frac{dz_j}{2\pi \mathrm{i} z_j} \int_{\mathbb{T}} \frac{dr}{2\pi \mathrm{i} r} \int_{\mathbb{T}} \frac{dw}{2\pi \mathrm{i} w} \\
& \times \frac{\Gamma(t^2 v x^{\pm 1}, t^2 v y^{\pm 2}, t^2 v z_1^{\pm 1} z_2^{\pm 1}, t^2 v r^{\pm 2}, t^2 v w^{\pm 1}; p, q)}{\Gamma(x^{\pm 1}, y^{\pm 2}, z_1^{\pm 1} z_2^{\pm 1}, r^{\pm 2}, w^{\pm 1}; p, q)} \\
& \times \Gamma\Big(\frac{t^2}{\sqrt{v}} y^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1}; p, q\Big)^2 \Gamma\Big(\frac{t^2}{\sqrt{v}} x^{\pm 1} y^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} w^{\pm 1}; p, q\Big) \\
& \times \prod_{j=1}^{2} \Gamma\Big(\frac{t^2}{\sqrt{v}} y^{\pm 1} z_j^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} z_j^{\pm 1}; p, q\Big), \qquad (61)
\end{aligned}
$$

where t is the same parameter as in $\mathcal{N}=4$ theories before and v is the chemical potential associated with some combination of the $U(2)_R$-group commuting R-charges. Introducing the variables $\alpha^2 = z_1 z_2,\ \beta^2 = z_1/z_2,\ \gamma^2 = x$ and $\delta^2 = w$, one can rewrite this integral as

$$
\begin{aligned}
I_M \;=\;& \frac{(p;p)_\infty^6 (q;q)_\infty^6}{64} \int_{\mathbb{T}} \frac{d\gamma}{2\pi \mathrm{i} \gamma} \int_{\mathbb{T}} \frac{dy}{2\pi \mathrm{i} y} \int_{\mathbb{T}} \frac{d\alpha}{2\pi \mathrm{i} \alpha} \int_{\mathbb{T}} \frac{d\beta}{2\pi \mathrm{i} \beta} \int_{\mathbb{T}} \frac{dr}{2\pi \mathrm{i} r} \int_{\mathbb{T}} \frac{d\delta}{2\pi \mathrm{i} \delta} \\
& \times \frac{\Gamma(t^2 v \gamma^{\pm 2}, t^2 v y^{\pm 2}, t^2 v \alpha^{\pm 2}, t^2 v \beta^{\pm 2}, t^2 v r^{\pm 2}, t^2 v \delta^{\pm 2}; p, q)}{\Gamma(\gamma^{\pm 2}, y^{\pm 2}, \alpha^{\pm 2}, \beta^{\pm 2}, r^{\pm 2}, \delta^{\pm 2}; p, q)} \qquad (62) \\
& \times \Gamma\Big(\frac{t^2}{\sqrt{v}} \gamma^{\pm 1} \gamma^{\pm 1} y^{\pm 1}, \frac{t^2}{\sqrt{v}} \delta^{\pm 1} \delta^{\pm 1} r^{\pm 1}, \frac{t^2}{\sqrt{v}} y^{\pm 1} \alpha^{\pm 1} \beta^{\pm 1}, \frac{t^2}{\sqrt{v}} r^{\pm 1} \alpha^{\pm 1} \beta^{\pm 1}; p, q\Big).
\end{aligned}
$$

The identity $I_E = I_M$ can be interpreted as the equality of superconformal indices following from a relation between particular $SO(4) \times SP(2)$ and $SU(2)$ $\mathcal{N}=2$ SYM generalized quiver theories. Although this is not the intrinsic electric-magnetic duality, we keep this terminology for indices. The "electric" part is an $SO(3) \times SP(2) \times SO(4) \times SP(2) \times SO(3)$ $\mathcal{N}=2$ SYM quiver and the "magnetic" part is the same theory rewritten as $SU(2)^6$-quiver, as illustrated in Fig. 9 of [32].

9. Conclusion

In this paper we have described superconformal indices for $\mathcal{N}=4$ SYM theories with simple non-Abelian gauge groups as elliptic hypergeometric integrals and analyzed some of their mathematical properties. In the case of G_2 and F_4 groups the equality of indices of S-dual theories follows from a simple change of variables in integrals which gives an additional test of these dualities.

For all classical simple gauge groups we have found particular limiting values of chemical potentials ($p \to 0$ followed by the $s_2 \to 0$ limit) for which $\mathcal{N}=4$ indices are computable exactly. According to the general ideology [7, 11, 17], exact computability of indices is associated with the confinement in the dual phase of the theory, since it provides a group-theoretical representation of indices without local gauge group symmetry. Therefore we conclude that there should exist some interesting supersymmetric (expectedly, three dimensional) field theories similar to the Wess-Zumino model whose superconformal indices are described by the right-hand sides of equalities (30), (32), (37), (42), (45), (48), (51), (54), and (57).

One of the initial motivations for consideration of superconformal indices in [5] was an analysis of the AdS/CFT correspondence for $\mathcal{N}=4$ SYM theory for $SU(N)$ gauge group which required consideration of the $N \to \infty$ limit. In this limit, the original index coming from the BPS states not forming long multiplets can be computed from the dual spectrum of gravitons appearing in the Type IIB supergravity compactified on $AdS_5 \times S^5$. It would be interesting to understand the meaning of the reduction $p \to 0$ from the AdS/CFT point of view on the level of graviton spectra in "parent" four dimensional theories. All our $p = s_2 = 0$ indices for gauge groups of arbitrary rank N are well defined in the limit $N \to \infty$ for $|s_1| < 1$ being given by curious explicit infinite products. So, we expect that there will be an essential simplification in the consideration of the corresponding gravitational duals for both finite and infitite N.

In [33, 34] the marginal deformations of superconformal field theories were studied and the importance of global symmetries for conformal manifold (a manifold of coupling constants of the theory where it stays conformal) is shown. A β-deformation of the $\mathcal{N}=4$ SYM theory [35] is obtained by introduction of a marginal deformation of the superpotential, which breaks $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$,

$$h\mathrm{Tr}\left(e^{\mathrm{i}\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-\mathrm{i}\pi\beta}\Phi_1\Phi_3\Phi_2\right),$$

where β is an arbitrary parameter and h is a Yukawa coupling. The parameter β may be complex and this does not break superconformal invariance of the theory [36]. The initial R-symmetry $SU(4)_R$ breaks to $U(1)_R$ with the additional global symmetry $U(1)_1 \times U(1)_2$ [35]. From the indices point of view the chemical potentials v and w introduced at the beginning play now the role of chemical potentials for the latter global group. As pointed to us by J. Maldacena, superconformal index for the β-deformed theory is the same as in the initial theory [5]. This means that these theories share essentially the same set of BPS states. In conclusion of [17] we discussed appearance of the $SO(3)$ $\mathcal{N}=4$ SYM theory from a $\mathcal{N}=1$ model after a superpotential deformation, such that both theories share the same superconformal index. Actually, superconformal indices of all exactly marginally deformed theories coincide, only the interpretation of chemical potentials is different, being tied to global groups of different meaning. Therefore these indices serve as invariants of the conformal manifold with their structure reflecting only a part of the global symmetries preserved by the superpotential.

As an example of different deformation of $\mathcal{N}=4$ theories we can mention the deformation to $\mathcal{N}=1$ SYM theory with two chiral superfields in the adjoint representation and an additional $U(1)$ global group (see [37] and references therein). This theory has an $SL(2,\mathbb{Z})$ electric-magnetic duality inherited from $\mathcal{N}=4$ SYM theory in its infrared fixed point. From the superconformal indices techniques viewpoint such a deformation is traced in a very simple way — it is just necessary to give a special value to one of the s_k-parameters, say, $s_3 = \sqrt{pq}$, in our integrals, which removes it completely.

The q-beta integrals with exact evaluations appearing from superconformal indices of all $\mathcal{N}=4$ SYM theories in the limit $p \to 0$, $s_2 \to 0$ are known to determine orthogonality measures for special cases of Macdonald and Koornwinder q-orthogonal polynomials (for E_6, E_7, and E_8 root systems these measures are generic [29]). We come thus to a natural question on whether one can give a similar meaning to general elliptic hypergeometric integrals describing $\mathcal{N}=4$ superconformal indices and construct corresponding biorthogonal functions (the first example of such biorthogonal functions in the univariate case has been found in [13] and for a particular $SP(2N)$-integral their multivariable generalization has been constructed in [15]). Note that for the exceptional root systems G_2, F_4, E_6, E_7, E_8 the indices define currently first examples of integrals at the elliptic hypergeometric level pretending to such a role.

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Appendix A. Characters of adjoint representations

Here we would like to list characters of the adjoint representations for all simple Lie groups G depending on complex variables z_j, $j = 1, \ldots, \texttt{rank}\, G$.

For $SU(N)$ group we have $N-1$ independent variables z_j and

$$\chi_{SU(N),adj}(z_1, \ldots, z_N) = \sum_{1\leq i<j\leq N} (z_i z_j^{-1} + z_i^{-1} z_j) + N - 1, \tag{63}$$

where $\prod_{j=1}^N z_j = 1$.

For $SO(2N+1)$ group of rank N the character is (no constraints on z_j)

$$\chi_{SO(2N+1),adj}(z) = \sum_{1\leq i<j\leq N} (z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1}) + \sum_{i=1}^N (z_i + z_i^{-1}) + N. \tag{64}$$

For $SP(2N)$ group of rank N the character is

$$\chi_{SP(2N),adj}(z) = \sum_{1\leq i<j\leq N} (z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1}) + \sum_{i=1}^N (z_i^2 + z_i^{-2}) + N. \tag{65}$$

For $SO(2N)$ group of rank N the character is

$$\chi_{SO(2N),adj}(z) = \sum_{1\leq i<j\leq N} (z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1}) + N. \tag{66}$$

The character for the adjoint representation of G_2 group is the symmetric polynomial of two parameters z_1 and z_2, but it is convenient to introduce the third variable using relation $z_1 z_2 z_3 = 1$. Then,

$$\chi_{G_2,adj}(z_1, z_2, z_3) = 2 + \sum_{1\leq i<j\leq 3} \left(z_i z_j + z_i^{-1} z_j + z_i z_j^{-1} + z_i^{-1} z_j^{-1} \right).$$

The exceptional F_4 group has rank four and the corresponding character is

$$\chi_{F_4,adj}(z_1, \ldots, z_4) = \sum_{i=1}^4 (z_i + z_i^{-1}) + \sum_{1\leq i<j\leq 4} (z_i z_j + z_i z_j^{-1} + z_i^{-1} z_j + z_i^{-1} z_j^{-1}) + (z_1^{1/2} + z_1^{-1/2})(z_2^{1/2} + z_2^{-1/2})(z_3^{1/2} + z_3^{-1/2})(z_4^{1/2} + z_4^{-1/2}) + 4. \tag{67}$$

Description of the root systems for the $E_{6,7,8}$ exceptional Lie groups can be found in [38]. The rank of E_6 group is equal to six and the character for the adjoint representation is

$$\chi_{E_6,adj}(z_1,\ldots,z_6) = 6 + \sum_{1\leq i<j\leq 5}(z_iz_j + z_i^{-1}z_j + z_iz_j^{-1} + z_i^{-1}z_j^{-1})$$
$$+ z_6^{3/2}\left(\prod_{i=1}^{5} z_i\right)^{-1/2}\left(1 + \sum_{1\leq i<j\leq 5} z_iz_j + \sum_{1\leq i<j<k<l\leq 5} z_iz_jz_kz_l\right) \tag{68}$$
$$+ z_6^{-3/2}\left(\prod_{i=1}^{5} z_i\right)^{1/2}\left(1 + \sum_{1\leq i<j\leq 5}(z_iz_j)^{-1} + \sum_{1\leq i<j<k<l\leq 5}(z_iz_jz_kz_l)^{-1}\right).$$

The rank of E_7 group is equal to seven and the needed character is

$$\chi_{E_7,adj}(z_1,\ldots,z_7) = 7 + \sum_{1\leq i<j\leq 6}(z_iz_j + z_i^{-1}z_j + z_iz_j^{-1} + z_i^{-1}z_j^{-1}) + z_7^2 + z_7^{-2} \tag{69}$$
$$+ (z_7 + z_7^{-1})\left(\left(\prod_{i=1}^{6} z_i\right)^{1/2}\sum_{i=1}^{6} z_i^{-1} + \left(\prod_{i=1}^{6} z_i\right)^{-1/2}\left(\sum_{i=1}^{6} z_i + \sum_{1\leq i<j<k\leq 6} z_iz_jz_k\right)\right).$$

The E_8 group is the biggest exceptional Lie group, it has rank eight and the character for the adjoint representation is

$$\chi_{E_8,adj}(z_1,\ldots,z_8) = 8 + \sum_{1\leq i<j\leq 8}(z_iz_j + z_i^{-1}z_j + z_iz_j^{-1} + z_i^{-1}z_j^{-1})$$
$$+ \prod_{i=1}^{8} z_i^{1/2}\left(1 + \sum_{1\leq i<j\leq 8}(z_iz_j)^{-1} + \sum_{1\leq i<j<k<l\leq 8}(z_iz_jz_kz_l)^{-1}\right)$$
$$+ \prod_{i=1}^{8} z_i^{-1/2}\left(1 + \sum_{1\leq i<j\leq 8} z_iz_j\right). \tag{70}$$

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