ON HOMOGENEOUS CONNECTIONS WITH EXOTIC HOLONOMY

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ABSTRACT.
In [Br], Bryant gave examples of torsion free connections on four-manifolds whose holonomy is *exotic*, i.e. is not contained on Berger's classical list of irreducible holonomy representations [Ber]. The holonomy in Bryant's examples is the irreducible four-dimensional representation of $Sl(2,\mathbb{R})$ ($Gl(2,\mathbb{R})$ resp.) and these connections are called H_3-connections (G_3-connections resp.).

In this paper we give a complete classification of homogeneous G_3-connections. The moduli space of these connections is four-dimensional, and the generic homogeneous G_3-connection is shown to be locally equivalent to a left-invariant connection on $U(2)$. Thus, we prove the existence of compact manifolds with G_3-connections. This contrasts a result in [Sch] which states that there are no compact manifolds with an H_3-connection.

§0 Introduction.

Since its introduction by Èlie Cartan, the *holonomy* of a connection has played an important role in differential geometry. Most of the classical results are concerned with the holonomy of Levi Civita connections of Riemannian metrics. In 1955, Berger [Ber] classified the possible irreducible Riemannian holonomies and much work has been done since to study these holonomies and their applications. See [Bes] and [Sa] for a historical survey and also [J] for more recent results.

At the same time, Berger also partially classified the possible non-Riemannian holonomies of torsion free connections. However, his classification omits a finite number of possibilities, which are referred to as *exotic holonomies*. Until now, the complete list of exotic holonomies is still not known.

The incompleteness of Berger's list and therefore the existence of exotic holonomies was shown by Bryant [Br]. He investigated the irreducible representations of $Sl(2,\mathbb{R})$. For each $d \geq 1$, we can regard $Sl(2,\mathbb{R})$ as a subgroup $H_d \subsetneq Gl(d+1,\mathbb{R})$ via the (unique) $(d+1)$-dimensional irreducible representation of $Sl(2,\mathbb{R})$. Moreover, if we let $G_d \subsetneq Gl(d+1,\mathbb{R})$ be the centralizer of H_d, then G_d may be regarded as a representation of $Gl(2,\mathbb{R})$. For $d \geq 3$, these representations do not occur on Berger's list of possible holonomies and are therefore candidates for exotic holonomies.

In his paper, Bryant showed that in the case $d = 3$ torsion free connections with holonomies H_3 and G_3 do exist. We shall refer to them as H_3-connections (G_3-connections resp.).

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The "moduli space" of H_3-connections is the union of a one-dimensional space and six points. Moreover, there is exactly one homogeneous (non-flat) H_3-connection with a five-dimensional symmetry group. For other global properties of H_3-connections see [Sch]. On the other hand, the moduli space of G_3-connections is infinite dimensional; namely, the "generic" G_3-connection depends on four functions of three variables.

In this paper, we investigate certain "non-generic" G_3-connections. The generic condition in [Br] implies that the connection does not admit any non-zero infinitesimal symmetries. In a sense, we assume the exact opposite and consider the question if there exist any *(locally) homogeneous* G_3-connections besides the flat and the (unique) homogeneous H_3-connection. The answer to this question which had been raised in [Br] is affirmative. In fact, we shall arrive at a complete classification of homogeneous G_3-connections.

At this point, we shall state some consequences of this classification.

Theorem 0.1. *Any homogeneous G_3-connection whose holonomy is not contained in H_3 is locally equivalent to a left-invariant connection on a four-dimensional Lie group.*

Theorem 0.2. *Up to isomorphism, there are twelve distinct possibilities for the Lie algebra of the symmetry group of a G_3-connection satisfying the hypothesis of Theorem 0.1. One of them is nilpotent, nine are solvable and the remaining two are* $\mathfrak{gl}(2,\mathbb{R})$ *and* $\mathfrak{u}(2)$.

Theorem 0.3. *The moduli space of homogeneous G_3-connections is four-dimensional. More specifically, the moduli space has one four-dimensional component, seven one-dimensional components and fourteen points, including the flat connection and the homogeneous H_3-connection.*

Theorem 0.4. *The reduced holonomy of a homogeneous G_3-connection is either trivial, equal to H_3 or equal to all of G_3.*

Here, the reduced holonomy stands for the identity component of the holonomy group. This result follows from a case-by-case investigation of all entries of the classification.

Theorem 0.5. *Generically, the symmetry group of a homogeneous G_3-connection has Lie algebra* $\mathfrak{u}(2)$, *i.e. the generic homogeneous G_3-connection is locally equivalent to a left-invariant connection on the (compact) Lie group $U(2)$.*

As a consequence, this yields the remarkable

Corollary 0.6. *There are G_3-connections on compact manifolds.*

Corollary 0.6. contrasts a result in [Sch] which states that there are no H_3-connections on compact four-manifolds.

In §1, we briefly recall the structure equations for a torsion free G_3-connection, following the notation of [Br].

In §2, the core of this paper, we first show that every connection other than the flat and the homogeneous H_3-connection has a symmetry group of dimension at most four. As a consequence, every homogeneous G_3-connection other than these two must be locally equivalent to a left-invariant connection on a four-dimensional Lie group. These connections are then shown to be in one-to-one correspondence with the orbit space of polynomials satisfying certain equations. Those polynomials can be completely classified.

Finally, in §3 we explicitly present the different polynomials that yield homogeneous G_3-connections. We also determine the Lie algebras of their symmetry groups which essentially determine, of course, the underlying manifolds.

The main part of the work presented here has been completed while the author was a visiting faculty member at Washington University in St. Louis, Mo, and he wishes to thank the department of Mathematics there for its hospitality.

§1 The structure equations.

We begin with a brief description of the irreducible $Sl(2,\mathbb{R})$-representations.

For $d \in \mathbb{N}$, let $\mathcal{V}_d \subseteq \mathbb{R}[x,y]$ be the $(d+1)$-dimensional subspace of homogeneous polynomials of degree d. There is an $Sl(2,\mathbb{R})$-action ($Gl(2,\mathbb{R})$-action resp.) on $\mathcal{V}_d$ induced by the transposed action of $Sl(2,\mathbb{R})$ ($Gl(2,\mathbb{R})$ resp.) on $\mathbb{R}^2$, i.e. if $p \in \mathcal{V}_d$ and $A \in Sl(2,\mathbb{R})$ ($A \in Gl(2,\mathbb{R})$ resp.) then

$$(A \cdot p)(x,y) := p(u,v) \quad \text{with} \quad (u,v) = (x,y)A.$$

It is well known that this action is irreducible for every d and moreover that - up to equivalence - this is the only irreducible $(d+1)$-dimensional representation of $Sl(2,\mathbb{R})$ ($Gl(2,\mathbb{R})$ resp.) (cf. [H]). We let $H_d \subseteq Gl(\mathcal{V}_d)$ ($G_d \subseteq Gl(\mathcal{V}_d)$ resp.) be the image of this representation and let $\mathfrak{h}_d \subseteq \mathfrak{gl}(\mathcal{V}_d)$ ($\mathfrak{g}_d \subseteq \mathfrak{gl}(\mathcal{V}_d)$ resp.) be the Lie algebra of H_d (G_d resp.).

The *Clebsch-Gordan formula* [H] describes the irreducible decomposition of a tensor product of irreducible $Sl(2,\mathbb{R})$-modules:

$$\mathcal{V}_m \otimes \mathcal{V}_n = \mathcal{V}_{|m-n|} \oplus \mathcal{V}_{|m-n|+2} \oplus \cdots \oplus \mathcal{V}_{m+n-2} \oplus \mathcal{V}_{m+n}.$$

A convenient tool to compute the decomposition of polynomials into their irreducible components are the bilinear pairings

$$\langle\ ,\ \rangle_p : V_n \otimes V_m \longrightarrow V_{n+m-2p}$$

$$\langle u,v\rangle_p = \frac{1}{p!}\sum_{k=0}^{p}(-1)^k \binom{p}{k} \frac{\partial^p u}{\partial^k x \partial^{p-k} y}\frac{\partial^p v}{\partial^{p-k} x \partial^k y} \quad \text{for} \quad u \in V_n, v \in V_m.$$

It can be shown that these pairings are $Sl(2,\mathbb{R})$-equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula.

Now we shall describe the structure equations for G_3-connections. Let M be a four-manifold and let $\pi : \mathfrak{F} \to M$ be the $\mathcal{V}_3$-coframe bundle, i.e. each $u \in \mathfrak{F}$ is a linear isomorphism $u : T_{\pi(u)}M \xrightarrow{\sim} \mathcal{V}_3$. Then $\mathfrak{F}$ is naturally a principal right $Gl(\mathcal{V}_3)$-bundle over M, where the right action $R_g : \mathfrak{F} \to \mathfrak{F}$ is defined by $R_g(u) = g^{-1} o u$. The *tautological 1-form* ω on $\mathfrak{F}$ with values in $\mathcal{V}_3$ is defined by letting $\omega(v) = u(\pi_*(v))$ for $v \in T_u\mathfrak{F}$. For ω, we have the $Gl(\mathcal{V}_3)$-equivariance $R_g^*(\omega) = g^{-1}\omega$.

A G_3-structure on M is, by definition, a G_3-subbundle $F \subseteq \mathfrak{F}$. For any G_3-structure, we will denote the restrictions of π and ω to F by the same letters.

We now turn to describe connections on F. Since G_3 is canonically isomorphic to $Gl(2,\mathbb{R})$, we may regard the $Gl(2,\mathbb{R})$-representations $\mathcal{V}_d$ equally well as G_3-representations. Moreover, it is easily seen that the map $\rho_d : \mathcal{V}_2 \oplus \mathcal{V}_0 \to \mathrm{End}(\mathcal{V}_d)$ defined by $\rho_d(a^2 + a^0)(a^d) := \langle a^0, a^d\rangle_0 + \langle a^2, a^d\rangle_1$ for all $a^i \in \mathcal{V}_i$ establishes an isomorphism $\mathcal{V}_2 \oplus \mathcal{V}_0 \stackrel{\sim}{\longleftrightarrow} \mathfrak{g}_d$. We will use this to regard a *connection* on F as a G_3-equivariant, $\mathcal{V}_2 \oplus \mathcal{V}_0$-valued 1-form $\varphi = \phi + \lambda$ on F where ϕ and λ take values in $\mathcal{V}_2$ and $\mathcal{V}_0$ resp. The *torsion* of φ is then represented by the $\mathcal{V}_3$-valued 2-form $T(\varphi) = d\omega + \langle\phi,\omega\rangle_1 + \langle\lambda,\omega\rangle_0$ and the *curvature* of φ by the $(\mathcal{V}_2 \oplus \mathcal{V}_0)$-valued 2-form $R(\varphi) = d\varphi + \frac{1}{2}\langle\varphi,\varphi\rangle_1 = d\lambda + d\phi + \frac{1}{2}\langle\phi,\phi\rangle_1$.

If we assume that φ describes a *torsion free* connection, then the *first structure equation* $T(\varphi) = 0$ reads

$$d\omega = -\lambda \wedge \omega - \langle\phi,\omega\rangle_1 . \tag{1}$$

Differentiating (1) yields

$$d\lambda \wedge \omega + \left\langle d\phi + \frac{1}{2}\langle\phi,\phi\rangle_1 , \omega\right\rangle_1 = 0.$$

This equation, which is the first Bianchi identity, can be solved to show that there is a $(\mathcal{V}_2 \oplus \mathcal{V}_4)$-valued function $\mathbf{a} = a^2 + a^4$ on F with $a^i : F \to \mathcal{V}_i$, such that the *second structure equations* hold:

$$\begin{aligned} d\lambda &= \langle a^4, \langle\omega,\omega\rangle_1\rangle_4 , \\ d\phi &= -\frac{1}{2}\langle\phi,\phi\rangle_1 + a^2\langle\omega,\omega\rangle_3 - \frac{1}{12}\langle a^2, \langle\omega,\omega\rangle_1\rangle_2 + \frac{1}{12}\langle a^4, \langle\omega,\omega\rangle_1\rangle_3 . \end{aligned} \tag{2}$$

Note that, in particular, we obtain as a formula for the curvature

$$R(\varphi) = \langle a^4, \langle\omega,\omega\rangle_1\rangle_4 + a^2\langle\omega,\omega\rangle_3 - \frac{1}{12}\langle a^2, \langle\omega,\omega\rangle_1\rangle_2 + \frac{1}{12}\langle a^4, \langle\omega,\omega\rangle_1\rangle_3 . \tag{3}$$

Differentiating these equations once again and solving for $\mathbf{a}$ we find that there is a $(\mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7)$-valued function $\mathbf{b} = b^1 + b^3 + b^5 + b^7$ on F with $b^i : F \to \mathcal{V}_i$, such that the *third structure equations* hold:

$$\begin{aligned} da^2 &= 2\lambda \wedge a^2 - \langle\phi, a^2\rangle_1 + 10\langle b^1,\omega\rangle_1 + \langle b^3,\omega\rangle_2 + 14\langle b^5,\omega\rangle_3 , \\ da^4 &= 2\lambda \wedge a^4 - \langle\phi, a^4\rangle_1 + 9\langle b^1,\omega\rangle_0 - 5\langle b^5,\omega\rangle_2 + \langle b^7,\omega\rangle_3 . \end{aligned} \tag{4}$$

The function $\mathbf{b}$ represents the covariant derivative ∇R of the curvature. In particular, we emphasize that $(\nabla R)(x) = 0$ at some $x \in M$ if and only if $\mathbf{b}(u) = 0$ for all $u \in \pi^{-1}(x)$.

We can also obtain formulas for $d\mathbf{b}$ by differentiating (4). Since these formulas are fairly involved we shall not write them in full. However, we can describe the G_3-equivariance of $\mathbf{b}$ by the equations

$$db^i \equiv 3\lambda \wedge b^i - \langle\phi, b^i\rangle_1 \quad \mathrm{mod}\ \omega, \quad \text{for } i = 1,3,5,7. \tag{5}$$

A *G_3-connection on M* is, by definition, a G_3-structure on M which carries a torsion free connection. A manifold M with a G_3-connection will be called a G_3-manifold.

§2 Homogeneous G_3-structures.

Throughout this section, we shall assume that M is a connected G_3-manifold. We begin with some definitions.

Definition 2.1. Let M be a connected G_3-manifold with connection ∇ and let $\pi : F \to M$ be the associated G_3-structure.

(1) A *(local) symmetry on* M is a (local) diffeomorphism $\underline{\alpha} : M \to M$ such that $\underline{\alpha}_*(\nabla_X Y) = \nabla_{\underline{\alpha}_*(X)}\underline{\alpha}_*(Y)$ for all $X, Y \in \mathfrak{X}(M)$.

(2) A *(local) symmetry on* F is a (local) diffeomorphism $\alpha : F \to F$ such that $\alpha^*(\omega) = \omega$ and $\alpha^*(\varphi) = \varphi$.

There is a one-to-one correspondence between symmetries on M and on F. In fact, given a (local) symmetry $\underline{\alpha}$ on M, there is a unique (local) symmetry α on F making the diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & F \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ M & \xrightarrow{\underline{\alpha}} & M \end{array}$$

commute, and vice versa.

Definition 2.2. Let M and $\pi : F \to M$ as before.

(1) An *infinitesimal symmetry on* M is a vector field $\underline{S} \in \mathfrak{X}(M)$ such that $\mathfrak{L}_{\underline{S}}\nabla = 0$, i.e. $[\underline{S}, \nabla_X Y] - \nabla_{[\underline{S},X]}Y - \nabla_X[\underline{S}, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$.

(2) An *infinitesimal symmetry on* F is a vector field $S \in \mathfrak{X}(F)$ such that $\mathfrak{L}_S\omega = \mathfrak{L}_S\theta = 0$.

Again, there is a one-to-one correspondence between infinitesimal symmetries on M and on F: in fact, given an infinitesimal symmetry $\underline{S} \in \mathfrak{X}(M)$, then there is a unique infinitesimal symmetry S on F s.th. $\underline{S} = \pi_*(S)$. Conversely, given an infinitesimal symmetry $S \in \mathfrak{X}(F)$ then the vector field $\pi_*(S)$ is well defined and is an infinitesimal symmetry on M.

Note that the flow along an infinitesimal symmetry on M (on F resp.) yields a one-parameter family of local symmetries on M (on F resp.). In fact, the infinitesimal symmetries form the Lie algebra of the group of (local) symmetries.

Due to the above mentioned one-to-one correspondences, we will frequently speak of symmetries (local, infinitesimal symmetries resp.) of the G_3-connection without specifying whether they are regarded as symmetries (local, infinitesimal symmetries resp.) on M or on F.

The group of (local) symmetries will be denoted by G and its Lie algebra of infinitesimal symmetries by $\mathfrak{g}$.

It is worth remarking that as a consequence of the structure equations (2) and (4) we have $d\mathbf{a}(S) = d\mathbf{b}(S) = 0$ for any infinitesimal symmetry S on F.

In this paper we will be concerned with *homogeneous* G_3*-manifolds*, i.e. those G_3-manifolds whose symmetry group acts transitively on M. First, we will prove a Lemma which will yield some relation between the isotropy and the curvature at a point of M.

Lemma 2.3. *Let M be a G_3-manifold, let $\pi : F \to M$ as before and let $x \in M$ be a point such that neither the curvature R nor its covariant derivative ∇R vanish at x, and let $\mathfrak{g}_x \subseteq \mathfrak{g}$ be the set of infinitesimal symmetries on M which vanish at x. If $\mathfrak{g}_x \neq 0$ then there exists a point $u_0 \in \pi^{-1}(x)$ such that*
either

$$a^2(u_0) = r_2 x^2, \quad a^4(u_0) = r_4 x^3 y,$$
$$b^1(u_0) = 0, \quad b^3(u_0) = s_3 x^3, \quad b^5(u_0) = s_5 x^4 y, \quad b^7(u_0) = s_7 x^5 y^2,$$

or

$$a^i(u_0) = r_i x^i, \quad \text{for } i = 2,4 \quad b^i(u_0) = s_i x^i, \quad \text{for } i = 1,3,5,7.$$

for some constants $r_i, s_i \in \mathbb{R}$.

Proof. The hypothesis that R and ∇R do not vanish at x implies that $\mathbf{a}(u) \neq 0$ and $\mathbf{b}(u) \neq 0$ for all $u \in \pi^{-1}(x)$.

Now let $0 \neq \underline{S} \in \mathfrak{g}_x$, and let $S \in \mathfrak{X}(F)$ be the corresponding infinitesimal symmetry on F. Clearly, $\pi_*(S_u) = 0$ and hence $\omega(S_u) = 0$ for all $u \in \pi^{-1}(x)$. Since $\varphi + \omega$ is a coframe on F and $S \neq 0$, we have $\varphi(S) \neq 0$. Moreover, since S is an infinitesimal symmetry, we also have $d\mathbf{a}(S) = d\mathbf{b}(S) = 0$.

Applying (4) and (5) to S, we obtain that at any point $u \in \pi^{-1}(x)$, we have

(6-1) $$2\lambda(S)a^i - \left\langle \phi(S), a^i \right\rangle_1 = 0 \quad \text{for } i = 2,4,$$

(6-2) $$3\lambda(S)b^i - \left\langle \phi(S), b^i \right\rangle_1 = 0 \quad \text{for } i = 1,3,5,7.$$

Now consider the following two cases:

(1) case 1: $\phi(S) \in \mathcal{V}_2$ factors into two independent linear factors over $\mathbb{C}$. Then there is some $g \in \mathrm{Sl}(2,\mathbb{C})$ such that $\rho_2^{\mathbb{C}}(g) \cdot \phi(S) = cxy$ for some $c \in \mathbb{C}$.

We deduce from (6-1) and $\mathbf{a} \neq 0$ that $2\lambda(S) = kc$ with $k \in \{0, \pm 2, \pm 4\}$. Likewise, from (6-2) and $\mathbf{b} \neq 0$ we deduce that $3\lambda(S) = kc$ with $k \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$.

The only possibility for these to hold simultaniously is that $\lambda(S) = \pm c$. In particular, $c \in \mathbb{R}$. From here we can conclude that $\phi(S)$ factors *over* $\mathbb{R}$, hence at some point $u_0 \in \pi^{-1}(x)$ we have $\phi(S_{u_0}) = \lambda(S_{u_0})xy$ and $\lambda(S_{u_0}) \neq 0$. From (6-1) and (6-2) we obtain that $\mathbf{a}(u_0)$ and $\mathbf{b}(u_0)$ are of the first form presented above.

(2) case 2: $\phi(S)$ is the square of a linear polynomial. Then there is some $u_0 \in \pi^{-1}(x)$ such that $\phi(S_{u_0}) = x^2$.

We deduce from (6-1) and $\mathbf{a} \neq 0$ that $\lambda(S_{u_0}) = 0$. Again, equations (6-1) and (6-2) imply that $\mathbf{a}(u_0)$ and $\mathbf{b}(u_0)$ are of the second form presented above.

(3) case 3: $\phi(S) = 0$. Then we deduce from (6-1) and $\mathbf{a} \neq 0$ that $\lambda(S) = 0$, i.e. $\varphi(S) = 0$ which is impossible. □

Now we obtain the following remarkable

Theorem 2.4. *Let M be a (locally) homogeneous G_3-manifold. Then either*

(1) *the G_3-connection is flat, or*
(2) *M is locally equivalent to the unique homogeneous H_3-manifold, or*
(3) *the isotropy group of the points of M is discrete, hence the (local) symmetry group has dimension four.*

Proof. Let G denote the group of symmetries and suppose that G acts transitively on M.

First of all, note that there cannot be a locally symmetric non-flat G_3-connection: the isotropy of a symmetric G_3-connection contains all of G_3. But the map $\mathbf{a} : F \to \mathcal{V}_2 \oplus \mathcal{V}_4$ must be invariant under the action of the isotropy group, therefore we must have $\mathbf{a} = 0$, i.e. the connection is *flat.*

We will now assume that the G_3-connection is neither flat nor locally symmetric and that the isotropy group at each point is at least one-dimensional. We shall conclude from these assumptions that the holonomy is contained in H_3, and this will complete the proof.

From Lemma 2.3. we conclude that there are $Gl(2,\mathbb{R})$-equivariant functions $v_i : F \to \mathcal{V}_1$ for $i = 1,2$ such that $\langle v_1, v_2\rangle_1 \equiv 1$ and functions $\underline{r}_i, \underline{s}_i : M \to \mathbb{R}$ such that

either

$$(*) \qquad \begin{aligned} a^2 = r_2 v_1^2, \quad a^4 = r_4 v_1^3 v_2, \\ b^1 = 0, \quad b^3 = s_3 v_1^3, \quad b^5 = s_5 v_1^4 v_2, \quad b^7 = s_7 v_1^5 v_2^2, \end{aligned}$$

or

$$(**) \qquad a^i = r_i v_1^i, \quad \text{for } i = 2,4 \quad b^i = s_i v_1^i, \quad \text{for } i = 1,3,5,7,$$

where $r_i = \underline{r}_i \circ \pi$ and $s_i = \underline{s}_i \circ \pi$.

Since the connection is *homogeneous* we may assume that v_i is G-invariant for $i = 1,2$ and that $\underline{r}_i, \underline{s}_i$ are *constant* for all i. Thus, so are r_i and s_i.

If (*) holds, then the structure equations (4) imply that

$$\begin{aligned} 2r_2 dv_1 = &\ 2r_2\lambda\wedge v_1 - 2r_2\langle\phi, v_1\rangle_1 \\ &+(3s_3 + 56s_5)v_2\langle v_1^3,\omega\rangle_3 + 3(28s_5 - s_3)v_1\langle v_1^2 v_2,\omega\rangle_3 \\ 2r_2r_4v_1dv_2 = &-2r_2r_4\lambda\wedge v_1v_2 - 2r_2r_4v_1\langle\phi, v_2\rangle_1 \\ &+(-9r_4s_3 - 60r_2s_5 - 168r_4s_5 + 20r_2s_7)v_2^2\langle v_1^3,\omega\rangle_3 \\ &+(9r_4s_3 + 20r_2s_5 - 252r_4s_5 + 40r_2s_7)v_1v_2\langle v_1^2v_2,\omega\rangle_3 \\ &+10r_2(4s_5 + s_7)v_1^2\langle v_1v_2^2,\omega\rangle_3 \end{aligned}$$

Taking the latter equation modulo v_1, we conclude that

$$-9r_4s_3 - 60r_2s_5 - 168r_4s_5 + 20r_2s_7 = 0.$$

If we furthermore assume that $r_2r_4 \neq 0$ then we get for the exterior derivatives

$$\begin{aligned} dv_1 = &\ \lambda\wedge v_1 - \langle\phi, v_1\rangle_1 + \frac{3s_3 + 56s_5}{2r_2}v_2\langle v_1^3,\omega\rangle_3 + 3\frac{28s_5 - s_3}{2r_2}v_1\langle v_1^2v_2,\omega\rangle_3 \\ dv_2 = &-\lambda\wedge v_2 - \langle\phi, v_2\rangle_1 + \frac{9r_4s_3 + 20r_2s_5 - 252r_4s_5 + 40r_2s_7}{2r_2r_4}v_2\langle v_1^2v_2,\omega\rangle_3 \\ &+5\frac{4s_5 + s_7}{r_4}v_1\langle v_1v_2^2,\omega\rangle_3 \end{aligned}$$

However, taking exterior derivatives of these equations we conclude that $r_2r_4 = 0$ which is impossible.

The remaining cases can be dealt with in a similar fashion: if $r_2 = 0$ then the above equations imply that $s_3 = s_5 = 0$. From (4) and the equation

$$0 = \langle dv_1, v_2 \rangle_1 + \langle v_1, dv_2 \rangle_1 ,$$

we can get explicit expressions for dv_1 and dv_2. If we take exterior derivatives then we conclude that $r_4 = 0$, i.e. the connection is *flat.*

If we assume that $r_4 = 0$ then we conclude that $s_5 = s_7 = 0$, and from there it follows that the holonomy group is contained in the subgroup $H_3 \subseteq G_3$.

Finally, if (**) holds then, by a similar analysis, we can conclude that the holonomy of the G_3-connection is contained in H_3. □

We turn now to the problem of classifying the homogeneous G_3-manifolds. By the preceding Theorem, it will suffice to consider left-invariant G_3-connections on four-dimensional Lie groups. In fact, if M is a (locally) homogeneous G_3-manifold with a four-dimensional (local) symmetry group G then, for some fixed point $p \in M$, the map $g \mapsto g \cdot p$ yields a local diffeomorphism from (an open subset of) G into M which can be used to define a left-invariant G_3-connection on G. By construction, this connection is locally equivalent to the connection on M.

Now let us describe left-invariant G_3-structures on a Lie group G.

Proposition 2.5. *Let G be a four-dimensional Lie group with Lie algebra $\mathfrak{g}$. Then there is a one-to-one correspondence between G_3-structures on G which are invariant under the natural left-action of G, and the set of equivalence classes of linear isomorphisms $\{\imath : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3\}/\sim$, where $\imath \sim g \circ \imath$ for all $g \in G_3$.*

Proof. Fix a linear isomorphism $\imath : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$. For any $p \in G$ and $g \in G_3$, we let $\alpha_{(p,g)} : T_pG \xrightarrow{\sim} \mathcal{V}_3$ be the linear isomorphism $\alpha_{(p,g)} := g^{-1} \circ \imath \circ (\omega_G)_p$, where ω_G denotes the Maurer-Cartan form of G. Then $F := \{\alpha_{(p,g)} : T_pG \xrightarrow{\sim} \mathcal{V}_3 | g \in G_3, p \in G\} \subseteq \mathfrak{F}$ defines a left-invariant G_3-structure on G. Note that if we replace the isomorphism $\imath : \mathfrak{g} \mapsto \mathcal{V}_3$ by $g \circ \imath$ for any $g \in G_3$ then the G_3-structure remains unchanged.

Conversely, given a left-invariant G_3-structure $\pi : F \to G$, pick any $\imath \in \pi^{-1}(e)$ and regard it as an isomorphism $\imath : T_eG = \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$. It is left to the reader to verify that this establishes the desired one-to-one correspondence. □

Now suppose that we are given a left-invariant G_3-connection on G. We want to find explicit expressions for the tautological and the connection 1-forms.

Let $\pi : F \to G$ be the underlying G_3-structure and let $\imath : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$ be a corresponding isomorphism. The map $\alpha : G \times G_3 \to F$ given by $(p, g) \mapsto \alpha_{(p,g)}$ is clearly a diffeomorphism, and we will use α as a coordinate system of F. Pulling back the Maurer-Cartan form on $G \times G_3$ to F via α^{-1}, we obtain a natural $\mathfrak{g} \oplus \mathfrak{g}_3$-valued coframe on F. Using the isomorphism $\imath + \rho_3^{-1} : \mathfrak{g} \oplus \mathfrak{g}_3 \xrightarrow{\sim} \mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$, with ρ_3 from the previous section, we get a $\mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$-valued coframe on F which we denote by $\underline{\omega} + \underline{\varphi}$ where $\underline{\omega}$ and $\underline{\varphi}$ take values in $\mathcal{V}_3$ and $\mathcal{V}_2 \oplus \mathcal{V}_0$ resp. We also let $\underline{\varphi} = \underline{\phi} + \underline{\lambda}$ be the decomposition of $\underline{\varphi}$ into its components. In this notation the tautological 1-form ω on F is given by $\omega = g^{-1}\underline{\omega}$.

The connection 1-form $\varphi := \phi + \lambda$ on F takes values in $\mathcal{V}_2 \oplus \mathcal{V}_0$, and $\omega + \phi + \lambda$ yields another $\mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$-valued coframe on F. In fact, the left-invariance of the connection implies that there is a linear map $\varphi_0 : \mathcal{V}_3 \to \mathcal{V}_2 \oplus \mathcal{V}_0$ such that at a point $\alpha_{(p,g)} \in F$, we have the relations

$$\begin{aligned} \omega &= g^{-1}\underline{\omega} \\ \varphi &= \underline{\varphi} + g^{-1}(\varphi_0 \circ \underline{\omega}). \end{aligned} \tag{7}$$

We can decompose $\varphi_0 = \phi_0 + \lambda_0$ with $\phi_0 \in \mathcal{V}_3^* \otimes \mathcal{V}_2 \cong \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5$ and $\lambda_0 \in \mathcal{V}_3^* \otimes \mathcal{V}_0 \cong \mathcal{V}_3$. It follows that there is a polynomial $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$ such that if we let $\mathbf{r} = r^1 + r^3 + r^5 + s^3$ with $r^i \in \mathcal{V}_i$ and $s^3 \in \mathcal{V}_3$ then $\phi_0(v) = \langle r^1, v\rangle_1 + \langle r^3, v\rangle_2 + \langle r^5, v\rangle_3$ and $\lambda_0(v) = \langle s^3, v\rangle_3$ for all $v \in \mathcal{V}_3$.

Note that if we replace the isomorphism $\imath : \mathfrak{g} \to \mathcal{V}_3$ by $g \circ \imath$ for $g \in G_3$ then the connection will be given by the polynomial $g \cdot \mathbf{r}$.

Let us now compute the torsion of the connection. We have

$$\begin{aligned} T(\varphi) &= d\omega + \phi \wedge \omega \\ &= g^{-1}(d\underline{\omega} + \langle \phi_0 \circ \underline{\omega}, \underline{\omega}\rangle_2 + \langle \lambda_0 \circ \underline{\omega}, \underline{\omega}\rangle_0) \\ &= g^{-1}(d\underline{\omega} + \langle \langle r^1, \underline{\omega}\rangle_1 + \langle r^3, \underline{\omega}\rangle_2 + \langle r^5, \underline{\omega}\rangle_3, \underline{\omega}\rangle_2 + \langle\langle s^3, \underline{\omega}\rangle_3, \underline{\omega}\rangle_0). \end{aligned}$$

Thus, the connection is *torsion free* if and only if

$$d\underline{\omega} + \langle \langle r^1, \underline{\omega}\rangle_1 + \langle r^3, \underline{\omega}\rangle_2 + \langle r^5, \underline{\omega}\rangle_3, \underline{\omega}\rangle_2 + \langle\langle s^3, \underline{\omega}\rangle_3, \underline{\omega}\rangle_0 = 0. \tag{8}$$

Taking the exterior derivative of (8), a calculation yields

$$\frac{1}{12}\langle t^0, \beta\rangle_0 + \frac{1}{900}\langle t^2, \beta\rangle_2 + \frac{1}{540}\langle t^4, \beta\rangle_4 + \frac{1}{900}\langle t^6, \beta\rangle_6 = 0, \tag{9}$$

where $\beta = \langle \underline{\omega}, \langle \underline{\omega}, \underline{\omega}\rangle_1\rangle_2$ and

$$\begin{aligned} t^0 &= \langle r^3, s^3\rangle_3, \\ t^2 &= -90(r^1)^2 - 15\langle r^1, 9r^3 - s^3\rangle_1 - 3\langle r^3, 3r^3 + s^3\rangle_2 - 7\langle r^5, 9r^3 - s^3\rangle_3 \\ &\qquad + 26\langle r^5, r^5\rangle_4, \\ t^4 &= -9r^1(5r^3 - s^3) + 3\langle r^5, 5r^3 - s^3\rangle_2 + \langle r^3, s^3\rangle_1, \\ t^6 &= 60r^1r^5 - 3r^3(3r^3 + s^3) + 2\langle r^5, 9r^3 - s^3\rangle_1 - 9\langle r^5, r^5\rangle_2. \end{aligned} \tag{10}$$

If we define the map

$$\begin{aligned} \tau : \quad \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5 &\longrightarrow \mathcal{V}_0 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \\ \mathbf{r} &\longmapsto t^0 + t^2 + t^4 + t^6 \end{aligned}$$

with t^i as in (10) then it is easy to see that (9) is satisfied if and only if $\tau(\mathbf{r}) = 0$.

Conversely, given $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$ with $\tau(\mathbf{r}) = 0$, then (8) determines a Lie algebra structure on $\mathcal{V}_3$ and ω and φ defined as in (7) establish a left-invariant G_3-connection on the G_3-structure $\pi : G \times G_3 =: F \to G$.

Thus, we have the following

Proposition 2.6. *There is a one-to-one correspondence between left-invariant G_3-connections on connected, simply connected four-dimensional Lie groups and the set of $Gl(2,\mathbb{R})$-orbits of $\tau^{-1}(0)$.*

Of course, the condition that the Lie group be simply connected is only imposed to make this correspondence one-to-one.

Let us now compute the curvature of the connection determined by $\mathbf{r}$. The second structure equation and a calculation yields

$$\begin{aligned} R(\varphi) &= d\varphi + \varphi \wedge \varphi \\ &= g^{-1}(d(\phi_0 \circ \underline{\omega}) + \frac{1}{2}\langle \phi_0 \circ \underline{\omega}, \phi_0 \circ \underline{\omega}\rangle_1) \\ &= g^{-1}(\left\langle \underline{a}^4, \langle \underline{\omega}, \underline{\omega}\rangle_1\right\rangle_4 + \underline{a}^2 \langle \underline{\omega}, \underline{\omega}\rangle_3 \\ &\qquad - \frac{1}{12}\left\langle \underline{a}^2, \langle \underline{\omega}, \underline{\omega}\rangle_1\right\rangle_2 + \frac{1}{12}\left\langle \underline{a}^4, \langle \underline{\omega}, \underline{\omega}\rangle_1\right\rangle_3 + \underline{T}), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \underline{a}^2 &= \frac{1}{10}(40(r^1)^2 + 5\left\langle r^1, 11r^3 - s^3\right\rangle_1 - 4\left\langle r^3, 7r^3 - s^3\right\rangle_2 \\ &\qquad + \left\langle 21r^3 + s^3, r^5\right\rangle_3 - 8\left\langle r^5, r^5\right\rangle_4), \\ \underline{a}^4 &= \frac{1}{12}(-6r^1 s^3 + \left\langle r^3, s^3\right\rangle_1 + 2\left\langle r^5, s^3\right\rangle_2), \quad \text{and} \\ \underline{T} &= \frac{15}{4}t^0 \langle \underline{\omega}, \underline{\omega}\rangle_3 + \frac{1}{20}t^2 \langle \underline{\omega}, \underline{\omega}\rangle_3 + \frac{1}{144}\left\langle t^4, \langle \underline{\omega}, \underline{\omega}\rangle_1\right\rangle_3 - \frac{1}{180}\left\langle t^6, \langle \underline{\omega}, \underline{\omega}\rangle_1\right\rangle_4. \end{aligned}$$

Clearly, if $\tau(\mathbf{r}) = 0$ then $\underline{T} = 0$. Also, comparing (11) with (3) yields

$$a^i = g^{-1}\underline{a}^i \quad \text{for } i = 2, 4. \tag{12}$$

As we mentioned earlier, the holonomy of the connection is contained in $H_3 \subsetneq G_3$ if and only if $a^4 \equiv 0$. Therefore, we have as a consequence of Proposition 2.6.

Corollary 2.7. *Let $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$ such that $\tau(\mathbf{r}) = 0$ and $-6r^1 s^3 + \left\langle r^3, s^3\right\rangle_1 + 2\left\langle r^5, s^3\right\rangle_2 \neq 0$. Then the holonomy of the G_3-connection defined by $\mathbf{r}$ is not contained in H_3. In particular, the connection is not flat.*

Thus, in order to classify the homogeneous G_3-connections we have to classify the $Gl(2,\mathbb{R})$-orbits of $\mathbf{r}$ which satisfy the two conditions of Corollary 2.7. This can be done by a careful case-by-case investigation.

The necessary calculations (all of which were performed by `MATHEMATICA`) are not presented here. However, the author shall provide the interested reader with copies of the `MATHEMATICA` files used to compile this classification. The results are presented in the following section.

§3 Classification of Homogeneous G_3-structures.

In this section we will state the result of the classification of homogeneous G_3-connections whose holonomy is not contained in H_3. As it turns out, this implies that the holonomy equals all of G_3.

Suppose that for a given $\mathbf{r} = r^1 + r^3 + r^5 + s^3$ with $r^i \in \mathcal{V}_i$, $s^3 \in \mathcal{V}_3$ we have $\tau(\mathbf{r}) = 0$ and $a^4 \neq 0$.

There are two cases to be distinguished.

Case A: $\mathbf{r} = r^1 + r^3 + r^5 + s^3$ and $s^3 \neq 5r^3$.

In this case, the orbits of $\mathbf{r}$ can be parametrized as follows:

	r^3	r^5	s^3
$(A1)$	$-\frac{6}{5}uv^2$	$\frac{1}{10}v^3(3u^2+tv^2)$	$-v^2(6u+v)$
$(A2)$	$\frac{3}{5}u(7v^2 \mp u^2)$	$-\frac{3}{10}u^2v(v^2 \pm u^2)$	$-15u(v^2 \pm u^2)$
$(A3)$	$\frac{1}{5}u(v^2 \mp 17u^2)$	$-\frac{3}{10}u^2v(v^2 \mp u^2)$	$3u(7v^2 \pm u^2)$
$(A4)$	$\frac{1}{5}u(-u^2 \pm 2uv + v^2)$	$-\frac{1}{10}u^2v(u+v)(u+3v)$	$u(u^2+6uv+3v^2)$
$(A5)$	$\frac{1}{15}(5u^3 - 45u^2v + 90uv^2 - 54v^3)$	$\frac{1}{10}u(v-u)(2v-u)(3v-2u)(3v-u)$	$5u^3 - 21u^2v + 30uv^2 - 18v^3$

Here we assume in each case that $u, v \in \mathcal{V}_1$ is a basis with $\langle u, v\rangle_1 = 1$, and also that $r^1 = v$.

The Lie algebras of the symmetry groups can be represented as follows:

$$(A1) \qquad \mathfrak{g} = \begin{pmatrix} \pm a & 6a & 0 & b \\ -4ta & \pm a & 12a & c \\ 0 & -2ta & \pm a & d \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A2) \qquad \mathfrak{g} = \begin{pmatrix} 3a & b & c \\ & a & d \\ & & 0 \end{pmatrix}$$

$$(A3) \qquad \mathfrak{g} = \begin{pmatrix} 5a & b & c \\ & 2a & d \\ & & 0 \end{pmatrix}$$

$$(A4), (A5) \qquad \mathfrak{g} = \begin{pmatrix} * & 0 & * \\ & * & * \\ & & 0 \end{pmatrix}$$

Thus, in all these cases the symmetry groups are *solvable*.

Case B: $\mathbf{r} = r^1 + r^3 + r^5 + 5r^3$.

In this case, $t^0 = t^4 = 0$ follows immediately. The orbits of these $\mathbf{r}$ with $\tau(\mathbf{r}) = 0$ and $a^4 \neq 0$ can be parametrized as follows:

	r^3	r^5
$(B1)$	$\pm\frac{1}{5}u^3$	$\frac{1}{30}v(v^2 \pm u^2)(2v^2 \pm 3u^2)$
$(B2)$	$\frac{3}{10}u(\pm u^2 - v^2)$	$-\frac{1}{10}u^2v(\pm u^2 + 3v^2)$
$(B3)$	$-\frac{3}{5}u(2v^2 \pm u^2)$	$\frac{3}{10}u^2v(v^2 \mp u^2)$
$(B4)$	$-\frac{6}{5}uv^2$	$\frac{3}{10}u^2v^3$
$(B5)$	$-\frac{1}{5}u(49u^2 + 36uv + 6v^2)$	$\frac{1}{30}u^2(8u + 3v)(47u^2 + 24uv + 3v^2)$
$(B6)$	$\frac{3}{5}uv^2$	$\frac{1}{10}u^2v^3$
$(B7)$	$\frac{3}{5}u(v^2 \pm u^2)$	$\frac{1}{10}u^2v(v^2 \pm 3u^2)$
$(B8)$	$\frac{3}{5}uv^2$	$\frac{1}{10}u^2(u^3 + v^3)$
$(B9)$	$\frac{3}{5}u(2cu^2 + v^2)$	$\frac{1}{10}u^2(6u^3 + 6cu^2v + v^3)$
$(B10)$	$\frac{3}{10}u((3c-2)u^2 - 3uv - v^2)$	$-\frac{3}{10}u^2(u + v)(cu^2 + uv + v^2)$
$(B11)$	$\frac{3}{5}u(cu^2 - 6uv - 2v^2)$	$\frac{1}{10}u^2(4u + 3v)(8u^2 + cu^2 + 4uv + v^2)$
$(B12)$	$\frac{1}{10}v(-3u^2 + (3-c)v^2)$	$\frac{1}{60c}(-u^2 - 2uv + (c-1)v^2)$ $(u^3 + 3u^2v + 3(c+1)uv^2 + (1-3c)v^3)$
$(B13)$	$\frac{3}{20}u((5+c)u^2 + 6uv + v^2)$	$\frac{1}{40c}((c+1)u^2 + 2uv + v^2)$ $((3-c)u^3 + 3(3+c)u^2v + 9uv^2 + 3v^3)$
$(B14)$	$\frac{12}{55c^2}u((28c^2 \mp 3)u^2$ $+3c^2uv - c^2v^2)$	$\frac{\pm 1}{1100c^2}((80c^2 \mp 12)u^2 + 40c^2uv + 5c^2v^2)$ $(32(\pm 1 - 10c^2)u^3 + 30(\pm 1 - 8c^2)u^2v$ $-60c^2uv^2 - 5c^2v^3)$
$(B15)$	$\frac{1}{1350c_2c_4^2}u(c_1^2(600c_3^2 + 360c_2c_3^2$ $-25c_2^2c_4 - 15c_2^3c_4 + 18c_2^4c_4)u^2$ $-180c_1(10 + 3c_2)c_3c_4uv$ $+1350c_4^2v^2)$	$\frac{1}{24300c_1^2c_2^3c_4^4}(c_1^2(40c_3^2 + 5c_2^2c_4 + 6c_2^3c_4)u^2$ $-120c_1c_3c_4uv + 90c_4^2v^2)$ $(10c_1^3c_3(-8c_3^2 + 3c_2^2c_4)u^3$ $+9c_1^2c_4(40c_3^2 - 5c_2^2c_4 + 3c_2^3c_4)u^2v)$ $-540c_1c_3c_4^2uv^2 + 270c_4^3v^3$

Again, we assume in each case that $u, v \in \mathcal{V}_1$ is a basis with $\langle u, v\rangle_1 = 1$, and that $r^1 = v$.

The Lie algebras of the symmetry groups in each case are as follows:

$$(B1) \qquad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } \pm =' +' \\ \mathfrak{gl}(2,\mathbb{R}) & \text{if } \pm =' -' \end{cases}$$

$$(B2),(B3) \qquad \mathfrak{g} = \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix}$$

$$(B4) \qquad \mathfrak{g} = \begin{pmatrix} 0 & a & b & c \\ & 0 & d & b \\ & & 0 & -a \\ & & & 0 \end{pmatrix}$$

$$(B5) \qquad \mathfrak{g} = \mathfrak{u}(2)$$

$$(B6)-(B11) \qquad \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$$

$$(B12) \qquad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } c \neq 18 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } c = 18 \end{cases}$$

$$(B13) \qquad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } c \neq -\frac{8}{9} \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } c = -\frac{8}{9} \end{cases}$$

$$(B14) \qquad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } 1210c^2 \neq \pm 189 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } 1210c^2 = \pm 189 \end{cases}$$

$$
(B15) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } 648c_3^2 + (3c_2 - 2)(6c_2 + 5)^2 c_4 \neq 0 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \begin{array}{c}\text{if } 648c_3^2 + (3c_2 - 2)(6c_2 + 5)^2 c_4 = 0 \\ \text{and } 6c_2 + 5 \neq 0\end{array} \\ \begin{pmatrix} a+b & 0 & c \\ 0 & a-b & d \\ 0 & 0 & a \end{pmatrix} & \text{if } c_3 = 6c_2 + 5 = 0 \text{ and } c_4 > 0 \\ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & a \end{pmatrix} & \text{if } c_3 = 6c_2 + 5 = 0 \text{ and } c_4 < 0 \end{cases}
$$

From this we can conclude that the "moduli space" of homogeneous G_3-connections has one four-dimensional component $(B15)$, seven one-dimensional components $(A1)$ and $(B9) - (B14)$, and fourteen points, including the flat connection and the homogeneous H_3-connection.

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