# SIMPLE FINITE SUBGROUPS OF THE CREMONA GROUP OF RANK 3

YURI PROKHOROV

ABSTRACT. We classify all finite simple subgroups of the Cremona group $\mathrm{Cr}_3(\mathbb{C})$.

## 1. INTRODUCTION

Let $\Bbbk$ be a field. The Cremona group $\mathrm{Cr}_d(\Bbbk)$ is the group of birational automorphisms of the projective space $\mathbb{P}^d_{\Bbbk}$, or, equivalently, the group of $\Bbbk$-automorphisms of the rational function field $\Bbbk(t_1, \ldots, t_d)$. It is well-known that $\mathrm{Cr}_1(\Bbbk) = \mathrm{PGL}_2(\Bbbk)$. For $d \geq 2$, the structure of $\mathrm{Cr}_d(\Bbbk)$ and its subgroups is very complicated. For example, the classification of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ is an old classical problem. Recently this classification has been completed by Dolgachev and Iskovskikh [DI06]. The following is a consequence of the list in [DI06].

**Theorem 1.1** ([DI06])**.** *Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a non-abelian simple finite subgroup. Then $G$ is isomorphic to one of the following groups:*

$$(1.2) \qquad\qquad \mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(7).$$

However, the methods and results of [DI06] show that one cannot expect a reasonable classification of all finite subgroups Cremona groups of in higher rank. In this paper we restrict ourselves with the case of simple finite subgroups of $\mathrm{Cr}_3(\mathbb{C})$. Our main result is the following:

**Theorem 1.3.** *Let $G \subset \mathrm{Cr}_3(\mathbb{C})$ be a non-abelian simple finite subgroup. Then $G$ is isomorphic to one of the following groups:*

$$(1.4) \qquad \mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathfrak{A}_7, \quad \mathrm{PSL}_2(7), \quad \mathrm{SL}_2(8), \quad \mathrm{PSp}_4(3).$$

*All the possibilities occur.*

In particular, we give the affirmative answer to a question of J.-P. Serre [Ser09, Question 6.0]: there are a lot of finite groups which do not admit any embeddings to $\mathrm{Cr}_3(\mathbb{C})$. More generally we classify simple finite subgroups in the group of birational automorphisms of an arbitrary three-dimensional

---

rationally connected variety and in many cases we determine all birational models of the action:

**Theorem 1.5.** *Let $X$ be a rationally connected threefold and let $G \subset \mathrm{Bir}(X)$ be a non-abelian simple finite subgroup. Then $G$ is isomorphic either to $\mathrm{PSL}_2(11)$ or to one of the groups in the list $(1.4)$. All the possibilities occur. Moreover, if $G$ does not admit any embeddings to $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem 1.1), then $G$ is conjugate to one of the following:*

- (i) $\mathfrak{A}_7$ *acting on some special smooth intersection of a quadric and a cubic $X_6' \subset \mathbb{P}^5$ (see Example 2.5),*
- (ii) $\mathfrak{A}_7$ *acting on $\mathbb{P}^3$ (see Theorem 3.3),*
- (iii) $\mathrm{PSp}_4(3)$ *acting on $\mathbb{P}^3$ (see Theorem 3.3),*
- (iv) $\mathrm{PSp}_4(3)$ *acting on the Burkhardt quartic $X_4^{\mathrm{b}} \subset \mathbb{P}^4$ (see Example 2.8),*
- (v) $\mathrm{SL}_2(8)$ *acting on some smooth Fano threefold $X_{12}^{\mathrm{m}} \subset \mathbb{P}^8$ of genus 7 (see Example 2.11),*
- (vi) $\mathrm{PSL}_2(11)$ *acting on the Klein cubic $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ (see Example 2.6),*
- (vii) $\mathrm{PSL}_2(11)$ *acting on some smooth Fano threefold $X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$ of genus 8 (see Example 2.9).*

However we should mention that in contrast with [DI06] for groups $\mathfrak{A}_5$, $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$ we do not describe their *actions*. We hope that by using our technique it is possible to describe birational models of actions all the groups in the above theorems but definitely this makes the paper much longer. We also do not answer to the question about conjugacy groups (iii)-(iv), (vi)-(vii), and (i)-(ii).

**Remark 1.6.** The cooresponding varieties in (ii)-(v) of the above theorem are rational. Hence these actions define embeddings of $G$ in $\mathrm{Cr}_3(\mathbb{C})$. Varieties $X_3^{\mathrm{k}}$ and $X_{14}'$ are birationally equivalent and non-rational (see Remark 2.10 and [CG72]). It is known that a general intersection of a quadric and a cubic is non-rational. As far as I know the non-rationality of all smooth threefolds in this family is still not proved.

**Remark 1.7.** (i) The orders of the above groups are as follows:

| $G$ | $\mathfrak{A}_5$ | $\mathfrak{A}_6$ | $\mathfrak{A}_7$ | $\mathrm{PSL}_2(7)$ | $\mathrm{SL}_2(8)$ | $\mathrm{PSp}_4(3)$ | $\mathrm{PSL}_2(11)$ |
|---|---|---|---|---|---|---|---|
| $|G|$ | 60 | 360 | 2520 | 168 | 504 | 25920 | 660 |

(ii) There are well-known isomorphisms $\mathrm{PSp}_4(3) \simeq \mathrm{SU}_4(2) \simeq \mathrm{O}_5(3)'$, $\mathfrak{A}_5 \simeq \mathrm{SL}_2(4) \simeq \mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7) \simeq \mathrm{GL}_3(2)$, and $\mathfrak{A}_6 \simeq \mathrm{PSL}_2(9)$ (see, e.g., [CCN$^+$85]).

The idea of the proof is quite standard. It follows the classical ideas (cf. [DI06]) but has much more technical difficulties. Here is an outline of our approach.

By running the equivariant Minimal Model Program we may assume that our group $G$ acts on a Mori-Fano fiber space $X/Z$. Since the group is simple, we may assume that $Z$ is a point. The latter means that $X$ is a $G\mathbb{Q}$-Fano threefold.

**Definition 1.8.** A *$G$-variety* is a variety $X$ provided with a biregular action of a finite group $G$. We say that a normal $G$-variety $X$ is $G\mathbb{Q}$-*factorial* if any $G$-invariant Weil divisor on $X$ is $\mathbb{Q}$-Cartier. A projective normal $G$-variety $X$ is called $G\mathbb{Q}$-*Fano* if it is $G\mathbb{Q}$-factorial, has at worst terminal singularities, $-K_X$ is ample, and $\operatorname{rk}\operatorname{Pic}(X)^G = 1$.

Thus the $G$-equivariant Minimal Model Program reduces our problem to the classification of finite simple subgroups in automorphism groups of $G\mathbb{Q}$-Fano threefolds. Smooth Fano threefolds are completely classified by Iskovskikh [Isk80] and Mori–Mukai [MM82]. To study the singular case we use estimates for the number of singular points and analyze the action of $G$ on the singular set.

The structure of the paper is as follows. In §2 we collect some examples and show that all the cases in our list really occur. Reduction to the case of $G\mathbb{Q}$-Fano threefolds is explained in §4. In §5 and §6 we study the cases where $X$ is Gorenstein and non-Gorenstein, respectively.

**Conventions.** All varieties are defined over the complex number field $\mathbb{C}$. $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric and the alternating groups, respectively. For linear groups over a field $\Bbbk$ we use the standard notations $\operatorname{GL}_n(\Bbbk)$, $\operatorname{SO}_n(\Bbbk)$, $\operatorname{Sp}_n(\Bbbk)$ etc. If the field $\Bbbk$ is finite and contains $q$ elements, then, for short, the above groups are denoted by $\operatorname{GL}_n(q)$, $\operatorname{SO}_n(q)$, $\operatorname{Sp}_n(q)$ etc. For a group $G$, we denote by $Z(G)$ and $[G,G]$ its center and derived subgroup, respectively. If the group $G$ acts on a set $\Omega$, then, for an element $P \in \Omega$, its stabilized is denoted by $G_P$. All simple groups are supposed to be non-abelian. The Picard number of a variety $X$ is denoted by $\rho(X)$. The a normal variety $X$, $\operatorname{Cl}(X)$ is the Weil divisor class group.

## 2. Examples

In this section we collect examples.

First of all, the group $\mathfrak{A}_5$ acts on $\mathbb{P}^1$ and $\mathbb{P}^2$. This gives a lot of embeddings to $\mathrm{Cr}_3(\mathbb{C})$ (by different actions on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^1$). The groups $\mathfrak{A}_6$ and $\mathrm{PSL}_2(7)$ admit embeddings to $\mathrm{Cr}_2(\mathbb{C})$, so they are also can be embedded to $\mathrm{Cr}_3(\mathbb{C})$.

**Example 2.1.** Consider the embedding of $\mathfrak{A}_5 \subset \mathrm{PGL}_2(\mathbb{C})$ as a binary icosahedron group. Let $H \subset \mathrm{PGL}_2(\mathbb{C})$ be another finite subgroup. Then there is an action of $\mathfrak{A}_5$ on a rational homogeneous variety $\mathrm{PGL}_2(\mathbb{C})/H$. This gives a series of embeddings of $\mathfrak{A}_5$ to $\mathrm{Cr}_3(\mathbb{C})$.

Trivial examples also provide subgroups of $\mathrm{PGL}_4(\mathbb{C})$ (see Theorem 3.3): $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathfrak{A}_7$, $\mathrm{PSL}_2(7)$, $\mathrm{PSp}_4(3)$. In the examples below we show that a finite simple group acts on a (possibly singular) Fano threefold. According to [Zha06] Fano varieties with log terminal singularities are rationally connected, so our constructions give embeddings of a finite simple group to the automorphism group of some rationally connected variety.

**Example 2.2.** The group $\mathfrak{A}_5$ acts on the smooth cubic $\{\sum_{i=1}^4 x_i^3 = 0\} \subset \mathbb{P}^4$ and on the smooth quartic $\{\sum_{i=1}^4 x_i^4 = 0\} \subset \mathbb{P}^4$. These varieties are not rational.

**Example 2.3.** The *Segre cubic* $X_3^{\mathrm{s}}$ is a subvariety in $\mathbb{P}^5$ given by the equations $\sum x_i = \sum x_i^3 = 0$. This cubic has 10 nodes, it is obviously rational, and $\mathrm{Aut}\, X_3^{\mathrm{s}} \simeq \mathfrak{S}_6$. In particular, alternating groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$ act on $X_3^{\mathrm{s}}$. In fact, this construction gives two embeddings of $\mathfrak{A}_5$ to $\mathrm{Cr}_3(\mathbb{C})$. We do not know if they are conjugate or not.

**Example 2.4.** Assume that $G$ acts on $\mathbb{C}^5$ so that there are (irreducible) invariants $\phi_2$ and $\phi_3$ of degree 2 and 3, respectively. Let $Y \subset \mathbb{P}^4$ be a (possibly singular) cubic surface given by $\phi_3 = 0$ and let $R \subset Y$ be the surface given by $\phi_2 = \phi_3 = 0$. Then $R \in |-K_Y|$. Consider the double cover $X \to Y$ ramified along $R$. Then $X$ is a Fano threefold. It can be realized as an intersection of a cubic and quadric in $\mathbb{P}^5$. The action of $G$ lifts to $X$. There are two interesting cases (cf. [Muk88b]):

    (a) $Y = \{\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0\} \subset \mathbb{P}^5$ is the Segre cubic, and $R$ is cut out by the equation $\sum_{i=0}^5 x_i^2 = 0$, $G = \mathfrak{A}_6$;

    (b) $Y = \{\sum_{i=0}^4 x_i = \sum_{i=0}^4 x_i^3 = 0\} \subset \mathbb{P}^5$ is a cubic cone, and $R$ is cut out by the equation $\sum_{i=0}^5 x_i^2 = 0$, $G = \mathfrak{A}_5$.

**Example 2.5.** Consider the subvariety in $X_6' \subset \mathbb{P}^6$ given by the equations $\sigma_1 = \sigma_2 = \sigma_3 = 0$, where $\sigma_i$ are symmetric polynomials in $x_1, \ldots, x_7$. Then $X_6'$ is Fano smooth threefold, an intersection of a quadric and a cubic in $\mathbb{P}^5$.

The alternating group $\mathfrak{A}_7$ naturally acts on $X'_6$. A general variety $X_6$ in this family is not rational [IP96]. The (non-)rationality of $X'_6$ is not known.

**Example 2.6.** The automorphism group of the cubic $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ given by the equation

$$(2.7) \qquad x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0$$

is isomorphic to $\mathrm{PSL}_2(11)$. This was discovered by F. Klein, see [Adl78] for a complete modern proof. [1]

**Example 2.8.** The *Burkhardt quartic* $X_4^{\mathrm{b}}$ is a subvariety in $\mathbb{P}^5$ given by $\sigma_1 = \sigma_4 = 0$, where $\sigma_i$ is $i$-th symmetric function in $x_1, \ldots, x_6$. The automorphism group of $X_4^{\mathrm{b}}$ is isomorphic to $\mathrm{PSp}_4(3)$, see [ST54]. [2]

**Example 2.9.** Let $W$ be a 5-dimensional irreducible representation of $\tilde{G} := \mathrm{SL}_2(11)$. Consider the following skew symmetric matrix whose entries are linear forms on $W$:

$$A := \begin{pmatrix} 0 & x_4 & x_5 & x_1 & x_2 & x_3 \\ -x_4 & 0 & 0 & x_3 & -x_1 & 0 \\ -x_5 & 0 & 0 & 0 & x_4 & -x_2 \\ -x_1 & -x_3 & 0 & 0 & 0 & x_5 \\ -x_2 & x_1 & -x_4 & 0 & 0 & 0 \\ -x_3 & 0 & x_2 & -x_5 & 0 & 0 \end{pmatrix}$$

The matrix $A$ can be regarded as a non-trivial $G$-equivariant linear map from $W$ to $\wedge^2 V$, where $V$ is a 6-dimensional irreducible representation of $\tilde{G}$, see [AR96, §47]. Thus the representation $\wedge^2 V$ is decomposed as $\wedge^2 V = W \oplus W^{\perp}$, where $\dim W^{\perp} = 10$. Let $X_{14}^{\mathrm{a}} := \mathbb{P}(W^{\perp}) \cap \mathrm{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$.

It is easy to check that $\mathrm{rk}\, A(w) \geq 4$ for any $w \in W$, $w \neq 0$. Thus $A$ is a regular net of skew forms in the sense of [Kuz04]. The Pfaffian of $A$ defines a cubic hypersurface $X_3 \subset \mathbb{P}^4$. This hypersurface $X_3$ is given by (2.7) because the action of $\mathrm{SL}_2(11)$ on $\mathbb{C}^5$ has only one invariant of degree 3 (see [Adl78]). So, $X_3 = X_3^{\mathrm{k}}$. Hence it is smooth and so is $X_{14}^{\mathrm{a}}$ by [Kuz04, Prop. A.4]. Then by the adjunction formula $X_{14}^{\mathrm{a}}$ is a Fano threefold of Picard number one and genus 8 [IP99]. By construction $X_{14}^{\mathrm{a}}$ admits a non-trivial action of $G$.

———————

[1] As pointed out by Prof. J. Ellenberg the existence of this action can also be seen from the fact that the cubic $X_3^{\mathrm{k}}$ is birational to $\mathcal{A}_{11}^{lev}$, the moduli space of abelian surfaces with $(1, 11)$-polarization and canonical level structure, see [M. Gross and S. Popescu. The moduli space of $(1, 11)$-polarized abelian surfaces is unirational. *Compositio Math.*, 126:1–23, 2001].

[2] Similar to the previous example the existence of this action follows also from an interpretation of $X_4^{\mathrm{b}}$ as a moduli space, see, e.g., [B. van Geemen. Projective models of Picard modular varieties. in *Classification of irregular varieties* (Trento, 1990), 68–99, Lecture Notes in Math., **1515**, Springer, Berlin, 1992]

**Remark 2.10.** It turns out that $X_{14}^{\mathrm{a}}$ and $X_3^{\mathrm{k}}$ are birationally equivalent (and not rational [CG72]), so our construction does not give an embedding of $G$ into $\mathrm{Cr}_3(\mathbb{C})$. The birational equivalence of $X_{14}^{\mathrm{a}}$ and $X_3^{\mathrm{k}}$ can be seen from the following construction. Given a smooth section $X_{14} = \mathrm{Gr}(2,6) \cap \mathbb{P}^9$, let $Y \subset \mathbb{P}^5$ be the variety swept out by lines representing points of $X_{14} \subset \mathrm{Gr}(2,6)$. Then $Y$ is a singular quartic fourfold. It is called the *Palatini quartic* of $X_{14}$. In our case $X_{14} = X_{14}^{\mathrm{a}}$ the equation of $Y$ is as follows [AR96, Cor. 50.2]:

$$x_0^4 + x_0(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1) +$$
$$+ x_1^2 x_3 x_5 + x_2^2 x_4 x_1 + x_3^2 x_5 x_2 + x_4^2 x_1 x_3 + x_5^2 x_2 x_4 = 0.$$

Let $H$ be a general hyperplane section of $Y$. Then $H$ is a quartic threefold with 25 singular points which is birational to both $X_{14}^{\mathrm{a}}$ and $X_3^{\mathrm{k}}$. Note however that our birational construction is not $G$-equivariant. We do not know whether our two embeddings of $G$ into $\mathrm{Bir}(X_{14}^{\mathrm{a}}) \simeq \mathrm{Bir}(X_3^{\mathrm{k}})$ are conjugate or not.

**Example 2.11** ([Muk92])**.** There is a curve $C^{\mathrm{m}}$ of genus 7 for which the Hurwitz bound of the automorphism group is achieved [Mac65]. In fact, $\mathrm{Aut}\, C^{\mathrm{m}} \simeq \mathrm{SL}_2(8)$. The "dual" Fano threefold of genus 7 has the same automorphism group. The construction due to S. Mukai [Muk92, Muk95] is as follows. Let $Q \subset \mathbb{P}^8$ be a smooth quadric. All 3-dimensional projective subspaces of $\mathbb{P}^8$ contained in $Q$ are parameterized by a smooth irreducible $\mathrm{SO}_9(\mathbb{C})$-homogeneous variety $\mathrm{LGr}(4,9)$, so-called, Lagrangian Grassmannian. In fact, $\mathrm{LGr}(4,9)$ is a Fano manifold of dimension 10 and Fano index 8 with $\rho(\mathrm{LGr}(4,9)) = 1$. The positive generator of $\mathrm{Pic}(\mathrm{LGr}(4,9)) \simeq \mathbb{Z}$ determines an embedding $\mathrm{LGr}(4,9) \hookrightarrow \mathbb{P}^{15}$. In fact, this embedding is given by the spinor coordinates on $\mathrm{LGr}(4,9)$. It is known that any smooth Fano threefold $X_{12}^{\mathrm{m}}$ of genus 7 with $\rho(X_{12}^{\mathrm{m}}) = 1$ is isomorphic to a section of $\mathrm{LGr}(4,9) \subset \mathbb{P}^{15}$ by a subspace of dimension 8 [Muk88a]. Similarly, any canonical curve $C$ of genus 7 is isomorphic to a section of $\mathrm{LGr}(4,9) \subset \mathbb{P}^{15}$ by a subspace of dimension 6 if and only if $C$ has no $g_4^1$ [Muk95]. The group $\mathrm{SL}_2(8)$ has a 9-dimensional representation $U$ and there is an invariant quadric $Q \subset \mathbb{P}(U)$. Hence $\mathrm{SL}_2(8)$ naturally acts on $\mathrm{LGr}(4,9)$. This action lifts to $\mathbb{P}^{15}$ so that there are two invariant subspaces $\Pi_1$ and $\Pi_2$ of dimension 6 and 8, respectively. The intersections $\mathrm{LGr}(4,9) \cap \Pi_1$ and $\mathrm{LGr}(4,9) \cap \Pi_2$ are our curve $C^{\mathrm{m}}$ and a smooth Fano threefold of genus 7 with $\rho = 1$. Recall that any smooth Fano threefold of genus 7 with $\rho = 1$ is rational (see, e.g., [IP99]). Therefore, the above construction provides an embedding of $\mathrm{SL}_2(8)$ into $\mathrm{Cr}_3(\mathbb{C})$.

## 3. FINITE LINEAR AND PERMUTATION GROUPS

**3.1. Finite linear groups.** Let $V$ be a vector space. An irreducible subgroup $G \subset \mathrm{GL}(V)$ is said to be *imprimitive* if it contains a non-trivial reducible normal subgroup $N$. In this case $G$ permutes $N$-invariant subspaces $V_i \subset V$ such that $V = \oplus V_i$. A group $G$ is said to be *primitive* if it is irreducible and not imprimitive. Clearly, a simple linear group has to be primitive.

All finite primitive linear groups of small degree are classified, see [Bli17], [Bra67], and [Lin71]. Basically we need only the list of the simple ones.

**Theorem 3.2** ([Bli17]). *Let $G \subset \mathrm{PGL}_3(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_3(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then only one of the following cases is possible:*

(i) *the icosahedral group, $G \simeq \tilde{G} \simeq \mathfrak{A}_5$;*
(ii) *the Valentiner group, $G \simeq \mathfrak{A}_6$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_3$;*
(iii) *the Klein group, $G \simeq \tilde{G} \simeq \mathrm{PSL}_2(7)$;*
(iv) *the Hessian group, $G \simeq (\boldsymbol{\mu}_3)^2 \rtimes \mathrm{SL}_2(3)$, $|G| = 216$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_3$;*
(v) *subgroups of the Hessian group of index 3 and 6.*

**Theorem 3.3** ([Bli17]). *Let $G \subset \mathrm{PGL}_4(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_4(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then one of the following cases is possible:*

(i) $G \simeq \tilde{G} \simeq \mathfrak{A}_5$,
(ii) $G \simeq \mathfrak{A}_6$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(iii) $G \simeq \mathfrak{A}_7$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(iv) $G \simeq \mathfrak{A}_5$, $\tilde{G} \simeq \mathrm{SL}_2(5)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(v) $G \simeq \mathrm{PSL}_2(7)$, $\tilde{G} \simeq \mathrm{SL}_2(7)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$,
(vi) $G \simeq \mathrm{PSp}_4(3)$, $\tilde{G} = \mathrm{Sp}_4(3)$, $Z(\tilde{G}) \simeq \boldsymbol{\mu}_2$.

**Theorem 3.4** ([Bra67]). *Let $G \subset \mathrm{PGL}_5(\mathbb{C})$ be a finite irreducible simple subgroup and let $\tilde{G} \subset \mathrm{SL}_5(\mathbb{C})$ be its preimage under the natural map $\mathrm{SL}_5(\mathbb{C}) \to \mathrm{PGL}_5(\mathbb{C})$ such that $[\tilde{G}, \tilde{G}] \supset Z(\tilde{G})$. Then $G \simeq \tilde{G}$ and only one of the following cases is possible:*

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathrm{PSL}_2(11), \quad \mathrm{PSp}_4(3).$$

**Theorem 3.5** ([Lin71]). *Let $G \subset \mathrm{GL}_6(\mathbb{C})$ be a finite irreducible simple subgroup. Then $G$ is isomorphic to one of the following groups:*

$$\mathfrak{A}_7, \quad \mathrm{PSL}_2(7), \quad \mathrm{PSp}_4(3), \quad \mathrm{SU}_3(3).$$

**Lemma 3.6.** *Let $G$ be a finite simple group. Assume that $G$ admits an embedding to $\mathrm{PSO}_n(\mathbb{C})$ with $n \leq 6$ and does not admit any embeddings to $\mathrm{Cr}_2(\mathbb{C})$. Then $n = 6$ and $G$ is isomorphic to $\mathfrak{A}_7$ or $\mathrm{PSp}_4(3)$.*

*Proof.* We may assume that $G \subset \mathrm{PSO}_6(\mathbb{C})$, i.e., $G$ acts faithfully on a smooth quadric $Q \subset \mathbb{P}^5$. It is well known that $Q$ contains two 3-dimensional families $F_1$, $F_2$ of planes [GH78, Ch. 6, §1]. Regarding $Q$ as the Grassmann variety $\mathrm{Gr}(2,4)$ we see $F_1 \simeq F_2 \simeq \mathbb{P}^3$ [GH78, Ch. 6, §2]. We get a nontrivial action of $G$ on $\mathbb{P}^3$. Now the assertion follows by Theorems 3.3 and 1.1. $\qquad\square$

**Transitive simple permutation groups.** Let $G$ be a group acting transitively on a finite set $\Omega$. A nonempty subset $\Omega' \subset \Omega$ is called a *block* for $G$ if for each $\delta \in G$ either $\delta(\Omega') = \Omega'$ or $\delta(\Omega') \cap \Omega' = \varnothing$. If $\Omega' \subset \Omega$ is a block for $G$, then for any $\delta \in G$ the image $\delta(\Omega')$ is also a block and the system of all such blocks forms a partition of $\Omega$. Moreover, the setwise stabilizer $G_{\Omega'}$ acts on $\Omega'$ transitively. The action of $G$ is said to be *imprimitive* if there is a block $\Omega' \subset \Omega$ containing more than one element. Otherwise the action is said to be *primitive*.

Below we list all finite simple transitive permutation groups acting on $n \leq 26$ symbols [DM96].

**Theorem 3.7.** *Let $G$ be a finite transitive permutation group acting on the set $\Omega$ with $|\Omega| \leq 26$. Assume that $G$ is simple and is not contained in the list (1.2). Then the action is primitive and we have one of the following cases:*

| $|\Omega|$ | $G$ | $\Omega$ | degrees of irreducible representations in the interval $[2,14]$ | $G_P$ |
|---|---|---|---|---|
| primitive groups | | | | |
| $n$ | $\mathfrak{A}_n$ | standard | 6, 10, 14  if $n = 7$<br>7, 14       if $n = 8$<br>$n-1$       if $n \geq 9$ | $\mathfrak{A}_{n-1}$ |
| 9 | $\mathrm{SL}_2(8)$ | $\mathbb{P}^1(\mathbb{F}_8)$ | 7, 8, 9 | $(\boldsymbol{\mu}_2)^3 \rtimes \boldsymbol{\mu}_7$ |
| 11 | $\mathrm{PSL}_2(11)$ | | 5, 10, 11, 12 | $\mathfrak{A}_5$ |
| 11 | $\mathrm{M}_{11}$ | standard | 10, 11 | $\mathrm{M}_{10}$ |
| 12 | $\mathrm{M}_{11}$ | | 10, 11 | $\mathrm{PSL}_2(11)$ |
| 12 | $\mathrm{M}_{12}$ | standard | 11 | $\mathrm{M}_{11}$ |
| 12 | $\mathrm{PSL}_2(11)$ | $\mathbb{P}^1(\mathbb{F}_{11})$ | 5, 10, 11, 12 | $\boldsymbol{\mu}_{11} \rtimes \boldsymbol{\mu}_5$ |

| $\lvert\Omega\rvert$ | $G$ | $\Omega$ | degrees of irreducible representations in the interval $[2,14]$ | $G_P$ |
|---|---|---|---|---|
| 13 | $\mathrm{SL}_3(3)$ | $\mathbb{P}^2(\mathbb{F}_3)$ | 12, 13 | $(\boldsymbol{\mu}_3)^2 \rtimes \mathrm{GL}_2(3)$ |
| 14 | $\mathrm{PSL}_2(13)$ | $\mathbb{P}^1(\mathbb{F}_{13})$ | 7, 12, 13, 14 | $\boldsymbol{\mu}_{13} \rtimes \boldsymbol{\mu}_6$ |
| 15 | $\mathfrak{A}_7$ | | 6, 10, 14 | $\mathrm{PSL}_2(7)$ |
| 15 | $\mathfrak{A}_8 \simeq \mathrm{SL}_4(2)$ | $\mathbb{P}^3(\mathbb{F}_2)$ | 7, 14 | $(\boldsymbol{\mu}_2)^3 \rtimes \mathrm{SL}_3(2)$ |
| 17 | $\mathrm{SL}_2(16)$ | $\mathbb{P}^1(\mathbb{F}_{16})$ | – | $(\boldsymbol{\mu}_2)^4 \rtimes \boldsymbol{\mu}_{15}$ |
| 18 | $\mathrm{PSL}_2(17)$ | $\mathbb{P}^1(\mathbb{F}_{17})$ | 9 | $\boldsymbol{\mu}_{17} \rtimes \boldsymbol{\mu}_8$ |
| 20 | $\mathrm{PSL}_2(19)$ | $\mathbb{P}^1(\mathbb{F}_{19})$ | 9 | $\boldsymbol{\mu}_{19} \rtimes \boldsymbol{\mu}_9$ |
| 21 | $\mathfrak{A}_7$ | | 6, 10, 14 | $\mathfrak{S}_5$ |
| 21 | $\mathrm{PSL}_3(4)$ | $\mathbb{P}^2(\mathbb{F}_4)$ | – | $(\boldsymbol{\mu}_2)^4 \rtimes \mathrm{SL}_2(4)$ |
| 22 | $\mathrm{M}_{22}$ | standard | – | $\mathrm{PSL}_3(4)$ |
| 23 | $\mathrm{M}_{23}$ | standard | – | $\mathrm{M}_{22}$ |
| 24 | $\mathrm{M}_{24}$ | standard | – | $\mathrm{M}_{23}$ |
| 24 | $\mathrm{PSL}_2(23)$ | $\mathbb{P}^1(\mathbb{F}_{23})$ | 11 | $\boldsymbol{\mu}_{23} \rtimes \boldsymbol{\mu}_{11}$ |
| 26 | $\mathrm{PSL}_2(25)$ | $\mathbb{P}^1(\mathbb{F}_{25})$ | 13 | $(\boldsymbol{\mu}_5)^2 \rtimes \boldsymbol{\mu}_{12}$ |
| imprimitive groups | | | | |
| 22 | $\mathrm{M}_{11}$ | | 10, 11 | $\mathfrak{A}_6$ |
| 26 | $\mathrm{SL}_3(3)$ | $\mathbb{A}^3(\mathbb{F}_3)\setminus\{0\}$ | 12, 13 | $(\boldsymbol{\mu}_3)^2 \rtimes \mathrm{SL}_2(3)$ |

Here $\mathrm{M}_k$ *denotes the Mathieu group,* $G_P$ *is the stabilizer of* $P \in \Omega$ *and* $\mathbb{P}^m(\mathbb{F}_q)$ *(resp.* $\mathbb{A}^m(\mathbb{F}_q)$*) denotes the projective (resp. affine) space over the finite field* $\mathbb{F}_q$*.*

**Remark 3.8.** We will show that the group $\mathfrak{A}_8$ cannot act non-trivially on a rationally connected threefold. Hence the same holds for all $\mathfrak{A}_n$ with $n \geq 8$. Therefore we can omit $\mathfrak{A}_n$ with $n \geq 9$ in Theorem 3.7.

*Proof.* All primitive permutation groups are taken from the book [DM96]. Their irreducible representations can be found in [CCN+85]. If the group $G$ is imprimitive, then $G$ acts on the system of blocks $\Lambda$, where $\lvert\Lambda\rvert = \lvert\Omega\rvert/m \leq 13$ and $m \geq 2$ is the number of elements in a block. Then $m \leq 3$, the action on $\Lambda$ is primitive, and for a block $\Omega'$, the setwise stabilizer $G_{\Omega'}$ acts on $\Omega'$ transitively. This gives us two possibilities: $\mathrm{M}_{11}$ and $\mathrm{SL}_3(3)$. $\square$

**Corollary 3.9.** *In notation of Theorem* 3.7 *the stabilizer* $G_P$ *has a faithful representation of degree* $\leq 4$ *only in the following cases:*

(i) $G \simeq \mathrm{PSL}_2(11)$, $\lvert\Omega\rvert = 11$, $G_P \simeq \mathfrak{A}_5$;

(ii) $G \simeq \mathfrak{A}_7, \quad |\Omega| = 15, \quad G_P \simeq \mathrm{PSL}_2(7)$;

(iii) $G \simeq \mathfrak{A}_7, \quad |\Omega| = 21, \quad G_P \simeq \mathfrak{S}_5$.

*Proof.* Clearly, in the above cases the group $G_P$ has a faithful representation of degree $\leq 4$. If $G_P \simeq \mathrm{M}_n$, $\mathrm{PSL}_3(4)$, or $\mathfrak{A}_{n-1}$ with $n \geq 7$, then $G_P$ has no such a representation (see, e.g., [CCN$^+$85]). In the remaining cases of Theorem 3.7 the group $G_P$ is a semi-direct product $A \rtimes B$, where $A$ is abelian. One can check that any non-trivial normal subgroup of $G_P$ contains $A$. Moreover, $A$ is a maximal abelian normal subgroup of $G_P$. Assume that $G_P$ has a faithful representation $V$ of degree $\leq 4$. By the above we may assume that $V$ is irreducible. Then the action of $G$ on $V$ is imprimitive and the induced action on eigenspaces $V_1, \ldots, V_n$ of $A$ induces a transitive embedding of $B$ to $\mathfrak{S}_n$ with $n \leq 4$. But in our cases $B$ is isomorphic to either $\mathrm{GL}_2(3)$, $\mathrm{SL}_3(2)$, $\mathrm{SL}_2(4)$, $\mathrm{SL}_2(3)$, or $B \simeq \boldsymbol{\mu}_l$, with $l \geq 5$. This group does not admit any embeddings to $\mathfrak{S}_4$, a contradiction. $\square$

## 4. Main reduction

**4.1. Terminal singularities.** Here we list only some of the necessary results on three-dimensional terminal singularities. For more complete information we refer to [Rei87]. Let $(X, P)$ be a germ of a three-dimensional terminal singularity. Then $(X, P)$ is isolated, i.e, $\mathrm{Sing}(X) = \{P\}$. The *index* of $(X, P)$ is the minimal positive integer $r$ such that $r K_X$ is Cartier. If $r = 1$, then $(X, P)$ is Gorenstein. In this case $\dim T_{P,X} = 4$, $\mathrm{mult}(X, P) = 2$, and $(X, P)$ is analytically isomorphic to a hypersurface singularity in $\mathbb{C}^4$. If $r > 1$, then there is a cyclic, étale outside of $P$ cover $\pi : (X^\sharp, P^\sharp) \to (X, P)$ of degree $r$ such that $(X^\sharp, P^\sharp)$ is a Gorenstein terminal singularity (or a smooth point). This $\pi$ is called the *index-one cover* of $(X, P)$. If $(X^\sharp, P^\sharp)$ is smooth, then the point $(X, P)$ is analytically isomorphic to a quotient $\mathbb{C}^3/\boldsymbol{\mu}_r$, where the weights $(w_1, w_2, w_3)$ of the action of $\boldsymbol{\mu}_r$ up to permutations satisfy the relations $w_1 + w_2 \equiv 0 \mod r$ and $\gcd(w_i, r) = 1$. This point is called a *cyclic quotient* singularity.

For any three-dimensional terminal singularity $(X, P)$ of index $r \geq 1$ there exists a one-parameter deformation $\mathfrak{X} \to \Delta \ni 0$ over a small disk $\Delta \subset \mathbb{C}$ such that the central fiber $\mathfrak{X}_0$ is isomorphic to $X$ and the general fiber $\mathfrak{X}_\lambda$ has only cyclic quotient terminal singularities $P_{\lambda,k}$. Thus, one can associate with a fixed threefold $X$ with terminal singularities a collection $\mathbf{B} = \{(\mathfrak{X}_\lambda, P_{\lambda,k})\}$ of cyclic quotient singularities. This collection is uniquely determined by the variety $X$ and is called the *basket* of singularities of $X$.

If $(X, P)$ is a singularity of index one, then it is an isolated hypersurface singularity. Hence $X \setminus \{P\}$ is simply-connected and the (local) Weil divisor class group $\mathrm{Cl}(X)$ is torsion free. If $(X, P)$ is of index $r > 1$, then the index one cover induces the topological universal cover $X^\sharp \setminus \{P^\sharp\} \to X \setminus \{P\}$.

**4.2. $G$-equivariant minimal model program.** Let $X$ be a rationally connected three-dimensional algebraic variety and let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. By shrinking $X$ we may assume that $G$ acts on $X$ biregularly. The quotient $Y = X/G$ is quasiprojective, so there exists a projective completion $\hat{Y} \supset Y$. Let $\hat{X}$ be the normalization of $\hat{Y}$ in the function field $\mathbb{C}(X)$. Then $\hat{X}$ is a projective variety birational to $X$ admitting a biregular action of $G$. There is an equivariant resolution of singularities $\tilde{X} \to \hat{X}$, see [AW97]. Run the $G$-equivariant minimal model program: $\tilde{X} \to \bar{X}$, see [Mor88, 0.3.14]. Running this program we stay in the category of projective normal varieties with at worst terminal $G\mathbb{Q}$-factorial singularities. Since $X$ is rationally connected, on the final step we get a Fano-Mori fibration $f : \bar{X} \to Z$. Here $\dim Z < \dim X$, $Z$ is normal, $f$ has connected fibers, the anticanonical Weil divisor $-K_{\bar{X}}$ is ample over $Z$, and the relative $G$-invariant Picard number $\rho(\bar{X})^G$ is one. Obviously, we have the following possibilities:

   (i) $Z$ is a rational surface and a general fiber $F = f^{-1}(y)$ is a conic;
   (ii) $Z \simeq \mathbb{P}^1$ and a general fiber $F = f^{-1}(y)$ is a smooth del Pezzo surface;
   (iii) $Z$ is a point and $\bar{X}$ is a $G\mathbb{Q}$-Fano threefold.

Now we assume that $G$ is a simple group. If $Z$ is not a point, then $G$ non-trivially acts either on the base $Z$ or on a general fiber. Both of them are rational varieties. Hence $G \subset \mathrm{Cr}_2(\mathbb{C})$ in this case. Thus we may assume that we are in the case (iii). Replacing $X$ with $\bar{X}$ we may assume that our original $X$ is a $G\mathbb{Q}$-Fano threefold.

In some statements below this assumption will be weakened. For example we will assume sometimes that $-K_X$ is just nef and big (not ample). We need this for some technical reasons (see §6).

The following is an easy consequence of the Kawamata-Viehweg vanishing theorem (see, e.g., [IP99, Prop. 2.1.2]).

**Lemma 4.3.** *Let $X$ be a variety with at worst (log) terminal singularities such that $-K_X$ is nef and big. Then $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ is torsion free. Moreover, the numerical equivalence of Cartier divisors on $X$ coincides with the linear one.*

**Corollary 4.4.** *Let $X$ be a threefold with at worst Gorenstein terminal singularities such that $-K_X$ is nef and big. Then the Weil divisor class group $\mathrm{Cl}(X)$ is torsion free.*

**Lemma 4.5.** *Let $X$ be a threefold with at worst terminal singularities and let $G \subset \mathrm{Aut}(X)$ be a finite simple group. If there is a $G$-fixed point $P$ on $X$, then $G$ is isomorphic to a subgroup of $\mathrm{Cr}_2(\mathbb{C})$.*

*Proof.* If $P \in X$ is Gorenstein, we consider the natural representation of $G$ in the Zariski tangent space $T_{P,X}$, so $G \subset \mathrm{GL}(T_{P,X})$, where $\dim T_{P,X} = 3$ or 4. Then by Theorems 3.2 and 3.3 the group $G$ is isomorphic to either $\mathfrak{A}_5$, $\mathfrak{A}_6$ or $\mathrm{PSL}_2(7)$. In these cases $G$ admit an embedding to $\mathrm{Cr}_2(\mathbb{C})$ (see Theorem 1.1).

Assume that $P \in X$ is non-Gorenstein of index $r > 1$. Take a small $G$-invariant neighborhood $P \ni U \subset X$ and consider the index-one cover $\pi \colon (U^\sharp, P^\sharp) \to (U, P)$ (see 4.1). Here $(U^\sharp, P^\sharp)$ is a Gorenstein terminal point and $U^\sharp \setminus \{P^\sharp\} \to U \setminus \{P\}$ is the topological universal cover. Let $\tilde{G} \subset \mathrm{Aut}(U^\sharp, P^\sharp)$ be the natural lifting of $G$. There is the following exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_r \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Since $G$ is a simple group, the above sequence is a central extension. If the representation of $G$ in $T_{P^\sharp, U^\sharp}$ is irreducible, then $\boldsymbol{\mu}_r$ must act on $T_{P^\sharp, U^\sharp}$ by scalar multiplications. According to the classification of terminal singularities [Rei87] this is possible only if $r = 2$ and $\dim T_{P^\sharp, U^\sharp} = 3$. Then we can apply Theorem 3.2. If the representation of $G$ in $T_{P^\sharp, U^\sharp}$ has a non-trivial irreducible subrepresentation $T \subset T_{P^\sharp, U^\sharp}$, then again we can apply Theorem 3.2 to the action on $T$. $\qquad\square$

**Lemma 4.6.** *Let $X$ be a $G$-threefold with at worst terminal singularities where $G$ is a finite simple group which does not admit an embedding to $\mathrm{Cr}_2(\mathbb{C})$. Assume that that $-K_X$ is nef and big. Let $S$ be a $G$-invariant effective integral Weil $\mathbb{Q}$-Cartier divisor numerically proportional to $-K_X$. Then $K_X + S$ is nef. Furthermore, if $K_X + S \sim 0$, then the pair $(X, S)$ is LC and the surface $S$ is reducible. If moreover $X$ is $G\mathbb{Q}$-factorial, then the group $G$ transitively acts on components of $S$.*

*Proof.* Assume that the divisor $-(K_X + S)$ is nef. If either $-(K_X + S)$ is ample or the pair $(X, S)$ is not LC, we can apply quite standard connectedness arguments of Shokurov [Sho93] (see, e.g., [MP09, Prop. 2.6]): for a suitable $G$-invariant boundary $D$, the pair $(X, D)$ is LC, the divisor $-(K_X + D)$ is ample, and the minimal locus $V$ of log canonical singularities is also $G$-invariant. Moreover, $V$ is either a point or a smooth rational curve. By Lemma 4.5 we may assume that $G$ has no fixed points. Hence, $G \subset \mathrm{Aut}(\mathbb{P}^1)$ and so $G \simeq \mathfrak{A}_5$, a contradiction. Thus we may assume that the pair $(X, S)$ is LC and $K_X + S \sim 0$.

If the pair $(X, S)$ is PLT, then by the Inversion of Adjunction [Sho93] the surface $S$ is normal and has only Du Val singularities. Moreover, $K_S \sim 0$ and $H^1(S, \mathscr{O}_S) = 0$. Let $\tilde{S} \to S$ be the minimal resolution. Then $\tilde{S}$ is a smooth K3 surface and $G$ naturally acts on $\tilde{S}$. Since $G$ is a simple group, this action is symplectic. According to [Muk88b] the group $G$ is isomorphic to one of the following: $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathrm{PSL}_2(7)$, so $G$ can be embedded to $\mathrm{Cr}_2(\mathbb{C})$.

Therefore, the pair $(X, S)$ is LC but not PLT. Assume that $S$ is irreducible and let $\nu\colon S' \to S$ be the normalization. If $S$ is rational, then we are in cases (1.2). So we assume that $S$ is not rational. Write $0 \sim \nu^*(K_X+S) = K_{S'}+D'$, where $D'$ is the different, see [Sho93], [Kaw07]. Here $D'$ is an effective reduced divisor and the pair is LC [Kaw07]. The group $G$ acts naturally on $S'$ and $\nu$ is $G$-equivariant. Now consider the minimal resolution $\mu\colon \tilde{S} \to S'$ and let $\tilde{D}$ be the crepant pull-back of $D'$, that is, $\mu_*\tilde{D} = D'$ and

$$K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0.$$

Here $\tilde{D}$ is again an effective reduced divisor. Hence $\tilde{S}$ is a ruled non-rational surface. Consider the Albanese map $\alpha\colon \tilde{S} \to C$. Clearly $\alpha$ is $G$-equivariant and the action of $G$ on $C$ is not trivial. Hence, $g(C) > 1$. Let $\tilde{D}_1 \subset \tilde{D}$ be a $\alpha$-horizontal component. By Adjunction $\tilde{D}_1$ is either a rational or elliptic curve. This contradicts, $g(C) > 1$.

Therefore the surface $S$ is reducible. If the action on components $S_i \subset S$ is not transitive and $X$ is $G\mathbb{Q}$-factorial, we have an invariant divisor $S' < S$ which should be $\mathbb{Q}$-Cartier. This contradicts the above considered cases. $\quad\square$

**Corollary 4.7.** *Let $X$ be a $G\mathbb{Q}$-factorial $G$-threefold with at worst terminal singularities where $G$ is a finite simple group which does not admit an embedding to $\mathrm{Cr}_2(\mathbb{C})$. Assume that $-K_X$ is nef and big. Let $\mathscr{H}$ be a $G$-invariant linear system such that $\dim \mathscr{M} > 0$ and $-(K_X + \mathscr{H})$ is nef. Then $\mathscr{H}$ has no fixed components.*

*Proof.* Assume the converse $\mathscr{H} = F + \mathscr{M}$, where $F$ is the fixed part and $\mathscr{M}$ is a linear system without fixed components. Then $F$ is an invariant divisor. This contradicts Lemma 4.6. $\quad\square$

**Lemma 4.8.** *Let $X$ be a $G\mathbb{Q}$-factorial $G$-threefold with at worst terminal singularities where $G$ is a finite simple group which does not admit an embedding to $\mathrm{Cr}_2(\mathbb{C})$. Assume that $-K_X$ is nef and big. Then $\dim H^0(X, -K_X)^G \le 1$.*

*Proof.* Assume that there is a pencil $\mathscr{H}$ of invariant anticanonical sections. By Corollary 4.7 $\mathscr{H}$ has no fixed components. We claim that a general member of $\mathscr{H}$ is irreducible. Indeed, otherwise $\mathscr{H} = m\mathscr{L}$, $m > 1$ and the pencil $\mathscr{L}$ determines a $G$-equivariant rational map $X \dashrightarrow \mathbb{P}^1$ so that the action on $\mathbb{P}^1$ is trivial. Hence, the fibers are $\mathbb{Q}$-Cartier divisors and $-K_X \sim m\mathscr{L}$, a contradiction. So, a general member $H \in \mathscr{H}$ is irreducible and $G$-invariant. This contradicts Lemma 4.6. $\quad\square$

## 5. Case: $X$ is Gorenstein

**Assumption 5.1.** In this section $X$ denotes a threefold with at worst terminal Gorenstein singularities such that the anticanonical divisor $-K_X$ is

nef and big. Let $G \subset \mathrm{Aut}(X)$ be a finite simple group which does not admit any embeddings to $\mathrm{Cr}_2(\mathbb{C})$. Write $-K_X^3 = 2g - 2$ for some $g$. This $g$ is called the *genus* of a Fano threefold. By Kawamata-Viehweg vanishing and Riemann-Roch we have $\dim | - K_X| = g + 1$. In particular, $g$ is an integer.

**Lemma 5.2.** *The linear system $| - K_X|$ is base point free.*

*Proof.* Assume that $\mathrm{Bs}\,| - K_X| \neq \varnothing$. If $\dim \mathrm{Bs}\,| - K_X| > 0$, then by [Shi89] $\mathrm{Bs}\,| - K_X|$ a smooth rational curve contained into the smooth locus of $X$. By Lemma 4.5 the action of $G$ on this curve is non-trivial. Hence $G \subset \mathrm{Aut}(\mathbb{P}^1)$ and so $G \simeq \mathfrak{A}_5$. This contradicts our assumption 5.1. Thus $\dim \mathrm{Bs}\,| - K_X| = 0$. Again by [Shi89] $\mathrm{Bs}\,| - K_X|$ is a single point, say $P$. Thus $G$ has a non-trivial representation in $T_{P,X}$, where $\dim T_{P,X} \leq 4$. Again this contradicts our assumption 5.1. $\square$

**Lemma 5.3.** *The linear system $| - K_X|$ determines a birational morphism $X \to \mathbb{P}^{g+1}$ whose image is a Fano threefold $\bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ with at worst canonical Gorenstein singularities. In particular, $g \geq 3$.*

*Proof.* Assume that the linear system $| - K_X|$ determines a morphism $\varphi \colon X \to \mathbb{P}^{g+1}$ and $\varphi$ is not an embedding. Let $Y = \varphi(X)$. Then $\varphi$ is a generically double cover and $Y \subset \mathbb{P}^{g+1}$ is a subvariety of degree $g - 1$ [Isk80], [PCS05]. First assume that $Y$ is a cone with vertex at a point, say $P$, over a surface $S$. Then $S$ is rational and $G$ acts non-trivially on $S$. Hence $G \subset \mathrm{Cr}_2(\mathbb{C})$. This contradicts our assumption 5.1. Assume that $Y$ is a cone with vertex along a line $L$ over a curve $C$. Then again $C$ is rational and $G \subset \mathrm{Aut}(C)$ or $G \subset \mathrm{Aut}(L)$, a contradiction.

Thus we assume that $Y$ is not a cone. According to the Enriques theorem the variety $Y \subset \mathbb{P}^{g+1}$ is one of the following (see, e.g., [Isk80, Th. 3.11]):

    (i) $\mathbb{P}^3$;
    (ii) a smooth quadric in $\mathbb{P}^4$;
    (iii) a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 3 vector bundle on $\mathbb{P}^1$.

By Lemma 4.6 the natural representation of $G$ in $H^0(X, -K_X) \simeq H^0(Y, \mathscr{O}_Y(1))$ is irreducible. In the first case $\varphi : X \to \mathbb{P}^3$ is a generically double cover with branch divisor $B \subset \mathbb{P}^3$ of degree 4. By our assumptions $B$ is irreducible (otherwise $G$ effectively acts on a rational component of $B$). If the singularities of $B$ are at worst Du Val, then the minimal resolution of $B$ is a K3 surface. By [Muk88b] $G$ is contained in the list (1.2). So, the singularities of $B$ are worse than Du Val. If $B$ is not normal, then $G$ effectively acts on a curve $C \subset \mathrm{Sing}(B)$ of genus $\leq 3$. By the Hurwitz bound $|G| \leq 168$, so $G$ admits an embedding to $\mathrm{Cr}_2(\mathbb{C})$, a contradiction. Therefore, $B \subset \mathbb{P}^3$ is a normal quartic having at least on non-Du Val singularity. By the connectedness principle [Sho93, Th. 6.9] $B$ has at most two non-Du Val points. Then $G$ has a fixed point on $B \subset \mathbb{P}^3$. This contradicts the irreducibility of the representation $H^0(Y, \mathscr{O}_Y(1))$.

The second case does not occur by Lemma 3.6. In the last case $\rho(Y) = 2$. Hence $G$ acts trivially on $\operatorname{Pic}(Y)$ and so the projection $Y \to \mathbb{P}^1$ is $G$-equivariant. We get an embedding of $G$ into $\operatorname{Aut}(\mathbb{P}^1)$ or $\operatorname{Aut}(F)$, where $F \simeq \mathbb{P}^2$ is a fiber. $\square$

**Lemma 5.4.** *In notation of Lemma* 5.3 *one of the following holds:*

(i) *the variety* $\bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ *is an intersection of quadrics* (*in particular,* $g \geq 5$);

(ii) $g = 3$, $\bar{X} = \bar{X}_4 \subset \mathbb{P}^4$ *is quartic, and* $G \simeq \operatorname{PSp}_4(3)$ (*see Example* 2.5);

(iii) $g = 4$, $\bar{X} = \bar{X}_6 \subset \mathbb{P}^5$ *is an intersection of a quadric and a cubic, and* $G \simeq \mathfrak{A}_7$ (*see Example* 2.5).

*Proof.* Assume that the linear system $|-K_X|$ determines a birational morphism but its image $\bar{X} = \bar{X}_{2g-2}$ is not an intersection of quadrics. Let $Y \subset \mathbb{P}^{g+1}$ be the variety that cut out by quadrics through $\bar{X}$. Then $Y$ is a four-dimensional irreducible subvariety in $\mathbb{P}^{g+1}$ of minimal degree [Isk80], [PCS05]. As in the proof of Lemma 5.3 we can use the Enriques theorem. Assume that $Y$ is a cone with vertex $L$ over $S$. Since $G$ is not contained in the list (1.2), $L$ is a point and $S$ is a three-dimensional variety of minimal degree (and $S \not\simeq \mathbb{P}^3$). We get a contradiction as in the proof of Lemma 5.3. Hence $Y$ is smooth and we have the following possibilities:

(i) $Y \simeq \mathbb{P}^4$;

(ii) $Y \subset \mathbb{P}^5$ is a smooth quadric;

(iii) a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 4 vector bundle on $\mathbb{P}^1$.

In the first case $g = 3$ and $\bar{X} = \bar{X}_4 \subset \mathbb{P}^4$ is a quartic. Consider the representation of $G$ in $H^0(\bar{X}, -K_{\bar{X}}) \simeq \mathbb{C}^5$. If this representation is reducible, then by our assumptions $\bar{X}$ has an invariant hyperplane section $S \in |-K_{\bar{X}}|$. Since $\deg S = 4$, this $S$ must be irreducible (otherwise $S$ has a $G$-invariant rational component). By Lemma 4.6 this is impossible. Then by Theorem 3.4 and our assumption 5.1 we have the case (ii) of the lemma or the group $G$ is isomorphic to $\operatorname{PSL}_2(11)$. On the other hand, the group $\operatorname{PSL}_2(11)$ has no invariant quartics (see [AR96, §29]), a contradiction.

In the second case $\bar{X} = \bar{X}_6 \subset \mathbb{P}^5$ is an intersection of a quadric and a cubic. By Lemma 3.6 we obtain either $G \simeq \mathfrak{A}_7$ or $\operatorname{PSp}_4(3)$. The second possibility is does not occur because the action of $\operatorname{PSp}_4(3)$ on $\mathbb{C}^6$ has no invariants of degree 3. (In fact, $\operatorname{PSp}_4(3)$ can be embedded into a group of order 51840 generated by reflections, see [ST54, Table VII, No. 35]). Thus $G \simeq \mathfrak{A}_7$. We get a situation of Example 2.5 because the group $\mathfrak{A}_7$ has only one irreducible representation of degree 6.

In the last case, as in Lemma 5.3, we have a $G$-equivariant contraction $Y \to \mathbb{P}^1$ whose fibers are isomorphic to $\mathbb{P}^3$. The restriction map $X \to \mathbb{P}^1$ is a fibration whose general fiber $F$ is a surface with big and nef anticanonical

divisor. Such a surface is rational. Hence either $G \subset \operatorname{Aut}(\mathbb{P}^1)$ or $G \subset \operatorname{Aut}(F)$. $\qquad\square$

**Corollary 5.5.** *In case* (i) *of Lemma 5.4 the variety* $\bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ *is an intersection of* $(g-2)(g-3)/2$ *quadrics.*

*Proof.* Let $S \subset \mathbb{P}^g$ be a general hyperplane section of $\bar{X}$ and let $C \subset \mathbb{P}^{g-1}$ be a general hyperplane section of $S$. Then $S$ is a smooth K3 surface and $C$ is a canonical curve of genus $g$. Let $\mathscr{I}_{\bar{X}}$ (resp. $\mathscr{I}_S$, $\mathscr{I}_C$) be the ideal sheaf of $\bar{X} \subset \mathbb{P}^{g+1}$ (resp. $S \subset \mathbb{P}^g$, $C \subset \mathbb{P}^{g-1}$). The space $H^0(\mathscr{I}_{\bar{X}}(2))$ is the space of quadrics in $H^0(\bar{X}, -K_{\bar{X}})$ passing through $\bar{X}$. The standard cohomological arguments (see, e.g., [Isk80, Lemma 3.4]) show that $H^0(\mathscr{I}_{\bar{X}}(2)) \simeq H^0(\mathscr{I}_S(2)) \simeq H^0(\mathscr{I}_C(2))$. This gives us

$$\dim H^0(\bar{X}, \mathscr{I}_{\bar{X}}(2)) = \frac{1}{2}(g-2)(g-3).$$

$\qquad\square$

**Theorem 5.6** ([Nam97]). *Let $X$ be a Fano threefold with terminal Gorenstein singularities. Then $X$ is smoothable, that is, there is a flat family $X_t$ such that $X_0 \simeq X$ and a general member $X_t$ is a smooth Fano threefold. Further, the number of singular points is bounded as follows:*

$$(5.7) \qquad |\operatorname{Sing}(X)| \leq 21 - \frac{1}{2}\operatorname{Eu}(X_t) = 20 - \rho(X_t) + h^{1,2}(X_t).$$

*where* $\operatorname{Eu}(X)$ *is the topological Euler number and* $h^{1,2}(X)$ *is the Hodge number.*

**Remark 5.8.**    (i) In the above notation we have $\rho(X_t) = \rho(X)$ and $-K_{X_t}^3 = -K_X^3$ (see [JR06, §1]).
  (ii) The estimate (5.7) is very far from being sharp. However it is enough for our purposes.

**Theorem 5.9** (see, e.g., [Isk80], [IP99]). *Let $X$ be a smooth Fano threefold with* $\operatorname{Pic}(X) = \mathbb{Z} \cdot (-K_X)$. *Then the possible values of its genus $g$ and Hodge numbers* $h^{1,2}(X)$ *are given by the following table:*

| $g$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| $h^{1,2}(X)$ | 52 | 30 | 20 | 14 | 10 | 7 | 5 | 3 | 2 | 0 |

**Assumption 5.10.** From now on and till the end of this section additionally to 5.1 we assume that $-K_X$ is ample, $X$ is $G\mathbb{Q}$-factorial, and $\rho(X)^G = 1$, i.e., $X$ is a Gorenstein $G\mathbb{Q}$-Fano threefold. Moreover, the anticanonical linear system determines an embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ and its image is an intersection of $(g-2)(g-3)/2$ quadrics.

**Lemma 5.11.** *Under the assumptions of* 5.10 *we have* $\rho(X) = 1$.

*Proof.* Assume that $\rho := \rho(X) > 1$. We have a natural action of $G$ on $\mathrm{Pic}(X) \simeq \mathbb{Z}^\rho$ such that $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$. In particular, there is a nontrivial representation $V \subsetneq \mathrm{Pic}(X) \otimes \mathbb{R}$. Hence $G$ admits an embedding to $\mathrm{PSO}_{\rho-1}(\mathbb{R})$. By Lemma 3.6 we have $\rho \geq 7$. Consider a smoothing $X_t$ of $X$. Here $X_t$ is a smooth Fano threefold with $\rho(X_t) = \rho(X)$ and $-K_{X_t}^3 = -K_X^3$. From the classification of smooth Fano threefolds with $\rho > 1$ [MM82] one can see that $X_t \simeq S \times \mathbb{P}^1$, where $S$ is a del Pezzo surface. Hence, $-K_X^3 = 6(11 - \rho)$. If $\rho = 10$, then $-K_X^3 = 6$ and $g = 4$. Then by Lemma 5.4 the variety $X$ is an intersection of a quadric and a cubic in $\mathbb{P}^5$. Hence, $\mathrm{Pic}(X) \simeq \mathbb{Z}$, a contradiction. Therefore, $(\rho, g) = (7, 13)$, $(8, 10)$ or $(9, 7)$. By (5.7) we have $|\mathrm{Sing}(X)| \leq 13$. For any $P \in \mathrm{Sing}(X)$, the stabilizer $G_P$ acts faithfully on $T_{P,X} \simeq \mathbb{C}^4$. The action of $G$ on $\mathrm{Sing}(X)$ induces a transitive embedding $G \subset \mathfrak{S}_{|\mathrm{Sing}(X)|}$. By Corollary 3.9 we have only one possibility: $G \simeq \mathrm{PSL}_2(11)$, $|\mathrm{Sing}(X)| = 11$. Then the representation of $G$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ has a trivial subrepresentation of degree $\geq 2$, a contradiction. $\square$

Recall that the Fano index of a Gorenstein Fano variety $X$ is the maximal positive integer dividing the class of $-K_X$ in $\mathrm{Pic}(X)$.

**Lemma 5.12.** *Under the assumptions of* 5.10 *we have either*
  (i) *the Fano index of $X$ is one, or*
  (ii) $G \simeq \mathrm{PSL}_2(11)$ *and* $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ *is the Klein cubic (see Example 2.6).*

*Proof.* Let $q$ be the Fano index of $X$. Write $-K_X = qH$, where $H$ is an ample Cartier divisor. Clearly, the class of $H$ is $G$-stable. Assume that $q > 1$. If $q > 2$, then $X$ is either $\mathbb{P}^3$ or a quadric in $\mathbb{P}^4$. Thus we may assume that $q = 2$. Below we use some facts on Gorenstein Fano threefolds of Fano index 2 with at worst canonical singularities, see [Isk80], [Shi89]. Denote $d = H^3$.

As in the proof of Lemma 5.11 there is a flat family $X_t$ such that $X_0 \simeq X$ and a general member $X_t$ is a smooth Fano threefold with the same Picard number, anticanonical degree, and Fano index. Since $\rho(X) = 1$, by the classification of smooth Fano threefolds [Isk80], [IP99] $d \leq 5$.

If $d = 1$, then $\mathrm{Bs}\,|H|$ is a single point contained into the smooth part of $X$. This point must be $G$-invariant. This contradicts Lemma 4.5. If $d = 2$, then the linear system $|H|$ determines a $G$-equivariant double cover $X \to \mathbb{P}^3$ with branch divisor $B = B_4 \subset \mathbb{P}^3$ of degree 4. Clearly, $B$ has only isolated singularities. If $B$ has at worst Du Val singularities, then according to [Muk88b] the group $G$ is isomorphic to one of the following: $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathrm{PSL}_2(7)$, so $G$ can be embedded to $\mathrm{Cr}_2(\mathbb{C})$, a contradiction. Hence $B$ is not Du Val. The non-Du Val locus of $B$ coincides with the locus of log canonical singularities $\mathrm{LCS}(\mathbb{P}^3, B)$ of the pair $(\mathbb{P}^3, B)$. By the connectedness principle

[Sho93, Th. 6.9] the set $\mathrm{LCS}(\mathbb{P}^3, B)$ is either connected or has two connected components. Then $G$ has a fixed point on $B$ and on $X$. This contradicts Lemma 4.5.

For $d > 2$, the linear system $|H|$ is very ample and determines a $G$-equivariant embedding $X \hookrightarrow \mathbb{P}^{d+1}$. Therefore, $G \subset \mathrm{PGL}_{d+2}(\mathbb{C})$. Take a lifting $\tilde{G} \subset \mathrm{GL}_{d+2}(\mathbb{C})$ so that $\tilde{G}/Z(\tilde{G}) \simeq G$ and $Z(\tilde{G}) \subset [\tilde{G}, \tilde{G}]$. We have a natural non-trivial representation of $\tilde{G}$ in $H^0(X, H)$, where $\dim H^0(X, H) = d + 2 \leq 7$. By Theorems 3.2, 3.3, and Lemma 4.6 this representation is irreducible.

Consider the case $d = 3$. Assuming that $G$ is not contained in $\mathrm{Cr}_2(\mathbb{C})$ by Theorem 3.4 we have either $G \simeq \mathrm{PSL}_2(11)$ or $G \simeq \mathrm{PSp}_4(3)$. In the first case, the only cubic invariant of this group is the Klein cubic (2.7), see [AR96, §29]. We get Example 2.6. The second case is impossible because the group $\mathrm{PSp}_4(3)$ has no invariants of degree 3, see [ST54].

Consider the case $d = 4$. Then $X = X_4 \subset \mathbb{P}^5$ is an intersection of two quadrics, say $Q_1$ and $Q_2$. The action of $G$ on the pencil generated by $Q_1$, $Q_2$ must be trivial. Hence $G$ acts on a degenerate quadric $Q' \in \langle Q_1, Q_2 \rangle$. In particular, $G$ acts on the singular locus of $Q'$ which is a linear subspace, a contradiction.

Consider the case $d = 5$. Then $X \subset \mathbb{P}^6$ is an intersection of 5 quadrics [Shi89]. Let $V = H^0(X, \mathscr{I}_X(2))$, where $\mathscr{I}_X$ be the ideal sheaf of $X$ in $\mathbb{P}^6$. The group $\tilde{G}$ naturally acts on $V$ so that the restriction of the action to $Z(\tilde{G})$ is by scalar multiplication. If the action of $\tilde{G}$ on $V$ is trivial, then, as above, there is a $G$-stable singular quadric $Q \subset \mathbb{P}^6$. But then the singular locus of $Q$ is a $G$-stable linear subspace in $\mathbb{P}^6$, a contradiction. Thus $G \subset \mathrm{PGL}_5(\mathbb{C})$. Assuming that $G$ is not contained in the list (1.2) by Theorems 3.3 and 3.4 the group $G$ is isomorphic to one of the following: $\mathrm{PSp}_4(3)$, $\mathfrak{A}_7$, $\mathrm{PSL}_2(11)$. In the last case, the Schur multiplier of $G$ is a group of order 2 and the covering group $\tilde{G}$ is isomorphic to $\mathrm{SL}_2(11)$, see [CCN+85]. Since the order of $\mathrm{SL}_2(11)$ is not divisible by 7, this group has no irreducible representations of degree 7, a contradiction. The same arguments give contradictions for groups $\mathrm{PSp}_4(3)$ and $\mathfrak{A}_7$. $\qquad\square$

**Assumption 5.13.** Thus in what follows additionally to 5.1 and 5.10 we assume that the Fano index of $X$ is one.

**Lemma 5.14.** *Under the assumptions of* 5.13 *we have* $H^0(X, -K_X)^G = 0$.

*Proof.* Assume that $G$ has an invariant hyperplane section $S$. By Lemma 4.6 the pair $(X, S)$ is LC, $S = \sum S_i$ and $G$ acts transitively on $\Omega := \{S_i\}$. Let $m := |\Omega|$. Recall that $4 \leq g \leq 12$ and $g \neq 11$. We have $m \deg S_i = 2g - 2 \leq 22$. Hence, $\deg S_i \leq 3$. The action of $G$ on $\Omega$ induces a transitive embedding $G \subset \mathfrak{S}_m$.

If $\deg S_i = 2$, then $m = g - 1 \leq 11$, $m \neq 10$. Recall that the natural representation of $G$ in $H^0(X, -K_X) = \mathbb{C}^{g+2} = \mathbb{C}^{m+3}$ has no two-dimensional trivial subrepresentations. Taking this into account and using table in Theorem 3.7 we get only one case: $m = 7$, $g = 8$, $G \simeq \mathfrak{A}_7$, and the action of $\mathfrak{A}_7$ on $\{S_1, \ldots, S_7\}$ is the standard one. Moreover, $S_i$ is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a quadratic cone $\mathbb{P}(1, 1, 2)$. Therefore the stabilizer $G_{S_i} \simeq \mathfrak{A}_6$ acts trivially on $S_i$. The ample divisor $\sum S_i$ is connected. Hence, $S_i \cap S_j \neq \varnothing$ for some $i \neq j$. Then the stabilizer $G_P$ of the point $P \in S_i \cap S_j$ contains the subgroup generated by $G_{S_i}$ and $G_{S_j}$. So, $G_P = G$. This contradicts Lemma 4.5.

Hence $\deg S_i \neq 2$. Then $\deg S_i$ is odd, $m$ is even, and $m \geq 8$. This implies that $\deg S_i = 1$, i.e., $S_i$ is a plane. Moreover, $m = 2g - 2 \leq 22$, $m \neq 20$. As above, using the fact that the representation of $G$ in $H^0(X, -K_X) = \mathbb{C}^{m/2+3}$ has no two-dimensional trivial subrepresentations and Theorem 3.7 we get only one case: $m = 8$, $g = 5$, and $G \simeq \mathfrak{A}_8$. Similar to the previous case we derive a contradiction. The lemma is proved. $\qquad \square$

**Corollary 5.15.** *If in the assumptions of 5.13 $g \leq 7$, then the representation of $G$ in $H^0(X, -K_X)$ is irreducible.*

*Proof.* Follows from Theorem 3.3 and Lemma 5.14. $\qquad \square$

**Lemma 5.16.** *Under the assumptions of 5.13 we have $g \geq 7$.*

*Proof.* Assume that $g = 5$. Then by Corollary 5.5 we have $\dim H^0(\mathscr{I}_X(2)) = 3$ and $X \subset \mathbb{P}^6$ is a complete intersection of three quadrics. The group $G$ acts on $H^0(\mathscr{I}_X(2)) \simeq \mathbb{C}^3$ and we may assume that this action is trivial (otherwise $G$ acts on $\mathbb{P}^2 = \mathbb{P}(H^0(\mathscr{I}_X(2)))$). Thus we have a net of invariant quadrics $\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$. In particular, there is an invariant degenerate quadric $Q' \in \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3$. By Lemma 3.6 $Q'$ is a cone with zero-dimensional vertex $P$. Thus $P \in \mathbb{P}^7$ is an invariant point and there is an invariant hyperplane section, a contradiction.

Now assume that $g = 6$. Again by Corollary 5.5 we have $\dim H^0(\mathscr{I}_X(2)) = 6$. If the action of $G$ on $\dim H^0(X, \mathscr{I}_X(2))^G > 1$, then $G$ acts on a singular irreducible 6-dimensional quadric $Q \subset \mathbb{P}^7$. In particular, the singular locus of $Q$, a projective space $L$ of dimension $\leq 4$ must be $G$-invariant. This contradicts the irreducibility of $H^0(X, -K_X)$. Therefore, $\dim H^0(X, \mathscr{I}_X(2))^G \leq 1$. In particular, $G$ acts on $H^0(X, \mathscr{I}_X(2)) \simeq \mathbb{C}^6$ non-trivially and so $G$ has an irreducible representation of degree 5 or 6. Since $G$ is simple and because we assume that $G$ is not contained in the list (1.2) by the classification theorems 3.4 and 3.5 we have only two possibilities: $G \simeq \mathfrak{A}_7$, $\mathrm{PSp}_4(3)$, $\mathrm{PSL}_2(11)$, or $\mathrm{SU}_3(3)$. But in all cases $G$ has no irreducible representations of degree 8 (see [CCN$^+$85]), a contradiction. $\qquad \square$

**Lemma 5.17.** *Under the assumptions of 5.13 the variety $X$ is smooth.*

*Proof.* Assume that $X$ is singular. Let $\Omega \subset \mathrm{Sing}(X)$ be a $G$-orbit and let $n := |\Omega|$. Let $x_1, \ldots, x_n \in H^0(X, -K_X)^*$ be the vectors corresponding to

19

the points of $\Omega$. By (5.7) we have $n \leq 26$. Let $P \in \mathrm{Sing}(X)$ and let $G_P$ be the stabilizer of $P$. Then the natural representation of $G_P$ in $T_{P,X}$ is faithful. On the other hand, by Corollary 3.9 the group $G_P$ has a faithful representation of degree $\leq 4$ only in the following cases:

(i) $G \simeq \mathrm{PSL}_2(11)$, $|\Omega| = 11$, $G_P \simeq \mathfrak{A}_5$;
(ii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 21$, $G_P \simeq \mathfrak{S}_5$;
(iii) $G \simeq \mathfrak{A}_7$, $|\Omega| = 15$, $G_P \simeq \mathrm{PSL}_2(7)$.

Locally near $P$ the singularity $X \ni P$ is given by a $G_P$-semi-invariant equation $\phi(x, \dots, t) = 0$. Write $\phi = \phi_2 + \phi_3 + \dots$, where $\phi_d$ is the homogeneous part of degree $d$. By the classification of terminal singularities, $\phi_2 \neq 0$. The last case $G_P \simeq \mathrm{PSL}_2(7)$ is impossible because, then the representation of $G_P$ in $T_{P,X}$ is reducible: $T_{P,X} = T_1 \oplus T_3$, where $T_3$ is an irreducible representation of degree 3. Since the action of $G_P$ on $T_3$ has no invariants of degree 2 and 3 (see [ST54]), we have $\phi_2 = \ell^2$ and $\phi_3 = \ell^3$, where $\ell$ is a linear form. But this contradicts the classification of terminal singularities [Rei87]. Therefore, $G_P \simeq \mathfrak{A}_5$ or $\mathfrak{S}_5$.

**Claim 5.17.1.** *If $X$ is singular, then $g = 8$.*

*Proof.* The natural representation of $G$ in $H^0(X, -K_X) \simeq \mathbb{C}^{g+2}$ has no trivial subrepresentations. Recall that $g = 7, 8, 9, 10,$ or $12$.

Consider the case $G \simeq \mathfrak{A}_7$. Then the degrees of irreducible representations in the interval $[2, 14]$ are 6, 10, 14 (see Theorem 3.7). Hence, $g = 8, 10,$ or 12. On the other hand, $X$ has at least 21 singular points. By (5.7) we have $h^{1,2}(X') \geq 2$. So, $g \neq 12$. Let $\chi$ be the character of $G$ on $H^0(X, -K_X)^*$. We need the character table for $G = \mathfrak{A}_7$ (see, e.g., [CCN$^+$85]):

(5.17.2)

| $G$ | $\mathcal{C}_1$ | $\mathcal{C}_2$ | $\mathcal{C}_3'$ | $\mathcal{C}_6$ | $\mathcal{C}_3''$ | $\mathcal{C}_4$ | $\mathcal{C}_5$ | $\mathcal{C}_7'$ | $\mathcal{C}_7''$ |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 6 | 2 | 3 | $-1$ | 0 | 0 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 10 | $-2$ | 1 | 1 | 1 | 0 | 0 | $\alpha$ | $\bar{\alpha}$ |
| $\chi_4$ | 10 | $-2$ | 1 | 1 | 1 | 0 | 0 | $\bar{\alpha}$ | $\alpha$ |
| $\dots$ | | | | | | | | | |

Here $\alpha = (-1 + \sqrt{-7})/2$. (We omit characters of degree $\geq 14$). Assume that $g = 10$. Then $\chi = \chi_2 \oplus \chi_2$. Thus, as $G$-module, $H^0(X, -K_X)^* = W \oplus W'$, where $W \simeq W'$ is a 6-dimensional representation. We can take this decomposition so that the first copy $W$ contains vector $x_1$ (corresponding to $P_1 \in \Omega$). Then $P_1 \in \mathbb{P}^5 = \mathbb{P}(W)$ and obviously $\Omega \subset \mathbb{P}(W)$. Consider the set $S := \mathbb{P}(W) \cap X$, the base locus of the linear system of hyperplane sectons passing through $\mathbb{P}(W)$. By Corollary 4.7 $\dim S \leq 1$. Assume that $\dim S = 0$. Take a general hyperplane section $H$ passing through $\mathbb{P}(W)$. By Bertini's theorem $H$ is a normal surface with isolated singularities. Moreover, $H$ is

singular at points of $\Omega$, so $|\operatorname{Sing}(H)| \geq |\Omega| = 21$. By the adjunction $K_H \sim 0$. Hence, by the connectedness principle [Sho93, Th. 6.9] $H$ has at most two non-Du Val singularities. Since $G$ has no fixed points on $X$, the surface $H$ has only Du Val singularities. Therefore, the minimal resolution $\tilde{H}$ of $H$ is a K3 surface. On the other hand, $\rho(\tilde{H}) > 21$, a contradiction. Thus $\dim S = 1$. Let $S'$ be the union of an orbit of a one-dimensional component. Since the representation $W$ is irreducible, $S'$ spans $\mathbb{P}(W)$. By Lemma 5.4 $S'$ is contained in an intersection of quadrics. In particular, $\deg S' \leq 16$. If $S'$ is reducible, then $G$ interchanges its components $S_i$. In this case, $\deg S_i \leq 2$. By Theorem 3.7 the number of components is either 7 or 15. The stabilizer $G_{S_i}$ ($\simeq \mathfrak{A}_6$ or $\operatorname{PSL}_2(7)$) acts on $S_i$ which is a rational curve, a contradiction. Therefore, $S' \subset \mathbb{P}^5$ is an irreducible curve contained in an intersection of quadrics. By the Castelnuovo bound $p_a(S') \leq 21$. On the other hand, by the Hurwitz bound $|G| \leq \operatorname{Aut}(S') \leq 84(p_a(S') - 1)$, a contradiction.

Now consider the case $G \simeq \operatorname{PSL}_2(11)$. As above, since the natural representation of $G$ in $H^0(X, -K_X) \simeq \mathbb{C}^{g+2}$ has no trivial subrepresentations, we have $g = 8$, 9, or 10 (see Theorem 3.7). Moreover, if $g = 10$, then the representation of $G$ in $H^0(X, -K_X)$ is irreducible. On the other hand, 11 points of the set $\Omega \subset \mathbb{P}(H^0(X, -K_X)^*) = \mathbb{P}^{11}$ generate and invariant subspace, a contradiction. If $g = 9$, then 11 points of the set $\Omega \subset \mathbb{P}(H^0(X, -K_X)^*)$ are in general position. Then the corresponding vectors $x_i \in H^0(X, -K_X)^*$ are linearly independent and the representation of $G$ in $H^0(C, -K_X)$ is induced from the trivial representation of $G_P$ in $\langle x_1 \rangle$. But in this case the $G$-invariant vector $\sum_{\delta \in G} \delta(x_1)$ is not zero, a contradiction. Thus $g = 8$. $\square$

**Claim 5.17.3.** *If $X$ is singular, then $G \not\simeq \mathfrak{A}_7$.*

*Proof.* Assume that $G \simeq \mathfrak{A}_7$. Then $G_P \simeq \mathfrak{S}_5$. We compare the character tables for $\mathfrak{A}_7$ (see (5.17.2)) and for $\mathfrak{S}_5$:

(5.17.4)

| $G_P$ | $\mathcal{C}_1$ | $\mathcal{C}_2'$ | $\mathcal{C}_2''$ | $\mathcal{C}_3$ | $\mathcal{C}_6$ | $\mathcal{C}_4$ | $\mathcal{C}_5$ |
|---|---|---|---|---|---|---|---|
| $\chi_1'$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_2'$ | 4 | $-2$ | 0 | 1 | 1 | 0 | $-1$ |
| $\chi_3'$ | 5 | $-1$ | 1 | $-1$ | $-1$ | 1 | 0 |
| $\chi_4'$ | 6 | 0 | $-2$ | 0 | 0 | 0 | 1 |
| $\chi_5'$ | 5 | 1 | 1 | $-1$ | 1 | $-1$ | 0 |
| $\chi_6'$ | 4 | 2 | 0 | 1 | $-1$ | 0 | $-1$ |
| $\chi_7'$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Let $\chi$ be the character of the representation of $G$ in $H^0(X, -K_X)^*$. By Lemma 5.14 and (5.17.2) $\chi$ is irreducible and either $\chi = \chi_3$ or $\chi = \chi_4$. In notations of (5.17.4) for the restriction $\chi|_{\mathfrak{S}_5}$ we have

$$\chi|_{\mathfrak{S}_5}(\mathcal{C}_1, \mathcal{C}_2', \mathcal{C}_2'', \mathcal{C}_3, \mathcal{C}_6, \mathcal{C}_4, \mathcal{C}_5) = (10, -2, -2, 1, 1, 0, 0).$$

Hence, $\chi|_{\mathfrak{S}_5} = \chi'_2 \oplus \chi'_4$. In particular, the representation of $G_P \simeq \mathfrak{S}_5$ in $H^0(X, -K_X)^*$ has no trivial subrepresentations, a contradiction. $\square$

Thus we may assume that $G \simeq \mathrm{PSL}_2(11)$ and $G_P \simeq \mathfrak{A}_5$.

**Claim 5.17.5.** *If $X$ is singular, then the natural representation of $G_P$ in $T_{P,X}$ is irreducible and $P \in X$ is an ordinary double point, that is, $\mathrm{rk}\,\phi_2 = 4$.*

*Proof.* Let $x \in H^0(X, -K_X)^*$ be a vector corresponding to $P$. There is a $G_P$-equivariant embedding $T_{P,X} \hookrightarrow H^0(X, -K_X)^*$ so that $x \notin T_{P,X}$. Thus $H^0(X, -K_X)^*$ has a trivial $G_P$-representation $\langle x \rangle$ which is not contained in $T_{P,X}$. Let $\chi$ be the character of $G$ on $H^0(X, -K_X)^*$. We need character tables for $G = \mathrm{PSL}_2(11)$ and $G_P = \mathfrak{A}_5$ (see, e.g., [CCN+85]):

| $G$ | $\mathcal{C}_1$ | $\mathcal{C}'_5$ | $\mathcal{C}''_5$ | $\mathcal{C}'_{11}$ | $\mathcal{C}''_{11}$ | $\mathcal{C}_2$ | $\mathcal{C}_3$ | $\mathcal{C}_6$ | | $G_P$ | $\mathcal{C}_1$ | $\mathcal{C}_2$ | $\mathcal{C}_3$ | $\mathcal{C}'_5$ | $\mathcal{C}''_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | $\chi'_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 5 | 0 | 0 | $\beta$ | $\bar\beta$ | 1 | $-1$ | 1 | | $\chi'_2$ | 3 | $-1$ | 0 | $\alpha$ | $\alpha^*$ |
| $\chi_3$ | 5 | 0 | 0 | $\bar\beta$ | $\beta$ | 1 | $-1$ | 1 | | $\chi'_3$ | 3 | $-1$ | 0 | $\alpha^*$ | $\alpha$ |
| $\chi_4$ | 10 | 0 | 0 | $-1$ | $-1$ | $-2$ | 1 | 1 | | $\chi'_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 10 | 0 | 0 | $-1$ | $-1$ | 2 | 1 | $-1$ | | $\chi'_5$ | 5 | 1 | $-1$ | 0 | 0 |
| $\dots$ | | | | | | | | | | | | | | | |

Here $\beta = (-1 + \sqrt{-11})/2$, $\alpha = (1 - \sqrt{5})/2$, and $\alpha^* = (1 + \sqrt{5})/2$. (We omit characters of degree $> 10$). Assume that the representation of $G_P$ in $T_{P,X}$ is reducible. Then the restriction $\chi|_{G_P}$ contains $\chi'_1$ with multiplicity $\geq 2$ and either $\chi'_2$ or $\chi'_3$. Comparing the above tables we see that the restrictions $\chi_2|_{G_P}$ and $\chi_3|_{G_P}$ are irreducible (and coincide with $\chi'_5$). Hence, $\chi = \chi_4$ or $\chi_5$. In particular, $\chi(\mathcal{C}'_5) = \chi(\mathcal{C}''_5) = 0$ and $\chi|_{G_P}$ contains both $\chi'_2$ and $\chi'_3$. Thus $\chi|_{G_P} = \chi'_2 + \chi'_3 + 4\chi'_1$ and so $\chi(\mathcal{C}_3) = 4$. This contradicts $\chi_4(\mathcal{C}_3) = \chi_4(\mathcal{C}_5) = 1$. Therefore the character of the representation of $G_P$ in $T_{P,X}$ coincides with $\chi'_4$ (and irreducible).

Then the vertex of the tangent cone $TC_{P,X} \subset T_{P,X}$ to $X$ at $P$ must be zero-dimensional. Hence, $TC_{P,X}$ a cone over a smooth quadric in $\mathbb{P}^3$. This shows that $P \in X$ is an ordinary double point (node). $\square$

Now we claim that $\mathrm{rk}\,\mathrm{Cl}(X) = 1$. Indeed, assume that $\mathrm{rk}\,\mathrm{Cl}(X) > 1$. Then we have a non-trivial representation of $G$ in $\mathrm{Cl}(X) \otimes \mathbb{Q}$ such that $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$. By [CCN+85] the group $G$ has no non-trivial rational representations of degree $< 10$. Hence, $\mathrm{rk}\,\mathrm{Cl}(X) \geq 11$. Let $F \subset X$ be a prime divisor and let $d := F \cdot K_X^2$ be its degree. Consider the $G$-orbit $F_1 = F, \dots, F_m$. Then $\sum F_i$ is a Cartier divisor on $X$. Hence, $\sum F_i \sim -rK_X$ for some $r$ and so $md = (2g-2)r = 14r$. Since $m$ divides $|G| = 660$, $d$ is divisible by 7. In particular, $X$ contains no surfaces of degree $\leq 6$. Then by [Kal, Cor. 3.11] $\mathrm{rk}\,\mathrm{Cl}(X) \leq 7$, a contradiction. Therefore, $\mathrm{rk}\,\mathrm{Cl}(X) = 1$. Then by

Claim 5.17.6 below the number of singular points of $X$ is at most 5. The contradiction proves the lemma. $\square$

**Claim 5.17.6.** *Let $X$ be a Gorenstein Fano threefold whose singularities are only (isolated) ordinary double points. Let $N$ be the number of singular points. Then*

$$N \leq \operatorname{rk}\operatorname{Cl}(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}) \leq \operatorname{rk}\operatorname{Cl}(X) - 1 + h^{1,2}(X'),$$

*where $X'$ is a smoothing of $X$ and $\hat{X} \to X$ is the blowup of singular points.*

*Proof.* Let $D \in |-K_X|$ be a general member, let $\tilde{X} \to X$ be a small (not necessarily projective) resolution, and let $\tilde{D} \subset \tilde{X}$ be the pull-back of $D$. By the proof of Theorem 13 in [Nam97] we can write

$$N \leq \dim \operatorname{Def}(X, D) - \dim \operatorname{Def}(\tilde{X}, \tilde{D}) = h^1(X', T_{X'}(-\log D')) -$$

$$- h^1(\tilde{X}, T_{\tilde{X}}(-\log \tilde{D})) = \tfrac{1}{2}\operatorname{Eu}(\tilde{X}) - \tfrac{1}{2}\operatorname{Eu}(X') = \tfrac{1}{2}\operatorname{Eu}(\hat{X}) - N - \tfrac{1}{2}\operatorname{Eu}(X'),$$
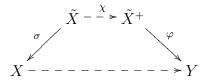
where $\operatorname{Def}(X, D)$ (resp. $\operatorname{Def}(\tilde{X}, \tilde{D})$) denotes the deformation space of the pair $(X, D)$ (resp. $(\tilde{X}, \tilde{D})$) and $(X', D')$ is a general member of the deformation family $\operatorname{Def}(X, D)$. Hence, $4N \leq \operatorname{Eu}(\hat{X}) - \operatorname{Eu}(X')$. Note that $\operatorname{rk}\operatorname{Cl}(X) = \rho(\hat{X}) - N$. Since both $X'$ and $\hat{X}$ are projective varieties with $H^i(X', \mathscr{O}_{X'}) = H^i(\hat{X}, \mathscr{O}_{\hat{X}}) = 0$, we get the disired inequality. $\square$

**Lemma 5.18.** *Under the assumptions of 5.13 we have $g \leq 8$.*

*Proof.* First we consider the case $g = 12$. Then the family of conics on $X$ is parameterized by the projective plane $\mathbb{P}^2$, see [KS04]. By our assumption the induced action of $G$ on $\mathbb{P}^2$ is trivial. Hence $G$ acts non-trivially on each conic, a contradiction.

Now assume that $g = 9$ or 10. We claim that in the case $g = 9$ the order of $G$ is divisible by 5 or 11. This follows from Theorem 3.3 whenever $G$ has an irreducible representation of degree 4. Otherwise the representation of $G$ in $H^0(X, -K_X) \simeq \mathbb{C}^{11}$ is either irreducible or has 5-dimensional irreducible subrepresentation. By Theorem 5.9 and our assumptions the action of $G$ on $H^{1,2}(X)$ is trivial, so is the action on $H^3(X, \mathbb{C})$. Let $\delta \in G$ be an element of prime order $p \geq 5$. If $g = 9$, then we take $p = 5$ or 11. Assume that $\delta$ has no fixed points. Then the quotient $X/\langle \delta \rangle$ is a smooth Fano threefold. On the other hand, Fano manifolds are simply-connected, a contradiction. Therefore, $\delta$ has at least one fixed point on $X$. By the Lefschetz fixed point formula we have $\operatorname{Lef}(X, \delta) = 4 - \dim H^3(X, \mathbb{C}) = 2g - 20$. If $g = 9$ or 10, then $\operatorname{Lef}(X, \delta) \leq 0$. Therefore, the set $\operatorname{Fix}(\delta)$ of $\delta$-fixed points has positive diminsion. Let $\Phi(X) \subset X$ be the surface swept out by lines. Then $\operatorname{Fix}(\delta) \cap \Phi(X) \neq \varnothing$. Take a point $P \in \operatorname{Fix}(\delta) \cap \Phi(X)$. Since $X$ is an intersection of quadrics, there are at most four lines passing through $P$, see [IP99, Prop. 4.2.2]. The group $\langle \delta \rangle$ cannot interchange these lines. Hence,

there is a $\langle\delta\rangle$-invariant line $\ell \subset X$. Now consider the double projection digram (see [Isk90], [IP99, Th. 4.3.3]):

$$\tilde{X} \overset{\chi}{\dashrightarrow} \tilde{X}^+$$

$$\sigma \swarrow \qquad \searrow \varphi$$

$$X \dashrightarrow Y$$

where $\sigma$ is the blowup of $\ell$ and $\chi$ is a flop. If $g \geq 9$, then $Y$ is a smooth Fano threefold and $\varphi$ is the blowup of a smooth curve $\Gamma \subset Y$. Moreover,

(i) if $g = 9$, then $Y \simeq \mathbb{P}^3$, $\Gamma \subset \mathbb{P}^3$ is a non-hyperelliptic curve of genus 3 and degree 7 contained in a unique irreducible cubic surface $F \subset \mathbb{P}^3$,

(ii) if $g = 10$, then $Y = Y_2 \subset \mathbb{P}^4$ is a smooth quadric, $\Gamma$ is a (hyperelliptic) curve of genus 2 and degree 7 contained in a unique irreducible surface $F \subset Y$ of degree 4.

Clearly, the above diagram is $\langle\delta\rangle$-equivariant. Since the linear span of $\Gamma$ coincides with $\mathbb{P}^3$ for $g = 9$ (resp. $\mathbb{P}^4$ for $g = 10$), the group $\langle\delta\rangle$ non-trivially acts on $\Gamma$. On the other hand, the action of $\langle\delta\rangle$ on $H^1(\Gamma, \mathbb{Z}) \simeq H^3(X, \mathbb{Z})$ is trivial. This contradicts the Lefschetz fixed point formula. $\qquad\square$

Now we are going to finish our treatment of the Gorenstein case. It remains to consider two cases: $g = 8$ and $g = 7$, where $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ is a smooth Fano threefold with $\mathrm{Pic}(X) = -K_X \cdot \mathbb{Z}$. Here we need the following result of S. Mukai.

**Theorem 5.19** ([Muk88a]).     (i) (*see also* [Gus83]) *Let* $X = X_{14} \subset \mathbb{P}^9$ *be a smooth Fano threefold of genus 8 with* $\rho(X) = 1$. *Then* $X$ *is isomorphic to a linear section of the Grassmannian* $\mathrm{Gr}(2, 6) \subset \mathbb{P}^{14}$ *by a subspace of codimension 5. Any isomorphism* $X = X_{14} \overset{\sim}{\longrightarrow}$ $X' = X'_{14}$ *of two such smooth sections is induced by an isomorphism of the Grassmannian* $\mathrm{Gr}(2, 6)$.

(ii) *Let* $X = X_{12} \subset \mathbb{P}^8$ *be a smooth Fano threefold of genus 7 with* $\rho(X) = 1$. *Then* $X$ *is isomorphic to a linear section of the Lagrangian Grassmannian* $\mathrm{LGr}(4, 9) \subset \mathbb{P}^{15}$ *by a subspace of dimension 8 (see Example 2.11). Any isomorphism* $X = X_{12} \overset{\sim}{\longrightarrow} X' = X'_{12}$ *of two such smooth sections is induced by an isomorphism of the Lagrangian Grassmannian* $\mathrm{LGr}(4, 9)$.

Consider the case $g = 8$. By the above theorem the group $G$ acts on $\mathrm{Gr}(2, 6)$ and on $\mathbb{P}^{14} = \mathbb{P}(\wedge^2\mathbb{C}^5) = \mathbb{P}(H^0(\mathrm{Gr}(2, 6), \mathscr{T}^*))$, where $\mathscr{T}$ is tautological rank two vector bundle on $\mathrm{Gr}(2, 6)$. The linear span of $X = X_{12}$ in $\mathbb{P}^{14}$ is a $G$-invariant $\mathbb{P}^9$. Let $\mathbb{P}^4 \subset \mathbb{P}^{14*} = \mathbb{P}(\wedge^2\mathbb{C}^{5*})$ be the $G$-invariant orthogonal subspace. The locus of all degenerate skew-forms is the Pfaffian cubic hypersurface $Y_3 \subset \mathbb{P}(\wedge^2\mathbb{C}^{5*})$. Put $X_3 = Y_3 \cap \mathbb{P}^4$. Then $X_3 \subset \mathbb{P}^4$ is a $G$-invariant cubic. Since the variety $X = X_{14}$ is smooth, so is our

cubic $X_3 \subset \mathbb{P}^4$, see [Kuz04, Prop. A.4]. Then by Lemma 5.12 we get $G \simeq \mathrm{PSL}_2(11)$, $X_3^{\mathrm{k}} \subset \mathbb{P}^4$ is the Klein cubic and we get Example 2.6.

Finally consider the case $g = 8$. The group $G$ acts on the Lagrangian Grassmannian $\mathrm{LGr}(4,9) \subset \mathbb{P}^{15}$. Let $C := \mathrm{LGr}(4,9) \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^{15}$ is the subspace orthogonal to $\mathbb{P}^8$ with respect to the $G$-invariant quadratic form on $\mathbb{P}^{14}$. Then $C \subset \mathbb{P}^6$ is a smooth canonical curve of genus 7 [IM04]. Hence $G \subset \mathrm{Aut}(C)$. On the other hand, by the Hurwitz bound we have $|G| \leq |\mathrm{Aut}(C)| \leq 504$. Furthermore, the group has an irreducible representation in $H^0(X, -K_X) \simeq \mathbb{C}^9$. Hence, $|G|$ is divisible by 9. Now it is an easy exercise to show that either $G \simeq \mathrm{SL}_2(8)$ or $G$ is contained in the list (1.2). For example, according to Theorem 3.7 we may assume that $G$ has no subgroups of index $\leq 26$, i.e., of order $\geq 19$. Hence $234 = 26 \cdot 9 \leq |G|$. Now we write the Hurwitz formula for the quotient $\pi : C \to C/G = C'$:

$$12 = 2g(C) - 2 = |G|(2g(C') - 2) + |G| \sum_{i=1}^{s}(1 - 1/a_i),$$

where $\sum(a_i-1)Q_i$ is the ramification divisor on $C'$. By the above $a_i \leq 18$ for all $i$. There are only two integer solutions: $|G| = 288$, $(a_1, \ldots, a_s) = (2,3,8)$ and $|G| = 504$, $(a_1, \ldots, a_s) = (2,3,7)$. In the first case the Sylow 17-subgroup has index 14 in $G$, a contradiction. In the second case the curve $C$ is unique up to isomorphism and $G \simeq \mathrm{PSL}_2(8)$, see [Mac65]. By the construction in Example 2.11 the threefold $X_{12}$ is uniquely determined by $C$, so $X_{12} = X_{12}^{\mathrm{m}}$. This finishes the treatment of the case of Gorenstein $X$.

## 6. Case: $X$ is not Gorenstein

In this section, as in §5, we assume that $G$ is a simple group which does not admit any embeddings to $\mathrm{Cr}_2(\mathbb{C})$. We assume $X$ is a $G\mathbb{Q}$-Fano threefold such that $K_X$ is not Cartier. Let $\Omega \subset \mathrm{Sing}(X)$ be the set of all non-Gorenstein points and let $n := |\Omega|$.

**Lemma 6.1.** *In the above assumptions the group $G$ transitively acts on $\Omega$, $n \geq 9$, and each point $P \in \Omega$ is a cyclic quotient singularity of index 2.*

*Proof.* Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ be the orbit decomposition, and let $n_i := |\Omega_i|$. For a point $P_i \in \Omega_i$, let $Q_{ij} \in \mathbf{B}$, $j = 1, \ldots, l_i$ be "virtual" points in the basket over $P_i$ and let $r_{ij}$ be the index of $Q_{ij}$. The orbifold Riemann-Roch and Myaoka-Bogomolov inequality give us (see [Kaw92], [KMMT00]) [3]

$$(6.2) \qquad 24 > \sum_{i=1}^{m} n_i \sum_{j=1}^{l_i}\left(r_{ij} - \frac{1}{r_{ij}}\right) \geq \frac{3}{2}\sum_{i=1}^{m} n_i.$$

---

[3]From [KMMT00] we have the inequality $\sum(r - 1/r) \leq 24$. The strict inequality follows from the proof in [Kaw92] because $\rho(X)^G = 1$. I would like to thank Professor Y. Kawamata for pointing me out this fact.

By Theorem 3.7 and our assumptions we have $n_1, \ldots, n_m \geq 7$.

Assume that $P_1 \in X$ is not a cyclic quotient singularity. Then over each $P_i \in \Omega_1$ there are at least two virtual points $Q_{ij}$, i.e, $l_1 > 1$. By (6.2) we have

$$24 > n_1 \sum_{j=1}^{l_1} \left( r_{1j} - \frac{1}{r_{1j}} \right) \geq 7 \sum_{j=1}^{l_1} \left( r_{1j} - \frac{1}{r_{1j}} \right).$$

There is only one possibility: $l_1 = 2$, $n_1 = 7$, and $r_{11} = r_{12} = 2$. In this case, by the classification [Rei87] the point $P_1 \in X$ is of type $\{xy + \phi(z^2, t)\}/\boldsymbol{\mu}_2(1, 1, 1, 0)$, where $\operatorname{ord} \phi(0, t) = 2$, or $\{x^2 + y^2 + \phi(z, t)\}/\boldsymbol{\mu}_2(0, 1, 1, 1)$ (because the "axial multiplicity" is equal to 2).

By Theorem 3.7 we have $G \simeq \mathfrak{A}_7$ and $G_P \simeq \mathfrak{A}_6$. As in the proof of Lemma 4.5 we have an embedding $\tilde{G}_P \subset \mathrm{GL}(T_{P^\sharp, U^\sharp})$, where $\dim T_{P^\sharp, U^\sharp} = 4$ and $\tilde{G}_P$ is a central extension of $G_P$ by $\boldsymbol{\mu}_2$. The action of $\tilde{G}_P$ preserves the tangent cone $TC_{P^\sharp, U^\sharp} \subset T_{P^\sharp, U^\sharp}$ which is given by a quadratic form of rank $\geq 2$. Since $G_P \simeq \mathfrak{A}_6$ cannot act non-trivially on a smooth quadric in $\mathbb{P}^3$, $\operatorname{rk} q \neq 4$. Hence, $\operatorname{rk} q = 2$ or $3$ and the representation of $\tilde{G}_P$ in $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^4$ is reducible: the singular locus of $TC_{P^\sharp, U^\sharp}$ is a $\tilde{G}_P$-invariant linear subspace. On the other hand, $\tilde{G}_P \simeq \mathfrak{A}_6$ has no faithful representations of degree $\leq 3$ (see, e.g., Theorem 3.2 or [CCN+85]), a contradiction.

Therefore, all the points in $\Omega$ are cyclic quotient singularities. Then (6.2) can be rewritten as follows:

$$(6.3) \qquad 24 > \sum_{i=1}^{m} n_i \left( r_i - \frac{1}{r_i} \right) \geq \frac{3}{2} \sum_{i=1}^{m} n_i,$$

where $r_i$ is the index of the point $P_i \in \Omega_i$. Assume that $n_1 \leq 8$, then by Theorem 3.7 $G \simeq \mathfrak{A}_n$ with $n = 7$ or $8$, and $G_P \simeq \mathfrak{A}_{n-1}$. As above $\tilde{G}_P \subset \mathrm{GL}(T_{P^\sharp, U^\sharp})$, where $\dim T_{P^\sharp, U^\sharp} = 3$ (because $U^\sharp$ is smooth) and $\tilde{G}_P$ is a central extension of $G_P$ by $\boldsymbol{\mu}_{r_1}$. Clearly, the representation $\tilde{G}_P$ in $\mathrm{GL}(T_{P^\sharp, U^\sharp})$ is irreducible. Hence $\boldsymbol{\mu}_{r_1}$ acts on $T_{P^\sharp, U^\sharp}$ by scalar multiplication. By the classification of terminal singularities (Terminal Lemma) [Rei87] we have $r_1 = 2$. But then the group $\tilde{G}_P$ has no non-trivial representations in $\mathbb{C}^3$ by Theorem 3.2. The contradiction shows that $n_1 \geq 9$ and, by symmetry, $n_i \geq 9$ for all $i$. Then by (6.3) we have $m = 1$ and $r_i = 2$ for all $i$. $\qquad \square$

**Lemma 6.4.** (i) $Z(G_P) = \{1\}$, $Z(\tilde{G}_P) = \boldsymbol{\mu}_2$.
  (ii) *The representation of $\tilde{G}_P$ in $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^3$ is irreducible.*
  (iii) *The action of $\tilde{G}_P$ on $T_{P^\sharp, U^\sharp} \simeq \mathbb{C}^3$ is primitive.*
  (iv) *The only possible case is $G \simeq \mathrm{PSL}_2(11)$, $n = 11$, $G_P \simeq \mathfrak{A}_5$.*

*Proof.* (i) follows from the explicit description of groups $G_P$ in Theorem 3.7.

26

(ii) Assume that $T_{P^\sharp, U^\sharp} = T_1 \oplus T_2$. Then the kernel of the homomorphism $\tilde{G}_P \to \mathrm{GL}(T_2)$ is contained into $Z(\tilde{G}_P)$. Hence, $\tilde{G}_P \to \mathrm{GL}(T_2)$ is injective and so $G_P$ effectively acts on $\mathbb{P}^1$, a contradiction.

(iii) Assume the converse. Then there is an abelian subgroup $\tilde{A} \subset \tilde{G}_P$ such that $\tilde{G}_P/\tilde{A} \simeq \mathfrak{A}_3$ or $\mathfrak{S}_3$. Hence there is an abelian subgroup $A \subset G_P$ such that $G_P/A \simeq \tilde{G}_P/\tilde{A} \simeq \mathfrak{A}_3$ or $\mathfrak{S}_3$. In particular, $G_P$ is not simple and its order is divisible by 3. Thus by Theorem 3.7 there are only three possibilities: $G \simeq \mathrm{SL}_3(3)$, $\mathrm{PSL}_2(13)$, and $\mathrm{SL}_4(2)$.

In the case $G \simeq \mathrm{PSL}_2(13)$ the group $G_P \simeq \boldsymbol{\mu}_{13} \rtimes \boldsymbol{\mu}_6$ has no surjective homomorphisms to $\mathfrak{S}_3$. So, $G_P/A \simeq \mathfrak{A}_3$ and $A \simeq \boldsymbol{\mu}_{26}$. On the other hand, $G_P$ contains no elements of order 26, a contradiction. Consider the case $G \simeq \mathrm{SL}_3(3)$. Then $G_P \supset \mathrm{GL}_2(3)$. Since $\mathrm{GL}_2(3)/Z(\mathrm{GL}_2(3)) \simeq \mathfrak{S}_4$, for $A \cap \mathrm{GL}_2(3)$ we have only one possibility: it is a group of order 8. But the group $A \cap \mathrm{GL}_2(3)$ is not abelian, a contradiction. Finally, in the case $G \simeq \mathrm{SL}_4(2)$ the group $\mathrm{SL}_3(2) \subset G_P$ is simple of order 168, a contradiction.

(iv) Follows from Theorem 3.2. $\qquad\square$

From now on we assume that $G \simeq \mathrm{PSL}_2(11)$ and $n = 11$.

**Lemma 6.5.** $\dim |-K_X| > 0$.

*Proof.* By [Kaw92] we have $-K_X \cdot c_2(X) = 24 - 3n/2$. Hence by the orbifold Riemann-Roch (see [Rei87])

$$\dim |-K_X| = \frac{1}{2}(-K_X)^3 - \frac{1}{12}K_X \cdot c_2(X) + \sum_{P \in \Omega} c_P(-K_X) =$$

$$= \frac{1}{2}(-K_X)^3 + 2 - \frac{n}{4} = \frac{1}{2}(-K_X)^3 - \frac{3}{4}.$$

Put $\dim |-K_X| = l$. Then $(-K_X)^3 = 2l + 3/2$. In particular, $l \geq 0$ and $|-K_X| \neq \varnothing$. Assume that $\dim |-K_X| = 0$. Then $(-K_X)^3 = 3/2$. Let $S \in |-K_X|$ be (a unique) member. By Lemma 4.6 the surface $S$ is reducible and $G$ transitively acts on its components. Write $S = \sum_{i=1}^m S_i$. Then $m(-K_X)^2 \cdot S_i = (-K_X)^3 = 3/2$. Since $2(-K_X)^2 \cdot S_i$ is an integer, we have $m \leq 3$, a contradiction. $\qquad\square$

**Lemma 6.6.** *The pair* $(X, |-K_X|)$ *is canonical.*

*Proof.* Put $\mathscr{H} := |-K_X|$. By Corollary 4.7 the linear system $\mathscr{H}$ has no fixed components. We apply a $G$-equivariant version of a construction [Ale94, §4]. Take $c$ so that the pair $(X, c\mathscr{H})$ is canonical but not terminal. By our assumption $0 < c < 1$. Let $f \colon (\tilde{X}, c\tilde{\mathscr{H}}) \to (X, c\mathscr{H})$ be a $G$-equivariant $G\mathbb{Q}$-factorial terminal modification (terminal model). We can write

$$\begin{aligned} K_{\tilde{X}} + c\tilde{\mathscr{H}} &= f^*(K_X + c\mathscr{H}), \\ K_{\tilde{X}} + \tilde{\mathscr{H}} + \sum a_i E_i &= f^*(K_X + \mathscr{H}) \sim 0, \end{aligned}$$

where $E_i$ are $f$-exceptional divisors and $a_i > 0$. Run $(\tilde{X}, c\tilde{\mathscr{H}})$-MMP:

$$(\tilde{X}, c\tilde{\mathscr{H}}) \dashrightarrow (\bar{X}, c\bar{\mathscr{H}}).$$

As in 4.2 $\bar{X}$ is a Fano threefold with $G\mathbb{Q}$-factorial terminal singularities and $\rho(\bar{X})^G = 1$. We also have $0 \sim K_{\bar{X}} + \bar{\mathscr{H}} + \sum a_i \bar{E}_i$. Here $\sum a_i \bar{E}_i$ is a non-trivial effective invariant divisor such that $-(K_{\bar{X}} + \sum a_i \bar{E}_i) \sim \bar{\mathscr{H}}$ is ample. This contradicts Lemma 4.6. $\qquad\square$

**Lemma 6.7.** *The image of the ($G$-equivariant) rational map $\phi : X \dashrightarrow \mathbb{P}^l$ given by the linear system $|-K_X|$ is three-dimensional.*

*Proof.* Let $Y := \phi(X)$. Since $X$ is rationally connected, $G$ acts trivially on $Y$. This contradicts Lemma 4.8. $\qquad\square$

Recall that non-Gorenstein points $P_1, \ldots, P_{11}$ of $X$ are of type $\frac{1}{2}(1,1,1)$. Let $f : \tilde{X} \to X$ be blow up of $P_1, \ldots, P_{11}$ and let $E_i = f^{-1}(P_i)$ be exceptional divisors. Then $\tilde{X}$ is smooth over $P_i$, it has at worst Gorenstein terminal singularities, $E_i \simeq \mathbb{P}^2$, and $\mathscr{O}_{E_i}(-K_{\tilde{X}}) = \mathscr{O}_{\mathbb{P}^2}(1)$. Put $\mathscr{H} := |-K_X|$ and let $\tilde{\mathscr{H}}$ be the birational transform. Since the pair $(X, \mathscr{H})$ is canonical, we have

$$K_{\tilde{X}} + \tilde{\mathscr{H}} \sim f^*(K_X + \mathscr{H}) \sim 0.$$

Hence, $|-K_{\tilde{X}}| = \tilde{\mathscr{H}}$.

**Lemma 6.8.** *The linear system $\tilde{\mathscr{H}}$ is base point free.*

*Proof.* Note that the restriction $\tilde{\mathscr{H}}|_{E_i} = |-K_{\tilde{X}}||_{E_i}$ is a (not necessarily complete) linear system of lines on $E_i \simeq \mathbb{P}^2$. Since this linear system is $G_{P_i}$-invariant, where $G_{P_i} \simeq \mathfrak{A}_5$, it is base point free. Hence $\mathrm{Bs}\, \tilde{\mathscr{H}} \cap E_i = \varnothing$. In particular, this implies that for any curve $C \subset \mathrm{Bs}\, \tilde{\mathscr{H}}$, we have $E_i \cdot C = 0$ and so $\tilde{\mathscr{H}} \cdot C = \mathscr{H} \cdot f(C) > 0$. Hence, $\tilde{\mathscr{H}}$ is nef. By Lemma 6.7 it is big. Then the assertion follows by Lemma 5.2. $\qquad\square$

Now we are in position to finish the proof of Theorems 1.3 and 1.5. Note that the divisors $E_i$ are linear independent elements of $\mathrm{Pic}(\tilde{X})$. Hence, $\rho(\tilde{X}) > 11$. If $\tilde{X}$ is a Fano threefold, then by Theorem 5.6 and Remark 5.8 there is a smoothing $\tilde{X}_t$ with $\rho(\tilde{X}_t) > 11$. This contradicts the classification of smooth Fano threefolds with $\rho > 1$ [MM82]. By Lemma 5.3 the linear system $|-K_{\tilde{X}}|$ determines a birational contraction $\varphi : \tilde{X} \to \bar{X} = \bar{X}_{2g-2} \subset \mathbb{P}^{g+1}$ whose image is an anticanonically embedded Fano threefold with at worst canonical singularities. Here $g$ in the genus of $\bar{X}$ (see 5.1). Since $\rho(\tilde{X})^G = 2$, $\rho(\bar{X})^G = 1$. Let $\bar{E}_i := \varphi(E_i)$.

Take a general member $\bar{H} \in |-K_{\bar{X}}|$. By Bertini's theorem $\bar{H}$ is a K3 surface with at worst Du Val singularities. Put $C_i := \bar{E}_i \cap \bar{H}$.

**Claim 6.8.1.** *$C_1, \ldots, C_{11}$ are disjointed smooth rational curves contained into the smooth locus of $\bar{H}$.*

*Proof.* Since $\bar{H}$ is Cartier, the number $\bar{E}_i \cdot \bar{E}_j \cdot \bar{H}$, where $1 \leq i, j \leq 11$, is well-defined and coincides with the intersection number $C_i \cdot C_j$ of curves $C_i := \bar{E}_i \cap \bar{H}$ and $C_j := \bar{E}_j \cap \bar{H}$ on $\bar{H}$. Clearly, the numbers $C_i^2 = \bar{E}_i^2 \cdot \bar{H}$ for $1 \leq i \leq 11$ do not depend on $i$. Since the action of $G$ on $\{\bar{E}_i\}$ is doubly transitive [CCN+85], the numbers $C_i \cdot C_j = \bar{E}_i \cdot \bar{E}_j \cdot \bar{H}$ for $1 \leq i \neq j \leq 11$ also do not depend on $i$, $j$.
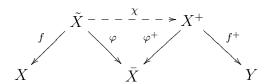
Since $(-K_{\tilde{X}})^2 \cdot E_i = 1$, the surfaces $\bar{E}_i$ are planes in $\mathbb{P}^{g+1}$ and every $C_i$ is a line on $\bar{E}_i$. If $C_i \cdot C_j > 0$ for some $i \neq j$, then $\bar{E}_i \cap \bar{E}_j$ is a line. Since $G$ acts doubly transitive on $\{\bar{E}_i\}$, the intersection $\bar{E}_i \cap \bar{E}_j$ is a line for all $i \neq j$. Hence, the linear span of $\bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_3$ is a three-dimensional projective subspace $\mathbb{P}^3 \subset \mathbb{P}^{g+1}$. In this case, $\bar{X} \cap \mathbb{P}^3$ cannot be an intersection of quadrics. This contradicts Lemma 5.4.

Thus we may assume that $C_i \cdot C_j = 0$ for all $i \neq j$. By the Hodge index theorem $C_k^2 \leq 0$ for all $k$. If $C_1^2 = 0$, then for some $m$ the linear system $|mC_1|$ determines an elliptic fibration $\psi : \bar{H} \to \mathbb{P}^1$ and all the curves $C_k$ are degenerate fibers of $\psi$. Let $\mu : \hat{H} \to \bar{H}$ be the minimal resolution, let $F_k := \mu^{-1}(C_k)$ be the degenerate fiber corresponding to $C_k$, and let $\hat{C}_k$ be the proper transform of $C_k$. Then $\hat{H}$ is a smooth K3 surface. Since $C_k$ is smooth, $\hat{C}_k \cdot (F_k - \hat{C}_k) = 1$. Using Kodaira's classification of degenerate fibers of elliptic fibrations we see that $F_k$ has at least three components. But then $\rho(\hat{H}) \geq 23$, a contradiction.

Therefore, $C_k^2 < 0$ for all $k$. In particular, $\operatorname{rk} \operatorname{Cl}(\bar{H}) \geq 12$. As above one can show that $\bar{H}$ cannot be singular near $C_k$. Hence all the $C_k$ are $(-2)$-curves contained into the smooth part of $\bar{H}$. $\qquad\square$

Clearly, fibers of $\varphi$ meet $\sum E_i$ (otherwise $\varphi$ is an isomorphism near $E_i$ and then $\rho(\bar{X})^G > 1$). Since $E_i \simeq \mathbb{P}^2$, $\varphi$ cannot contract divisors to points. Assume that $\varphi$ contracts divisors $D_l$ to curves $\Gamma_l$. Then $\Gamma_l \subset E_i$ for some $i$. Since $\varphi$ is $K$-trivial, $\bar{X}$ is singular along $\Gamma_l$ and $\bar{H}$ is singular at point $\Gamma_l \cap \bar{H}$. Since $\Gamma_l \cap \bar{H} \subset C_i$, we get a contradiction with the above claim.

Therefore $\varphi$ does not contract any divisors, i.e., it contracts only a finite number of curves. Then $\bar{X}$ is a Fano threefold with Gorenstein terminal (but not $G\mathbb{Q}$-factorial) singularities. Consider the following diagram (cf. [IP99, Ch. 4]):

$$\tilde{X} \dashrightarrow^{\chi} X^+$$

$$\begin{array}{ccccc} & \tilde{X} & \dashrightarrow^{\chi} & X^+ & \\ {}^{f}\swarrow & {}^{\varphi}\downarrow & {}^{\varphi^+}\downarrow & & \searrow^{f^+} \\ X & & \bar{X} & & Y \end{array}$$

Here $\chi$ is a $G$-equivariant flop, $\varphi^+$ is a small modification, and $f^+$ is a $K$-negative $G$-equivariant $G$-extremal contraction. As in 4.2 we may assume that $Y$ is $G\mathbb{Q}$-Fano threefold with $\rho(Y)^G = 1$. Moreover by the results of §5 we may assume that $Y$ is not Gorenstein. Therefore, $Y$ is of the same

type as $X$. In particular, $Y$ has 11 cyclic quotient singular points of index 2. Let $D_1, \ldots, D_{11}$ be the $f^+$-exceptional divisors and let $D := \sum D_i$. We can write

$$-K_{\tilde{X}}^3 = -K_{X^+}^3 = -K_{\bar{X}}^3 = 2g - 2,$$

$$(-K_{\tilde{X}})^2 \cdot E = (-K_{X^+})^2 \cdot E^+ = (-K_{\bar{X}})^2 \cdot \bar{E} = 11,$$

$$-K_{\tilde{X}} \cdot E^2 = -K_{X^+} \cdot E^{+2} = -K_{\bar{X}} \cdot \bar{E}^2 = -22.$$

Further, $D \sim -\alpha K_{X^+} - \beta E^+$ for some $\alpha, \beta \in \mathbb{N}$. This gives us

$$(-K_{\bar{X}})^2 \cdot D = 11 = (2g - 2)\alpha - 11\beta,$$

$$-K_{\bar{X}} \cdot D^2 = -22 = (2g - 2)\alpha^2 - 22\alpha\beta - 22\beta^2.$$

In particular, either $g - 1$ or $\alpha$ is divisible by 11. Assume that $g - 1 = 11k$, $k \in \mathbb{N}$. Then the above equalities can be rewritten as follows:

$$\beta = 2k\alpha - 1,$$

$$0 = -1 - k\alpha^2 + \alpha\beta + \beta^2.$$

Eliminating $\beta$ we get

$$0 = -1 - k\alpha^2 + \alpha(2k\alpha - 1) + (2k\alpha - 1)^2 = (\alpha + 4k)(k\alpha - 1).$$

Since $\alpha, k > 0$ we get $k = 1$ and $g = 12$. Hence $\dim H^0(\tilde{X}, -K_{\tilde{X}}) = 14$ and so $\dim H^0(\tilde{X}, -K_{\tilde{X}})^G \geq 2$ (because the degrees of irreducible representations of $G = \mathrm{PSL}_2(11)$ are 1, 5, 5, 10, 10, 11, 12, 12). This contradicts Lemma 4.8.

Therefore, $\alpha = 11k$, $k \in \mathbb{N}$. Then, as above,

$$\beta = 2(g - 1)k - 1,$$

$$0 = -1 - 11(g - 1)k^2 + 11k\beta + \beta^2.$$

Thus

$$0 = -1 - 11(g - 1)k^2 + 11k(2(g - 1)k - 1) + (2(g - 1)k - 1)^2 =$$
$$= (11 + 4(g - 1))((g - 1)k - 1).$$

Since $g > 2$ (see Lemma 5.3) we have a contradiction. This finishes our proof of Theorems 1.3 and 1.5.

## REFERENCES

[Adl78]    A. Adler. On the automorphism group of a certain cubic threefold. *Amer. J. Math.*, 100(6):1275–1280, 1978.

[Ale94]    V. Alexeev. General elephants of **Q**-Fano 3-folds. *Compositio Math.*, 91(1):91–116, 1994.

[AR96]    A. Adler and S. Ramanan. *Moduli of abelian varieties*, volume 1644 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[AW97]     D. Abramovich and J. Wang. Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4(2-3):427–433, 1997.

[Bli17]     H. F. Blichfeldt. *Finite collineation groups.* The Univ. Chicago Press, Chicago, 1917.

[Bra67]     R. Brauer. Über endliche lineare Gruppen von Primzahlgrad. *Math. Ann.*, 169:73–96, 1967.

[CCN⁺85]    J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson. *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.* Oxford: Clarendon Press. XXXIII, 252 p., 1985.

[CG72]     H. Clemens and P. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[DI06]     I. Dolgachev and V. Iskovskikh. Finite subgroups of the plane cremona group. to appear in *Algebra, Arithmetic, and Geometry In Honor of Yu. I. Manin*, Progr. Math., 269–270, Birkhäuser, 2009.

[DM96]     J. D. Dixon and B. Mortimer. *Permutation groups.* Graduate Texts in Mathematics. 163. New York, NY: Springer-Verlag. xii, 346 p., 1996.

[GH78]     P. Griffiths and J. Harris. *Principles of algebraic geometry.* Pure and Applied Mathematics. A Wiley-Interscience Publication. New York etc.: John Wiley &amp; Sons. XII, 813 p., 1978.

[Gus83]     N. P. Gushel'. Fano varieties of genus 8. *Uspekhi Mat. Nauk*, 38(1(229)):163–164, 1983.

[IM04]     A. Iliev and D. Markushevich. Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7. *Adv. Geom.*, 4(3):287–318, 2004.

[IP96]     V. A. Iskovskikh and A. V. Pukhlikov. Birational automorphisms of multi-dimensional algebraic manifolds. *J. Math. Sci.*, 82(4):3528–3613, 1996.

[IP99]     V. A. Iskovskikh and Yu. Prokhorov. *Fano varieties. Algebraic geometry. V.*, volume 47 of *Encyclopaedia Math. Sci.* Springer, Berlin, 1999.

[Isk80]     V.A. Iskovskikh. Anticanonical models of three-dimensional algebraic varieties. *J. Sov. Math.*, 13:745–814, 1980.

[Isk90]     V.A. Iskovskikh. A double projection from a line on Fano threefolds of the first kind. *Math. USSR, Sb.*, 66:265–284, 1990.

[JR06]     P. Jahnke and I. Radloff. Terminal Fano threefolds and their smoothings. preprint arXiv: 0601769, 2006.

[Kal]      A.-S. Kaloghiros. The defect of Fano 3-folds. preprint arXiv:0711.2186.

[Kaw92]     Y. Kawamata. Boundedness of $\mathbf{Q}$-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, Contemp. Math. **131**, 439–445, Providence, RI, 1992. Amer. Math. Soc.

[Kaw07]     M. Kawakita. Inversion of adjunction on log canonicity. *Invent. Math.*, 167:129–133, 2007.

[KMMT00]   J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi. Boundedness of canonical $\mathbf{Q}$-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(5):73–77, 2000.

[KS04]     J. Kollár and F.-O. Schreyer. Real Fano 3-folds of type $V_{22}$. In *The Fano Conference*, 515–531. Univ. Torino, Turin, 2004.

[Kuz04]     A. G. Kuznetsov. Derived categories of cubic and $V_{14}$ threefolds. *Proceedings of the Steklov Institute of Mathematics*, 246:171–194, 2004.

[Lin71]     J. H. Lindsey, II. Finite linear groups of degree six. *Canad. J. Math.*, 23:771–790, 1971.

[Mac65]     A. M. Macbeath. On a curve of genus 7. *Proc. London Math. Soc. (3)*, 15:527–542, 1965.

[MM82]      S. Mori and S. Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82. Erratum: Manuscripta Math. 110 (2003), 407.

[Mor88]     S. Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.

[MP09]      S. Mori and Yu. Prokhorov. Multiple fibers of del Pezzo fibrations. *Proc. Steklov Inst. Math.*, 264(1):131–145, 2009.

[Muk88a]    S. Mukai. Curves, $K3$ surfaces and Fano 3-folds of genus $\leq 10$. In *Algebraic geometry and commutative algebra, Vol. I*, 357–377. Kinokuniya, Tokyo, 1988.

[Muk88b]    S. Mukai. Finite groups of automorphisms of K3 surfaces and the Mathieu group. *Invent. Math.*, 94(1):183–221, 1988.

[Muk92]     S. Mukai. Curves and symmetric spaces. *Proc. Japan Acad., Ser. A*, 68(1):7–10, 1992.

[Muk95]     S. Mukai. Curves and symmetric spaces. I. *Am. J. Math.*, 117(6):1627–1644, 1995.

[Nam97]     Y. Namikawa. Smoothing Fano 3-folds. *J. Algebraic Geom.*, 6(2):307–324, 1997.

[PCS05]     V. V. Przhiyalkovskij, I. A. Chel'tsov, and K. A. Shramov. Hyperelliptic and trigonal Fano threefolds. *Izv. Math.*, 69(2):365–421, 2005.

[Rei87]     M. Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, *Proc. Sympos. Pure Math.*, **46**, 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Ser09]     J.-P. Serre. A Minkowski-style bound for the order of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. preprint arXiv: 0903.0514, 2009.

[Shi89]     K.-H. Shin. 3-dimensional Fano varieties with canonical singularities. *Tokyo J. Math.*, 12(2):375–385, 1989.

[Sho93]     V. V. Shokurov. 3-fold log flips. *Russ. Acad. Sci., Izv., Math.*, 40(1):95–202, 1993.

[ST54]      G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[Zha06]     Q. Zhang. Rational connectedness of log **Q**-Fano varieties. *J. Reine Angew. Math.*, 590:131–142, 2006.

Yuri Prokhorov, Department of Higher Algebra, Faculty of Mathematics and Mechanics, Moscow State Lomonosov University, Vorobievy Gory, Moscow, 119 899, RUSSIA

*E-mail address*: `prokhoro@gmail.com`