# Instability Phenomena for the Fourier Coefficients

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**Abstract**

Let $P$ be an elliptic differential operator on a non-compact connected manifold $X$; suppose that both X and the coefficients of $P$ are real analytic. Given a pair of open sets $\mathcal{D}$ and $\sigma$ in $X$ with $\sigma \subset\subset \mathcal{D} \subset\subset X$, we fix a sequence $\{e_\nu\}$ of solutions to $Pu = 0$ in $\mathcal{D}$ which are pairwise orthogonal under integration over both $\mathcal{D}$ and $\sigma$. By orthogonality is meant the orthogonality in the corresponding Sobolev spaces; we also assume a completeness of the system on $\sigma$. For a fixed $y \in X \setminus \overline{\sigma}$, denote by $k_\nu(y)$ the Fourier coefficients of a fundamental solution $\Phi(\cdot, y)$ to $P$ with respect to the restriction of $\{e_\nu\}$ to $\sigma$. Suppose $K$ is a compact set in $\mathcal{D} \setminus \overline{\sigma}$, and let $f$ be a distribution with support on $K$. In this paper we show, under appropriate conditions on $K$, that if the moments $\langle f, k_\nu \rangle$ decrease sufficiently rapidly in a certain precise sense, then these moments vanish identically. In the most favorable cases, it is then possible to conclude that $f = 0$. This phenomenon was previously noticed by the first author and L.Zalcman for analytic and harmonic moments of $f$.

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# Contents



# 1 Introduction

Let $K$ be a compact set in the complex plane which does not contain the origin, and let $m$ be a finite complex measure on $K$ with moments

$$m_\nu = \int_K \frac{dm(\zeta)}{\zeta^{\nu+1}}, \qquad \nu = 0, 1, \dots. \tag{1.1}$$

Aizenberg and Zalcman [2] proved that if

$$\limsup_{\nu \leftarrow \infty} \sqrt[\nu]{|m_\nu|} < \frac{1}{\max_K |z|} \tag{1.2}$$

and $K$ does not separate 0 from $\infty$ (i.e., 0 belongs to the unbounded component of $\mathbf{C} \setminus K$), then $m_\nu = 0$ for all $\nu = 0, 1, \dots$.

Moreover, if $K$ does separate 0 from $\infty$, then for each sequence $\{m_\nu\}$ satisfying (1.2) there is a measure $m$ on $K$ having $\{m_\nu\}$ as its moments, i.e., such that (1.1) holds.

In this paper we explain this "instability phenomenon" in the context of the so-called *bases with double orthogonality.*

In Section 2 we briefly recall the concept of a basis with double orthogonality and show the conditions under which such bases exist. Sections 3 and 4 contain some explicit examples of bases with double orthogonality. In Section 5 we discuss a Liouville type theorem for solutions of elliptic equations. In Section 6 we state and prove our main result on the instability of the Fourier coefficients with respect to bases with double orthogonality. In Sections 7 and 8 we restrict our attention to the results of Aizenberg and Zalcman [2] and show how these results follow from ours (with the exception of Section 2 in [2] devoted to holomorphic moments in $\mathbf{C}^n$, $n > 1$). Finally, in the last section a new interesting example is indicated (of course, there are many other examples of the instability of the Fourier coefficients in various situations).

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# 2 Bases with double orthogonality

Let $P$ be an elliptic differential operator of order $p$ on a non-compact connected manifold $X$; suppose that both $X$ and the coefficients of $P$ are real analytic.

By a classical solution of $Pu = 0$ in an open set $U \subset X$ is meant any function $u \in C^p_{loc}(U)$ satisfying this equation pointwise in $U$. By a theorem of Petrovskii, any classical solution to $Pu = 0$ is in fact a real analytic function in $U$.

Moreover, if $u \in \mathcal{D}'(U)$ satisfies $Pu = 0$ weakly in $U$, then $u$ is induced by a classical solution of this equation (*Weil's Lemma*).

Given an open set $U \subset X$, we denote by $\mathrm{Sol}(U)$ the space of all classical solutions to the equation $Pu = 0$ in $U$. Moreover, let $\mathrm{Sol}(\overline{U})$ stand for the spase of $C^p$ functions $u$ which are solutions of the equation $Pu = 0$ in at least some neighborhood of the closure of $U$.

We will be interested in the subspaces of $\mathrm{Sol}(U)$ which possess Hilbert structures. Such a structure may be induced by the Hilbert structure of a Sobolev space $H^s(U)$, where $s$ is a non-negative integer.

Of course, $H^s(U)$ has no canonical Hilbert structure unless $U$ is a coordinate patch in $X$. The inner product of $H^s(U)$ does depend on the particular choice of the covering of (the closure of) $U$ by coordinate patches. However, if $U$ is a relatively compact open subset of $X$, then the topology in $H^s(U)$ is actually independent of the coverings.

From what has already been said it follows that the subspace of $H^s(U)$ which consists of classical solutions to $Pu = 0$ in $U$ is closed. Thus, the intersection $H^s(U) \cap \mathrm{Sol}(U)$ is a Hilbert space with the inner product inherited from $H^s(U)$.

We now fix two relatively compact domains $\mathcal{D}$ and $\sigma$ in $X$ such that $\sigma \subset \mathcal{D}$. In what follows we assume that both $\mathcal{D}$ and $\sigma$ have strong cone property. This ensures the equivalence of two possible definitions of Sobolev spaces on these domains, namely internal and external spaces (see Example 1.4.24 in [16]).

**Definition 2.1** *A system $e_\nu$, $\nu = 1, 2, \ldots$, in $Sol(\mathcal{D})$ is said to be a basis with double orthogonality if it is an orthonormal basis in $H^s(\mathcal{D}) \cap Sol(\mathcal{D})$ and its restriction to $\sigma$ is an orthogonal basis in $H^s(\sigma) \cap Sol(\sigma)$.*

We are going to prove that such bases always exist unless $\sigma$ has "holes" in $\mathcal{D}$. Moreover, we show an explicit way of constructing bases with the property of double orthogonality.

To this end, set $L_1 = H^s(\mathcal{D})$ and $L_2 = H^s(\sigma)$, so that $L_1$ and $L_2$ are separable Hilbert spaces.

The mapping $T : L_1 \to L_2$ is defined to be the restriction from $\mathcal{D}$ to $\sigma$, i.e., $Tu = u|_\sigma$ for $u \in L_1$. Then $T$ is a continuous linear operator with norm $\|T\|_{\mathcal{L}(L_1 \to L_2)} \leq 1$.

Further, we distinguish in $L_1$ the subspace $H_1$ which is formed by classical solutions of the equation $Pu = 0$ in $\mathcal{D}$. As above, $H_1$ is a closed subspace of $L_1$, and so $H_1$, when endowed with the induced Hermitian structure, becomes a Hilbert space.

Letting $H_2$ denote the subspace of $L_2$ consisting of classical solutions of the equation $Pu = 0$ in $\sigma$, we see that the restriction of $T$ to $H_1$ maps to $H_2$. However, it is not evident that the image of $H_1$ by $T$ is dense in $H_2$.

**Lemma 2.2** *If the complement of $\sigma$ has no compact connected components belonging to $\mathcal{D}$, then the operator $T : H_1 \to H_2$ has dense range.*

**Proof.** Our task is to prove that restrictions to $\sigma$ of elements in $H^s(\mathcal{D}) \cap Sol(\mathcal{D})$ are dense in $H^s(\sigma) \cap Sol(\sigma)$ in the $H^s(\sigma)$-norm. Since $\sigma$ has strong cone property, Theorems 8.1.2 and 8.4.1 in Tarkhanov [16] show that the subspace $Sol(\overline{\sigma})$ is dense in $H^s(\sigma) \cap Sol(\sigma)$ in the $H^s(\sigma)$-norm. On the other hand, as the complement of $\sigma$ has no compact connected components in $\mathcal{D}$, the subspace $Sol(\overline{\mathcal{D}})$ is, by the *Runge Theorem*, dense in $Sol(\overline{\sigma})$ in the inductive limit topology of $C^\infty(\overline{\sigma})$. Since $Sol(\overline{\mathcal{D}}) \subset H^s(\mathcal{D}) \cap Sol(\mathcal{D})$ and the inductive limit topology in $Sol(\overline{\sigma})$ is stronger than the topology induced by the $H^s(\sigma)$-norm, we obtain even more than we wanted to prove.

□

To describe the adjoint mapping for $T : H_1 \to H_2$, we denote by $T^*$ the adjoint for $T$ acting on the whole $L_1$ and by $\pi$ the orthogonal projection of $L_1$ onto $H_1$. It follows from Subsection 4.2.8 of [15] that $H_1$ is a *Hilbert space with reproducing kernel.* Therefore, $\pi$ is an integral operator whose kernel is the reproducing kernel $\mathcal{K}$ of the domain $\mathcal{D}$ with respect to $H^s(\mathcal{D}) \cap Sol(\mathcal{D})$.

**Lemma 2.3** *For any $u_0 \in L_2$, we have*

$$\pi T^* u_0(x) = (u_0, \mathcal{K}(x, \cdot))_{H^s(\sigma)}, \qquad x \in \mathcal{D}.$$

**Proof.** Indeed,

$$\begin{aligned} \pi T^* u_0 &= (T^* u_0, \mathcal{K}(x, \cdot))_{L_1} \\ &= (u_0, T\mathcal{K}(x, \cdot))_{L_2}, \end{aligned}$$

which establishes the formula.

□

A trivial verification shows that the adjoint mapping for $T : H_1 \to H_2$ is given by the restriction of the operator $\pi T^*$ to $H_2$. Hence, the composition $\pi T^* T$, when restricted to $H_1$, is a selfadjoint operator in this space. (If $s = 0$, then $\pi T^* T$ is a *Toeplitz operator* in $L_1$.)

As follows, the bases with double orthogonality are complete systems of eigenfunctions of the operator $\pi T^* T$ in $H_1$.

To handle the corresponding eigenvalue problem, let us look more closely at the properties of the restriction of $T$ to $H_1$.

**Lemma 2.4** *The operator $T: H_1 \to H_2$ is injective.*

**Proof.** Let $u \in H_1$ and $Tu = 0$. This means that $u$ is a solution of the equation $Pu = 0$ in the domain $\mathcal{D}$ vanishing on the nonempty open subset $\sigma$ of $\mathcal{D}$. Hence the real analyticity of $u$ implies that $u \equiv 0$ everywhere in $\mathcal{D}$, as desired.

□

However, the most important property of $T$ is the following.

**Lemma 2.5** *The operator $T: H_1 \to H_2$ is compact.*

**Proof.** The task is to show, given any bounded set $B \subset H_1$, that the image of $B$ by the mapping $T$ is relatively compact in $H_2$.

Let $B$ be a bounded subset of $H_1$, i.e., there is a constant $R > 0$ such that $\|u\|_{L_1} \leq R$ for all $u \in B$. The image of $B$ by $T$ (denoted by $T(B)$) is relatively compact in $H_2$ if from any sequence $\{u_{0,i}\} \subset T(B)$ one can extract a subsequence $\{u_{0,i_\nu}\}$ converging in $H_2$.

However, if $\{u_{0,i}\} \subset T(B)$, then $u_{0,i} = u_i|_\sigma$, where $\{u_i\} \subset B$. As the sequence $\{u_i\}$ is bounded in the Hilbert space $H_1$, it contains a subsequence $\{u_{i_\nu}\}$ which converges weakly to some element $u \in H_1$. Clearly, $\{u_{i_\nu}\}$ converges to $u$ in the topology of the space of distributions in $\mathcal{D}$.

We now invoke the *Stieltijes-Vitali Theorem* (see Subsection 2.1.5 in [15]) to conclude that $\{u_{i_\nu}\}$, converges to $u$ together with all derivatives uniformly on compact subsets of $\mathcal{D}$. Setting $u_0 = u|_\sigma$ and $u_{0,i_\nu} = u_{i_\nu}|_\sigma$, we can assert that $u_0 \in H_2$ and $\{u_{0,i_\nu}\}$ converges to $u_0$ in $H_2$.

This is the desired conclusion.

□

We can now formulate the main result concerning the existence of bases with double orthogonality.

**Theorem 2.6** *If $\sigma$ is a subdomain of $\mathcal{D}$ such that $\mathcal{D}\setminus\sigma$ has no compact connected components, then there is an orthonormal basis $\{e_\nu\}$ in $H^s(\mathcal{D}) \cap Sol(\mathcal{D})$ whose restriction to $\sigma$ is an orthogonal basis in $H^s(\sigma) \cap Sol(\sigma)$.*

**Proof.** Consider the operator $\pi T^* T$ in $H_1$. This operator is selfadjoint, injective and compact. According to the *Spectral Theorem*, $\pi T^* T$ has a complete system of normalized eigenfunctions $\{e_\nu\}_{\nu=1,2,\dots}$ corresponding to eigenvalues $\{\lambda_\nu\} \subset (0,1]$. An easy computation shows that $(Te_\mu, Te_\nu)_{L_2} = \lambda_\mu \, (e_\mu, e_\nu)_{L_1}$, and so the system $\{Te_\nu\}$ is orthogonal in $L_2$. Since $T: H_1 \to H_2$ has dense range, $\{Te_\nu\}$ is an orthogonal basis in $H_2$, which is our claim.

□

The concept of sequences of analytic functions which are pairwise orthogonal simultaneously in two domains, one of which contains the other, is due to Bergman (see [4], p. 14–20). Shapiro [10] is convinced that Bergman knew well that the phenomenon of double orthogonality was of general character. Krasichkov [9] showed that a simple application of the spectral theorem leads to an abstract Bergman theorem on the existence of bases with double orthogonality. Our account in this section reproduces Bergman's concept in general (see also Shlapunov and Tarkhanov [12] and Tarkhanov [16, Ch.12]).

# 3 A basis of holomorphic monomials

Let $P = \partial/\partial\bar{z}$ be the Cauchy-Riemann operator in the complex plane $\mathbf{C} \cong \mathbf{R}^2$.

Denote by $B(0,R)$ the disk of center 0 and radius $R$ in the plane, and by $Hol(B(0,R))$ the space of holomorphic functions in the disk.

**Lemma 3.1** *For any $0 < R < \infty$, the system $\left\{\frac{1}{\sqrt{\pi}}\frac{\sqrt{1+\nu}}{R^{1+\nu}}\, z^\nu\right\}_\nu$ $\nu = 0,1,\ldots$, is an orthonormal basis in $L^2(B(0,R)) \cap Hol(B(0,R))$, and an orthogonal basis in $L^2(B) \cap Hol(B)$ where $B$ is an arbitrary disk with center at the origin.*

**Proof.** We begin by proving that the system $\{z^\nu\}$ is orthogonal in any ball $B(0,R)$. For this purpose, we use the polar coordinates $z = re^{\imath\varphi}$ in obtaining

$$
\begin{aligned}
(z^\mu, z^\nu)_{L^2(B(0,R))} &= \int_{B(0,R)} z^\mu\, \overline{z^\nu}\, dv(z) \\
&= \int_0^R r^{1+\mu+\nu}\, dr \int_0^{2\pi} e^{\imath(\mu-\nu)\varphi}\, d\varphi \\
&= \frac{R^{2+\mu+\nu}}{2+\mu+\nu} \int_0^{2\pi} e^{\imath(\mu-\nu)\varphi}\, d\varphi,
\end{aligned}
$$

from which the desired conclusion follows.

We shall have established the lemma if we prove the following: for any disk $B$, the restriction of the system $\{z^\nu\}$ to $B$ is complete in the space $L^2(B) \cap Hol(B)$. However, combining the contraction $u(z) \mapsto u(t\,z)$, $t \in [0,1]$, with the *Runge Theorem* we deduce that the holomorphic polynomials, when restricted to $B$, are dense in $L^2(B) \cap Hol(B)$. On the other hand, any holomorphic polynomial is a finite linear combination of the monomials $z_\nu$. Hence our assertion follows.

□

We fix positive $r < R$, and set $\mathcal{D} = B(0,R)$, $\sigma = B(0,r)$ so that $\sigma \subset \mathcal{D}$. Then Lemma 3.1 just amounts to saying that the restriction of the system

$$
e_\nu = \frac{1}{\sqrt{\pi}}\frac{\sqrt{1+\nu}}{R^{1+\nu}}\, z^\nu, \qquad \nu = 0,1,\ldots,
$$

to $\mathcal{D}$ is a *basis with double orthogonality* in $L^2(\mathcal{D}) \cap Hol(\mathcal{D})$.

# 4 A basis of harmonic polynomials

Assume now that $P = \Delta$ is the Laplace operator in the space $\mathbf{R}^n$.

Let $\left\{h_\nu^{(j)}\right\}$ be a set of the homogeneous harmonic polynomials in $\mathbf{R}^n$ which form a complete orthonormal system in the Lebesgue space $L^2$ on the unit sphere $\{|y| = 1\}$ (*spherical harmonics*). The index $\nu$ means the degree of homogeneity, and the index $j$ runs through the number of polynomials of the degree $\nu$ in the basis. The dependence of the range of $j$ on $\nu$ is well-known, namely $j = 1, \ldots, J(\nu)$ where

$$J(\nu) = \frac{(n + 2\nu - 2)(n + \nu - 3)!}{\nu!(n-2)!}$$

for $n > 2$. If $n = 2$, then, obviously, $J(0) = 1$ and $J(\nu) = 2$ for $\nu \geq 1$ (see Sobolev [13, p. 453]).

**Example 4.1** If $n = 2$, then, as a system of homogeneous harmonic polynomials $\left\{h_\nu^{(j)}\right\}$, we can take the system $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} z^\nu, \frac{1}{\sqrt{2\pi}} \bar{z}^\nu\right\}$ where $z = x_1 + i x_2$. □

Denote by $B(0, R)$ the ball of center 0 and radius $R$ in the space, and by $Harm(B(0, R))$ the space of harmonic functions in the ball. The main property of the system $\left\{h_\nu^{(j)}\right\}$ is established by our next lemma.

**Lemma 4.2** *For any* $0 < R < \infty$, *the system* $\left\{\sqrt{\frac{n+2\nu}{R^{n+2\nu}}}\, h_\nu^{(j)}\right\}$ *is an orthonormal basis in* $L^2(B(0,R)) \cap Harm(B(0,R))$, *and an orthogonal basis in* $L^2(B) \cap Harm(B)$ *where* $B$ *is an arbitrary ball with center at zero.*

**Proof.** We begin by proving that the system $\left\{h_\nu^{(j)}\right\}$ is orthogonal in any ball $B(0, R)$. For this purpose, we write

$$\begin{aligned}
\left(h_\mu^{(j)}, h_\nu^{(i)}\right)_{L^2(B(0,R))} &= \int_{B(0,R)} h_\mu^{(j)}(x)\, \overline{h_\nu^{(i)}(x)}\, dx \\
&= \int_0^R r^{n-1+\mu+\nu}\, dr \int_{|z|=1} h_\mu^{(j)}(z)\, \overline{h_\nu^{(i)}(z)}\, ds \\
&= \frac{R^{n+\mu+\nu}}{n+\mu+\nu} \left(h_\mu^{(j)}, h_\nu^{(i)}\right)_{L^2(\partial B(0,1))},
\end{aligned}$$

which is the desired conclusion.

We shall have established the lemma if we prove the following: for any ball $B$, the restriction of the system $\left\{h_\nu^{(j)}\right\}$ to $B$ is complete in the space $L^2(B) \cap Harm(B)$. However, combining the contraction $u(x) \mapsto u(t\,x)$, $t \in [0, 1]$, with the *Runge Theorem* we deduce that the harmonic polynomials, when restricted to $B$, are dense in $L^2(B) \cap Harm(B)$. On the other hand, any harmonic polynomial is the sum of a finite number of homogeneous harmonic polynomials, and any homogeneous harmonic polynomial of degree $\nu$ is a linear combination of the polynomials $h_\nu^{(1)}, \dots, h_\nu^{(J(\nu))}$ (cf. Sobolev [13]). Hence our assertion follows.

□

We fix $0 < r < R$, and set $\mathcal{D} = B(0,R)$, $\sigma = B(0,r)$ so that $\sigma \subset \mathcal{D}$. Then Lemma 4.2 just amounts to saying that the restriction of the system

$$e_\nu^{(j)} = \sqrt{\frac{n+2\nu}{R^{n+2\nu}}}\, h_\nu^{(j)}, \qquad \nu = 0,1,\dots;\ j = 1,\dots,J(\nu),$$

to $\mathcal{D}$ is a *basis with double orthogonality* in $L^2(\mathcal{D}) \cap Harm(\mathcal{D})$.

# 5 A Liouville theorem

From now on, fix a fundamental solution $\Phi$ of the differential operator $P$. That such a solution exists follows from a theorem of Malgrange (see also Theorem 4.4.3 in Tarkhanov [16]. Moreover, $\Phi$ is a pseudodifferential operator of order $-p$ on $X$; its Schwartz kernel $\Phi(x,y)$ is a distribution in the product $X \times X$ real analytic away from the diagonal $\{x = y\}$.

We are aimed in extending *Liouville's Theorem* to solutions of the equation $Pu = 0$ in $X$. To this end, we need a suitable concept of the solution to $Pu = 0$ at infinity. However, this depends on a compactification of $X$.

We shall use the so-called one-point, or Aleksandrov, compactification of $X$, denoted by $\widehat{X}$. This means that $\widehat{X}$ is the union of $X$ and the symbolic point $\infty$, and the topology of $\widehat{X}$ is given by the following neighborhoods bases:

- If $x \in X$, then we take the usual basis of neighborhoods of $x$ (for example, the family of all balls centered at $x$).
- If $x = \infty$, then the basis of neighborhoods of $x$ is defined to be the family $\{U \cup \infty\}$, where $U$ is an open subset of $X$ with compact complement.

If $f$ is a distribution with compact support in $X$, then the potential $u = \Phi(f)$ satisfies $Pu = 0$ away from the support of $f$.

**Definition 5.1** *Let $u$ be a solution of $Pu = 0$ on an open set $U$ in $X$ with compact complement. Then $u$ is said to be regular at infinity if in a neighborhood of $\infty$ we have $u = \Phi(f)$, where $f \in \mathcal{E}'(X)$.*

We emphasize that this definition depends in an essential way on the choice of the fundamental solution $\Phi$ to $P$.

**Lemma 5.2** *If $u$ satisfies $Pu = 0$ on the whole manifold $X$ and is regular at infinity, then $u \equiv 0$.*

This result is actually due to Grothendieck (see also Section 5.4 in Tarkhanov [15]). Because the proof is simple, we present it for the convenience of the reader.

**Proof.** Let $f \in \mathcal{E}'(X)$ be such that $u = \Phi(f)$ in the complement of a compact subset of $X$. Then $u' = u - \Phi(f)$ is a compactly supported distribution in $X$, and so $u' = \Phi(Pu')$. Hence it follows that $u = \Phi(f + Pu')$. Applying the differential operator $P$ to both parts of this equality yields $0 \equiv f + Pu'$ on $X$. Therefore, $u \equiv 0$ on $X$, as desired.

□

# 6 Instability phenomena

In what follows, we assume that $\mathcal{D}$ is a relatively compact domain in $X$ with connected boundary.

Let $\sigma$ be a subdomain of $\mathcal{D}$ whose complement has no compact connected components in $\mathcal{D}$. (Under the assumption above, the last condition just amounts to saying that $\mathcal{D} \setminus \sigma$ is connected.)

By Theorem 2.6, there is an orthonormal basis $\{e_\nu\}$ in $H^s(\mathcal{D}) \cap Sol(\mathcal{D})$ whose restriction to $\sigma$ is an orthogonal basis in $H^s(\sigma) \cap Sol(\sigma)$. We fix such a basis.

For fixed $y \in X \setminus \overline{\sigma}$, the fundamental solution $\Phi(\cdot, y)$ satisfies $P\,\Phi(\cdot, y) = 0$ in a neighborhood of the closure of $\sigma$. We denote by $k_\nu(y)$ the Fourier coefficients for the restriction of $\Phi(\cdot, y)$ to $\sigma$ with respect to the orthogonal system $\{e_\nu\}$ in $H^s(\sigma) \cap Sol(\sigma)$, i.e.,

$$k_\nu(y) = \frac{(\Phi(\cdot, y), e_\nu)_{H^s(\sigma)}}{\|e_\nu\|^2_{H^s(\sigma)}}, \qquad \nu = 1, 2, \ldots. \tag{6.1}$$

**Lemma 6.1** *For $\nu = 1, 2, \ldots$, the coefficients $k_\nu$ satisfy the transposed equation $P'\,k_\nu = 0$ away from the closure of $\sigma$ on $X$.*

**Proof.** This is obvious because of the equality $P'(y, D)\,\Phi(x, y) = \delta_x(y)$ on $X$, where $\delta_x$ is the Dirac delta-function supported at $x \in X$.

□

Given a compact set $K \subset \mathcal{D} \setminus \overline{\sigma}$ and a distribution $f$ with support on $K$, we consider the moments

$$c_\nu = \int k_\nu(y)\, f(y)\, dv(y), \qquad \nu = 1, 2, \ldots. \tag{6.2}$$

The set $K$ is said *to do not separate $\sigma$ from $\infty$* if $\sigma$ belongs to the component of $\widehat{X} \setminus K$ which contains $\infty$.

**Theorem 6.2** *If*

$$\sum_{\nu=1}^{\infty} |c_\nu|^2 < \infty, \tag{6.3}$$

*and $K$ does not separate $\sigma$ from $\infty$, then $c_\nu = 0$ for all $\nu$.*

*Moreover, if $K$ does separate $\sigma$ from $\infty$, then for each sequence $\{c_\nu\}$ satisfying (6.3) there is a distribution $f$ with support on $K$ having $\{c_\nu\}$ as its moments, i.e., such that (6.2) holds.*

**Proof.** Suppose that $K$ does not separate $\sigma$ and $\infty$. Consider the potential

$$u'(x) = \int \Phi(x,y)\, f(y)\, dv(y), \qquad x \in X \setminus K.$$

Clearly, $u'$ satisfies $Pu' = 0$ in the complement of $K$ in $X$, in particular, in a neighborhood of the closure of $\sigma$. Expanding the restriction of $u'$ to $\sigma$ in a Fourier series with respect to the orthogonal basis $\{e_\nu\}$ in $H^s(\sigma) \cap Sol(\sigma)$, we find

$$u'(x) = \sum_{\nu=1}^{\infty} c_\nu\, e_\nu(x), \qquad x \in \sigma,$$

where the $c_\nu$ are given by (6.2).

If (6.3) holds, this series converges, by the *Fischer-Riesz Theorem*, in the norm of $H^s(\mathcal{D})$ to a function $u''$ satisfying $Pu'' = 0$ in $\mathcal{D}$. Letting $U$ denote the component of $\widehat{X} \setminus K$ which contains $\infty$, we consider the function

$$u(x) = \begin{cases} u'(x), & \text{when } x \in U; \\ u''(x), & \text{when } x \in \mathcal{D} \setminus U. \end{cases}$$

Obviously, $u$ is an analytic extension of $u'$ to the whole manifold $X$. It follows that $Pu = 0$ on $X$.

On the other hand, $u$ is regular at infinity. By Lemma 5.2, $u$ vanishes identically on $X$.

So $u' \equiv 0$ and $c_\nu = 0$ for $\nu = 1, 2, \ldots$. This proves the first part of the theorem.

Now suppose that $K$ separates $\sigma$ from $\infty$. We denote by $U$ the complement of the closure of $\sigma$ in $\widehat{X}$. Then $U$ is a subdomain of $\widehat{X}$ containing $K$.

Define the space $C^{p-1}(K)$ as the inductive limit of the sequence $C^{p-1}_{loc}(U_i)$, where $\{U_i\}$ is a sequence of neighborhoods of $K$ with $U_{i+1} \subset\subset U_i$ and $\cap_i U_i = K$. It is easy to see that the dual space for $C^{p-1}(K)$ can be identified with the subspace of distributions of order $p-1$ on $X$ consisting of those supported on $K$.

Let $Sol(U, P')$ be the space of all classical solutions to the transposed equation $P'g = 0$ on $U$ which are regular at infinity (with respect to the fundamental solution $\Phi'$ of $P'$). We endow this space with the topology induced by $C^{p-1}(K)$.

Assume that $\{c_\nu\}$ is a sequence of complex numbers which satisfies (6.3). By the *Fischer-Riesz Theorem*, the series $u = \sum_{\nu=1}^{\infty} c_\nu\, e_\nu$ converges in the norm of $H^s(\mathcal{D})$ to a function $u$ satisfying $Pu = 0$ in $\mathcal{D}$.

Fix a Green operator $G_P$ for $P$ on $X$, i.e., a bidifferential operator of order $p-1$ with values in the space of differential forms of degree $n-1$ such that $d\,G_P(g,u) = (g\,Pu - P'g\,u)dv$ pointwise on $X$, for all $g$ and $u$ smooth enough. Now set

$$\langle \mathcal{F}_u, g\rangle = -\int_{\partial \mathcal{O}} G_P(g,u), \qquad g \in Sol(U, P'), \tag{6.4}$$

where $\mathcal{O}$ is a relatively compact subdomain of $\mathcal{D}$ with piecewise smooth boundary such that $\overline{\sigma} \subset \mathcal{O}$.

It follows from Stokes' formula that the functional $\mathcal{F}_u$ is independent of the particular choice of the domain $\mathcal{O}$. In particular, we may assume that $K \subset \mathcal{O}$, for if not, then we enlarge $\mathcal{O}$. So, applying Green's formula for solutions of $P'g = 0$ regular at infinity shows that $\mathcal{F}_u$ is a continuous linear functional on $Sol(U, P')$.

By the *Hahn-Banach Theorem*, $\mathcal{F}_u$ can be extended to the whole space $C^{p-1}(K)$. Hence there is a distribution $f$ of order $p-1$ with support on $K$ such that

$$\langle \mathcal{F}_u, g\rangle = \int g\, f\, dv \qquad \text{for all } g \in Sol(U, P'). \tag{6.5}$$

For every $\nu = 1, 2, \ldots$, the function $k_\nu(y)$ is, by Lemma 6.1, in $Sol(U, P')$. Thus, applying (6.5) we get

$$\langle \mathcal{F}_u, k_\nu\rangle = \int k_\nu(y)\, f(y)\, dv(y), \qquad \nu = 1, 2, \ldots. \tag{6.6}$$

On the other hand, (6.4) and Green's formula for $u$ yield

$$\begin{aligned} \langle \mathcal{F}_u, k_\nu\rangle &= -\int_{\partial\mathcal{O}} G_P(k_\nu, u) \\ &= \frac{\left(-\int_{\partial\mathcal{O}} G_P(\Phi(\cdot, y), u(y)), e_\nu\right)_{H^s(\sigma)}}{\|e_\nu\|^2_{H^s(\sigma)}} \\ &= \frac{(u, e_\nu)_{H^s(\sigma)}}{\|e_\nu\|^2_{H^s(\sigma)}} \\ &= c_\nu, \end{aligned}$$

for all $\nu = 1, 2, \ldots$.

Comparing the latter equalities with (6.6), we obtain the desired conclusion.

□

In the most favorable cases, it is possible to conclude from (6.3) that $f = 0$.

**Corollary 6.3** *Let $K$ be a compact subset of $\mathcal{D} \setminus \overline{\sigma}$ of zero measure, and let the complement of $K$ is connected. For any distribution $f$ of order $p-1$ with support on $K$, the condition (6.3) implies $f \equiv 0$.*

**Proof.** Consider the potential

$$u'(x) = \int \Phi(x, y)\, f(y)\, dv(y), \qquad x \in X \setminus K.$$

As above,

$$u'(x) = \sum_{\nu=1}^{\infty} c_\nu\, e_\nu(x), \qquad x \in \sigma,$$

where the $c_\nu$ are given by (6.2).

Since $K$ does not separate $\sigma$ from $\infty$, Theorem 6.2 shows that $c_\nu = 0$ for all $\nu$. It follows that $u' \equiv 0$ in $\sigma$.

However, combining Theorems 5.3.2 and 6.3.1 in Tarkhanov [16] we deduce that finite linear combination of the potentials $\Phi(x, \cdot)$, where $x \in \sigma$, are dense in $C^{p-1}(K)$. For this reason, the equality $u' \equiv 0$ in $\sigma$ implies $f \equiv 0$ on $X$, as desired.

□

We finish this section with an example showing that a regularity condition for $f$ is necessary for Corollary 6.3 be valid.

**Example 6.4** Given a smooth closed surface $S$ in $\mathcal{D} \setminus \overline{\sigma}$, denote by $\delta_S$ the functional of integration over $S$ (*surface layer on* $S$). This is a measure with support $S$ on $X$, and so $f = P\delta_S$ is well-defined to be a distribution supported in $S$. Since $\Phi(f) = \delta_S$, the potential $u' = \Phi(f)$ vanishes away from $S$. It follows that all of the moments (6.2) are zero for $f$ while $f \neq 0$.

□

# 7 Holomorphic moments

Assume that $P$ is the Cauchy-Riemann operator in the complex plane, and $\{e_\nu\}$ the basis with double orthogonality in $L^2(B(0,R)) \cap Hol(B(0,R))$ constructed in Section 3.

The standard fundamental solution of the Cauchy-Riemann operator in $\mathbf{C}$ is $\Phi(z,\zeta) = \frac{1}{\pi}\frac{1}{z-\zeta}$.

The Fourier coefficients of the restriction of $\Phi(z,\zeta)$ to $B(0,r)$, for fixed $\zeta$ away from the closure of $B(0,r)$, are given in our next lemma which is elementary.

**Lemma 7.1** *If* $|\zeta| > r$, *then*

$$k_\nu(\zeta) = -\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+\nu}}\left(\frac{R}{\zeta}\right)^{1+\nu}, \qquad \nu = 0,1,\ldots. \tag{7.1}$$

**Proof.** Use the fact that in the cone $C = \{(z,\zeta) \in \mathbf{C} \times \mathbf{C} : \ |\zeta| > |z|\}$ one has

$$\frac{1}{\pi}\frac{1}{z-\zeta} = -\frac{1}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{\zeta^{\nu+1}}z^\nu;$$

the series converges together with all the derivatives in $z$ and $\zeta$ absolutely and uniformly on closed subsets of $C$.

□

Given a distribution $f$ with a compact support $K$ in $B(0,R) \setminus \overline{B(0,r)}$, define the *holomorphic moments* of $f$ by

$$c_\nu = \int k_\nu(\zeta)\, f(\zeta)\, dv(\zeta), \qquad \nu = 0,1,\ldots.$$

Set

$$m_\nu = \int f(\zeta)\,\frac{1}{\zeta^{\nu+1}}\, dv(\zeta), \qquad \nu = 0,1,\ldots,$$

just as in (1.1). Then condition (1.2) is written as

$$\limsup_{\nu\to\infty} \sqrt[\nu]{|m_\nu|} < \frac{1}{R'}, \tag{7.2}$$

where $R' = \max_K |z|$.

On the other hand, condition (6.3) becomes

$$\sum_{\nu=0}^{\infty} \frac{R^{2(1+\nu)}}{1+\nu} |m_\nu|^2 < \infty \tag{7.3}$$

because of (7.1).

Let us now compare Theorem 6.2 in the case of holomorphic moments under consideration to the result of Aizenberg and Zalcman [2] cited in Section 1.

If (7.3) holds, then there is a constant $c$ such that

$$|m_\nu| \le c \frac{\sqrt{1+\nu}}{R^{1+\nu}}, \qquad \nu = 0, 1, \ldots,$$

and so

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|m_\nu|} < \frac{1}{R}.$$

Since $R' < R$, condition (7.2) follows.

Conversely, if (7.2) is fulfilled, then

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|m_\nu|} < \frac{1}{R' + \varepsilon}.$$

for some $\varepsilon > 0$ small enough. Pick $R$ such that $R' < R < R' + \varepsilon$. For such an $R$, the general term in (7.3) can be estimated by

$$\frac{R^{2(1+\nu)}}{1+\nu} |m_\nu|^2 < c' \frac{1}{1+\nu} \left( \frac{R}{R' + \varepsilon} \right)^{2(1+\nu)},$$

and hence (7.3) holds.

Therefore, the mentioned result of Aizenberg and Zalcman [2] is a consequence of our Theorem 6.2.

# 8 Harmonic moments

Assume that $P$ is the Laplace operator in the space $\mathbf{R}^n$, and $\left\{ e_\nu^{(j)} \right\}$ the basis with double orthogonality in $L^2(B(0,R)) \cap Harm(B(0,R))$ constructed in Section 4.

The standard fundamental solution of the Laplace operator in $\mathbf{R}^n$ is

$$\Phi(x,y) = \begin{cases} \frac{1}{2\pi} \log |x-y| & \text{when } n = 2; \\ \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{1}{2-n} \frac{1}{|x-y|^{n-2}} & \text{when } n > 2, \end{cases}$$

The Fourier coefficients of the restriction of $\Phi(x,y)$ to $B(0,r)$, for fixed $y$ away from the closure of $B(0,r)$, are given in our next lemma.

**Lemma 8.1** *If* $|y| > r$, *then*

$$k_\nu^{(j)}(y) = -\frac{1}{n+2\nu-2} \sqrt{\frac{R^{n+2\nu}}{n+2\nu}} \frac{\overline{h_\nu^{(j)}(y)}}{|y|^{n+2\nu-2}}, \qquad \nu = 0, 1, \ldots; \ j = 1, \ldots, J(\nu). \tag{8.1}$$

**Proof.** Use the fact that in the cone $C = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \ |y| > |x|\}$ one has

$$\Phi(x,y) = \Phi(0,y) - \sum_{\nu=1}^{\infty} \sum_{j=1}^{J(\nu)} \frac{1}{n+2\nu-2} \frac{\overline{h_\nu^{(j)}(y)}}{|y|^{n+2\nu-2}} h_\nu^{(j)}(x); \tag{8.2}$$

the series converges together with all the derivatives in $x$ and $y$ absolutely and uniformly on closed subsets of $C$.

□

Expansion (8.2) has been frequently used, beginning with the paper of Deny [8]. There may well be an earlier source.

Given a distribution $f$ with compact support in $B(0,R) \setminus \overline{B(0,r)}$, define the *harmonic moments* of $f$ by

$$c_\nu^{(j)} = \int k_\nu^{(j)}(y)\, f(y)\, dv(y), \qquad \nu = 0, 1, \ldots;\ j = 1, \ldots, J(\nu).$$

The condition

$$\limsup_{\nu \to \infty} \max_j \sqrt[\nu]{\left| \int f(y) \frac{\overline{h_\nu^{(j)}(y)}}{|y|^{n+2\nu-2}}\, dv(y) \right|} < \frac{1}{R}$$

is easily seen to imply

$$\sum_{\nu=0}^{\infty} \sum_{j=1}^{J(\nu)} |c_\nu^{(j)}|^2 < \infty.$$

Thus, applying Theorem 6.2 yields the following result due to Aizenberg and Zalcman [2]. (They proved it for $f$ being a measure.)

Let $K$ be a compact set in $\mathbf{R}^n$ which does not contain the origin, and let $f$ be a distribution supported on $K$ with moments

$$m_\nu^{(j)} = \int f(y) \frac{\overline{h_\nu^{(j)}(y)}}{|y|^{n+2\nu-2}}\, dv(y), \qquad \nu = 0, 1, \ldots;\ j = 1, \ldots, J(\nu). \tag{8.3}$$

**Corollary 8.2** *If*

$$\limsup_{\nu \to \infty} \max_j \sqrt[\nu]{\left| m_\nu^{(j)} \right|} < \frac{1}{\max_K |x|} \tag{8.4}$$

*and $K$ does not separate 0 from $\infty$, then $m_\nu^{(j)} = 0$ for all $\nu$, $j$.*

*Moreover, if $K$ does separate 0 from $\infty$, then for each sequence $\{m_\nu^{(j)}\}$ satisfying (8.4) there is a distribution $f$ with support on $K$ having $\{m_\nu^{(j)}\}$ as its moments, i.e., such that (8.3) holds.*

# 9 Factorizations

Suppose that $P$ is a first order homogeneous differential operator with constant coefficients in $\mathbf{R}^n$ satisfying $P^*P = -\Delta$, where $P^*$ is the formal adjoint for $P$. Such differential operators are known as *factorizations* of the Laplace operator in $\mathbf{R}^n$.

The standard fundamental solution $\Phi(x,y)$ for $P$ is obtained by applying the differential operator $-P^{*\prime}$ in the variable $y$ to the standard fundamental solution of $\Delta$.

If $Pu = 0$ in a ball $B \subset \mathbf{R}^n$, then $u$ is obviously a harmonic function in $B$. For this reason, $H^s(B) \cap Sol(B)$ is a closed subspace of $H^s(B) \cap Harm(B)$.

In the recent paper of Shlapunov [11], there was constructed an orthonormal basis $\left\{ e_\nu^{(j)} \right\}$ in $H^s(B(0,R)) \cap Harm(B(0,R))$ consisting of homogeneous harmonic polynomials.

More precisely, each $e_\nu^{(j)}$ is an eigenfunction of Green's integral

$$u \mapsto -\int_{\partial B(0,R)} G_P(\Phi(x,\cdot),u)$$

on $H^s(B(0,R))$ corresponding to an eigenvalue $0 \le \lambda_\nu^{(j)} \le 1$. (It is worth pointing out that if an $e_\nu^{(j)}$ corresponds to $\lambda_\nu^{(j)} = 0$, then

$$P\,\frac{e_\nu^{(j)}(y)}{|y|^{n+2\nu-2}} = 0$$

away from the origin.)

From what has already been proved, it follows that $\left\{ e_\nu^{(j)} \right\}$ is a basis with double orthogonality in $H^s(B(0,R)) \cap Harm(B(0,R))$, with $\sigma$ an arbitrary ball $B(0,r)$ of smaller radius.

Those $e_\nu^{(j)}$, which correspond to the eigenvalues $\lambda_\nu^{(j)} > 0$, form a basis with double orthogonality in $H^s(B(0,R)) \cap Sol(B(0,R))$.

Our next objective is to evaluate the Fourier coefficients of the restriction of $\Phi(x,y)$ to $B(0,r)$, for fixed $y$ away from the closure of $B(0,r)$.

**Lemma 9.1** *If $|y| > r$, then*

$$k_\nu^{(j)}(y) = \frac{1}{n+2\nu-2}\sqrt{\frac{R^{n+2\nu}}{n+2\nu}}\;\overline{P\,\frac{e_\nu^{(j)}(y)}{|y|^{n+2\nu-2}}}, \qquad \nu = 0,1,\ldots;\ j = 1,\ldots,J(\nu), \tag{9.1}$$

*up to a multiple of $\nu^s$ (the constant being in a range depending only on $n$ and $s$).*

**Proof.** It is sufficient to apply the differential operator $-P^{*\prime}$ in the variable $y$ to both sides of decomposition (8.2) and make use of Lemma 1.4 in Shlapunov [11] which states that

$$c_1\,\nu^{2s}\,\|e_\nu^{(j)}\|_{L^2(B(0,R))} \le \|e_\nu^{(j)}\|_{H^s(B(0,R))} \le c_2\,\nu^{2s}\,\|e_\nu^{(j)}\|_{L^2(B(0,R))},$$

with $c_1$ and $c_2$ constants depending only on $n$, $s$.

□

For a distribution $f$ with compact support in $B(0,R)\setminus\overline{B(0,r)}$, we consider the *moments*

$$c_\nu^{(j)} = \int k_\nu^{(j)}(y)\, f(y)\, dv(y), \qquad \nu = 0,1,\ldots;\ j = 1,\ldots,J(\nu).$$

The condition

$$\limsup_{\nu\to\infty} \max_j \sqrt[\nu]{\left|\int f(y)\, \overline{P\,\frac{e_\nu^{(j)}(y)}{|y|^{n+2\nu-2}}}\, dv(y)\right|} < \frac{1}{R}$$

is easily verified to imply

$$\sum_{\nu=0}^{\infty}\sum_{j=1}^{J(\nu)} \left|c_\nu^{(j)}\right|^2 < \infty.$$

Thus, Theorem 6.2 leads to the following result which sheds some new light on Corollary 8.2.

Let $K$ be a compact set in $\mathbf{R}^n$ which does not contain the origin, and let $f$ be a distribution supported on $K$ with moments

$$m_\nu^{(j)} = \int f(y)\, \overline{P\,\frac{e_\nu^{(j)}(y)}{|y|^{n+2\nu-2}}}\, dv(y), \qquad \nu = 0,1,\ldots;\ j = 1,\ldots,J(\nu). \tag{9.2}$$

**Corollary 9.2** *If*

$$\limsup_{\nu\to\infty} \max_j \sqrt[\nu]{\left|m_\nu^{(j)}\right|} < \frac{1}{\max_K |x|} \tag{9.3}$$

*and $K$ does not separate $0$ from $\infty$, then $m_\nu^{(j)} = 0$ for all $\nu$, $j$.*

*Moreover, if $K$ does separate $0$ from $\infty$, then for each sequence $\{m_\nu^{(j)}\}$ satisfying (9.3) there is a distribution $f$ with support on $K$ having $\{m_\nu^{(j)}\}$ as its moments, i.e., such that (9.2) holds.*

# References

[1] L. A. Aizenberg. *Carleman's Formulas in Comlex Analysis. Theory and Applications.* Kluwer Academic Publishers, Dordrecht, 1993.

[2] L. Aizenberg and L. Zalcman. Instability phenomena for the moment problem. *Preprint*, Bar-Ilan Univ., Ramat-Gan, 1994, 14 pp.

[3] N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68: 337–404, 1950.

[4] S. Bergman. *The kernel function and conformal mapping.* Mathematical surveys (No. 5). AMS, Providence, R.I., second (revised) edition, 1970.

[5] N. Bourbaki. *Topological vector spaces.* Elements of mathematics. Springer-Verlag, Berlin et al., 1987.

[6] M. Brelot. Sur l'approximation et la convergence dans la theorie de fonctions harmoniques et holomorphes. *Bull. Soc. Math. France*, 73: 55–70, 1945.

[7] P. Davis. An application of doubly orthogonal functions to a problem of approximation in two regions. *Trans. Amer. Math. Soc.*, 72: 104–137, 1952.

[8] J. Deny. Systèmes totaux de fonctions harmoniques. *Ann. Inst. Fourier (Grenoble)*, 1: 103–113, 1949.

[9] I. F. Krasichkov. Systems of functions with the property of double orthogonality. *Mat. Zam.*, 4(5): 551–556, 1968 (Russian).

[10] H. S. Shapiro. Stefan Bergman's theory of doubly orthogonal functions; an operator-theoretic approach. *Proc. Royal Irish Acad.*, 79 A(6): 49–58, 1979.

[11] A. Shlapunov. Spectral decomposition of Green's integrals and existence of $W^{s,2}$-solutions of matrix factorizations of the Laplace operator in a ball. *Preprint di Matematica*, Scuola Normale Superiore di Pisa, 1995, 20 pp.

[12] A. A. Shlapunov and N. N. Tarkhanov. Bases with double orthogonality in the Cauchy problem for systems with injective symbols. *Proc. London Math. Soc.*, 71(1), 1995.

[13] S. L. Sobolev. *Introduction to the theory of cubature formulae.* Nauka, Moscow, 1974 (Russian).

[14] N. N. Tarkhanov. Approximation on compact sets by solutions of systems with surjective symbols. *Russian Math. Surveys*, 48, 1993.

[15] N. N. Tarkhanov. *Complexes of Differential Operators.* Kluwer Academic Publishers, Dordrecht, 1995.

[16] N. N. Tarkhanov. *The Cauchy Problem for Solutions of Elliptic Equations.* Akademie-Verlag, Berlin, 1995.