Real analytic Jacobi Eisenstein series and Dirichlet series attached to three Jacobi forms

By T. Arakawa and B. Heim

Abstract

In this paper, the iterated Petersson scalar product of a diagonal-restricted real analytic Jacobi Eisenstein series of degree $(3,1)$ against three elliptic Jacobi cusp form is determined. This scalar product gives an integral representation of a certain Dirichlet series, which involves the Fourier coeffiencts of the three forms. Moreover new results and some partly proven conjectures towards analytic properties of real analytic Jacobi Eisenstein series of degree $(n,1)$ are given.

Introduction

P. Garrett studied the iterated Petersson scalar product of a diagonal-restricted Siegel Eisenstein series of degree 3 against three elliptic cusp forms [Ga87], which eventually led him to an integral representation of the Rankin triple L-function. Then analytic and arithmetic properties of the Siegel Eisenstein series had been used to get the meromorphic continuation, funqtional equation and 'special value' properties of the L-function from his approach. But to generalize and to use Garrett's interpretation of the Rankin-Selberg method for other L-functions or at least Dirichlet series, seems to be a very hard and delicate problem in modern number theory. Nevertheless I. Piatetski-Shapiro and S. Rallis [P-R87] succeded in refining Garrett's work to get an integral representation of the "internal tensor product L-function" of an automorphic form on a semi-simple cubic abelian algebra over a number field (see also [Ga85]). Recently, B. Heim [He97] used a diagonal restricted Eisenstein series of degree 5 to get an integral representation of the twisted spinor L-function.

In this paper we start with a new generalization. We study the iterated Petersson scalar product of a diagonal-restricted real analytic Jacobi Eisenstein series of degree $(3,1)$ against three elliptic Jacobi cusp form. We show that this leads to an integral representation of a certain Dirichlet series, which involves the Fourier coeffiencts of the three forms.

We explain this more precisely. Let k,n,t be positive integers and let $I\!H_n$ be Siegel's upper-half-space of degree n. We denote by $\mathcal{J}_{k,t}^{\mathsf{cusp}}$ the space of Jacobi cusp forms of weight k and index t on $I\!H_1 \times \mathbb{C}$. Let $\Phi \in \mathcal{J}_{k,t}^{\mathsf{cusp}}$. Then it has the Fourier expansion

$$\Phi(\tau,z) = \sum_{\substack{r\in\mathbb{Z},\, D\in\mathbb{N}\\ -D\equiv r^2 \bmod (4t)}} c_r^{\Phi}(D)\; e\!\left(\tfrac{r^2+D}{4t}\tau + rz\right), \tag{1}$$

where $(\tau, z) \in \mathbb{H}_1 \times \mathbb{C}$ and $e(z) = e^{2\pi i z}$. For $(\tilde{\tau},\tilde{z}) \in \mathbb{H}_n \times \mathbb{C}^{1,n}$ and $s \in \mathbb{C}$, we denote by $E_n^{k,t}\big((\tilde{\tau},\tilde{z}); s\big)$ the real analytic Jacobi Eisenstein series (for more details see Definition 1.4 and Remark 1.5). Let further $\langle\, ; \,\rangle_J$ denote the Petersson scalar product of Jacobi forms.

Theorem *Let k be an even integer with $k + 2\,Re(s) > 4$ and let t be a natural number. Let $\Phi_1, \Phi_2, \Phi_3 \in \mathcal{J}_{k,t}^{cusp}$ and let Φ_3 also be a Hecke Jacobi eigenform. Then*

$$(2) \quad \langle\langle\langle E_3^{k,t}\left(\begin{pmatrix}\tau_1 & & 0\\ & \tau_2 & \\ 0 & & \tau_3\end{pmatrix}, (z_1 z_2 z_3); s\right) ; \Phi_1(\tau_1, z_1)\, \Phi_2(\tau_2, z_2)\Phi_3(\tau_3, z_3)\rangle_J\rangle_J\rangle_J$$
$$= \gamma(k, t, s)\, \Gamma_\infty(s)\, Z_1^J(2s + k, \Phi_3)\, D^*_{1,2;3}(s).$$

Here $\gamma(k, t, s)$ depends only on k, t, s and is never vanishing, $Z_1^J(s, \Phi_3)$ is the standard L-function of Φ_3 (c.f. section 1.3) and

$$\begin{aligned}\Gamma_\infty(s) &= \frac{\Gamma(2k + s - 3)\, \Gamma(s + k - 3/2)^3}{\Gamma(2k + 2s - 3)\, \Gamma(s + k - 1/2)}\\ D^*_{1,2;3}(s) &= \sum_{r_0=0}^{2t-1} \sum_{\substack{D\in\mathbb{N}\\ -D\equiv r_0^2 \ mod\ 4t}} \sum_{\substack{\varepsilon,\nu=1\\ (\varepsilon,\nu)=1}}^{\infty} \overline{c_{r_0}^{\Phi_3}(D_0)}\ \overline{c_{r_0\varepsilon}^{\Phi_1}(D_0\varepsilon^2)}\ \overline{c_{r_0\nu}^{\Phi_2}(D_0\nu^2)}\\ &\quad (\varepsilon\nu)^{-(2s+2k-3)} D_0^{-(s+2k-3)}.\end{aligned}$$

Now the meromorphic continuation of the Dirichlet $D^*_{1,2;3}(s)$ series on the whole complex plane follows from [Ar94] and [He97]. We also would like to obtain a functional equation of this Dirichlet series, but this would require more knowledge about the analytic properties of real analytic Jacobi Eisenstein series, which is of course of interest in its own right. However, in §2 of this paper, we develop a strategy to obtain the functional equation of real analytic Jacobi Eisenstein series $E_n^{k,S}((\tau, z); s)$ of general degree n and of index S (see Definition 1.4 again). In this connection our goal will be to prove the following conjecture (Conjecture 2.4) which states the explicit description of the functional equation satisfied by those Eisenstein series.

Conjecture [Main Conjecture] *Under a certain maximality condition on S (see (2.9)) the Eisenstein series $E_n^{k,S}((\tau, z); s)$ will satisfy the functional equation*

$$E_n^{k,S}((\tau, z); s - \kappa) = \Phi^{(n)}(s) E_n^{k,S}\left((\tau, z); \frac{n+1}{2} - s - \kappa\right),$$

where $\kappa = (k - l/2)/2$ and where $\Phi^{(n)}(s)$ is a meromorphic function of s which is constructed from the 'constant' term of the Eisenstein series. Moreover $\Phi^{(n)}(s)$ will *satisfy the functional equation*

$$\Phi^{(n)}(s)\Phi^{(n)}\left(\frac{n+1}{2}-s\right)=1.$$

In the case of $l=1$ and $S=1$ the explicit form of $\Phi^{(n)}(s)$ has been almost determined in Remark 2.7.

Acknowledgement The second author would like to thank G. Faltings for his support and the Max-Planck Institut für Mathematik in Bonn for hospitality.

Notation: For an associative ring R with identity element, we denote by $R^{\times}$ the group of all invertible elements of R and by $R^{m,n}$ the module of all $m\times n$ matrices with entries in R. If M is a matrix, M^t, $\det(M)$, and $\mathrm{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R)=R^{n,n}$, $Gl_n(R)=M_n(R)^{\times}$, $Sl_n(R)=\{M\in Gl_n(R)|\det(M)=1\}$ (when R is commutative) and $S_n(R)=\{M\in M_n(R)\mid M=M^t\}$. The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ then $Sp_n(R)=\{M\in Gl_{2n}(R)|M^tJ_nM=J_n\}$.

$$\begin{aligned}
P_{n,r}(R) &= \left\{\begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix}\in Sp_n(R)\right\},\\
P^J_{n,r}(R) &= \left\{\begin{pmatrix} 1_{n-r} & * \\ 0_{n+r,n-r} & * \end{pmatrix}\in Sp_n(R)\right\}.
\end{aligned}$$

For convenience let $\Gamma_n=Sp_n(\mathbb{Z})$, $P_{n,r}=P_{n,r}(\mathbb{Z})$, $\Gamma=\Gamma_1$ and $\Gamma_\infty=P_{1,0}$. For real symmetric matrices A and B, we write $A>B$ ($A\geq B$) to indicate that $A-B$ is positive definite (positive semi-definite) and $A[B]=B^tAB$ if A,B are suitable. If $A_1,A_2,\ldots,A_n$ are square matrices, $[A_1,A_2,\ldots,A_n]$ denotes the matrix with $A_1,A_2,\ldots,A_n$ in the diagonal blocks and 0 in all other blocks. Let A_n, A_n^{sp}, and A_n^+ denote the set of all semi-integral symmetric matrices, the set of all semi-integral positive semi-definite symmetric matrices, and the set of all semi-integral positive definite symmetric matrices of size n, respectively. The disjoint union of sets is denoted by $\bigsqcup$. Let $Z\in\mathbb{C}^{n,n}$, then we put $e\{Z\}=e^{2\pi i\,(\mathrm{tr}\,Z)}$ and $\mathrm{Re}(Z),\mathrm{Im}(Z)$ for the real and imaginary part of Z and for $M\in\mathbb{C}^{m,n}$ we put occasionally $x_M=\mathrm{Re}(M)$ and $y_M=\mathrm{Im}(M)$. Further let $\delta(Z)=\det(\mathrm{Im}\,(Z))$ and $S_n(\mathbb{R})^+$ be the set of all positive definite real symmetric matrices of size n.

1 Background on Jacobi forms

In this section we first fix notation regarding Jacobi forms, and recall some important facts on analytic Eisenstein series. Then a Jacobi-version of Garrett-Böcherer's Theorem is given.

1.1 The space of Jacobi forms

It is well known that the symplectic group $\mathsf{Sp}_n(\mathbb{R})$ acts on Siegel's upper-half-space $I\!H_n$ of degree n as a group of biholomorphic automorphism by

$$(M,Z) \longmapsto M(Z) = (AZ+B)(CZ+D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{Sp}_n(\mathbb{R})$ with $A, B, C, D \in \mathbb{R}^{n,n}$. Let us put $J(M,Z) = CZ + D$ and $j(M,Z) = \det(CZ+D)$. More general, let $(n_j)_{j=1}^q$ be a partition of n and $M_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right) \in \mathsf{Sp}_{n_j}(\mathbb{R})$, then we denote by $M_1 \times \ldots \times M_q$ the matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{Sp}_n(\mathbb{R})$ with $A = [a_1, \ldots, a_q]$, $B = [b_1, \ldots, b_q]$, $C = [c_1, \ldots, c_q]$, $D = [d_1, \ldots, d_q]$. These matrices form a subgroup of $\mathsf{Sp}_n(\mathbb{R})$, which acts on $I\!H_{n_1} \times \ldots \times I\!H_{n_q}$, identified with $[I\!H_{n_1} \times \ldots \times I\!H_{n_q}]$, via

$$(3) \qquad \Big(\Big(M_1 \times \ldots \times M_q\Big), (z_1 \times \ldots \times z_q)\Big) \longmapsto [M_1(z_1), \ldots, M_q(z_q)]\,.$$

Hence $J(M, [z_1, \ldots, z_q]) = [J(M_1, z_1), \ldots, J(M_q, z_q)]$. Let l be a natural number, then we regard the **Jacobi group** $G^J_{n,l}(\mathbb{R}) := P^J_{n+l,l}(\mathbb{R})$ as a subgroup of $\mathsf{Sp}_{n+l}(\mathbb{R})$. Let $g \in G^J_{n,l}(\mathbb{R})$, then

$$((\lambda, \mu, \rho), M) \in \Big(\big(\mathbb{R}^{l,n} \times \mathbb{R}^{l,n} \times \mathbb{R}^{l,l}\big) \times \mathsf{Sp}_n(\mathbb{R})\Big)$$

determined by

$$(4) \qquad g = \begin{pmatrix} 1_l & 0 & \rho & \mu \\ 0 & 1_n & \mu^t & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_l & \lambda & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & -\lambda^t & 1_n \end{pmatrix} \Big(1_{2l} \times M\Big),$$

gives a convenient parameterization of g. If we regard $H_{n,l}(\mathbb{R}) := \mathbb{R}^{l,n} \times \mathbb{R}^{l,n} \times \mathbb{R}^{l,l}$ and $\mathsf{Sp}_n(\mathbb{R})$ as subgroups of $G^J_{n,l}(\mathbb{R})$ with respect to the embeddings

$$(5) \qquad \big(\lambda, \mu, \rho\big) \mapsto \Big(\big(\lambda, \mu, \rho\big), 1_{2n}\Big) \text{ and } M \mapsto \Big((0,0,0), M\Big),$$

then the Jacobi group can be interpreted as the semi-direct product of the normal subgroup $H_{n,l}(\mathbb{R})$, the Heisenberg group, and $\mathsf{Sp}_n(\mathbb{R})$. The operation of $\mathsf{Sp}_{n+l}(\mathbb{R})$ on $I\!H_{n+l}$ leads to an operation of $G^J_{n,l}(\mathbb{R})$ on $\mathcal{D}_{n,l} = I\!H_n \times \mathbb{C}^{n,l}$, given by

$$(6) \qquad g(\tau, z) = \big(M(\tau), zJ(M,\tau)^{-1} + \lambda M(\tau) + \mu\big)\,,$$

where $g = ((\lambda, \mu, \rho), M)$ and $(\tau, z) \in \mathcal{D}_{n,l}$.
For each $r \in \mathbb{N}_0$ with $r \leq n$, we define a subgroup of $G^J_{n,l}(\mathbb{R})$ given by

(7)
$$G^J_{n,l,r}(\mathbb{R}) = \left\{ \Big((0\lambda_2, \mu, \rho), M\Big) \,\Big|\, \lambda_2 \in \mathbb{R}^{l,r}, \mu \in \mathbb{R}^{l,n}, \rho \in \mathbb{R}^{l,l} \text{ and } M \in P_{n,r}(\mathbb{R}) \right\}$$

and the map

$$(8) \qquad *_{J,r} : \left\{ \begin{array}{ccc} \mathcal{D}_{n,l} & \longrightarrow & \mathcal{D}_{r,l} \\ (\tau, z) & \longmapsto & \left(\tau \begin{bmatrix} 0 \\ 1_r \end{bmatrix}, z \begin{pmatrix} 0 \\ 1_r \end{pmatrix}\right) \end{array} \right. ,$$

which is a generalization of the projection $* : I\!\!H_n \to I\!\!H_r$, $\tau \mapsto \tau_* = \tau\left[\begin{smallmatrix} 0 \\ 1_r \end{smallmatrix}\right]$.

Definition 1.1 *Let $k, l, n \in \mathbb{N}$ and $S \in A_l^+$. The factor of automorphy $j_{k,S}$ of index S and weight k for $G^J_{n,l}(\mathbb{R})$ is the map*

$$(9) \qquad j_{k,S} : \left\{ \begin{array}{ccc} G^J_{n,l}(\mathbb{R}) \times D_{n,l} & \longrightarrow & D_{n,l} \\ (g, (\tau, z)) & \mapsto & j_{k,S}(g, (\tau, z)) \end{array} \right. ,$$

where

$$(10) \quad \begin{aligned} j_{k,S}(g, (\tau, z)) \;=\; & j(M, \tau)^k e\{-S\rho\} \\ & e\{-S[\lambda]\, M(\tau) - 2\lambda^t S z J(M, \tau)^{-1} + S[z] J(M, \tau)^{-1} c\} \end{aligned}$$

for $g = ((\lambda, \mu, \rho), M)$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $F : D_{n,l} \longrightarrow \mathbb{C}$, then the generalized Petersson slash operator $|_{J,k}$ is given by

$$(11) \qquad (F|_{k,S})\,(\tau, z) := j_{k,S}(g, (\tau, z))^{-1} F(g(\tau, z)).$$

It follows from our construction that $\Gamma^J_{n,l,r} := G_{n,l,r}(\mathbb{Z})$ and $H_{n,l} := H_{n,l}(\mathbb{Z})$ are groups. For convenience we put $\Gamma^J_{n,l} = \Gamma^J_{n,l,0}$.

Definition 1.2 *Let $k, l \in \mathbb{N}$ and $S \in A_l^+$. We denote by $\mathcal{J}^n_{k,S}$ the $\mathbb{C}-$vector space consisting of holomorphic functions $F(\tau, z)$ on $\mathcal{D}_{n,l}$ satisfying:*

(i) $(F|_{k,S}\, g) = F$ *for all* $g \in \Gamma^J_{n,l}$,

(ii) F *has a Fourier expansion of the form*

$$(12) \qquad F(\tau, z) = \sum_{\left(\begin{smallmatrix} N & R^t/2 \\ R/2 & S \end{smallmatrix}\right) \in A_{n+l}} c(N, R)\; e\{N\tau + R^t z\}.$$

Each element of $\mathcal{J}^n_{k,S}$ is called Jacobi form of weight k and index S with respect to the Jacobi group $\Gamma^J_{n,l}$. Let $(\tau,z)\in\mathcal{D}_{n,l}$ and Im$(\tau,z)=(\tau_y,z_y)$, further we put

$$\mathcal{C}_{k,S}(\tau,z)=\delta(\tau)^k\, e\{2iS[z_y]\tau_y^{-1}\}, \tag{13}$$

then $F\in\mathcal{J}^n_{k,S}$ is said to be a cusp form, if it satisfies the following condition:

$$\mathcal{C}_{k,S}(\tau,z)^{1/2}|F(\tau,z)| \text{ is bounded on } \mathcal{D}_{n,l}. \tag{14}$$

Denote by $\mathcal{J}^{n,cusp}_{k,S}$ the subspace of $\mathcal{J}^n_{k,S}$, consisting of all cusp forms. Supplementary we put $\mathcal{J}^{0,cusp}_{k,S}=\mathbb{C}$.

Remark 1.3 *Let F be a cusp form, then $c(N,R)=0$ if $T\notin A^+_{n+l}$.*

1.2 Jacobi Eisenstein series and Petersson scalar products

Let us begin with the definition of an analytic Jacobi Eisenstein series.

Definition 1.4 *Let $k,l,n\in\mathbb{N}$ with k even, $0\le r\le n$ and $S\in A^+_l$. To $\Phi\in\mathcal{J}^{r,cusp}_{k,S}$ we attach an analytic Jacobi Eisenstein series of Klingen type on $\mathcal{D}_{n,l}$ defined by:*

(15)

$$E^{k,S}_{n,r}((\tau,z),\Phi);s)=\sum_{\gamma\in\Gamma^J_{n,l,r}\backslash\Gamma^J_{n,l}}\Phi\big(\gamma(\tau,z)_{*J}\big)\ j_{k,S}(\gamma,(\tau,z))^{-1}\left(\frac{\delta(M(\tau))}{\delta(M(\tau)_*)}\right)^s,$$

here $\gamma=(h,M)$ and $s\in\mathbb{C}$ with Re(s) sufficiently large. If $r=0$, we denote by $E^{k,S}_n((\tau,z);s)=E^{k,S}_{n,0}((\tau,z),1;s)$ the Siegel-Jacobi Eisenstein series.

Remark 1.5
a) The Eisenstein series is absolutely convergent for $k+2Re(s)>n+r+l+1$. If $k>n+r+l+1$, then $E^{k,S}_{n,r}(\Phi,(\tau,z);0)\in\mathcal{J}^n_{k,S}$.
b) Let us put

$$\chi^{k,s;J}_\mu(g,(\tau,z))=j_{k,S}(g,(\tau,z))^{-1}|j(M,\tau)|^{-2s}, \tag{16}$$

then we have the cocycle relation

$$\chi^{k,s;J}_\mu(gh,(\tau,z))=\chi^{k,s;J}_\mu(g,h(\tau,z))\,\chi^{k,s;J}_\mu(h,(\tau,z)) \tag{17}$$

and

$$E^{k,S}_n((\tau,z),s)=\delta(\tau)^s\sum_{\gamma\in\Gamma^J_{n,l,0}\backslash\Gamma^J_{n,l}}\chi^{k,s;J}_\mu(\gamma,(\tau,z)).$$

c) The series $E^{k,S}_{n,r}((\tau,z),\Phi);s)$ is absolutely convergent. Moreover, from [He97] (Theorem 4.1) we know that $E^{k,S}_n((\tau,z),s)$ admits a meromorphic continuation on the whole $s-$space. From there we have a precise knowledge about the location of possible poles. In the case $n=l=1$ also a functional equation is available [Ar90].

Let $(\tau, z) \in \mathcal{D}_{n,l}$. Then we denote by $d\omega_{n,l}(\tau, z)$ the $\Gamma^J_{n,l}$- invariant measure on $\mathcal{D}_{n,l}$ (for an explicit definition we refer to [Ar94](2.2)). For continues functions $F, G : \mathcal{D}_{n,l} \to \mathbb{C}$ satisfying item (i) in Definition 1.2, we define the Petersson scalar product $\langle F \, , \, G\rangle_J$ by

$$(18) \qquad \langle F \, , \, G\rangle_J = \int_{\Gamma^J_{n,l}\backslash\mathcal{D}_{n,l}} F(\tau, z)\overline{G(\tau, z)} \; \mathcal{C}_{k,S}(\tau, z) \; d\omega_{n,l}(\tau, z) \;\; ,$$

if the integral on the right side exist. This is for example the case if FG is a cusp form.

1.3 The Theorem of Garrett and Böcherer for Jacobi forms

As already mentioned, we can consider $\mathsf{Sp}_n(\mathbb{R})$ as a subgroup of $G^J_{n,l}(\mathbb{R})$. Then it is easy to see, that every $\Gamma^J_{n,l}$-double coset attached to $\alpha \in \mathsf{Sp}_n(\mathbb{Q})$ decomposes into finitely many $\Gamma^J_{n,l}$- left cosets:

$$(19) \qquad \Gamma^J_{n,l} \, \alpha \, \Gamma^J_{n,l} = \sqcup \; \Gamma^J_{n,l} \; g_j.$$

Hence for Φ on $\mathcal{D}_{n,l}$ satisfying the condition $\Phi|_{k,S} \; g = \Phi$ for all $g \in \Gamma^J_{n,l}$ and $\alpha \in \mathsf{Sp}_n(\mathbb{Q})$, we can define the Hecke-Jacobi operators

$$(20) \qquad \Phi\big|[\alpha]_J := \sum_{j=1}^{h} \Phi|_{k,S} \; g_j.$$

A Jacobi form Φ which is a common eigen form under all the Hecke-Jacobi operators $[\alpha]_J$ is simply called a *weak Hecke-Jacobi eigen form.* The eigen values are denoted by $\lambda(\Phi, \alpha)$. We know from the elementary divisor theorem that $\Gamma_n\alpha\Gamma_n = \Gamma_n[D, D^{-1}]\Gamma_n$, where $D = [d_1, \ldots, d_n]$, $d_j \in \mathbb{N}$ and $d_j | d_{j+1}$. Thus '$\lambda(\Phi, D) = \lambda(\Phi, \alpha)$' by abuse of notation. Let D_n be the set of all elementary matrixes and $\mathsf{D}^{(p)}_n$ the subset, where the elements have to form $[p^{e_1}, \ldots, p^{e_n}]$. From Arakawa [Ar94] we know that

$$(21) \qquad Z^J_n(s, \Phi) := \sum_{D \in D_n(\mathbb{Z})} \lambda(\Phi, D) \, \mathsf{det}(D)^{-s}$$

is absolutely convergent for $\mathsf{Re}(s) > 2n + l + 1$ and has the Euler product expansion

$$(22) \qquad Z^J_n(s, \Phi) = \prod_p Z^J_{n,p}(s, \Phi),$$

where

$$(23) \qquad Z^J_{n,p}(s, \Phi) = \sum_{D \in D^{(p)}_n(\mathbb{Z})} \lambda(\Phi, D) \, \mathsf{det}(D)^{-s}.$$

To state the Theorem of Garrett and Böcherer in the frame work of Jacobi forms ([Ar94], Theorem 2.9) we still need some more notation, in peculiar we need a generalization of the operation given in (3).
Let $(n_j)_{j=1}^q$ be a partition of n and $g_j = ((\lambda_j, \mu_j, \rho_j), M) \in G^J_{n_j,l}(\mathbb{R})$. Then we denote by $g_1 \times \ldots g_q$ the element

$$g = \left(\left((\lambda_1 \ldots \lambda_q), (\mu_1 \ldots \mu_q), \sum \rho_j \right), M \right) \in G^J_{n,l}(\mathbb{R}). \tag{24}$$

All elements of this type form a subgroup, which operates on $\mathcal{D}_{n_1} \times \ldots \mathcal{D}_{n_q}$ (identified with $(I\!H_{n_1} \times \ldots I\!H_{n_q}) \times (\mathbb{C}^{l,n_1} \ldots \mathbb{C}^{l,n_q}) \subseteq \mathcal{D}_{n,l}$), by

$$\Big((g_j)_j, (\tau_j, z_j) \Big) \longmapsto \Big(g_j(\tau_j, z_j) \Big)_j.$$

Theorem 1.6 *(Version of [Ar94], Theorem 2.9)*
Let $m \geq n$ and k be an even natural number with $k > m + n + l + 1$. Let $\Phi \in \mathcal{J}^{n,cusp}_{k,S}$ and $k + 2Re(s) > m + n + l + 1$, then for any $(\tau, z) \in \mathcal{D}_{m,l}$:

$$\Big\langle E^{k,S}_{m+n}\left((\tau, z) \times (\tau', z'), s \right), \Phi(\tau', z') \Big\rangle_J \tag{25}$$

$$= \mu(k, S, n, s)\ Z^J_n(2s + k, \Phi)\ E^{k,S}_{m,n}(\Phi^\sharp, (\tau, z), s). \tag{26}$$

In particular if $m = n$, then the right side of this equation is equal to

$$\mu(k, S, n, s) Z^J_n(2s + k, \Phi)\ \ \Phi^\sharp(\tau, z).$$

Here $\Phi^\sharp(\tau, z) = \sum \overline{c(N, R)}\ e\{N\tau + R^t z\}$ and μ is a constant depending only on k, S, n and s ([Ar94] (2.4)).

For simplification we put $\mathcal{J}^{\text{cusp}}_{k,S} = \mathcal{J}^{1,\text{cusp}}_{k,S}$, $I\!H = I\!H_1$, $\mathcal{D} = \mathcal{D}_{1,1}$, $\Gamma = \Gamma_1$ and $\Gamma_\infty = P_{1,0}$. If $n = l = 1$, then $Z^J_1(s, \Phi)$ is essentially the Mellin transform of a certain elliptic cusp form. More precisely:
Let f be a elliptic cusp form of weight k (i.e $f : I\!H \to \mathbb{C}$ holomorphic satisfying $f(\gamma(\tau)) = j(\gamma, \tau)^k f(\tau)$ for all $\gamma \in \Gamma$ and vanishing at the cusps). We denote by S_k the space of such forms. To a normalized Hecke eigen form $f(\tau) = \sum_{n=1}^\infty a(n)\, e\{n\tau\} \in S_k$ one assoziates the L-function $L_f(s)$ given by

$$L_f(s) = \prod_p L_{f,p}(p^{-s})^{-1} \qquad \text{for } \mathrm{Re}(s) > \tfrac{k+1}{2}, \tag{27}$$

where $L_{f,p}(X) = 1 - a(p)X + p^{k-1}X^2$.

Proposition 1.7 *Let k be even and $\Phi \in \mathcal{J}_{k,S}^{cusp}$ be a (weak) Hecke-Jacobi eigen form and φ the unique elliptic cusp form of weight $2k-2$ corresponding to Φ by the Shimura correspondance. Then*

$$Z_1^J(s,\Phi) = \frac{L_\varphi(s+k-3)}{\zeta(2s-2)}. \tag{28}$$

Proof:
[Ar94] (section 6.1).

$\square$

In this paper we are mainly interested in a $n=2$ and $m=l=1$ version of Theorem 1.6.

2 The 'constant' term of Jacobi Eisenstein series

In this subsection we compute the 'constant' term of the Eisenstein series $E_n^{k,S}((\tau,z);s)$ and present a conjecture on the explicit form of the functional equation satisfied by them. We assume that k is an even positive integer throughout this section. It is also convenient to use $e(t)=e^{2\pi i\, t}$ for $t \in \mathbb{C}$.

2.1 Fourier expansion of Jacobi Eisenstein series

For each $\mu \in \mathbb{Z}^{l,n}$, we define the theta series $\theta_\mu(\tau,z)$ to be the sum

$$\sum_{\lambda \in \mathbb{Z}^{l,n}} e\left(\mathrm{tr}\Big(S[\lambda+(2S)^{-1}\mu]\tau + 2^t(\lambda+(2S)^{-1}\mu)Sz\Big)\right), \qquad ((\tau,z)\in \mathcal{D}_{n,l}).$$

Then, $\theta_\mu(\tau,z)$ depends only on $\mu \bmod (2S)\mathbb{Z}^{l,n}$.

We recall the definition of the real analytic Eisenstein series:

$$E_n^{k,S}((\tau,z);s) := \sum_{\gamma \in \Gamma_{n,l,0}^J \backslash \Gamma_{n,l}^J} j_{k,S}(\gamma,(\tau,z))^{-1}\delta(M(\tau))^s.$$

For each ν $(0 \le \nu \le n)$, the sub series $E_n^{k,S}((\tau,z);s)^{(\nu)}$ of $E_n^{k,S}((\tau,z);s)$ is given by

$$E_n^{k,S}((\tau,z);s)^{(\nu)} = \sum_{\substack{\gamma \in \Gamma_{n,l,0}^J \backslash \Gamma_{n,l}^J \\ \mathrm{rank}\ c(M)=\nu}} j_{k,S}(\gamma,(\tau,z))^{-1}\delta(M(\tau))^s,$$

where M is a Sp_n-part of γ and $c(M)$ is a left-lower block of M. In particular if $\nu=0$, then

$$E_n^{k,S}((\tau,z);s)^{(0)} = \delta(\tau)^s\theta_0(\tau,z).$$

Set $\Delta_n = GL_n(\mathbb{Z})$ and for each ν $(0 \le \nu \le n)$ denote by $\Delta_{\nu,n-\nu}$ the subgroup of Δ_n consisting of matrices whose right upper $\nu \times n-\nu$ blocks vanish:

$$\Delta_{\nu,n-\nu} = \left\{ \begin{pmatrix} * & 0_{\nu,n-\nu} \\ & * \end{pmatrix} \in \Delta_n \right\}.$$

Set $\Gamma_n^* = \{M \in \Gamma_n \mid c(M) \ne 0\}$ and $P_{n,0}^J = P_{n,0}^J(\mathbb{Z})$ (see for $P_{n,0}^J(R)$ the Notation). For each $T = \begin{pmatrix} N & r^t/2 \\ r/2 & S \end{pmatrix} \in A_{n+l}$, let $\zeta_n(T,s)$ be a singular series defined by

$$\zeta_n(T,s) = \sum_{M \in P_{n,0}\backslash \Gamma_n^*/P_{n,0}^J} \ \sum_{\lambda \in \mathbb{Z}^{l,n}/\mathbb{Z}^{l,n}c^t} |\det c|^{-s} \times e(\mathrm{tr}(S[\lambda]ac^{-1} + Nc^{-1}d - r^t\lambda(c^{-1})^t)),$$

where we write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We refer to [Ar93] on this singular series.

Let $\xi_n(\eta,u;\alpha,\beta)$ denote Shimura's confluent hypergeometric functions ([Sh82]):

$$\xi_n(\eta,u;\alpha,\beta) = \int_{S_n(\mathbb{R})} e\Big(-\mathrm{tr}(ux)\Big) \det(x+i\eta)^{-\alpha} \det(x-i\eta)^{-\beta} dx$$

for $u \in S_n(\mathbb{R}),\ \eta \in S_n(\mathbb{R})^+,\ \alpha,\ \beta \in \mathbb{C}$. We use the ordinary one for the branch of $\det(Z)^{-\alpha}$ $(Z \in \mathbb{H}_n)$. This integral converges absolutely for $\mathrm{Re}(\alpha+\beta) > n$.

Proposition 2.1 *Let $1 \le \nu \le n$. We have*

$$E_n^{k,S}((\tau,z);s)^{(\nu)} = i^{-\nu l/2}(\det 2S)^{-\nu/2}\delta(\tau)^s \times \sum_{r \in \mathbb{Z}^{l,\nu}/(2S)\mathbb{Z}^{l,\nu}} \ \sum_{V \in \Delta_{\nu,n-\nu}\backslash\Delta_n}$$

$$\left(\sum_{N \in A_\nu} \zeta_\nu(T,2s+k)\xi_\nu\left(\widehat{\eta}_1, \widetilde{T}, s+k-\frac{l}{2}, s\right) e\big(\mathrm{tr}(\widetilde{T}\widehat{\xi}_1)\big) \right) \times \theta_{(r,0)V}(\tau,z)$$

where for each r, N, we put $T = \begin{pmatrix} N & r^t/2 \\ r/2 & S \end{pmatrix}$, $\widetilde{T} = N - \frac{1}{4}r^tS^{-1}r$, and for each V, $\widehat{\tau}_1 = (V\tau V^t)\begin{bmatrix} 1_\nu \\ 0 \end{bmatrix}$, $\widehat{\eta}_1 = \mathrm{Im}(\widehat{\tau}_1)$, $\widehat{\xi}_1 = \mathrm{Re}(\widehat{\tau}_1)$.

Proof:
The formula in the proposition corresponds to that of [Ma71], p.306 in the Siegel Eisenstein series case (see also [Mi93]).

First we discuss the case of $\nu = n$. It is easily proved in a manner similar to that of Proposition 4.2 in [Ar93] that

$$(2.1) \qquad E_n^{k,S}((\tau,z);s)^{(n)} = \sum_{M\in P_{n,0}\backslash\Gamma_n^*/P_{n,0}^J} \sum_{\lambda\in\mathbb{Z}^{l,n}/\mathbb{Z}^{l,n}c^t}$$

$$(\det c)^{-k}|\det c|^{-2s} \times e(\mathrm{tr}(S[\lambda]ac^{-1})) \times F_{k,S,n}(\tau + c^{-1}d, z-\lambda(c^{-1})^t; s),$$

where we set

$$F_{k,S,n}(\tau,z;s) = \delta(\tau)^s \sum_{p\in S_n(\mathbb{Z})}\sum_{q\in\mathbb{Z}^{l,n}} \det(\tau+p)^{-k}|\det(\tau+p)|^{-2s}\times e(-\mathrm{tr}(S[z+q](\tau+p)^{-1}).$$

This infinite series is absolutely convergent for $k + 2\mathrm{Re}(s) > n + l + 1$. The Poisson summation formula yields the following identity for $F_{k,S,n}(\tau,z;s)$ which will coincide with that of Proposition 4.1, (i) in [Ar93] if s is evaluated at $s = 0$. If $k+2\mathrm{Re}(s) > n+l+1$, then

$$(2.2) \quad F_{k,S,n}(\tau,z;s) = i^{-nl/2}(\det 2S)^{-n/2}\delta(\tau)^s \sum_{x\in A_n}\sum_{q\in\mathbb{Z}^{l,n}} e\left(\mathrm{tr}\left(\left(x - \frac{1}{4}S^{-1}[q]\right)\xi\right)\right)$$

$$\times\xi_n\left(\eta, x - \frac{1}{4}S^{-1}[q], s+k-\frac{l}{2}, s\right) e\left(\mathrm{tr}\left(\frac{1}{4}S^{-1}[q]\tau + q^t z\right)\right),$$

where $\eta = \mathrm{Im}(\tau)$, $\xi = \mathrm{Re}(\tau)$. It follows from (2.1), (2.2) that

$$E_n^{k,S}((\tau,z);s)^{(n)} = i^{-nl/2}(\det 2S)^{-n/2}\delta(\tau)^s \sum_{N\in A_n}\sum_{r\in\mathbb{Z}^{l,n}} \zeta_n(T;2s+k)$$

$$\times\xi_n\left(\eta, \widetilde{T}, s+k-\frac{l}{2}, s\right) e\left(\mathrm{tr}\left(\widetilde{T}\xi\right)\right) e\left(\mathrm{tr}\left(\frac{1}{4}S^{-1}[r]\tau + r^t z\right)\right),$$

where for each r, N, we put $T = \begin{pmatrix} N & {}^t r/2 \\ r/2 & S\end{pmatrix}$ and $\widetilde{T} = N - \frac{1}{4}S^{-1}[r]$. Further replacing r with $r + (2S)\lambda$ ($r \in \mathbb{Z}^{l,n}/(2S)\mathbb{Z}^{l,n}$, $\lambda \in \mathbb{Z}^{l,n}$), we have

$$(2.3) \qquad E_n^{k,S}((\tau,z);s)^{(n)} = i^{-nl/2}(\det 2S)^{-n/2}\delta(\tau)^s \left(\sum_{N\in A_n}\sum_{r\in\mathbb{Z}^{l,n}/(2S)\mathbb{Z}^{l,n}}\right.$$

$$\left.\zeta_n(T;2s+k)\xi_n\left(\eta, \widetilde{T}, s+k-\frac{l}{2}, s\right) e\left(\mathrm{tr}(\widetilde{T}\xi)\right)\right)\cdot\theta_r(\tau,z),$$

which completes the proof in the case of $\nu = n$.

Next let $1 \leq \nu < n$. In a manner similar to the proof in Proposition 4.3 of [Ar93] we infer that

$$E_n^{k,S}((\tau,z);s)^{(\nu)} = \delta(\tau)^s \sum_{V,\,\lambda_2} E_\nu^{k,S}((\widehat{\tau}_1,\widehat{z}_1+\lambda_2\widehat{\tau}_{21});s)^{(\nu)}\delta(\widehat{\tau}_1)^{-s}e\Big(\mathrm{tr}(S[\lambda_2]\widehat{\tau}_2+2\lambda_2^t S\widehat{z}_2)\Big),$$

where V (resp. λ_2) runs over a complete set of representatives of $\Delta_{\nu,n-\nu}\backslash\Delta_n$ (resp. all elements of $\mathbb{Z}^{l,n-\nu}$), and where we put, for each $V \in \Delta_n$,

$$(\widehat{\tau},\widehat{z}) = (V\tau V^t, zV^t)$$

and write

$$\widehat{\tau} = \begin{pmatrix} \widehat{\tau}_1 & {}^t\widehat{\tau}_{21} \\ \widehat{\tau}_{21} & \widehat{\tau}_2 \end{pmatrix}, \quad \widehat{z} = (\widehat{z}_1,\widehat{z}_2) \qquad \text{with} \quad (\widehat{\tau}_1 \in I\!H_\nu,\ \widehat{z}_1 \in \mathbb{C}^{l,\nu}).$$

By using the result in the case of $\nu = n$, the right hand side of the above identity equals

$$i^{-\nu l/2}(\det 2S)^{-\nu/2}\delta(\tau)^s \sum_{V,\,\lambda_2}\ \sum_{r_1\in\mathbb{Z}^{l,\nu}/(2S)\mathbb{Z}^{l,\nu}}\ \sum_{N_1\in A_\nu} \zeta_\nu(T_1;2s+k)$$

$$\times\xi_\nu\left(\widetilde{\eta}_1,\widetilde{T}_1,s+k-\frac{l}{2},s\right)e\left(\mathrm{tr}(\widetilde{T}_1\xi)\right)\cdot\theta_{r_1}(\widehat{\tau}_1,\widehat{z}_1+\lambda_2\widehat{\tau}_{21})\cdot e\Big(\mathrm{tr}(S[\lambda_2]\widehat{\tau}_2+2\lambda_2^t S\widehat{z}_2)\Big),$$

where $T_1 = \begin{pmatrix} N_1 & {}^t r_1/2 \\ r_1/2 & S \end{pmatrix}$. Here we note that

$$\begin{aligned}
&\sum_{\lambda_2} \theta_{r_1}(\widehat{\tau}_1,\widehat{z}_1+\lambda_2\widehat{\tau}_{21})\cdot e\Big(\mathrm{tr}(S[\lambda_2]\widehat{\tau}_2+2\lambda_2^t S\widehat{z}_2)\Big)\\
&= \theta_{(r_1,0)}(\widehat{\tau},\widehat{z})\\
&= \theta_{(r_1,0)V}(\tau,z).
\end{aligned}$$

Therefore we obtain the identity for $1 \leq \nu < n$ in the introduction.

□

2.2 The 'constant' term

In the above Fourier expansion in Proposition 2.1 the sub series corresponding to those T with $\widetilde{T} = 0$ may be called the 'constant' term of $E_n^{k,S}((\tau,z);s)^{(\nu)}$. We write

$c^{(\nu)}((\tau, z); s)$ for this 'constant' term. Immediately by Proposition 2.1, we have

$$c^{(\nu)}((\tau, z); s) = c(\nu)\ \delta(\tau)^s \sum_{\substack{r \in \mathbb{Z}^{l,\nu}/(2S)\mathbb{Z}^{l,\nu} \\ \frac{1}{4}r^t S^{-1} r \equiv 0 \bmod A_\nu}} \sum_{V \in \Delta^{\infty}_{\nu, n-\nu} \backslash \Delta_n} \zeta_\nu(T, 2s+k) \times$$

$$\xi_\nu\left(\widehat{\eta}_1, 0, s+k-\frac{l}{2}, s\right) \times \theta_{(r,0)V}(\tau, z),$$

where $c(\nu) = i^{-\frac{\nu l}{2}}(\det 2S)^{-\frac{\nu}{2}}$ and $T = \begin{pmatrix} \frac{1}{4}r^t S^{-1} r & r^t/2 \\ r/2 & S \end{pmatrix}$ for each r. Note that up to the factors $\theta_{(r,0)V}(\tau, z)$, the 'constant' term $c^{(\nu)}((\tau, z); s)$ is a function of $\eta = \mathrm{Im}(\tau)$. In particular if $\nu = 0$,

$$c^{(0)}((\tau, z); s) = \delta(\tau)^s \theta_0(\tau, z).$$

Set

$$c_n^{k,S}((\tau, z), s) = \sum_{\nu=0}^{n} c^{(\nu)}((\tau, z); s),$$

which is the 'constant' term of $E_n^{k,S}((\tau, z); s)$.

For each prime integer p let $\Gamma_n^{*(p)}$ denote the subset of Γ_n^* consisting of all elements $M \in \Gamma_n^*$ such that $\det c(M) = \pm p^m$ with some non-negative integers m. We recall from [Ar93] that

$$\begin{aligned} \zeta_n(T, s) &= \sum_{M \in P_{n,0} \backslash \Gamma_n^* / P_{n,0}^J} \sum_{\lambda \in \mathbb{Z}^{l,n}/\mathbb{Z}^{l,n} c} |\det c|^{-s} \times e\left(\mathrm{tr}\left(T \begin{bmatrix} 1_n \\ \lambda \end{bmatrix} c^{-1} d\right)\right) \\ &= \prod_{p < \infty} \zeta_{n,p}(T; s), \end{aligned}$$

where

$$\zeta_{n,p}(T; s) = \sum_{M \in P_{n,0} \backslash \Gamma_n^{*(p)} / P_{n,0}^J} \sum_{\lambda \in \mathbb{Z}^{l,n}/\mathbb{Z}^{l,n} c} |\det c|^{-s} e\left(\mathrm{tr}\left(T \begin{bmatrix} 1_n \\ \lambda \end{bmatrix} c^{-1} d\right)\right).$$

Let $|\ |_p$ denote the normalized p-adic valuation of $\mathbb{Q}_p$. For each $a \in \mathbb{Q}_p^\times$ denote by $\nu_p(a)$ the integer ν by which a can be written in the form $a = p^\nu a'$, $a' \in \mathbb{Z}_p^\times$.

Obviously, $|a|_p = p^{-\nu_p(a)}$. For each $x \in S_n(\mathbb{Q}_p)$ choose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n(\mathbb{Z}_p)$ with $x = c^{-1}d$. Set

$$\nu(x) = |\det c|_p^{-1} = p^{\nu_p(\det c)},$$

which is independent of the choice of $M \in Sp_n(\mathbb{Z}_p)$ satisfying $x = c^{-1}d$. We define an additive function e_p on $\mathbb{Q}_p$ by putting

$$e_p(y) = e(y')$$

with $y' \in \mathbb{Q}$ being chosen so that $y - y' \in \mathbb{Z}_p$.

Proposition 2.2 *Let $p > 2$ be a prime not dividing $\det(2S)$ and $T = \begin{pmatrix} N & {}^t r/2 \\ r/2 & S \end{pmatrix} \in A_{n+l}$ with $\widetilde{T} = N - \frac{1}{4} {}^t r S^{-1} r = 0$.*

(i) *If l is even, then*

$$\zeta_{n,p}(T;s) = \frac{1 - \varepsilon p^{\frac{l}{2}-s}}{1 - \varepsilon p^{n+\frac{l}{2}-s}} \prod_{i=1}^{n} \frac{1 - p^{2i+l-2s}}{1 - p^{n+i+l-2s}},$$

where $\varepsilon = \left(\dfrac{(-1)^{l/2}\det(2S)}{p}\right)$.

(ii) *If l is odd, then*

$$\zeta_{n,p}(T;s) = \prod_{i=1}^{n} \frac{1 - p^{2i-1+l-2s}}{1 - p^{n+i+l-2s}}.$$

Proof:
The proof of Proposition 2.2 is based on some results of Shimura [Sh83].
By a little computation, one gets

$$e\left(\operatorname{tr}\left(T\begin{bmatrix} 1_n \\ \lambda \end{bmatrix} c^{-1}d\right)\right) = e\left(\operatorname{tr}\left(S[\lambda + (2S)^{-1}r]c^{-1}d\right) + \operatorname{tr}\left(\widetilde{T}c^{-1}d\right)\right).$$

Note that $|\det c|$ is a power of p. Since the map $\lambda \longmapsto \lambda + (2S)^{-1}r$ induces an automorphism of the quotient module $\mathbb{Z}_p^{l,n}/\mathbb{Z}_p^{l,n}c$, one has

$$\sum_{\lambda \in \mathbb{Z}_p^{l,n}/\mathbb{Z}_p^{l,n}c} e_p\left(\operatorname{tr}\left(T\begin{bmatrix} 1_n \\ \lambda \end{bmatrix} c^{-1}d\right)\right) = \sum_{\lambda \in \mathbb{Z}_p^{l,n}/\mathbb{Z}_p^{l,n}c} e_p\left(\operatorname{tr}(S[\lambda]c^{-1}d)\right) \cdot e_p(\operatorname{tr}(\widetilde{T}c^{-1}d)).$$

We choose a positive integer m such that $p^m\mathbb{Z}_p^{l,n} \subset \mathbb{Z}_p^{l,n}c$. Then we have, for the group indices,

$$[\mathbb{Z}_p^{l,n}c : p^m\mathbb{Z}_p^{l,n}] = \frac{[\mathbb{Z}_p^{l,n} : p^m\mathbb{Z}_p^{l,n}]}{[\mathbb{Z}_p^{l,n} : \mathbb{Z}_p^{l,n}c]} = \frac{p^{mln}}{|\det c|^l}.$$

Set $x = c^{-1}d$. Consequently,

$$\begin{aligned}
\sum_{\lambda \in \mathbb{Z}_p^{l,n}/\mathbb{Z}_p^{l,n} c} e_p\left(\mathrm{tr}(S[\lambda]x)\right) &= \frac{1}{[\mathbb{Z}_p^{l,n} c : p^m \mathbb{Z}_p^{l,n}]} \sum_{\lambda \in \mathbb{Z}_p^{l,n}/p^m \mathbb{Z}_p^{l,n}} e_p\left(\mathrm{tr}(S[\lambda]x)\right) \\
&= \frac{|\det c|^l}{p^{mln}} \cdot p^{mln} \nu(x)^{-l/2} \cdot \chi_{0,p}(x) \\
&= |\det c|^{l/2} \cdot \chi_{0,p}(x),
\end{aligned}$$

where $\chi_{0,p}(x)$ is given as follows:

$$\sum_{\lambda \in \mathbb{Z}_p^{l,n}/p^m \mathbb{Z}_p^{l,n}} e_p((\mathrm{tr}(S[\lambda]x)) = \nu(x)^{-l/2} p^{mln} \chi_{0,p}(x).$$

It is easy to see that $\chi_{0,p}(x)$ depends only on x and is independent of the choice of sufficiently large m. Viewing the above arguments, we have, if $p > 2$ does not divide $\det(2S)$,

$$\begin{aligned}
\zeta_{n,p}(T;s) &= \sum_{M \in P_{n,0} \backslash \Gamma_n^{*(p)} / P_{n,0}^J} |\det c|^{-(s-l/2)} \chi_{0,p}(x) e(\mathrm{tr}(\widetilde{T}x)) \\
&= \sum_{x \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \nu(x)^{-(s-l/2)} \chi_{0,p}(x) e(\mathrm{tr}(\widetilde{T}x)).
\end{aligned}$$

If $\widetilde{T} = 0$, then

$$\zeta_{n,p}(T;s) = S_{n,l}(p^{-s}), \tag{2.4}$$

where we put

$$S_{n,l}(t) = \sum_{x \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \chi_{0,p}(x) (p^{l/2} t)^{\nu_p(\nu(x))}.$$

Following Shimura [Sh83], Lemma 4.4, we define a function ω on $S_n(\mathbb{Q}_p)$ by putting

$$\sum_{\mu \in \mathbb{Z}_p^{n,1}/p^m \mathbb{Z}_p^{n,1}} e_p\big(\big(\mathrm{tr}(\lambda^t y \lambda)\big) = \nu(y)^{-1/2} p^{mn} \omega(y) \qquad (y \in S_n(\mathbb{Q}_p))$$

for sufficiently large positive integer m. Then, $\omega(y)$ is independent of m. Shimura [Sh83] moreover defined the following Siegel singular series:

$$\alpha_k(y,t) = \sum_{x \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \omega(x)^k e_p(\mathrm{tr}(xy)) t^{\nu_p(\nu(x))},$$

where $k = 0$ or 1, and $y \in S_n(\mathbb{Q}_p)$ ([Sh83], (4.8a).

Under the assumption that $p > 2$ does not divide $\det(2S)$, it is not difficult to see that

$$\chi_{0,p}(x) = \omega(x)^l \left(\frac{(-1)^l \det(S)}{p}\right)^{\nu_p(\nu(x))},$$

where $\left(\dfrac{a}{p}\right)$ denotes the Legendre quadratic residue symbol. If l is even, then by the identity $\omega(x)^2 = \left(\dfrac{-1}{p}\right)^{\nu_p(\nu(x))}$ in Lemma 4.4 of [Sh83],

$$\chi_{0,p}(x) = \left(\frac{(-1)^{l/2} \det(2S)}{p}\right)^{\nu_p(\nu(x))}.$$

In this case

$$S_{n,l}(t) = \sum_{x \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} (\varepsilon p^{l/2} t)^{\nu_p(\nu(x))},$$

where $\varepsilon = \left(\frac{(-1)^{l/2}\det(2S)}{p}\right)$.

Similarly if l is odd, then

$$\chi_{0,p}(x) = \omega(x) \left(\frac{(-1)^{(l+1)/2} \det(S)}{p}\right)^{\nu_p(\nu(x))}.$$

Then,

$$S_{n,l}(t) = \sum_{x \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \omega(x)(\varepsilon' p^{l/2} t)^{\nu_p(\nu(x))},$$

where $\varepsilon' = \left(\frac{(-1)^{(l+1)/2}\det(S)}{p}\right)$. Summing up with the use of $\alpha_k(y,t)$, we conclude that, if l is even,

$$S_{n,l}(t) = \alpha_0(0, \varepsilon p^{l/2} t) \tag{2.5}$$

and that, if l is odd,

$$S_{n,l}(t) = \alpha_1(0, \varepsilon' p^{l/2} t). \tag{2.6}$$

The exact values of the Siegel singular series $\alpha_k(y,t)$ at $y = 0$ have been computed by Shimura [Sh83], Proposition 5.1. If l is even, then

$$\alpha_0(0,t) = \frac{1-t}{1-p^n t} \cdot \prod_{i=1}^{[n/2]} \frac{1-p^{2i}t^2}{1-p^{2n+1-2i}t^2} = \frac{1-t}{1-p^n t} \cdot \prod_{i=1}^{n} \frac{1-p^{2i}t^2}{1-p^{n+i}t^2}. \tag{2.7}$$

If l is odd, then

$$\alpha_1(0,t) = \prod_{i=1}^{[(n-1)/2]} \frac{1-p^{2i+1}t^2}{1-p^{2n-2i}t^2} = \prod_{i=1}^{n} \frac{1-p^{2i-1}t^2}{1-p^{n+i}t^2}. \tag{2.8}$$

Taking the identities from (2.4) to (2.8) into account of, we obtain the assertions (i), (ii) of the proposition.

□

To compute $c^{(\nu)}((\tau,z);s)$ we need an explicit value of $\xi_m(\eta,0;\alpha,\beta)$. Set

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma\left(s - \frac{j}{2}\right).$$

By Shimura [Sh82], p.275, (1.31), we get

$$\xi_m(\eta,0;\alpha,\beta) = i^{m(\beta-\alpha)} 2^{-\frac{m(m-1)}{2}} \cdot (2\pi)^{\frac{m(m+1)}{2}} \cdot \frac{\Gamma_m\left(\alpha+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\alpha)\Gamma_m(\beta)} \cdot \det(2\eta)^{\frac{m+1}{2}-\alpha-\beta}.$$

Thus,

$$\begin{aligned}\xi_\nu\left(\widehat{\eta}_1, 0, s+k-\frac{l}{2}, s\right) &= i^{\nu(\frac{l}{2}-k)} 2^\nu \pi^{\frac{\nu(\nu+1)}{2}} \cdot \frac{\Gamma_\nu\left(2s+k-\frac{l}{2}-\frac{\nu+1}{2}\right)}{\Gamma_\nu(s+k-\frac{l}{2})\Gamma_\nu(s)} \\ &\quad \times \det(2\widehat{\eta}_1)^{\frac{l}{2}+\frac{\nu+1}{2}-k-2s}.\end{aligned}$$

Now we assume a maximality condition on S:

(2.9) If $S[x] \in \mathbb{Z}$ for $x \in (2S)^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x \in M_{l,1}(\mathbb{Z})$.

This condition ensures the uniqueness of the Eisenstein series.
Under this condition the 'constant' term $c^{(n)}((\tau,z);s)$ is expressed as

$$\begin{aligned}c^{(n)}((\tau,z);s) &= i^{-nk}(\det 2S)^{-\frac{n}{2}} \cdot 2^n \pi^{\frac{n(n+1)}{2}} \cdot 2^{n(\frac{l+n+1}{2}-k-2s)} \times \\ &\zeta_n(T_0, 2s+k) \cdot \frac{\Gamma_n\left(2s+k-\frac{l+n+1}{2}\right)}{\Gamma_n(s+k-\frac{l}{2})\Gamma_n(s)} (\det \eta)^{\frac{l+n+1}{2}-k-s} \cdot \theta_0(\tau,z),\end{aligned}$$

T_0 being $\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \in A_{n+l}$.
Set $\kappa = \frac{k-l/2}{2}$. Then the above formula changes into the following

Proposition 2.3 *We have*

$$c^{(n)}((\tau,z);s-\kappa)=\Phi^{(n)}(s)\delta(\tau)^{\frac{n+1}{2}-s-\kappa}\theta_0(\tau,z),$$

where we set

$$\Phi^{(n)}(s)=i^{-nk}\det(2S)^{-\frac{n}{2}}2^n\pi^{\frac{n(n+1)}{2}}\cdot 2^{n(\frac{n+1}{2}-2s)}\frac{\Gamma_n\left(2s-\frac{n+1}{2}\right)}{\Gamma_n(s+\kappa)\Gamma_n(s-\kappa)}\cdot\zeta_n\left(T_0,2s+\frac{l}{2}\right).$$

2.3 Some conjectures on the functional equation

It has been proved in [Ar90], [Ar92] that, if $n=1$, then the Eisenstein series $E_1^{k,S}((\tau,z);s)$ can be continued to a meromorphic function of s in the whole complex plane satisfying a certain functional equation. For instance if $l=1$ and S is a square free positive integer, then the explicit functional equation is given by

$$\begin{aligned} E_1^{k,S}((\tau,z);s-\kappa) &= \Phi(s)E_1^{k,S}((\tau,z);1-s-\kappa), \\ \Phi(s)\Phi(1-s) &= 1 \end{aligned}$$

where $\kappa=(k-1/2)/2$ and

$$\Phi(s)=\frac{e^{-\pi ik/2}}{\sqrt{2S}}\cdot\frac{2^{2-2s}\pi\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)}\cdot\frac{\zeta(4s-2)}{\zeta(4s-1)}\prod_{p\,|\,S}\frac{1+p^{3/2-2s}}{1+p^{1/2-2s}} \tag{2.10}$$

(see [Ar90], Proposition 3.5). In this case the 'constant' term $c_1^{k,S}((\tau,z);s-\kappa)$ of $E_1^{k,S}((\tau,z);s-\kappa)$ is given by

$$c_1^{k,S}((\tau,z);s-\kappa)=(\eta^{s-\kappa}+\eta^{1-s-\kappa}\Phi(s))\theta_0(\tau,z)$$

([Ar90], Proposition 3.3, (3.15)). We are able to know the exact functional equations of $E_1^{k,S}((\tau,z);s-\kappa)$ and $\Phi(s)$ from the explicit expression of the 'constant' term.

Now we go gack to the case of degree n. Assume that S satisfies the condition (2.9). Set

$$\kappa=(k-l/2)/2.$$

We guess that among the 'constant' terms $c^{(\nu)}((\tau,z);s-\kappa)$ of $E_n^{k,S}((\tau,z);s-\kappa)^{(\nu)}$ $(0\le\nu\le n)$ the first term

$$c^{(0)}((\tau,z);s-\kappa)=\delta(\tau)^{s-\kappa}\theta_0(\tau,z)$$

and the last term

$$c^{(n)}((\tau,z);s-\kappa)=\Phi^{(n)}(s)\delta(\tau)^{(n+1)/2-s-\kappa}\theta_0(\tau,z)$$

will be replaced with each other up to some constant factor. Namely we guess that

$$c^{(0)}((\tau, z); s-\kappa) = \Phi^{(n)}(s)c^{(n)}((\tau, z); (n+1)/2 - s - \kappa).$$

Our precise conjecture is as follows.

Conjecture 2.4 (Main Conjecture) *Under the maximality condition* (2.9) *on* S *the function* $\Phi^{(n)}(s)$ *will satisfy the functional equation*

$$\Phi^{(n)}(s)\Phi^{(n)}\left(\frac{n+1}{2} - s\right) = 1.$$

Moreover the Eisenstein series $E_n^{k,S}((\tau, z); s)$ *will satisfy the functional equation*

$$E_n^{k,S}((\tau, z); s-\kappa) = \Phi^{(n)}(s)E_n^{k,S}\left((\tau, z); \frac{n+1}{2} - s - \kappa\right).$$

Remark 2.5 **(i)** *If* $n = 1$, *Conjecture 2.4 holds true. The case of* $l = 1$ *has been discussed in [Ar90], while in [Ar92] the conjecture for* $n = 1$ *and arbitrary* l *has been proved.*

(ii) *If the general theory of Langlands for the Eisenstein series of semi-simple Lie groups is available to our Jacobi Eisenstein series, then we have only to determine the explicit form of* $\Phi^{(n)}(s)$.

Finally we consider the case of $l = 1$ and $S = 1$ (the simplest but the most important case). In view of Proposition 2.2 we present a conjecture on $\zeta_{n,2}(T_0; s)$ which will lead us to the explicit computation of $\Phi^{(n)}(s)$.

Conjecture 2.6 *Let* $l = 1$ *and* $S = 1$. *The assertion* (ii) *in Proposition* 2.2 *holds true also for* $p = 2$. *Namely, if* $\widetilde{T} = 0$ *(namely* $T = T_0$*), then*

$$\zeta_{n,2}(T; s) = \prod_{i=1}^{n} \frac{1 - 2^{2i-2s}}{1 - 2^{n+i+1-2s}}.$$

Therefore,

$$\zeta_n(T; s) = \prod_{i=1}^{n} \frac{\zeta(2s-n-1-i)}{\zeta(2s-2i)}.$$

Remark 2.7 **(i)** *If* $n = 1$, *then Conjecture 2.6 can be easily shown from the definition of* $\zeta_{1,2}(T_0, s)$ *directly.*

(ii) *Let $l = 1$ and $S = 1$. Then under Conjecture 2.6 we have, for arbitrary n,*

$$\Phi^{(n)}(s) = i^{-nk}\pi^{\frac{n(n+1)}{2}} \times 2^{n(\frac{n+2}{2}-2s)} \frac{\Gamma_n\left(2s - \frac{n+1}{2}\right)}{\Gamma_n(s+\kappa)\Gamma_n(s-\kappa)} \times \prod_{i=1}^{n} \frac{\zeta(4s-n-1-i)}{\zeta(4s+1-2i)},$$

which is consistent with (2.10) if $n = 1$. From this expression we can prove directly the functional equation

$$\Phi^{(n)}(s)\Phi^{(n)}\left(\frac{n+1}{2} - s\right) = 1.$$

3 A Pullback formula

In this section we prove a pullback formula of a Jacobi Eisenstein series of Klingen type of genus two. Let us assume $l = 1$.

Let $\mathcal{U}_\perp = \{\left(\begin{smallmatrix} * & * \\ \varepsilon & \nu \end{smallmatrix}\right) \in P^J_{1,0}\backslash\Gamma \,|\, \varepsilon, \nu > 0\}$, $\Phi \in \mathcal{J}^{\text{cusp}}_{k,t}$ and $(\tau_1, z_1), (\tau_2, z_2) \in \mathcal{D}$. Then we put

(29)
$$\psi_\Phi^{\varepsilon,\nu}\Big((\tau_1,z_1),(\tau_2,z_2);s\Big) = \mathrm{Im}\left(\tau_1\tau_2(\varepsilon^2\tau_1+\nu^2\tau_2)^{-1}\right)^s\ \Phi\big(\varepsilon^2\tau_1+\nu^2\tau_2,\ \varepsilon z_1+\nu z_2\big).$$

Theorem 3.1 *Let $k, t \in \mathbb{N}$ and k even. If $\Phi \in \mathcal{J}^{cusp}_{k,t}$ and $k + 2Re(s) > 4$, then*

$$(30)\ E^{k,t}_{2,1}\Big(\Phi,(\tau_1,z_1)\times(\tau_2,z_2)),s\Big)$$

$$\begin{aligned}(31)\quad = \quad & E^{k,t}_1((\tau_1,z_1);s)\ \Phi(\tau_2,z_2) + E^{k,t}_1((\tau_2,z_2);s)\ \Phi(\tau_1,z_1) \\ & + \sum_{\gamma_1,\gamma_2\in\Gamma_\infty\backslash\Gamma} \ \sum_{\left(\begin{smallmatrix} a & b \\ \varepsilon & \nu \end{smallmatrix}\right)\in\mathcal{U}_\perp} \ \sum_{\lambda\in\mathbb{Z}} \\ & \qquad \left(\psi_\Phi^{\varepsilon,\nu}\big|_{k,t,1}\ ((\lambda a,0,0),\gamma_1)\ \big|_{k,t,2}\big((\lambda b,0,0),\gamma_2\big)\right)\ ((\tau_1,z_1),(\tau_2,z_2);s).\end{aligned}$$

Here $|_{k,t,j}$ is the operator $|_{k,t}$ corresponding to (τ_j, z_j).

Proof:

First we have to choose an appropriate representative system for $\Gamma^J_{2,1,1}\backslash\Gamma^J_2$. Let us take the set $\{(\lambda 0,0,0), M \in \Gamma^J_2 \,|\, \lambda \in \mathbb{Z}$ and $M \in P_{2,1}\backslash\Gamma_2\}$, with the decomposition $R = R_1 \sqcup R_2 \sqcup R_3$ of $P_{2,1}\backslash\Gamma_2$, given by

$$(32)\qquad \big(\Gamma_\infty\backslash\Gamma\times 1_2\big)\sqcup\pi\Big(1_2\times\Gamma_\infty\backslash\Gamma\Big)\sqcup T(\mathcal{U}_\perp)\Big(\Gamma_\infty\backslash\Gamma\times\Gamma_\infty\backslash\Gamma\Big).$$

Here $T(\mathcal{U}_\perp) = \{T_\perp(u) = \left(\begin{smallmatrix} u & 0 \\ 0 & * \end{smallmatrix}\right) \in \Gamma_2 \mid u \in \mathcal{U}_\perp\}$ and

$$\pi = \left(\begin{array}{cc} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array} \right).$$

The Theorem can be proven in the same style as Theorem 2.3 in [He96], with only minor changes. Thus we only give some useful formulas. Let g be an element of R_3:

$$g = \Big((\lambda 0, 0, 0), T \begin{pmatrix} a & b \\ \varepsilon & \nu \end{pmatrix} (\gamma_1 \times \gamma_2) \Big).$$

Then we have:

$$\Phi\Big(g\big((\tau_1, z_1) \times (\tau_2, z_2)\big)_{*_J} \Big) = \Phi\Big(\varepsilon^2 \gamma_1(\tau_1) + \nu^2 \gamma_2(\tau_2), \varepsilon z_1 J(\gamma_1, \tau_1)^{-1} + \nu z_2 J(\gamma_2, \tau_2)^{-1} + \lambda(a\varepsilon\gamma_1(\tau_1) + b\nu\gamma_2(\tau_2)) \Big),$$

(33)
$$\left(\frac{\delta\left(T\left(\begin{smallmatrix} a & b \\ \varepsilon & \nu \end{smallmatrix}\right)(\gamma_1 \times \gamma_2)(\tau_1 \times \tau_2)\right)}{\delta\left(T\left(\begin{smallmatrix} a & b \\ \varepsilon & \nu \end{smallmatrix}\right)(\gamma_1 \times \gamma_2)(\tau_1 \times \tau_2)\right)_*} \right)^s = \left(\prod_{j=1}^{2} \delta(\gamma_j(\tau_j))^s \right) \delta\Big(\varepsilon^2 \gamma_1(\tau_1) + \nu^2 \gamma_2(\tau_2) \Big)^{-s},$$

(34)
$$\begin{aligned} & j_{k,t}\Big(g, (\tau_1, z_1) \times (\tau_2, z_2) \Big)^{-1} \\ = \;& j_{k,t}\Big((\lambda a, 0, 0)\gamma_1, (\tau_1, z_1) \Big)^{-1} j_{k,t}\Big((\lambda b, 0, 0)\gamma_2, (\tau_2, z_2) \Big)^{-1}. \end{aligned}$$

$\square$

The formula (31) in Theorem 3.1 and the results of Theorem 1.6 make it transparent, that pullbacks of Jacobi Eisenstein series and Eisenstein series attached to the Siegel modular group have similar properties. The main point is here, that this is true in the non-holomorphic case !.

4 Certain Dirichlet series attached to three Jacobi forms

In this section we calculate the iterated scalar product of a diagonal-restricted Jacobi Eisenstein series against three Jacobi cusp forms. The main ingredient is the pullback formula (31) developed in the last section.

Let us first simplify and specialize our notation a little more. Moreover we fix the weight k even and the index $t \in A_1^+ = \mathbb{N}$ of our forms. We denote by $\tau = \tau_x + i\,\tau_y$ and $z = z_x + i\,z_y$ the decomposition of $(\tau, z) \in \mathcal{D}$ in real- and imaginary part. Hence the $G_{1,1}^J(\mathbb{R})-$ invariant volume element on $\mathcal{D}$ has the form

$$d\omega_{1,1}(\tau, z) = \tau_y^{-3} d\tau_x\, d\tau_y\, dz_x\, dz_y$$

and the function $\mathcal{C}_{k,t}$ is equal to $\tau_y^k\, e\{2i\,t\,z_y^2 \tau_y^{-1}\}$.

Let $\Phi_1, \Phi_2, \Phi_3 \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 5$. We are mainly interested in calculating the expression

(35)

$$D_{1,2;3}(s) := \Big\langle \Big\langle E_{2,1}^{k,t}\Big(\Phi_3^\sharp, \big((\tau_1, z_1) \times (\tau_2, z_2)\big), s\Big),\ \Phi_1(\tau_1, z_1)\, \Phi_2(\tau_2, z_2)\Big\rangle_J \Big\rangle_J.$$

A fundamental domain $\mathcal{F}$ of $(\Gamma_\infty \times H_{1,1}) \setminus I\!H \times \mathbb{C}$ is given by

(36)

$$\mathcal{F} = \Big\{(\tau, z) \in \mathcal{D}\ \Big|\ 0 \le \tau_x < 1;\ 0 < \tau_y < \infty \text{ and } 0 \le z_x < 1;\ 0 \le z_y < \tau_y\Big\}.$$

After some elementary calculations we get the

Proposition 4.1 *Let $\Phi_1, \Phi_2, \Phi_3 \in \mathcal{J}_{k,t}^{\text{cusp}}$. Then*

$$\begin{aligned}
D_{1,2;3}(s) \;=\; & \int_0^1 d\tau_{1,x} \int_0^1 d\tau_{2,x} \int_0^\infty d\tau_{1,y} \int_0^\infty d\tau_{2,y} \int_0^1 dz_{1,x} \int_0^1 dz_{2,x} \\
& \int_0^{\tau_{1,y}} dz_{1,y} \int_0^{\tau_{2,y}} dz_{2,y} \sum_{\substack{n,\,r \\ 4nt>r^2}} \sum_{\substack{n_1,\,r_1 \\ 4n_1t>r_1^2}} \sum_{\substack{n_2,\,r_2 \\ 4n_2t>r_2^2}} \sum_{\lambda\in\mathbb{Z}} \sum_{\left(\begin{smallmatrix} ab \\ \varepsilon\nu \end{smallmatrix}\right)\in\mathcal{U}_\perp} \\
& \overline{c_1(n_1, r_1) c_2(n_2, r_2) c_3(n, r)}\ \tau_{1,y}^{s+k-3} \tau_{2,y}^{s+k-3} (\varepsilon^2 \tau_{1,y} + \nu^2 \tau_{2,y})^{-s} \\
& \times\ e\{-n_1\tau_{1,x} + in_1\tau_{1,y} - r_1 z_{1,x} + ir_1 z_{1,y}\} \\
& \times\ e\{-n_2\tau_{2,x} + in_1\tau_{2,y} - r_1 z_{2,x} + ir_2 z_{2,y}\} \\
& \times\ e\{n\varepsilon^2\tau_{1,x} + in\varepsilon^2\tau_{1,y} + n\nu^2\tau_{2,x} + in\nu^2\tau_{2,y}\} \\
& \times\ e\{r\varepsilon z_{1,x} + ir\varepsilon z_{1,y} + r\nu z_{2,x} + i\nu r z_{2,y} \\
& \qquad + r\lambda a\varepsilon\tau_{1,x} + ir\lambda a\varepsilon\tau_{1,y} + r\lambda b\nu\tau_{2,x} + ir\lambda b\nu\tau_{2,y}\} \\
& \times\ e\{\lambda^2 a^2 t\tau_{1,x} + i\lambda^2 a^2 t\tau_{1,y} + 2\lambda at z_{1,x} + 2i\lambda at z_{1,y}\} \\
& \times\ e\{\lambda^2 b^2 t\tau_{2,x} + i\lambda^2 b^2 t\tau_{2,y} + 2\lambda bt z_{2,x} + 2i\lambda bt z_{2,y} \\
& \times\ e\{2i\pi t z_{1,y}^2/\tau_{1,y}\}\ e\{2i\pi t z_{2,y}^2/\tau_{2,y}\}.
\end{aligned}$$

Before we state the main theorem of this section we have to make several remarks concerning the notation and properties of Fourier coefficients of Jacobi cusp forms. Let

$$\Phi(\tau,z) = \sum_{\left(\begin{smallmatrix} n & r/2 \\ r/2 & t \end{smallmatrix}\right)\in A_2^+} c^{\Phi}(n,r)\ e\{n\tau + rz\} \in \mathcal{J}_{k,t}^{\text{cusp}}. \tag{37}$$

As a simple consequence of the invariance of Φ with respect to the Heisenberg group, one has

$$c^{\Phi}(n,r) = c^{\Phi}(n + r\lambda + t\lambda^2,\ r + 2t\lambda) \quad \text{for all } \lambda \in \mathbb{Z}. \tag{38}$$

Hence $c^{\Phi}(n,r) = c^{\Phi}(n',r')$, whenever $r \equiv r' (\operatorname{mod} 2t)$ and $4n\,t - r^2 = 4n'\,t - (r')^2$. Thus the Fourier coefficients depend only on the 'discriminant' $D = 4n\,t - r^2$ and $r\,(\operatorname{mod} 2t)$. Moreover, because we have choosen k even, $c^{\Phi}(n,r) = c^{\Phi}(n,-r)$ and if $t = 1$ or equal to a prime p, then the Fourier coefficients depend only on the discriminant ([E-Z85], Theorem 2.2). Let $c_r^{\Phi}(D) = c^{\Phi}(\frac{r^2+D}{4t}, r)$, then

$$\Phi(\tau,z) = \sum_{\substack{r\in\mathbb{Z},\, D\in\mathbb{N} \\ -D\equiv r^2 \operatorname{mod}(4t)}} c_r^{\Phi}(D)\ e\{\tfrac{r^2+D}{4t}\tau + rz\}. \tag{39}$$

For $\Phi_j \in \mathcal{J}_{k,t}^{\text{cusp}}$ we put $c^j(n,r) = c^{\Phi_j}(n,r)$ and $c_r^j(D) = c_r^{\Phi_j}(D)$.

Theorem 4.2 *Let k,t be natural integers with k even and $k + 2Re(s) > 4$ for $s \in \mathbb{C}$. Let $\Phi_1, \Phi_2, \Phi_3 \in \mathcal{J}_{k,t}^{cusp}$, then*

$$D_{1,2;3}(s) = a(k,s,t)\ \Gamma_{\infty}^*(s)\ D_{1,2;3}^*(s), \tag{40}$$

here $a(k,s,t) = 4^{-1}\pi^{-(s+2k-3)}t^{s+2k-4}$,

$$\begin{aligned} \Gamma_{\infty}^*(s) &= \frac{\Gamma(2k+s-3)\Gamma(s+k-3/2)^2}{\Gamma(2k+2s-3)} \\ D_{1,2;3}^*(s) &= \sum_{r_0=0}^{2t-1} \sum_{\substack{D\in\mathbb{N} \\ -D\equiv r_0^2 \text{ mod } 4t}} \sum_{\substack{\varepsilon,\nu=1 \\ (\varepsilon,\nu)=1}}^{\infty} \\ &\quad \overline{c_{r_0}^3(D_0)}\ \overline{c_{r_0\varepsilon}^1(D_0\varepsilon^2)}\ \overline{c_{r_0\nu}^2(D_0\nu^2)}(\varepsilon\nu)^{-(2s+2k-3)}D_0^{-(s+2k-3)}. \end{aligned}$$

Proof:
First we employ the integration with respect to $d\tau_{1,x}$; $d\tau_{2,x}$; $dz_{1,x}$ and $dz_{2,x}$. This leads

to

$$
\begin{aligned}
D_{1,2;3}(s) &= \sum_{\substack{n,r\in\mathbb{Z}\\4nt>r^2}} \sum_{\lambda=-\infty}^{\infty} \sum_{\left(\begin{smallmatrix}a b\\ \varepsilon\nu\end{smallmatrix}\right)\in\mathcal{U}_\perp} \overline{c^1(n_1,r_1)}\;\overline{c^2(n_2,r_2)}\;\overline{c^3(n,r)} \\
&\quad \int_0^\infty\int_0^\infty d\tau_{1,y}\; d\tau_{2,y} \int_0^{\tau_{1,y}} dz_{1,y} \int_0^{\tau_{2,y}} dz_{2,y}\;\; \tau_{1,y}^{s+k-3}\tau_{2,y}^{s+k-3}(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y})^{-s} \\
&\quad \times\; e\{2it(z_{1,y}^2/\tau_{1,y}+r_1/t\; z_{1,y}+n_1/t)\} \\
&\quad \times\; e\{2it(z_{2,y}^2/\tau_{2,y}+r_2/t\; z_{2,y}+n_2/t)\}.
\end{aligned}
$$

Here n_1, n_2, r_1 and r_2 are parameterized by $\begin{pmatrix} a & b\\ \varepsilon & \nu\end{pmatrix}$ and n, r, λ in the following way:

$$
\begin{aligned}
n_1 &= (\lambda a)^2 t + n\varepsilon^2 + r\lambda a\varepsilon, \\
n_2 &= (\lambda b)^2 t + n\nu^2 + r\lambda b\nu
\end{aligned}
$$

and

$$r_1 = r\varepsilon + 2\lambda a t \text{ and } r_2 = r\nu + 2\lambda b t.$$

The invariance properties of the Fourier coefficients lead to the useful identity

$$
\begin{aligned}
c^1(n_1,r_1)\; c^2(n_2,r_2)\; c^3(n,r) &= c^1(n\varepsilon^2, r\varepsilon)\; c^2(n\nu^2, r\nu)\; c^3(n,r) \\
&= c^1_{r\varepsilon}(D\varepsilon^2)\; c^2_{r\nu}(D\nu^2)\; c^3_r(D),
\end{aligned}
$$

here $D = 4nt - r^2$.

Substituting $z_{j,y}$ by $p_j\tau_{j,y}$ and p_j by $\tilde{p}_j - \frac{r_j}{2t}$ leads to

$$
\begin{aligned}
D_{1,2;3}(s) &= \sum_{\substack{n,r\in\mathbb{Z}\\4nt>r^2}} \sum_{\left(\begin{smallmatrix}a b\\ \varepsilon\nu\end{smallmatrix}\right)\in\mathcal{U}_\perp} \overline{c^1_{r\varepsilon}(D\varepsilon^2)}\;\overline{c^2_{r\nu}(D\nu^2)}\;\overline{c^3_r(D)} \\
&\quad \int_0^\infty\int_0^\infty d\tau_{1,y}\, d_{\tau_{2,y}}\; \tau_{1,y}^{s+k-2}\;\tau_{2,y}^{s+k-2}\;\; (\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y})^{-s} e^{-4\pi D t^{-1}(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y})} \\
&\quad \sum_{\lambda=-\infty}^{\infty} \int_{r_1/2t}^{(r_1+2t)/2t}\int_{r_2/2t}^{(r_2+2t)/2t} d\tilde{p}_1\; d\tilde{p}_2\; e^{-4\pi\, t(\tau_{1,y}\tilde{p}_1^2+\tau_{2,y}\tilde{p}_2^2)}.
\end{aligned}
$$

Let $\mathcal{D} = \{(D,r)\in\mathbb{N}\times\mathbb{Z}\,|\,-D\equiv r^2 \bmod 4t\}$ and $\mathcal{R}_0 = \{0,1,\ldots,2t-1\}$. Then we have

$$\mathcal{D} = \bigsqcup_{r_0\in\mathcal{R}_0} \mathcal{D}_{r_0}, \qquad \text{here } \mathcal{D}_{r_0} = \{(D,r)\in\mathcal{D}\,|\,r\equiv r_0 \bmod 2t\}.$$

Hence we have

$$D_{1,2;3}(s) \;=\; \sum_{r_0\in\mathcal{R}_0} \sum_{\substack{D\in\mathbb{N}\\ -D\equiv r_0^2 \bmod 4t}} \overline{c^1_{r_0\varepsilon}(D\varepsilon^2)}\; \overline{c^2_{r_0\nu}(D\nu^2)}\; \overline{c^3_{r_0}(D)}$$
$$\int_0^\infty \int_0^\infty d\tau_{1,y}\, d_{\tau_{2,y}}\; \tau_{1,y}^{s+k-2}\, \tau_{2,y}^{s+k-2}\; \left(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y}\right)^{-s} e^{-4\pi Dt^{-1}(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y})}$$
$$\sum_{\lambda,u\in\mathbb{Z}} \int_{\frac{r_0\varepsilon}{2t}+u\varepsilon+\lambda\, a}^{\frac{r_0\varepsilon}{2t}+u\varepsilon+\lambda\, a+1} \int_{\frac{r_0\nu}{2t}+u\nu+\lambda\, b}^{\frac{r_0\nu}{2t}+u\nu+\lambda\, b+1} d\tilde{p}_1\; d\tilde{p}_2\; e^{-4\pi\, t(\tau_{1,y}\tilde{p}_1^2+\tau_{2,y}\tilde{p}_2^2)}.$$

But this can easily be simplified to

$$D_{1,2;3}(s) \;=\; \sum_{r_0\in\mathcal{R}_0} \sum_{\substack{D\in\mathbb{N}\\ -D\equiv r_0^2 \bmod 4t}} \overline{c^1_{r_0\varepsilon}(D\varepsilon^2)}\; \overline{c^2_{r_0\nu}(D\nu^2)}\; \overline{c^3_{r_0}(D)} \int_0^\infty \int_0^\infty d\tau_{1,y}\, d_{\tau_{2,y}} \int_{-\infty}^\infty \int_{-\infty}^\infty dx\, dy$$
$$\tau_{1,y}^{s+k-2}\, \tau_{2,y}^{s+k-2}\; \left(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y}\right)^{-s} e^{-4\pi Dt^{-1}(\varepsilon^2\tau_{1,y}+\nu^2\tau_{2,y})} e^{-4\pi t(\tau_{1,y}x^2++\tau_{2,y}y^2)}$$

Using $\int_{-\infty}^\infty e^{-ab^2}db = \left(\frac{\pi}{a}\right)^{1/2}$ and obvious substitutions give the desired result.

□

Theorem 4.3 *Let k,t be natural integers with k even and $\Phi_1,\Phi_2,\Phi_3\in\mathcal{J}_{k,t}^{cusp}$, then the Dirichlet series $D^*_{1,2;3}(s)$ has a meromorphic continuation to the whole complex plane.*

Proof:

It follows from the work of Arakawa [Ar90], [Ar92], [Ar94] and Heim [He97], that all Jacobi Eisenstein series have a meromorphic continuation to the whole complex plane.

□

Remark 4.4 *The results given in [Ar94] and [He97] (e.g. Theorem 4.1) can be used to determine the location of the possible poles of real analytic Jacobi Eisenstein series of arbitary degree and hence the analytical behaviour of $D^*_{1,2;3}(s)$.*
And moreover, the program started in §2 of this paper, will finally give us a functional equation of this series.

References

[Ar90] **T. Arakawa**, Real Analytic Eisenstein Series for the Jacobi Group, *Abh. Math. Sem.Univ. Hamburg 60* (1990), 131-148

[Ar92] **T. Arakawa**, Selberg zeta functions and Jacobi forms, *Advanced Stud. in Pure Math. 21* (1992), 181-218

[Ar93] **T. Arakawa**, Siegel's formula for Jacobi forms, *Internat. Journ. of. Math. 4* (1993), 689-719

[Ar94] **T. Arakawa**, Jacobi Eisenstein series and a basis problem for Jacobi forms, *Comm. Mathematici Universitatis Sancti Pauli Vol. 43 No.2* (1994), 181-216

[Be94] **R. Berndt**, L-functions for Jacobi forms a la Hecke, *manuscripta math. 84,* (1994)

[Boe85b] **S. Böcherer**, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, *J. reine angew. Math. 362* (1985), 146-168

[Du97] **J. Dulinski**, L-Functions for Jacobi Forms on $\mathbb{H} \times \mathbb{C}$, *Result. Math. 31* (1997), 75-94

[E-Z85] **M. Eichler and D. Zagier**, The theory of Jacobi forms, *Progress in Mathematics Vol. 55, Birkhäuser Verlag* (1985)

[Ga84] **P. Garrett**, Pullbacks of Eisenstein series; Applications. In: Automorphic forms of several variables. *Taniguchi symposium 1983, Birkhäuser* (1984)

[Ga85] **P. Garrett**, Integral representations of certain L-functions attached to one, two and three modular forms, Preprint (1985)

[Ga87] **P. Garrett**, Decomposition of Eisenstein series: triple product L-functions, *Ann. math. 125* (1987), 209-235 Eisenstein series on Shimura varieties, *Ann. Math. 119* (1984), 59-94

[He96] **B. Heim**, Pullbacks of Eisenstein series, Hecke-Jacobi theory and automorphic L-functions, *to appear in the Proceedings of Symposia of Pure Mathematics, in honor of G. Shimura* 1996

[He97] **B. Heim**, Analytic Jacobi Eisenstein series and the Shimura method, *MPI preprint* 1998 (1)

[Ma71] **H. Maass**, Siegel's modular forms and Dirichlet series, *Springer LNM 216* (1971)

[Mi93] **S. Mizumoto, S** Eisenstein for Siegel modular groups, *Math. Ann. 297* (1993), 581-625; Corrections, ibid. 307 (1997), 169-171

[Mu89] **A. Murase**, L-functions attached to Jacobi forms of degree n, Part I. The basic identity, *J. reine Math. 401* (1989), 122-156

[P-R87] **I. I. Piatetski-Shapiro and S. Rallis**, Rankin triple L-functions, *Compositio Math., 64* (1987), 31-115

[Sh73] **G. Shimura**, On modular forms of half integral weight, *Ann. of Math. 97* (1973), 440-448

[Sh82] **G. Shimura**, Confluent hypergeometric functions on tube domains, *Math. Ann. 260* (1982), 269-302

[Sh83] **G. Shimura**, On Eisenstein series, *Duke Math. J. 50* (1983), 417-476

[Sh85] **G. Shimura**, On Eisenstein series of half-integral weight, *Duke Math. J.52* (1985), 281-324

[Za77] **D. Zagier**, Modular forms whose Fourier coefficients involve zeta functions of quadratic fields, in Modular Functions of One Variable VI, *Springer Lecture Notes 627, Springer Verlag, Berlin - Heidelberg - New York* (1977), 105 -169

[Zi89] **C. Ziegler**, Jacobi forms of higher degree, *Abh. M. Sem. Univ. Hamburg 59* (1989), 191-224

Tsuneo Arakawa
Department of Mathematics,
Rikkyo University,
Nishi-Ikebukuro, Tokyo 171, Japan
arakawa@rkmath.rikkyo.ac.jp

Bernhard Heim
Max-Planck Institut für Mathematik
Gottfried-Claren Strasse 26
53225 Bonn, Germany
heim@mpim-bonn.mpg.de