A TORELLI TYPE THEOREM FOR THE MODULI SPACE OF PARABOLIC VECTOR BUNDLES OF RANK TWO OVER CURVES

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Abstract. Let S (respectively, S') be a finite subset of a compact connected Riemann surface X (respectively, X') of genus at least two. Let $\mathcal{M}$ (respectively, $\mathcal{M}'$) denote a moduli space of parabolic stable bundles of rank two over X (respectively, X') with fixed determinant of degree one, having a nontrivial quasi-parabolic structure over each point of S (respectively, S'), and of parabolic degree less than two. It is proved that $\mathcal{M}$ is isomorphic to $\mathcal{M}'$ if and only if there is an isomorphism of X with X' taking S to S'.

1. Introduction

Let X and X' be two compact connected Riemann surfaces of genus g. The classical Torelli theorem says that if their Jacobians $J(X)$ and $J(X')$ are isomorphic as polarized varieties, with the polarization defined using the *theta line bundle*, then X and X' are isomorphic.

D. Mumford and P. Newstead proved the following Torelli type theorem. Assuming $g \geq 2$, let $\mathcal{N}_X$ (respectively, $\mathcal{N}_{X'}$) be a moduli space of fixed determinant semistable vector bundles over X (respectively, X') of rank two and degree one. With this notation, if the two projective varieties $\mathcal{N}_X$ and $\mathcal{N}_{X'}$ are isomorphic, then X is isomorphic to X' [MN].

Our aim in this paper is to extend the above theorem of [MN], which ensures the recovery of a Riemann surface from its moduli space of stable bundles, to the more general situation of n-pointed Riemann surfaces. Given a finite subset $S \subset X$ of a compact Riemann surface, after fixing some numerical data, known as the *parabolic data*, we consider the moduli space, $\mathcal{M}$, of fixed determinant parabolic semistable bundles with S as the parabolic divisor. One might hope that the moduli space $\mathcal{M}$ determines the pair (X, S). In other words, given another such pair (X', S'), if the corresponding moduli space of parabolic semistable bundles for (X', S') is isomorphic to $\mathcal{M}$, then one might hope that there is an isomorphism of X with X' which takes S to S'.

Here we consider the simplest type of parabolic data.

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Let E be a parabolic stable bundle of rank two and degree one over X such that the sum of all the parabolic weights is less than one. In such a situation the vector bundle E must be stable in the usual sense. Therefore, the corresponding moduli space $\mathcal{M}$ of parabolic bundles is simply a fiber bundle over a moduli space of usual stable vector bundles, with a product of copies of $\mathbb{CP}^1$ as the typical fiber. In other words, $\mathcal{M}$ is the total space of the fiber product of the projective bundles over the usual moduli space, $\mathcal{N}_X$, obtained by restricting the universal projective bundle over $X \times \mathcal{N}_X$ to the subvarieties $s \times \mathcal{N}_X$, where $s \in S$.

For a parabolic data of the above type we prove that $\mathcal{M}$ is isomorphic to $\mathcal{M}'$ if and only if there is an isomorphism of X with X' which takes the subset S to S' [Theorem 3.2].

The main idea in the proof of the above result is the observation that a certain natural class of line bundles over $\mathcal{M}$ determine the numerically effective cone in $\mathrm{Pic}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{R}$. The natural class of line bundles in question is obtained from a construction known as the Hecke correspondence.

A result of Mumford and Newstead in [MN], which says that the intermediate Jacobian of the rank two fixed determinant coprime moduli space is isomorphic to the Jacobian of the Riemann surface, forms a key input in the proof.

In the final section we show that an infinitesimal version of the Torelli theorem for the moduli space $\mathcal{M}$ is also valid. The infinitesimal Torelli in question says that the space of all infinitesimal deformations of the pointed curve (X, S) is naturally isomorphic to the space of all infinitesimal deformations of the variety $\mathcal{M}$ [Proposition 4.2].

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2. The case of one point

Let X be a connected smooth projective curve over $\mathbb{C}$, or equivalently a compact connected Riemann surface, of genus g, with $g \geq 2$. Fix a holomorphic line bundle $\mathcal{L}$ over X of degree one. Let $\mathcal{N}(2, \mathcal{L})$ denote the moduli space of stable vector bundles E over X of rank two and $\bigwedge^2 E = \mathcal{L}$. It is known that there is a universal vector bundle $\mathcal{E}$ over $X \times \mathcal{N}(2, \mathcal{L})$ [MN]. The universal property of $\mathcal{E}$ is that for every $p \in \mathcal{N}(2, \mathcal{L})$, the vector bundle over X obtained by restricting $\mathcal{E}$ to $X \times p$ is represented by the point p of the moduli space. The universal projective bundle $\mathbb{P}(\mathcal{E})$, consisting of all lines in the fibers of $\mathcal{E}$, does not depend on the choice of the universal vector bundle. Indeed, any two universal vector bundles differ by tensoring with the pull back of a line bundle over $\mathcal{N}(2,\mathcal{L})$.

Fix a point $x \in X$. Let $\mathbb{P}(\mathcal{E}_x)$ denote the projective bundle over $\mathcal{N}(2,\mathcal{L})$ obtained by restricting $\mathbb{P}(\mathcal{E})$ to $x \times \mathcal{N}(2,\mathcal{L})$. The dimension of the total space of $\mathbb{P}(\mathcal{E})$ is $3g-2$.

Take another compact connected Riemann surface Y of genus g. Fix a line bundle $\mathcal{L}'$ over Y of degree one, and also fix a point $y \in Y$. Consider the moduli space $\mathcal{N}_Y(2,\mathcal{L}')$, and define the projective bundle $\mathbb{P}(\mathcal{E}'_y)$ over $\mathcal{N}_Y(2,\mathcal{L})$, as before, by restricting the universal projective bundle to $y \times \mathcal{N}_Y(2,\mathcal{L}')$.

The following theorem shows that the total space $\mathbb{P}(\mathcal{E}_x)$ of the projective bundle determines the pair (X,x).

Theorem 2.1. *Let the two projective manifolds $\mathbb{P}(\mathcal{E}_x)$ and $\mathbb{P}(\mathcal{E}'_y)$ be isomorphic. Then there is an isomorphism $h : X \longrightarrow Y$ which takes the point x to y.*

Proof. Our first step in the proof will be to determine the numerically effective cone in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x)) \bigotimes_{\mathbb{Z}} \mathbb{R}$. The tensor product operation on the Picard group will be written additively. Denoting the space of ample line bundles in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x))$ by A, the interior of the numerically effective cone is precisely the closure of the image of the cone $A \bigotimes_{\mathbb{N}} \mathbb{R}^+$ inside $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x)) \bigotimes_{\mathbb{Z}} \mathbb{R}$ generated by A, where $\mathbb{R}^+$ denotes the positive real numbers.

Let Θ denote the line bundle over $\mathcal{N}(2,\mathcal{L})$ whose fiber over the point represented by a vector bundle E is naturally isomorphic to

$$\Big(\bigwedge^{\mathrm{top}} H^0(X,\,E)^* \bigotimes \bigwedge^{\mathrm{top}} H^1(X,\,E)\Big)^{\otimes 2} \bigotimes \Big(\bigwedge^{2} E_x\Big)^{\otimes (3-2g)}$$

It is known that the Picard group of $\mathcal{N}(2,\mathcal{L})$ is isomorphic to $\mathbb{Z}$, and the ample generator of $\mathrm{Pic}(\mathcal{N}(2,\mathcal{L}))$ is Θ [B].

This implies that the Picard group of $\mathbb{P}(\mathcal{E}_x)$ is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z}$, and it is generated by $\mathrm{Pic}(\mathcal{N}(2,\mathcal{L}))$ together with the tautological line bundle. Therefore, the numerically effective cone in the vector space $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x)) \bigotimes_{\mathbb{Z}} \mathbb{R}$ is determined by two half-lines. We note that the tautological line bundle depends on the choice of the universal vector bundle.

Let $p : \mathbb{P}(\mathcal{E}_x) \longrightarrow \mathcal{N}(2,\mathcal{L})$ denote the natural projection. Since Θ is ample, the pull back $p^*\Theta$ is numerically effective but not ample. Furthermore, the line bundle $p^*\Theta$ is nontrivial. Therefore, the half-line $p^*\Theta \cdot \mathbb{R}^+$ in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x)) \bigotimes_{\mathbb{Z}} \mathbb{R}$, defined by $p^*\Theta$, is one of the two boundary half-lines of the numerically effective cone.

Take a line $L \subset E_x$ of the fiber E_x over x of a vector bundle $E \in \mathcal{N}(2,\mathcal{L})$. Consider the vector bundle V over X that fits in the following exact sequence of sheaves over X

$$0 \longrightarrow V \longrightarrow E \longrightarrow \frac{E_x}{L} \longrightarrow 0$$

The vector bundle V is semistable. This is an immediate consequence of the fact that the degree of the line subbundle of E generated by a line subbundle ξ of V is at least $\deg(\xi)$.

Let L_0 denote the line bundle $\mathcal{L}\bigotimes\mathcal{O}_X(-x)$ over X. Let

$$\psi\,:\,\mathbb{P}(\mathcal{E}_x)\,\longrightarrow\,\mathcal{N}(2,L_0) \tag{2.2}$$

be the morphism to the moduli space $\mathcal{N}(2,L_0)$ that sends any pair (E,L) to the vector bundle V constructed in the above fashion.

Let Θ_0 denote the line bundle over $\mathcal{N}(2,L_0)$ whose fiber over the point represented by a vector bundle V is naturally isomorphic to

$$\bigwedge^{\mathrm{top}} H^0(X,\,V)^*\bigotimes\bigwedge^{\mathrm{top}} H^1(X,\,V)\bigotimes\Big(\bigwedge^2 V_x\Big)^{\otimes(1-g)}$$

The Picard group $\mathrm{Pic}(\mathcal{N}(2,L_0))$ is isomorphic to $\mathbb{Z}$ and Θ_0 is the ample generator [B].

Therefore, the line bundle $\psi^*\Theta_0$ over $\mathbb{P}(\mathcal{E}_x)$ is numerically effective but not ample, and also it is nontrivial. This immediately implies that $\psi^*\Theta_0$ generates a boundary half-line of the numerically effective cone in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x))\bigotimes_{\mathbb{Z}}\mathbb{R}$.

Now $\mathcal{N}(2,\mathcal{L})$ is a smooth variety, and $\mathcal{N}(2,L_0)$ is a singular variety if $g\geq 3$. Furthermore, if $g=2$, then $\mathcal{N}(2,\mathcal{L})$ is known to be not isomorphic to $\mathcal{N}(2,L_0)$. Therefore, as both Θ and Θ_0 are ample, we conclude that $p^*\Theta$ is a not a rational multiple of $\psi^*\Theta_0$. Another way to see that $p^*\Theta$ is a not a rational multiple of $\psi^*\Theta_0$ is to consider the restrictions of both $p^*\Theta$ and $\psi^*\Theta_0$ to a fiber of the morphism p. The restriction of $p^*\Theta$ is trivial, while the restriction of $\psi^*\Theta_0$ has degree one. The fact that the restriction of $\psi^*\Theta_0$ to a fiber of p has degree one, actually implies that $p^*\Theta$ and $\psi^*\Theta_0$ together generate the Picard group $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x))$ of the projective bundle.

We will show that $(p^*\Theta)^{\otimes a}\bigotimes(\psi^*\Theta_0)^{\otimes b}$ is ample if both a and b are strictly positive integers. Since both $p^*\Theta$ and $\psi^*\Theta_0$ are numerically effective, it is enough to show that $p^*\Theta\bigotimes\psi^*\Theta_0$ is ample. Consider the direct image

$$p_*(p^*\Theta\otimes\psi^*\Theta_0)\;=\;\Theta\otimes p_*(\psi^*\Theta_0)$$

over $\mathcal{N}(2,\mathcal{L})$. Since Θ is ample and $\psi^*\Theta_0$ is numerically effective, we conclude that the vector bundle $\Theta\bigotimes p_*(\psi^*\Theta_0)$ is ample [Vi, Proposition 2.9]. This implies that $p^*\Theta\bigotimes\psi^*\Theta_0$ is ample.

The above observations immediately imply that the numerically effective cone in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x))\bigotimes_{\mathbb{Z}}\mathbb{R}$ is bounded by the two half-lines generated by $p^*\Theta$ and $\psi^*\Theta_0$ respectively.

As the next step towards the recovery of the pointed Riemann surface (X,x) from the given projective manifold $\mathbb{P}(\mathcal{E}_x)$, we will now consider the linear systems corresponding to the line bundles in the two boundary half-lines of the numerically effective cone in $\mathrm{Pic}(\mathbb{P}(\mathcal{E}_x))\bigotimes_{\mathbb{Z}}\mathbb{R}$.

If we take two sufficiently divisible nontrivial line bundles ξ_1 and ξ_2 in the respective two half-lines, then they are base point free, and the corresponding two morphisms

$$q_1 : \mathbb{P}(\mathcal{E}_x) \longrightarrow Z_1 \quad \text{and} \quad q_2 : \mathbb{P}(\mathcal{E}_x) \longrightarrow Z_2 \tag{2.3}$$

have the property that Z_1 (respectively, Z_2) is isomorphic to $\mathcal{N}(2,\mathcal{L})$ (respectively, $\mathcal{N}(2,L_0)$). This evidently is an immediate consequence of the fact that the line bundle Θ (respectively, Θ_0) is ample over $\mathcal{N}(2,\mathcal{L})$ (respectively, $\mathcal{N}(2,L_0)$).

For two Riemann surfaces, of genus at least two, if the corresponding two moduli spaces of rank two degree one and fixed determinant stable vector bundles are isomorphic, then the two Riemann surfaces must be isomorphic [MN, page 1201]. So the two Riemann surfaces X and Y in the statement of the theorem are isomorphic. In order to complete the proof of the theorem we need to ensure that the isomorphism can be so chosen that it sends x to y.

The existence of an isomorphism taking x to y will be proved using the characteristic class of the projective bundle $\mathbb{P}(\mathcal{E}_x)$ over $\mathcal{N}(2,\mathcal{L})$ defined by p.

Since $\mathrm{Pic}(\mathcal{N}(2,\mathcal{L}))=\mathbb{Z}$, it is possible to fix a universal vector bundle E in a unique fashion. More precisely, there is exactly one universal vector bundle $\mathcal{E}$ over $X\times\mathcal{N}(2,\mathcal{L})$ such that the restriction of the line bundle $\bigwedge^{\otimes 2}\mathcal{E}$ to $z\times\mathcal{N}(2,\mathcal{L})$, where $z\in X$, is isomorphic to Θ [R]. Henceforth, unless explicitly stated otherwise, by a universal vector bundle we will always mean the above uniquely determined vector bundle.

Let

$$c_2(\mathcal{E}) \in H^4_{\mathcal{D}}(X\times\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2)) \tag{2.4}$$

be the second Chern class of $\mathcal{E}$ in the *Deligne-Beilinson cohomology*. We recall that the Deligne-Beilinson cohomology $H^k_{\mathcal{D}}(X\times\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(j))$ is the k-th hypercohomology of the complex

$$0 \longrightarrow (2\pi\sqrt{-1})^j\cdot\mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{j-1} \longrightarrow 0\,,$$

where the constant sheaf $(2\pi\sqrt{-1})^j\cdot\mathbb{Z}$ is at the 0-th position. We also recall that for a connected smooth projective variety M, there is a natural exact sequence [EV, (7.9), page 86]

$$0 \longrightarrow J^k(M) \longrightarrow H^{2k}_{\mathcal{D}}(M,\,\mathbb{Z}(k)) \longrightarrow \mathrm{Hg}^k(Y) \longrightarrow 0\,, \tag{2.5}$$

where $J^k(Y)$ is the k-th *intermediate Jacobian* and $\mathrm{Hg}^k(Y)$ denotes the Hodge cycles. The basic definitions and constructions about Deligne-Beilinson cohomology can be found in [EV].

Using $c_2(\mathcal{E})$ as the correspondence, a homomorphism

$$F : H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1)) \longrightarrow H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2)) \tag{2.6}$$

is obtained which sends any class c to $p_{2*}(c_2(\mathcal{E}) \cdot p_1^* c)$, where p_1 (respectively, p_2) is the projection of $X \times \mathcal{N}(2,\mathcal{L})$ onto X (respectively, $\mathcal{N}(2,\mathcal{L})$).

The following proposition is very useful for the proof of the theorem.

Proposition 2.7. *The homomorphism F defined in (2.6) is injective.*

Proof of Proposition 2.7. The restriction of F to $\mathrm{Pic}^0(X)$, which is the connected component of the zero element of the group $H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1))$, is injective [MN, Proposition 1, page 1204]. On the other hand, for a point $z \in X$, the image of $c_2(\mathcal{E}_z)$ in $H^4(\mathcal{N}(2,\mathcal{L}),\, (2\pi\sqrt{-1})^2 \cdot \mathbb{Z}) = \mathbb{Z}^{\oplus 2}$ is nonzero [N2], [KN]. This nonzero element in $H^4(\mathcal{N}(2,\mathcal{L}),\, (2\pi\sqrt{-1})^2 \cdot \mathbb{Z})$ is the image of the oriented generator of $H^2(X,\, 2\pi\sqrt{-1} \cdot \mathbb{Z}) = \mathbb{Z}$ by the homomorphism $H^2(X,\, 2\pi\sqrt{-1}\cdot\mathbb{Z}) \longrightarrow H^4(\mathcal{N}(2,\mathcal{L}),\, (2\pi\sqrt{-1})^2 \cdot \mathbb{Z})$ defined using the usual Chern class $c_2(\mathcal{E}) \in H^4(X \times \mathcal{N}(2,\mathcal{L}),\, (2\pi\sqrt{-1})^2 \cdot \mathbb{Z})$ as the correspondence. Now, since

$$\frac{H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1))}{\mathrm{Pic}^0(X)} = H^2(X,\, 2\pi\sqrt{-1}\cdot\mathbb{Z}),$$

the homomorphism F must be injective. □

Let

$$\tau : X \longrightarrow H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1)) \tag{2.8}$$

denote the Abel-Jacobi embedding which sends any point $z \in X$ to $c_1(\mathcal{O}_X(z)) \in H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1))$.

The following simple proposition is the key to the use of Proposition 2.7.

Proposition 2.9. *For any $z \in X$, let $\mathcal{E}_z$ denote the vector bundle over $\mathcal{N}(2,\mathcal{L})$ obtained by restricting $\mathcal{E}$ to $z \times \mathcal{N}(2,\mathcal{L})$. Then the element $F \circ \tau(z) \in H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$ coincides with $c_2(\mathcal{E}_z)$.*

Proof of Proposition 2.9. The Deligne-Beilinson cohomology class $F \circ \tau(z)$ coincides with the pull back of $c_2(\mathcal{E})$ to $\mathcal{N}(2,\mathcal{L})$ by the embedding $\mathcal{N}(2,\mathcal{L}) \longrightarrow X \times \mathcal{N}(2,\mathcal{L})$ defined by $E \longmapsto (z, E)$. Now the proposition follows from the fact that the Chern classes are compatible with the pull back operation. □

Starting with the projective manifold $\mathbb{P}(\mathcal{E}_x)$, we consider the projection q_1 defined in (2.3). Let $\overline{\mathcal{E}}$ denote the unique vector bundle over Z_1 such that the projective bundle $\mathbb{P}(\overline{E})$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E}_x)$ defined by q_1 and $\bigwedge^2 \overline{E} = \overline{\Theta}$, where $\overline{\Theta}$ is the ample generator of $\mathrm{Pic}(Z_1) = \mathbb{Z}$.

Now, although the projective bundle defined by q_1 in (2.3) is isomorphic to the one over $\mathcal{N}(2,\mathcal{L})$ defined by p, the identification of Z_1 with $\mathcal{N}(2,\mathcal{L})$ is not unique. Therefore, we need to determine the action of the automorphism group, $\mathrm{Aut}(\mathcal{N}(2,\mathcal{L}))$, of $\mathcal{N}(2,\mathcal{L})$ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(2))$.

The proof of the theorem will be completed by establishing the following statement : the orbit of the point $c_2(\mathcal{E}_x)$ under the natural action of $\mathrm{Aut}(\mathcal{N}(2,\mathcal{L}))$ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(2))$ is contained in the image $F\circ\tau(C_x)$, where $C_x\subset X$ is the orbit of the point x for the tautological action of the group of automorphisms of X on X.

Let $\mathrm{Aut}(X)$ denote the group of all holomorphic automorphisms of X. Let Γ be the set of all pairs (ζ,g), where $\zeta\in\mathrm{Pic}^0(X)$ and $g\in\mathrm{Aut}(X)$, such that the line bundle $\zeta^{\otimes 2}$ is isomorphic to $\mathcal{L}\bigotimes(g^{-1})^*\mathcal{L}^*$. Define a group operation on Γ as follows : $(\zeta,g)\circ(\zeta',g')=(\zeta\bigotimes(g^{-1})^*\zeta',gg')$. Note that Γ is a finite group of order $2^{2g}\cdot\#\mathrm{Aut}(X)$, where $\#\mathrm{Aut}(X)$ is the order of the group $\mathrm{Aut}(X)$.

Let $J_2(X)\subset\mathrm{Pic}^0(X)$ denote the group of all two torsion points of the Jacobian. The group Γ is evidently an extension of $\mathrm{Aut}(X)$ by $J_2(X)$.

The group Γ acts on $\mathcal{N}(2,\mathcal{L})$ as follows :

$$(\zeta,g)\circ E=(g^{-1})^*E\otimes\zeta\,.$$

If $g\geq 3$, then Kouvidakis and Pantev proved that the homomorphism from Γ to $\mathrm{Aut}(\mathcal{N}(2,\mathcal{L}))$ defined above is surjective [KP, Theorem B, page 228]. Note that for a rank two vector bundle E, from the isomorphism $E^*=E\bigotimes\bigwedge^2E^*$ it follows that the automorphisms of the type $E\longmapsto E^*\bigotimes\zeta$ in the statement of Theorem B of [KP] are already covered by Γ.

If $g=2$, then the surjectivity of the above homomorphism from Γ to $\mathrm{Aut}(\mathcal{N}(2,\mathcal{L}))$ was proved by Newstead, [N1, Theorem 3], in which case he gave a very explicit description of the moduli space.

Let

$$\rho\,:\,\Gamma\longrightarrow\mathrm{Aut}(X)\tag{2.10}$$

be the natural projection which sends any (ζ,g) to g.

The action of Γ on $\mathcal{N}(2,\mathcal{L})$ obviously induces an action of Γ on the Deligne-Beilinson cohomology group $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(2))$.

The proof of the theorem will be completed using the following proposition.

Proposition 2.11. *The kernel of the homomorphism ρ, defined in (2.10), acts trivially on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(2))$ for the natural action of Γ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\,\mathbb{Z}(2))$.*

Proof of Proposition 2.11. The kernel of ρ is $J_2(X)$, the group of two torsion points in $\mathrm{Pic}^0(X)$.

It is known that the group of two torsion points in $\mathrm{Pic}^0(X)$, namely $J_2(X)$, acts trivially on the cohomology ring $H^*(\mathcal{N}(2,\mathcal{L}),\, \mathbb{C})$ [N2, Theorem 1], [HN, Theorem 1]. So, in particular, the action of $J_2(X)$ on any intermediate Jacobian $J^k(\mathcal{N}(2,\mathcal{L}))$ or Hodge cycle $\mathrm{Hg}^k(\mathcal{L}(2,\mathcal{L}))$ is trivial. Now, using the exact sequence (2.5) it follows that $J_2(X)$ acts trivially on the Deligne-Beilinson cohomology of $\mathcal{N}(2,\mathcal{L})$. This completes the proof of the proposition. □

Now to complete the proof of the theorem, we recall an earlier comment that in order to complete the proof of the theorem it suffices to establish the following statement : the orbit of the point $c_2(\mathcal{E}_x)$ under the natural action of $\mathrm{Aut}(\mathcal{N}(2,\mathcal{L}))$ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$ is contained in the image $F \circ \tau(C_x)$, where $C_x \subset X$ is the orbit of the point x for the natural action of $\mathrm{Aut}(X)$ on X; the maps F and τ are defined in (2.6) and (2.8) respectively.

To see that the above statement indeed completes the proof of the theorem, first note that the Abel-Jacobi embedding τ is equivariant for the natural actions of $\mathrm{Aut}(X)$ on X and $H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1))$ respectively. Using Proposition 2.11, the natural action of Γ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$ induces an action of $\mathrm{Aut}(X)$ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$. The correspondence homomorphism F in (2.6) is evidently equivariant for the actions of $\mathrm{Aut}(X)$. Now, in view of Proposition 2.9 and the fact that any two identifications of $\mathcal{N}(2,\mathcal{L})$ with $\mathcal{N}_Y(2,\mathcal{L}')$ differ by an automorphism of $\mathcal{N}(2,\mathcal{L})$, the above statement is equivalent to the assertion that there is an isomorphism between X and Y which takes x to y, where (Y, y) is as in the statement of the theorem.

On the other hand, the validity of the above statement is obvious. To see it in more details, for any $\gamma = (\zeta, g) \in \Gamma$, let $\overline{g}$ denote the automorphism $g \times Id$ of $X \times \mathcal{N}(2,\mathcal{L})$. From Proposition 2.11 and its proof it follows that for any $\alpha \in H^2_{\mathcal{D}}(X,\, \mathbb{Z}(1))$, the action of the element $\rho(\gamma) \in \mathrm{Aut}(X)$, where ρ is defined in (2.10), on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$ sends the point $F(\alpha)$ to the image of α by the correspondence map constructed using $c_2(\overline{g}^*\mathcal{E}) \in H^4_{\mathcal{D}}(X \times \mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$. Now, $c_2(\overline{g}^*\mathcal{E}) = \overline{g}^* c_2(\mathcal{E})$. Therefore, the orbit of $c_2(\mathcal{E}_x)$ for the action of Γ on $H^4_{\mathcal{D}}(\mathcal{N}(2,\mathcal{L}),\, \mathbb{Z}(2))$ is the image $F \circ \tau(C_x)$, where τ is the Abel-Jacobi map and C_x is the orbit of the point x for the action of $\mathrm{Aut}(X)$ on X. This completes the proof of the theorem. □

The diagram of morphisms

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_x) & \xrightarrow{\psi} & \mathcal{N}(2, L_0) \\ \Big\downarrow{\scriptstyle p} & & \\ \mathcal{N}(2,\mathcal{L}) & & \end{array}$$

is called the *Hecke correspondence*. The above theorem shows that the Hecke correspondence is completely determined by the variety $\mathbb{P}(\mathcal{E}_x)$.

In the next section we will use Theorem 2.1 in order to obtain a more general result described in the introduction that involves reconstructing n-pointed curves.

3. A generalization to multi-pointed curves

We continue with the notation of the previous section.

Fix n distinct points $\{x_1, x_2, \ldots, x_n\}$ of the Riemann surface X, where $n \geq 2$. Let

$$\mathbb{P} = \mathbb{P}(\mathcal{E}_{x_1}) \times_{\mathcal{N}(2,\mathcal{L})} \mathbb{P}(\mathcal{E}_{x_2}) \times_{\mathcal{N}(2,\mathcal{L})} \ldots \times_{\mathcal{N}(2,\mathcal{L})} \mathbb{P}(\mathcal{E}_{x_n}) \tag{3.1}$$

be the fiber product over $\mathcal{N}(2,\mathcal{L})$ of the projective bundles $\mathbb{P}(\mathcal{E}_{x_i}) \longrightarrow \mathcal{N}(2,\mathcal{L})$ obtained by restricting the universal projective bundle $\mathbb{P}(\mathcal{E})$ to $x_i \times \mathcal{N}(2,\mathcal{L})$.

Let Y be another compact connected Riemann surface of genus g, and let $\{y_1, \ldots, y_n\}$ be n distinct points on it. Construct the fiber product $\mathbb{P}'$ as done in (3.1) for the n-pointed Riemann surface Y.

The following theorem is a generalization of Theorem 2.1.

Theorem 3.2. *Let the two projective manifolds $\mathbb{P}$ and $\mathbb{P}'$ be isomorphic. Then there is an isomorphism $h : X \longrightarrow Y$ such that h takes the subset $\{x_1, \ldots, x_n\} \subset X$ to $\{y_1, y_2, \ldots, y_n\} \subset Y$.*

Proof. The first step will be to determine the numerically effective cone in $\mathrm{Pic}(\mathbb{P}) \bigotimes_{\mathbb{Z}} \mathbb{R}$.

Let $f_i : \mathbb{P} \longrightarrow \mathbb{P}(\mathcal{E}_{x_i})$ denote the projection onto the i-th factor. The projection of $\mathbb{P}$ onto $\mathcal{N}(2,\mathcal{L})$ will be denoted by f.

Let $\psi_i : \mathbb{P}(\mathcal{E}_{x_i}) \longrightarrow \mathcal{N}(2, L_i)$ be the morphism obtained by replacing x with x_i in the construction of ψ done in (2.2). The line bundle $\mathcal{L} \bigotimes \mathcal{O}_X(-x_i)$ is denoted by L_i.

Denoting the ample generator of the Picard group of the singular moduli space $\mathcal{N}(2, L_i)$ by Θ_i, define

$$\theta_i = (\psi_i \circ f_i)^* \Theta_i \tag{3.3}$$

to be the line bundle over $\mathbb{P}$. The line bundle $f^*\Theta$ over $\mathbb{P}$, where Θ, as before, is the ample generator of $\mathrm{Pic}(\mathcal{N}(2,\mathcal{L}))$, will be denoted by θ_0.

Since $\mathrm{Pic}(\mathbb{P}) = \mathbb{Z}^{\oplus(n+1)}$, at least $n+1$ points are required in order to generate the numerically effective cone in $\mathrm{Pic}(\mathbb{P}) \bigotimes_{\mathbb{Z}} \mathbb{R}$.

The following lemma describes the numerically effective cone in $\mathrm{Pic}(\mathbb{P}) \bigotimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 3.4. *The numerically effective cone in $Pic(\mathbb{P}) \bigotimes_{\mathbb{Z}} \mathbb{R}$ is generated by the $n+1$ line bundles θ_i, where $0 \leq i \leq n$.*

Proof of Lemma 3.4. Since $\mathbb{P}$ is a fiber product of $\mathbb{CP}^1$ bundles over $\mathcal{N}(2, \mathcal{L})$, using induction on n we conclude that $\{\theta_i\}_{0\leq i\leq n}$ generate $\mathrm{Pic}(\mathbb{P})$. It is obvious that each θ_i, where $0 \leq i \leq n$, is numerically effective. It is straight-forward to check that if the tensor product $\bigotimes_{0\leq i\leq n}\theta_i^{a_i}$ is isomorphic to the trivial line bundle, then $a_i = 0$ for all $i \in [0, n]$.

Following the argument given in Section 2 proving that $(p^*\Theta)^{\otimes a} \bigotimes(\psi^*\Theta_0)^{\otimes b}$ is ample if $a, b > 0$, and using induction on n, we conclude that the tensor product

$$\bigotimes_{i\in[0,n]} \theta_i^{\otimes a_i}$$

is ample if $a_i > 0$ for every $i \geq [0, n]$.

In order to complete the proof of the lemma it suffices to show that for every $j \in [0, n]$, the following tensor product of line bundles

$$\xi_j = \bigotimes_{i\in[0,n]-\{j\}} \theta_i$$

is not ample.

Take any $j \in [1, n]$. Consider the natural projection pr_j of $\mathbb{P}$ to the fiber product $\prod_{i\in[1,n]-j} \mathbb{P}(\mathcal{E}_{x_i})$ which forgets the j-th factor of the fiber product. The line bundle ξ_j is the pull back of a line bundle over $\prod_{i\in[1,n]-j} \mathbb{P}(\mathcal{E}_{x_i})$ using the projection pr_j. Therefore, ξ_j cannot be ample.

We need to show that ξ_0 is not ample to complete the proof of the lemma. This will done by comparing $\mathbb{P}$ with a certain moduli space of vector bundles with parabolic structure. The moduli space in question will be described now.

We consider rank two vector bundles with parabolic structures over the points $\{x_1, x_2, \ldots , x_n\}$. The parabolic structure at x_i on a vector bundle E is given by line $F_i \subset E_{x_i}$. The parabolic weights at any x_i are $\{1/n, 0\}$.

Let $\mathcal{M}$ denote the moduli space of parabolic semistable bundles $(E, \{F_1, F_2, \ldots , F_n\})$ over X of the above type and satisfying the condition $\bigwedge^2 E = \mathcal{L}$.

The projective variety $\mathcal{M}$ is known to be irreducible and normal [S]. We will show that $\mathcal{M}$ is not smooth.

Consider the direct sum $E = \mathcal{O}_X \bigoplus \mathcal{L}$. The parabolic flag at x_i is $(\mathcal{O}_X)_{x_i} \subset E_{x_i}$. The parabolic weights at x_i are $\{1/n, 0\}$. The parabolic bundle obtained this way is a direct sum of parabolic line bundles of parabolic degree one. Therefore, this rank two parabolic bundle is semistable but not stable. It can be checked that the point of the moduli space $\mathcal{M}$ represented by this parabolic bundle is singular. In other words, the variety $\mathcal{M}$ is singular. In fact, the argument presented in [NR, Remark 2.4, page 573] shows that $\mathcal{M}$ is not locally factorial.

Note that $\mathbb{P}$ is a moduli space of parabolic semistable bundles with sufficiently small parabolic weights. Indeed, If the sum of the parabolic weights is less than one, then such a moduli space of parabolic bundles with nontrivial quasi-parabolic structures over $\{x_1, \ldots, x_n\}$ is naturally identified with $\mathbb{P}$.

There is a Zariski open subset U_1 (respectively, U_2) of $\mathbb{P}$ (respectively, $\mathcal{M}$), with the complement of U_1 (respectively, U_2) being of codimension at least two, such that U_1 is isomorphic to U_2, with the isomorphism given by identifying the underlying quasi-parabolic bundles [NR, Section 6d].

Consider the generalized theta line bundle $\Theta_{\mathcal{M}}$ defined over the moduli space of parabolic bundles $\mathcal{M}$ [NR, page 572]. It is easy to check that the restriction $\xi_0|_{U_1}$ coincides with the restriction $(\Theta_{\mathcal{M}})^{\otimes n}|_{U_2}$ by the above mentioned identification between U_1 and U_2. This follows immediately by comparing the definitions. It is useful in making the comparison that the relative tangent line bundle over $\mathbb{P}$ for the projection pr_j defined earlier coincides with the line bundle $\theta_j^{\otimes 2} \otimes \theta_0^*$.

The line bundle $\Theta_{\mathcal{M}}$ over $\mathcal{M}$ is ample [NR, Theorem 1(B), page 572]. From this it can be deduced that ξ_0 is not ample. Indeed, as the complements of U_1 and U_2 are of codimension at least two, the spaces of sections of $\xi_0^{\otimes m}$ and $(\Theta_{\mathcal{M}})^{\otimes mn}$ are naturally identified. More precisely, these spaces of sections are identified with the space of sections of $(\xi_0|_{U_1})^{\otimes m}$ over U_1, or equivalently those of $(\Theta_{\mathcal{M}})^{\otimes mn}|_{U_2}$. Furthermore, assuming that ξ_0 is ample, for m sufficiently large both $\xi_0^{\otimes m}$ and $(\Theta_{\mathcal{M}})^{\otimes mn}$ are very ample. Since both $\mathbb{P}$ and $\mathcal{M}$ are the closure of the image of the same open set by the same embedding in the same projective space, $\mathbb{P}$ would be isomorphic to $\mathcal{M}$. However, we already observed that $\mathcal{M}$ is singular. So it cannot be isomorphic to the smooth variety $\mathbb{P}$.

Therefore, ξ_0 is not ample. This completes the proof of the lemma. $\square$

Starting with the given projective manifold $\mathbb{P}$, we first consider the $n+1$ half-lines in $\mathrm{Pic}(\mathbb{P}) \otimes_{\mathbb{Z}} \mathbb{R}$ defining the numerically effective cone. Any nontrivial line bundle in these half-lines is base point free if it is sufficiently divisible.

Let $\overline{\theta}_i$, where $i \in [0, n]$, be a sufficiently divisible line bundle in the i-th half-line among the $n+1$ half-lines defining the numerically effective cone in $\mathrm{Pic}(\mathbb{P}) \otimes_{\mathbb{Z}} \mathbb{R}$. So we have morphisms

$$h_i : \mathbb{P} \longrightarrow Z_i$$

given by the base point free line bundle $\overline{\theta}_i$. From our earlier observations we know that exactly one of the Z_i, say Z_0, is smooth and the rest are singular varieties.

As already noted in the previous section, there is exactly one Riemann surface X such that the moduli space of vector bundles over X is isomorphic to Z_0 [MN, page 1201].

Fix once and for all an isomorphism H of Z_0 with a moduli space $\mathcal{N}(2, \mathcal{L})$ of stable vector bundles over X.

For any $j \in [1, n]$, consider the linear system corresponding to a sufficiently large multiple of $\overline{\theta}_0 \otimes \overline{\theta}_j$, which is base point free. Let $\overline{h}_j : \mathbb{P} \longrightarrow \overline{Z}_j$ be the corresponding morphism. Now we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P} & = & \mathbb{P} \\ \Big\downarrow \overline{h}_j & & \Big\downarrow h_0 \\ \overline{Z}_j & \xrightarrow{f_j} & Z_0 \end{array}$$

The above projection f_j is a projective bundle over Z_0. Consider this as a projective bundle over the moduli space $\mathcal{N}(2, \mathcal{L})$ by using the isomorphism H between Z_0 and $\mathcal{N}(2, \mathcal{L})$. As in the proof of Theorem 2.1, this projective bundle determines a unique point x_j of X by considering the second Chern class of the unique rank two vector bundle E_j over $\mathcal{N}(2, \mathcal{L})$ which satisfies the following two conditions:

1. $\bigwedge^2 E_j$ is the ample generator of $\mathrm{Pic}(\mathcal{N}(2, \mathcal{L}))$;
2. the projective bundle $\mathbb{P}(E_j)$ over $\mathcal{N}(2, \mathcal{L})$ is isomorphic to the one defined by f_j.

The set of n points of X obtained this way from the n projective bundles $\{f_i\}_{1 \leq i \leq n}$ clearly coincide with the set of n points of X chosen originally to construct $\mathbb{P}$. This completes the proof of the theorem. $\square$

Theorem 3.2 can be generalized to the case of higher rank bundles. More precisely, let $\mathcal{N}$ denote the moduli space of fixed determinant stable bundles over X of rank r and degree one. Then a pair (X, S), where $S \subset X$ is a finite subset, is determined by the total space of the fiber product over $\mathcal{N}$ of the restrictions of the universal projective bundle corresponding to the points of S.

4. An infinitesimal Torelli condition

Let D denote the divisor $x_1 + x_2 + \ldots + x_n$ on X. The space of all infinitesimal deformations of the pointed curve $(X, \{x_1, x_2, \ldots, x_n\})$ is parametrized by the cohomology group $H^1(X,\, T_X \otimes \mathcal{O}_X(-D))$. Consider the exact sequence of cohomologies

$$0 \longrightarrow \bigoplus_{i=1}^{n} T_{x_i} X \longrightarrow H^1(X,\, T_X \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(X,\, T_X) \longrightarrow 0\,. \tag{4.1}$$

induced by the exact sequence

$$0 \longrightarrow T_X \otimes \mathcal{O}_X(-D) \longrightarrow T_X \longrightarrow \bigoplus_{i=1}^{n} T_{x_i}X \longrightarrow 0\,.$$

The projection in (4.1) corresponds to sending an infinitesimal deformation of the pointed curve to the infinitesimal deformation of X given by forgetting the points, and the inclusion in (4.1) corresponds to the deformations of the points of the fixed curve X.

Theorem 3.2 implies that the natural homomorphism

$$\delta \,:\, H^1(X,\, T_X \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(\mathbb{P},\, T_{\mathbb{P}})$$

is injective for the generic point $(X, \{x_1, x_2, \ldots, x_n\})$ of the moduli space of n-pointed curves.

However, the homomorphism δ is always an isomorphism, as shown by the following proposition. In other words, the infinitesimal Torelli is valid.

Proposition 4.2. *The homomorphism* $\delta : H^1(X,\, T_X \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(\mathbb{P},\, T_{\mathbb{P}})$ *is an isomorphism.*

Proof. Consider the direct image $f_*T_{\mathbb{P}}$, of the tangent sheaf $T_{\mathbb{P}}$, where f, as before, is the natural projection of $\mathbb{P}$ onto $\mathcal{N}(2, \mathcal{L})$. All the higher direct images of $T_{\mathbb{P}}$ evidently vanish [NaRa], and therefore,

$$H^1(\mathbb{P},\, T_{\mathbb{P}}) \,=\, H^1(\mathcal{N}(2, \mathcal{L}),\, f_*T_{\mathbb{P}})\,. \tag{4.3}$$

On the other hand, $f_*T_{\mathbb{P}}$ fits into the exact sequence of vector bundles

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Ad}(\mathcal{E}_{x_i}) \longrightarrow f_*T_{\mathbb{P}} \longrightarrow T_{\mathcal{N}(2,\mathcal{L})} \longrightarrow 0\,, \tag{4.4}$$

induced by the differential $df : T_{\mathbb{P}} \longrightarrow f^*T_{\mathcal{N}(2,\mathcal{L})}$, and the fact that the direct image p_*T_{rel} of the relative tangent bundle, where the projection p is defined in Section 2, coincides with the vector bundle $\mathrm{Ad}(\mathcal{E}_x) \subset \mathrm{End}(\mathcal{E}_x)$ consisting of trace zero endomorphisms.

It is known that $H^1(\mathcal{N}(2, \mathcal{L}),\, \mathrm{Ad}(\mathcal{E}_{x_i})) = T_{x_i}X$, and $H^2(\mathcal{N}(2, \mathcal{L}),\, \mathrm{Ad}(\mathcal{E}_{x_i})) = 0$ [NaRa, Theorem 2, page 392]. The infinitesimal deformation map for the family of vector bundles $\{\mathrm{Ad}(\mathcal{E}_z)\}_{z\in X}$, parametrized by X, induces the above identification of $H^1(\mathcal{N}(2, \mathcal{L}),\, \mathrm{Ad}(\mathcal{E}_{x_i}))$ with $T_{x_i}X$.

Now, using the isomorphism (4.3), the long exact sequence of cohomologies corresponding to (4.4) yields the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} T_{x_i}X \longrightarrow H^1(\mathbb{P},\, T_{\mathbb{P}}) \longrightarrow H^1(\mathcal{N}(2, \mathcal{L}),\, T_{\mathcal{N}(2,\mathcal{L})}) \longrightarrow 0\,. \tag{4.5}$$

Also, the natural homomorphism $H^1(X, T_X) \longrightarrow H^1(\mathcal{N}(2, \mathcal{L}), T_{\mathcal{N}(2,\mathcal{L})})$ is an isomorphism [NaRa].

Now from the compatibility of the exact sequences (4.1) and (4.5) with the homomorphism δ we conclude that δ must be an isomorphism. This completes the proof of the proposition. $\square$

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