ON THOM SPACES, MASSEY PRODUCTS AND NON-FORMAL SYMPLECTIC MANIFOLDS

Yuli Rudyak and Aleksy Tralle

July 6, 1999

Abstract. We suggest a simple general method of constructing of non-formal manifolds. In particular, we construct a large family of non-formal symplectic manifolds. Here we detect non-formality via non-triviality of rational Massey products. In fact, we analyze the behavior of Massey products of closed manifolds under the blow-up construction. In this context Thom spaces play the role of a technical tool which allows us to construct non-trivial Massey products in an elegant way.

Introduction

In this paper we suggest a simple general method of constructing of non-formal manifolds. In particular, we construct a large family of non-formal symplectic manifolds. Here we detect non-formality via non-triviality of rational Massey products. In this context Thom spaces play the role of a technical tool which allows us to construct non-trivial Massey products in an elegant way, Lemmas 3.4 and 3.5.

In greater detail, we analyze the behavior of Massey products of closed manifolds under the blow-up construction. The knowledge of various homotopic properties of blow-ups is important in many areas and, in particular, in symplectic geometry [Go, M, MS, BT, TO]. In the article we present several results (Theorems 4.4, 4.5 and 5.2) which can be regarded as a qualitative description of what happens to the non-formality under the blow-up. More precisely, the non-triviality of Massey products in the ruled submanifold yields a non-trivial Massey products in the resulting manifold.

It is clear that the blow-up procedure enables us to construct a large class of manifolds and, on the other hand, many classes of manifolds (complex, Kähler, symplectic, etc) are invariant with respect to blow-up. Because of this, our approach turns out to be useful in constructing of non-formal manifolds with certain structures or other prescribed properties.

For example, these results have a nice application to symplectic topology. We suggest a simple way to construct a large family of new examples of closed symplectic non-formal (and, hence, non-Kähler) manifolds, including new simply-connected examples. Note that the problem of constructing symplectic closed non-Kähler manifolds (the *Weinstein-Thurston problem*) was and still remains of substantial interest in symplectic geometry [BT, CFG, FMG, FLS, Go], see [TO] for a detailed survey. It is also worth mentioning that usually the blow-up procedure does not

1991 *Mathematics Subject Classification.* Primary 55S30, 53C15, Secondary 55P62, 57R19.

change the fundamental group of ambient manifold(s), and so our approach is applicable to manifolds with arbitrary fundamental groups and, in particular, can be used in order to produce families of simply-connected non-formal symplectic manifolds (this case in known to be the most difficult, this is the *Lupton-Oprea problem* [LO,TO]) and the first simply connected examples have very recently appeared in [BT].

As another example, we mention that our method enables us to construct a large family of algebraic varieties and Kähler manifolds with non-trivial Massey products in $\mathbb{Z}/p$-cohomology. (Such examples were already known, [E], we just emphasize that our method yields a simple construction of a large family.)

It seems that this nice (and a bit surprising) application of Thom spaces is conceptually interesting in its own right. We want also to mention that our research was originally initiated by ideas of Gitler[G], who have used Thom spaces in studying of homology of (complex) blow-ups.

We use the term "F-fibration" for any (Hurewicz) fibration with the (homotopy) fiber F. Also, when we write "an F-fibration $F \to E \to B$", it means that $E \to B$ is a fibration with the (homotopy) fiber F.

Throughout the paper we fix a commutative ring R with the unit, and $H^*(X)$ always denotes the singular cohomology $H^*(X;R)$ unless something other is said explicitly.

Acknowledgment. The authors are grateful to Ivan Babenko, Hans Baues, Sam Gitler, Manfred Stälzer and Iskander Taimanov for valuable discussions.

This research was supported by the Max Planck Institut für Mathematik and by the Polish Research Committee (KBN).

1. Preliminaries on Massey products

Throughout the section we fix a differential graded associative algebra (A,d). We denote by $H(A)$ the cohomology ring of (A,d). Given an element $a \in A$ with $da = 0$, we denote by $[a]$ the cohomology class of a. So, $[a] \in H(A)$.

Given a homogeneous element $a \in A$, we set $\overline{a} = (-1)^{|a|}a$.

1.1. Definition ([K]). Given homogeneous elements $\alpha_1, \dots, \alpha_n \in H(A)$, a *defining system* for $\alpha_1, \dots, \alpha_n$ is a family $\mathcal{X} = \{x_{ij}\}$ of elements of A, $1 \leq i < j \leq n+1$, with the following properties:

(1) $[x_{i,i+1}] = \alpha_i$ for every i;

(2) $dx_{ij} = \displaystyle\sum_{r=i+1}^{j-1} \overline{x}_{ir}x_{rj}$ if $i+1 < j < n+1$.

Consider the element $c(\mathcal{X}) := \displaystyle\sum_{r=2}^{n} \overline{x}_{1r}x_{r,n+1}$. One can prove that it is a cocycle, and so we have the class $[c(\mathcal{X})] \in H^*(\mathcal{A})$. We define

$$\langle \alpha_1, \dots, \alpha_n \rangle := \big\{[c(\mathcal{X})] \bigm| \mathcal{X} \text{ runs over all defining systems}\big\} \subset H(A).$$

The family $\langle \alpha_1, \dots, \alpha_n \rangle$ is called the *Massey n-tuple product* of $\alpha_1, \dots, \alpha_n$.

The *indeterminacy* of the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is the subgroup

$$\mathrm{Indet}\langle V_1, \dots, V_n \rangle := \{x - y \mid x, y \in \langle V_1, \dots, V_n \rangle\}.$$

of $H^*(X)$. So, $\langle \alpha_1, \dots, \alpha_n \rangle$ is contained in a coset with respect to Indet.

We say that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is *trivial* if $0 \in \langle \alpha_1, \dots, \alpha_n \rangle$, i.e., if there is a defining system $\mathcal{X}$ with $[c(\mathcal{X})] = 0$; otherwise we say that the Massey product is *non-trivial*.

For conveniences of references we fix the following proposition. The proof follows from the definition directly.

1.2. Proposition. *Let the classes $\alpha_1, \dots, \alpha_n \in H(A)$ be such that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined. Then for every morphism $f : (A, d) \to (A', d')$ of DG-algebras the Massey product $\langle f_*\alpha_1, \dots, f_*\alpha_n \rangle$ is defined, and $f_*\langle \alpha_1, \dots, \alpha_n \rangle \subset \langle f_*\alpha_1, \dots, f_*\alpha_n \rangle$. In particular, if $0 \notin \langle f_*\alpha_1, \dots, f_*\alpha_n \rangle$ then $0 \notin \langle \alpha_1, \dots, \alpha_n \rangle$.* □

Now let us consider a special case of Massey *triple* products $\langle \alpha, \beta, \gamma \rangle$. The above definition leads to the following description. Let $a, b, c \in A$ be such that $\alpha = [a], \beta = [b]$ and $\gamma = [c]$. Consider $x, y \in A$ such that $dx = \overline{a}b$ and $dy = \overline{b}c$. Then $\langle \alpha, \beta, \gamma \rangle$ consists of all classes of the form $[\overline{a}y + \overline{x}c]$. Furthermore, the indeterminacy of $\langle \alpha, \beta, \gamma \rangle$ is the set of elements of the form $\alpha u + v\beta$ where $u, v \in H^*(A)$ are arbitrary elements with $|\alpha u| = |v\beta| = |\alpha| + |\beta| + |\gamma| - 1$. In other words, $\langle \alpha, \beta, \gamma \rangle$ is trivial if there are x, y as above with $[\overline{a}x + \overline{x}c] \in (\alpha, \beta)$.

There are three important examples. First, given a space X and a commutative ring (associative and possessing the unit) R, let $C^*(X; R)$ be the singular cochain complex of X. We equip $C^*(X; R)$ with the standard (Alexander–Whitney) associative cup product pairing. Then $(C^*(X; R), d)$ turns into an associative differential graded algebra. Now, according to what we said above, we can consider Massey products $\langle \alpha_1, \dots, \alpha_n \rangle$ where $\alpha_i \in H^*(X; R)$.

Sullivan minimal models give us the second example. In greater details, if X is a simply connected (or, more generally, nilpotent) space then there is a natural commutative differential graded algebra $(\mathcal{M}_X, d)$ *over the field of rational numbers* $\mathbb{Q}$ which is a homotopic invariant and which completely determines the rational homotopy type of X, see [L].

A space X is called *formal*, if there exists a DGA-morphism

$$\rho : (\mathcal{M}_X, d) \to (H^*(X), 0)$$

inducing isomorphism on the cohomology level. Formality is an important homotopic property, since the rational homotopy type of any formal space can be reconstructed by some "formal" procedure from its cohomology algebra. Kähler manifolds are formal [DGMS]. Examples of formal and non-formal manifolds occurring in various geometric situations can be found in [TO]. However, not every space is formal. It is known [DGMS, TO] that a formal space does not possess any non-trivial rational Massey products. In other words, a space X is not formal if we can find a non-trivial Massey product in $H^*(X; \mathbb{Q})$.

The third example is the de Rham algebra $((DR(X), d)$ of differential forms on a smooth manifold X. Then $H(DR(X), d) = H^*(X; \mathbb{R})$, and we can compute the corresponding Massey products in $H^*(X; \mathbb{R})$.

It is easy to see that the Sullivan model (tensored by $\mathbb{R}$) and the de Rham complex lead to the same Massey triple products because both DGA-algebras are commutative, see [ABCKT, Lemma 3.28]. Unexpectedly, we cannot prove that $C^*(X; \mathbb{R})$ leads to the same Massey product that $DR(X)$ does. However, all the theorems are true if you keep the algebra throughout the theorem. Certainly, when we talk about formality, the reader must keep in mind the Sullivan model or de Rham algebra.

2. $\mathbb{C}P^k$-fibrations

Recall that, for every map $p : E \to B$, the pairing

$$H^*(E) \otimes H^*(B) \to H^*(E), \quad a \otimes \lambda \mapsto ap^*(\lambda)$$

turns $H^*(E)$ into a (right) $H^*(B)$-module.

2.1. Theorem. *Let*

$$\mathbb{C}P^k \xrightarrow{j} E \xrightarrow{p} B$$

be a $\mathbb{C}P^k$-fibration, and let $\xi \in H^2(E)$ be an element such that $j^*(\xi) \neq 0 \in H^2(\mathbb{C}P^n)$. Then $H^*(E)$ is a free $H^*(B)$-module with generators $1, \xi, \dots, \xi^k$.

Proof. We set $\omega = j^*\xi \in H^2(\mathbb{C}P^k)$, $\omega \neq 0$. It is well known that ω generates the $H^*(\mathrm{pt})$-algebra $H^*(\mathbb{C}P^k)$, and so the homomorphism $j^* : H^*(E) \to H^*(\mathbb{C}P^k)$ is an epimorphism. Now the result follows from the Leray–Hirsh Theorem (see e.g. [S, Theorem 15.47]) because $H^*(\mathbb{C}P^k)$ is a free $H^*(\mathrm{pt})$-module with generators $1, \omega, \dots, \omega^k$. $\square$

2.2. Corollary. *Let $\mathbb{C}P^k \xrightarrow{j} E \xrightarrow{p} B$ be a $\mathbb{C}P^k$-fibration over a path connected base B, and let $\alpha, \beta, \gamma \in H^*(B)$ be such that the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined and non-trivial. Let $\xi \in H^2(E)$ be an element with $j^*\xi \neq 0$. Then:*

(i) *if $k \geq 1$ then each of the Massey products*

$$\langle \xi p^*\alpha, p^*\beta, p^*\gamma \rangle, \ \langle p^*\alpha, \xi p^*\beta, p^*\gamma \rangle, \ \langle p^*\alpha, p^*\beta, \xi p^*\gamma \rangle$$

is defined and non-trivial;

(ii) *if $k \geq 2$ then each of the Massey products*

$$\langle \xi p^*\alpha, \xi p^*\beta, p^*\gamma \rangle, \ \langle p^*\alpha, \xi p^*\beta, \xi p^*\gamma \rangle, \ \langle \xi p^*\alpha, p^*\beta, \xi p^*\gamma \rangle$$

is defined and non-trivial;

(iii) *if $k \geq 3$ then the Massey product*

$$\langle \xi p^*\alpha, \xi p^*\beta, \xi p^*\gamma \rangle$$

is defined and non-trivial.

Proof. We prove the first case of (ii) only, all the other cases can be considered similarly. We consider the case of the DGA-algebra $C^*(X)$, all the other cases can be considered without any changes. Let $p^{\#} : C^*(B) \to C^*(E)$ denote the induced homomorphism of singular cochains. Take elements $a, b, c \in C^*(B)$ with $[a] = \alpha, [b] = \beta, [c] = \gamma$. Then there are cochains $x, y \in C^*(B)$ with $dx = \overline{a}b, dy = \overline{b}c$ and such that $[\overline{a}y + \overline{x}c] \notin (\alpha, \beta)$. Take a cochain $s \in C^2(E)$ with $[s] = \xi$. It is clear that $d(s^2 p^{\#} x) = (sp^{\#}\overline{a})(sp^{\#}b)$ and that $d(sp^{\#}(y) = (sp^{\#}\overline{b})(p^{\#}c)$. So, it suffices to prove that $[s^2(p^{\#}(\overline{a}y + \overline{x}c)] \notin (p^*\alpha, p^*\gamma)$, i.e, that $\xi^2 p^*[\overline{a}x + \overline{y}c] \notin (p^*\alpha, p^*\gamma)$.

Suppose the contrary. Then

$$\begin{aligned}\xi^2 p^*[\overline{a}y + \overline{x}c] &= (\xi p^*\alpha)(u_0 + \xi u_1 + \ldots + \xi^k u_k) + (\xi p^*\gamma)(v_0 + \xi v_1 + \ldots + \xi^k v_k)\\ &= (p^*\alpha)(\xi u_0 + \xi^2 u_1 + \ldots + \xi^{k+1} u_k) + (p^*\gamma)(\xi v_0 + \xi^2 v_1 + \ldots + \xi^{k+1} v_k)\end{aligned}$$

where $u_i, v_i \in p^* H^*(B)$. By expanding $\xi^{k+1} = \sum_{i=0}^{k} \xi^i \lambda_i$ with $\lambda_i \in H^*(B)$, we have

$$\xi^2 p^*[\overline{a}y + \overline{x}c] = (p^*\alpha)(a_0 + \xi a_1 + \ldots + \xi^k a_k) + (p^*\gamma)(b_0 + \xi b_1 + \ldots + \xi^k b_k)$$

where $a_i, b_i \in p^* H^*(B)$. Now, because of 2.1 and since $k \geq 2$, we conclude that $p^*[\overline{a}y + \overline{x}c] = p^*(\alpha)a_2 + p^*(\gamma)b_2$, and so $[\overline{a}y + \overline{x}c] \in (\alpha, \gamma)$. This is a contradiction. □

3. Thom spaces and Massey products

We need some preliminaries on Thom spaces of normal bundles. Standard references are [B], [R]. Let M, X be two closed smooth manifolds, and let $i : M \to X$ be a smooth embedding. Let ν be a normal bundle of $i : M \subset X$, $\dim \nu = d$, and let $T\nu$ be the Thom space of ν. We assume that ν is orientable, choose an orientation of ν and denote by $U \in H^d(T\nu)$ the Thom class of ν.

Let N be a closed tubular neighborhood of $i(M)$ in X. Let V be the interior of N, and set $\partial N = N \setminus V$. The Thom space $T\nu$ can be identified with $X/X \setminus V = N/\partial N$. We denote by

$$c : X \xrightarrow{\text{quotient}} X/(X \setminus V) = T\nu \tag{3.1}$$

the standard collapsing map.

Recall that, for every two pairs (Y, A), (Y, B) of topological spaces, there is a natural cohomology pairing

$$H^i(Y, A) \otimes H^j(Y, B) \to H^{i+j}(Y, A \cup B)$$

see [D]. In particular, we have the pairings

$$\varphi : H^i(T\nu) \otimes H^j(M) \to H^{i+j}(T\nu)$$

of the form

$$H^j(N, \partial N) \otimes H^i(N) \to H^{i+j}(N, \partial N)$$

and the pairing

$$\psi : H^i(T\nu) \otimes H^j(X) \to H^{i+j}(T\nu)$$

of the form

$$H^j(X, X \setminus V) \otimes H^i(X) \to H^{i+j}(X, X \setminus V)$$

It is well known and easy to see that the diagram

$$\begin{array}{ccc} H^*(T\nu) \otimes H^*(M) & \xrightarrow{\varphi} & H^*(T\nu) \\ {\scriptstyle 1\otimes i^*}\big\uparrow & & \big\| \\ H^*(T\nu) \otimes H^*(X) & \xrightarrow{\psi} & H^*(T\nu) \\ {\scriptstyle 1\otimes c^*}\big\downarrow & & \big\downarrow{\scriptstyle c^*} \\ H^*(X) \otimes H^*(X) & \xrightarrow{\Delta} & H^*(X) \end{array} \tag{3.2}$$

commutes; here $\Delta = \Delta_X$ is induced by the diagonal $X \to X \times X$. As usual, for the sake of simplicity we denote each of the products $\varphi(a \otimes b)$, $\psi(a \otimes b)$ and $\Delta(a \otimes b)$ by ab.

Finally, we recall that the Euler class $\chi = \chi(\nu)$ of ν is defined as $\chi := \mathfrak{z}^* U \in H^d(M)$ where $\mathfrak{z} : M \to T\nu$ is the zero section of the Thom space. Furthermore,

$$\mathfrak{z}^*(Ua) = \chi a \quad \text{for every } a \in H^*(M) \tag{3.3}$$

see e.g. [R, Prop. V.1.27].

3.4. Lemma. *Let $\alpha, \beta, \gamma \in H^*(M)$ be such that the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined. Then the Massey product $\langle \chi\alpha, \chi\beta, \chi\gamma \rangle$ is defined. Furthermore, if $0 \notin \langle \chi\alpha, \chi\beta, \chi\gamma \rangle$ then there are $u, v, w \in H^*(X)$ such that the Massey product $\langle u, v, w \rangle$ is defined and $0 \notin \langle u, v, w \rangle$.*

Proof. Clearly, $\chi\alpha\chi\beta = 0 = \chi\beta\chi\gamma$, and so the Massey product $\langle \chi\alpha, \chi\beta, \chi\gamma \rangle$ is defined. Set $u := c^*(U\alpha)$, $v := c^*(U\beta)$, $w := c^*(U\gamma)$ and prove that $uv = 0 = vw$. Consider the diagram (where H denotes H^*)

$$\begin{array}{ccccc} H(T\nu) \otimes H(M) \otimes H(T\nu) \otimes H(M) & \xrightarrow{\varphi\otimes\varphi} & H(T\nu) \otimes H(T\nu) & \xrightarrow{\Delta} & H(T\nu) \\ \big\downarrow{\scriptstyle T} & & & & \big\| \\ H(T\nu) \otimes H(T\nu) \otimes H(M) \otimes H(T\nu) & \xrightarrow{\Delta\otimes\Delta} & H(T\nu) \otimes H(M) & \xrightarrow{\varphi} & H(T\nu) \end{array}$$

where $T(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$. This diagram commutes up to sign, and so

$$\begin{aligned} (U\alpha)(U\beta) &= \Delta((U\alpha) \otimes (U\beta) = \Delta \circ (\varphi \otimes \varphi)(U \otimes \alpha \otimes U \otimes \beta) \\ &= \pm\varphi(\Delta \otimes \Delta)(U \otimes U \otimes \alpha \otimes \beta) = \pm\varphi((UU)(\alpha\beta)) = 0 \end{aligned}$$

since $\alpha\beta = 0$. So, $uv = c^*((U\alpha)(U\beta)) = 0$.

Similarly, $vw = 0$, and so the Massey product $\langle u, v, w \rangle$ is defined. Furthermore, the map

$$M \xrightarrow{i} X \xrightarrow{c} T\nu$$

coincides with the zero section $\mathfrak{z} : M \to T\nu$. Now,

$$0 \notin \langle \chi\alpha, \chi\beta, \chi\gamma \rangle = \langle \mathfrak{z}^*(U\alpha), \mathfrak{z}^*(U\beta), \mathfrak{z}^*(U\gamma) \rangle = \langle i^*u, i^*v, i^*w \rangle,$$

and the result follows from 1.2. $\square$

3.5. Lemma. *Let $\alpha, \beta \in H^*(M)$ and $w \in H^*(X)$ be such that the Massey product $\langle \alpha, \beta, i^*w \rangle$ is defined. Then the Massey product $\langle \chi\alpha, \chi\beta, i^*w \rangle$ is defined. Furthermore, if $0 \notin \langle \chi\alpha, \chi\beta, i^*w \rangle$ then there are $u, v \in H^*(X)$ such that the Massey product $\langle u, v, w \rangle$ is defined and $0 \notin \langle u, v, w \rangle$.*

Proof. Clearly, the Massey product $\langle \chi\alpha, \chi\beta, i^*w \rangle$ is defined. As in the proof of 3.4, we set $u := c^*(U\alpha)$, $v := c^*(U\beta)$. We have proved in 3.4 that $uv = 0$. Now we prove that $vw = 0$. i.e., that the Massey product $\langle u, v, w \rangle$ is defined. Consider the commutative diagram

$$\begin{array}{ccc} H^*(T\nu) \otimes H^*(M) \otimes H^*(M) & \xrightarrow{1\otimes\Delta} & H^*(T\nu) \otimes H^*(M) \\ {\scriptstyle \varphi\otimes 1}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ H^*(T\nu) \otimes H^*(M) & \xrightarrow{\varphi} & H^*(T\nu) \end{array}$$

Notice that

$$\varphi(U\beta \otimes i^*w) = \varphi \circ (\varphi \otimes 1)(U \otimes \beta \otimes i^*w) = \varphi(1 \otimes \Delta)(U \otimes \beta i^*w) = 0. \tag{3.6}$$

Furthermore, in the diagram (3.2) we have

$$vw = \Delta(v \otimes w) = \Delta(c^*(U\beta) \otimes v) = c^*\psi((U\beta) \otimes w) = c^*\varphi(U\beta \otimes i^*w) = 0$$

the last equality follows from (3.6). Now the proof can be completed just as the proof of 3.4. $\square$

3.7. Remark. Certainly, in 3.5 we can consider the Massey product $\langle i^*w, \alpha, \beta \rangle$, resp. $\langle \alpha, i^*w, \beta, \rangle$, and get the non-trivial Massey product $\langle w, u, v \rangle$, resp. $\langle u, w, v, \rangle$. We leave it to the reader to formulate and prove the corresponding parallel results.

4. Applications to blow-up

4.1. Recollection–Construction–Definition. Let ζ be a $(k+1)$-dimensional complex vector bundle over a space Y, and let $\mathrm{Prin}(\zeta) = \{P \to Y\}$ be the corresponding principal $U(k+1)$-bundle. Recall that the *projectivization* of ζ is a locally trivial $\mathbb{C}P^k$-bundle

$$\widetilde{Y} \to Y$$

where $\widetilde{Y} := P \times_{U(k+1)} \mathbb{C}P^k$ and the $U(k+1)$-action on $\mathbb{C}P^k$ is induced by the canonical $U(k+1)$-action on $\mathbb{C}^{k+1}$.

Now we define a canonical complex line bundle λ_ζ over $\widetilde{Y}$ as follows. The total space L of the canonical line bundle η_k over $\mathbb{C}P^k$ has the form

$$L = \{(z, l) \in \mathbb{C}^{k+1} \times \mathbb{C}P^k \bigm| z \in l\}.$$

In particular, the projection $\pi : L \to \mathbb{C}P^k$, $(z, l) \mapsto l$ is an $U(k+1)$-equivariant map. We define λ_ζ to be the induced map

$$1 \times \pi : P \times_{U(k+1)} L \to P \times_{U(k+1)} \mathbb{C}P^k = \widetilde{Y}.$$

Notice that if Y is the one-point space then $\widetilde{Y} = \mathbb{C}P^k$ and the canonical bundle λ coincides with η_k.

The construction λ_ζ is natural in the following sense. Let ζ be a complex vector bundle over Y, and let $f : Z \to Y$ be an arbitrary map. Then the obvious map $I : f^*(\operatorname{Prin}\zeta) \to \operatorname{Prin}\zeta$ yields a commutative diagram

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{Y} \arrow[d] \\
Z \arrow[r, "f"] & Y
\end{tikzcd}$$

where $\widetilde{Z} \to Z$ is the projectivization of $f^*\zeta$ and $\widetilde{f}$ is induced by I.

4.2. Proposition. *$\widetilde{f}^*\lambda_\zeta$ is naturally isomorphic to $\lambda_{f^*\zeta}$.* $\square$

4.3. Corollary. *If $\mathbb{C}P^k \xrightarrow{j} \widetilde{Y} \to Y$ is the projectivization of a complex vector bundle ζ then $j^*\lambda_\zeta$ is isomorphic to the canonical line bundle η_k over $\mathbb{C}P^k$.* $\square$

4.4. Definition. Let M and X be two closed connected smooth manifolds, and let $i : M \to X$ be a smooth embedding of codimension $2k+2$. A *blow-up along i* is a commutative diagram of smooth manifolds and maps

$$\begin{tikzcd}
\mathbb{C}P^k \arrow[d, "j"'] & \\
\widetilde{M} \arrow[r, "\widetilde{i}"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "q"] \\
M \arrow[r, "i"] & X
\end{tikzcd} \tag{4.5}$$

such that the following holds:

(i) $\mathbb{C}P^k \xrightarrow{j} \widetilde{M} \xrightarrow{p} M$ is a locally trivial bundle (with the fiber $\mathbb{C}P^k$) which is a projectivization of a complex $(k+1)$-dimensional vector bundle ζ;

(ii) $\widetilde{M}$ and $\widetilde{X}$ are closed connected manifolds, $\widetilde{i} : \widetilde{M} \to \widetilde{X}$ is a smooth embedding of codimension 2, and the line bundle λ_ζ is isomorphic (as a real vector bundle) to the normal bundle ν of $\widetilde{i}$;

(iii) there is a closed tubular neighborhood N of $i(M)$ such that $\widetilde{N} := q^{-1}(N)$ is a a tubular neighborhood of $\widetilde{i}(\widetilde{M})$ and

$$q|_{\widetilde{X}\setminus \operatorname{Int}\widetilde{N}} : \widetilde{X} \setminus \operatorname{Int}\widetilde{N} \to X \setminus \operatorname{Int} N$$

is an isomorphism.

By 4.4(ii), the normal bundle ν of $\widetilde{i}$ is isomorphic to λ_ζ, and hence ν is orientable. Take an orientation of ν and consider the Euler class $\chi(\nu) \in H^2(\widetilde{M})$. Since, by 4.4(ii) and 4.3, $j^*\nu$ is isomorphic to η_k, we conclude that

$$j^*\chi(\nu) = \chi(\eta_k) \neq 0. \tag{4.6}$$

4.7. Proposition. *Consider a blow-up diagram* (4.5)*. If $k \geq 2$ then $q_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ is an isomorphism.*

Proof. Let V denote the interior of N and set $\partial N = N \setminus V$. Similarly, $\widetilde{V} := q^{-1}(V)$ and $\partial \widetilde{N} := \widetilde{N} \setminus \widetilde{V}$, Since $k \geq 2$, the inclusion $X \setminus V \subset X$ induces an isomorphism of fundamental groups. So, we must prove that the inclusion $\widetilde{X} \setminus \widetilde{V} \subset \widetilde{X}$ induces an isomorphism of fundamental groups. But this follows from the van Kampen Theorem, since the inclusion $\partial \widetilde{N} \subset \widetilde{N}$ induces an epimorphism $\pi_1(\partial \widetilde{N}) \to \pi_1(\widetilde{N})$. The last claim holds, in turn, because $\partial \widetilde{N} \to \widetilde{M}$ is a locally trivial bundle with connected fiber. □

4.8. Remark. Usually the term "blow-up" is reserved for a canonical procedure which, in particular, leads to a diagram like (4.5), cf. [G], [M], [MS]. In Definition 4.4 we just axiomatized certain useful (for us) properties.

4.9. Theorem. *Consider a blow-up diagram* (4.5)*. Suppose that $k \geq 3$ and that M possesses a non-trivial triple Massey product* (*i.e., there exist $\alpha, \beta, \gamma \in H^*(M)$ with $0 \notin \langle \alpha, \beta, \gamma \rangle$*)*. Then $\widetilde{X}$ possesses a non-trivial triple Massey product.*

Proof. Let χ be the Euler class of the normal bundle of $\widetilde{i}$. Then, by 4.6 and 2.2(iii),

$$0 \notin \langle \chi p^* \alpha, \chi p^* \beta, \chi p^* \gamma \rangle.$$

Thus, by 3.4, $\widetilde{X}$ possesses a non-trivial triple Massey product. □

Sometimes it is useful to replace the assumption $k \geq 3$ in 4.9 by a more weak assumption $k \geq 2$. We do it as follows.

4.10. Theorem. *Consider a blow-up diagram as in* 4.5*. Suppose that $k \geq 2$ and that there are elements $\alpha, \beta \in H^*(M)$ and $w \in H^*(X)$ such that at least one of the triple Massey product $\langle \alpha, \beta, i^*w \rangle$, $\langle \alpha, i^*w, \beta, \rangle$, $\langle i^*w, \alpha, \beta, \rangle$ is defined and non-trivial. Then $\widetilde{X}$ possesses a non-trivial triple Massey product.*

Proof. We consider the case when the Massey product $\langle \alpha, \beta, i^*w \rangle$ is defined and non-trivial, all the other cases can be considered similarly. Let χ be the Euler class of the normal bundle of $\widetilde{i}$. Then, by 4.6 and 2.2(ii), the Massey product $\langle \chi p^* \alpha, \chi p^* \beta, \widetilde{i}^* q^* w \rangle$ is defined, and

$$0 \notin \langle \chi p^* \alpha, \chi p^* \beta, \widetilde{i}^* q^* w \rangle$$

Thus, by 3.5, $\widetilde{X}$ possesses a non-trivial Massey product of the form $\langle u, v, q^* w \rangle$. □

5. Application to symplectic manifolds

5.1. Theorem (McDuff [M]). *Let M and X be two closed symplectic manifolds, and let $i : M \to X$ be a symplectic embedding. Then there exists a blow-up diagram where the map $\widetilde{i}$ is a symplectic embedding. In particular, $\widetilde{X}$ is a symplectic manifold.*

We define a *symplectic blow-up* to be a blow-up with i and $\widetilde{i}$ symplectic.

Actually, McDuff [M] suggested a canonical construction of symplectic blow-up. In particular, the bundle ζ in 4.4(i) turns out to be the normal bundle of the embedding i. A detailed exposition of this construction can be found in [MS], [TO].

This theorem allows to use (apply) the above Theorems 4.9, 4.10 in order to construct non-formal symplectic manifolds. To start with, we must have at least one symplectic manifold with a non-trivial triple Massey product.

5.2. Example (Kodaira–Thurston)**.** Let H be the Heisenberg group, i.e., the group of the 3×3-matrices of the form

$$\alpha = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$, and let Γ be the subgroup of H with integer entries. We set

$$K := (H/\Gamma) \times S^1.$$

So, the Kodaira–Thurston manifold K is a 4-dimensional nil-manifold. We have $H^1(K;\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}$ and $H^2(K;\mathbb{Q}) = \mathbb{Q}\oplus\mathbb{Q}$. Moreover, there are classes $\alpha,\beta \in H^1(K;\mathbb{Q})$ and $\gamma \in H^2(K;\mathbb{Q})$ such that the Massey product $\langle\alpha,\beta,\gamma\rangle$ is defined and

$$0 \notin \langle\alpha,\beta,\gamma\rangle. \tag{5.3}$$

Furthermore, K possesses a symplectic form ω, and the class

$$\gamma\otimes 1 \in H^*(K;\mathbb{Q})\otimes\mathbb{R} = H^*(K;\mathbb{R})$$

coincides with the de Rham cohomology class of ω. See [TO] for details.

5.4. Theorem. *Consider a symplectic embedding $i : K \to X$ and any symplectic blow-up*

$$\begin{tikzcd} \mathbb{C}P^k \arrow[d, "j"'] & \\ \widetilde{K} \arrow[r, "\widetilde{i}"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "q"] \\ K \arrow[r, "i"] & X \end{tikzcd}$$

along i. If $k \geq 2$ then $\widetilde{X}$ possesses a non-trivial rational Massey triple product. In particular, $\widetilde{X}$ is not a formal space.

Proof. For $k \geq 3$ this follows from 4.9, because K possesses a non-trivial triple Massey product. Let $k = 2$ and consider the symplectic form ω_X on X. Then $i^*\omega_X = \omega_K$ where ω_K is the symplectic form on K. Hence, $i^* : H^2(X;\mathbb{R}) \to H^2(K;\mathbb{R})$ is an epimorphism, and so $i^* : H^2(X;\mathbb{Q}) \to H^2(K;\mathbb{Q})$ is an epimorphism. So, according to 5.3, $0 \notin \langle\alpha,\beta,i^*w\rangle$ for some $w \in H^2(X;\mathbb{Q})$. Now, the result follows from 4.10. $\square$

In particular, if the initial ambient space X was simply connected, then the space $\widetilde{X}$ in 5.4 gives us an example of non-formal simply connected symplectic manifold.

It is well known that, for every $n \geq 5$, there exists a symplectic embedding $K \to \mathbb{C}P^n$, [Gr], [T]. Consider a (the) symplectic blow-up along such an embedding, and let $\widetilde{\mathbb{C}P^n}$ denote the corresponding space (in the top right corner of the diagram (4.5)). Then 5.2 yields the following Corollary.

5.5. Corollary (Babenko–Taimanov [BT]). *Every space $\widetilde{\mathbb{C}P^n}$, $n \geq 5$ possesses a non-trivial triple Massey product, and hence it is not a formal space.* □

5.6. Remarks. 1. Generalizing 5.2, consider the so-called Iwasawa manifolds. Namely, we set

$$I(p,q) = (H(1,p) \times H(1,q))/\Gamma$$

where $H(1,p)$ consists of all matrices of the form

$$\alpha = \begin{pmatrix} I_p & A & C \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

It is proved in [CFG] that $I(p,q)$ always possesses a non-trivial rational triple Massey product. So, we can use these manifolds for construction of families of non-formal symplectic manifolds. In particular, if both p and q are even then $I(p,q)$ is symplectic, and so in this way we can also get a large family of non-formal (and so non Kähler) symplectic manifolds.

2. Certainly, the blow-up construction can be iterated, i.e, starting from $M \subset X$ and getting $\widetilde{X}$, we can embed $\widetilde{X}$ into another manifold and construct a new blow-up, and so on. So, here we really have a lot of possibilities to construct manifolds with non-trivial Massey products and, in particular, non-formal manifolds.

6. Massey products in algebraic and Kähler manifolds

According to [DMGS], every Kähler manifold M is formal, and so every Massey product in $H^*(M;\mathbb{Q})$ is trivial. However, there are complex projective algebraic varieties (and in particular Kähler manifolds) S with non-trivial Massey products in $H^*(S;\mathbb{Z}/p)$. Indeed, Kraines [K] proved that, for every prime $p > 2$ and every $x \in H^{2n+1}(X;\mathbb{Z}/p)$ we have

$$\beta P^n(x) \in -\langle \underbrace{x, x, \dots, x}_{p \text{ times}} \rangle$$

where $\beta : H^i(X;\mathbb{Z}/p) \to H^{i+1}(X;\mathbb{Z}/p)$ is the Bockstein homomorphism and

$$P^n : H^i(X;\mathbb{Z}/p) \to H^{i+2n(p-1)}(X;\mathbb{Z}/p)$$

denotes the reduced Steenrod power. In particular, if $H_1(X;Z) = \mathbb{Z}/3$ then $H^1(X;\mathbb{Z}/3) = \mathbb{Z}/3$ and $\beta(x) \neq 0$ for every $x \in H^1(X;\mathbb{Z}/3), x \neq 0$. So, the Massey product $\langle x,x,x\rangle$ has zero indeterminacy, and $\langle x,x,x\rangle \neq 0$ provided $x \neq 0$.

So, if $H_1(X;\mathbb{Z}) = \mathbb{Z}/3$, then $H^*(X;\mathbb{Z}/3)$ possesses a non-trivial Massey triple product. Certainly, one can find many examples of algebraic varieties and Kähler manifolds with such homology groups, see e.g. [ABCKT]. Now, since the classes of Kähler manifolds and algebraic varieties are invariant under the blow-up construction, we can use Theorems 4.9 and 4.10 and construct a large class of Kähler manifolds and algebraic varieties with non-trivial triple Massey $\mathbb{Z}/3$-products, including simply connected objects as well.

Perhaps, the following observation looks interesting. If we have a Kähler manifold or algebraic variety X_0 with $u_0 \in H^1(X_0;\mathbb{Z}/3)$ such that $0 \notin \langle u_0, u_0, u_0\rangle$, then we can perform a blow-up along X_0 (in the corresponding category) and get a resulting object X_1, and if $\dim X_1 - \dim X_0 \geq 6$ then there is an element $u_1 \in H^3(X_1;\mathbb{Z}/3)$ with $0 \notin \langle u_1, u_1, u_1\rangle$. In particular, $\beta P^1(u_1) \neq 0$. Similarly, performing an obvious induction, we can construct X_n (algebraic or Kähler) and an element $u_n \in H^{2n+1}(X_n;\mathbb{Z}/3)$ with $\beta P^n(u) \neq 0$.

References

[ABCKT] J. Amorós, M. Burger, K. Corlette, D. Kotschik, D. Toledo, *Fundamental groups of compact Kähler manifolds*, Amer. Math. Soc, 1996.

[BT] I. Babenko and I. Taimanov, *On non-formal simply-connected symplectic manifolds*, preprint, math.SG/9811167 v2, Dec.,1, 1998.

[B] W. Browder, *Surgery on Simply-Connected Manifolds*, Springer, Berlin, 1972.

[CFG] L. Cordero, M. Fernandez and A. Gray, *Symplectic manifolds with no Kähler structure*, Topology **25** (1986), 375-380.

[DGMS] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245-274.

[D] A. Dold, *Lectures on Algebraic Topology*, Springer, Berlin, 1972,.

[FMG] M. Fernandez, J. Morgan and A. Gray, *Compact symplectic manifolds with free circle actions and Massey products*, Michigan Math. J. **38** (1991), 271-290.

[FLS] M. Fernandez, M. de Leon and M. Saralegi, *A six dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), 18-39.

[E] T. Ekedahl, *Two examples of smooth projective varieties with non-zero Massey products*, Algebra, Algebraic Topology and Their Interactions, Proc. Conf. Stockholm, 1983, Lect. Notes Math. 1183, Springer, Berlin, 1986, pp. 128-132.

[G] S. Gitler, *The cohomology of blow-ups*, Bol. Soc. Mat. Mexicana **37** (1982), 167-175.

[Go] R. Gompf, *A new construction of symplectic manifolds*, Annals of Math. **142** (1995), 527-595.

[Gr] M. Gromov, *Partial Differential Relations*, Springer, Berlin, 1986.

[K] D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431-449.

[L] D. Lehmann, *Théorie homotopique des forms différentielles (d'aprés D. Sullivan)*, Asterisque 45 (1977).

[LO] G. Lupton and J. Oprea, *Symplectic manifolds and formality*, J. Pure & Appl. Algebra **91** (1994), 193-207.

[M] D. McDuff, *Examples of symplectic simply connected manifolds with no Kähler structure*, J. Diff. Geom. **20** (1984), 267-277.

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press, Oxford, 1995.

[R] Yu. Rudyak, *On Thom Spectra, Orientability and Cobordism*, Springer, Berlin, 1998.

[S] R. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer, Berlin, 1975.

[T] D. Tischler, *Closed 2-forms and an embedding theorem for symplectic manifolds*, J. Diff. Geom. **12** (1977), 229-235.

[TO] A. Tralle and J. Oprea, *Symplectic Manifolds with no Kähler Structure*, Lecture Notes in Math. **1661**, Springer, Berlin, 1997.

FB6/Mathematik, Universität Siegen, 57068 Siegen
E-mail address: `rudyak@mathematik.uni-siegen.de, july@mathi.uni-heidelberg.de`

University of Olsztyn
Current address: Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `tralle@tufi.wsp.olsztyn.pl, tralle@mpim-bonn.mpg.de`