SYLVESTER'S FORMULAS FOR EUCLIDEAN DIVISION AND SCHUR MULTIFUNCTIONS

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(01.04.2001)

Introduction and statement of the main result.

In the present paper we address the following problem of classical algebra. Let $P(x)$, $Q(x)$ be polynomials in one variable with coefficients in the field of complex numbers, for simplicity. Suppose that $m = \deg(P) \geqslant n = \deg(Q)$.

Let us look at the iterated division of P and Q :

$$P = *\, Q + c_1\mathcal{R}_1 \quad , \quad Q = *\, \mathcal{R}_1 + c_2\mathcal{R}_2 \quad , \quad \mathcal{R}_1 = *\, \mathcal{R}_2 + c_3\mathcal{R}_3 \quad , \quad \ldots$$

The successive coefficients $(*)$ are the unique polynomials such at $n > \deg \mathcal{R}_1 > \deg \mathcal{R}_2 > \deg \mathcal{R}_3 > \cdots$. Instead of the usual Euclidean division, where $c_1 = c_2 = c_3 = \cdots = 1$, we shall choose the constants c_i in such a way that the remainders are polynomials in the roots of P and Q, instead of being rational functions. We shall fix later normalizations (cf. Sect.2). We are interested in the problem of expressing $\mathcal{R}_1$, $\mathcal{R}_2$ ... as polynomials in the roots of P and Q, generalizing the expression of the resultant as a product of differences.

An interesting solution to this problem was proposed by Sylvester[1] about 150 years ago, who found "The successive residues, divested of their allotrious factors ..." that is, who also normalized the remainders in such a way as to obtain polynomials in the roots, the last "residue" being the resultant.

To state Sylvester's result, we need the following notation. For two finite sets $\mathbb{A}$ and $\mathbb{B}$ of elements in a commutative ring, we set $R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A}, b\in\mathbb{B}}(a-b)$. Let now $\mathbb{A} = \{a_1, \ldots , a_m\}$, $\mathbb{B} = \{b_1, \ldots , b_n\}$ be two sequences of (commuting) variables. Let $0 \leqslant p \leqslant m$, $0 \leqslant q \leqslant n$ be two integers. Define after Sylvester,

$$Sylv^{p,q} = Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x) := \sum_{\mathbb{A}'\subset\mathbb{A}} \sum_{\mathbb{B}'\subset\mathbb{B}} R(x, \mathbb{A}')R(x, \mathbb{B}')\frac{R(\mathbb{A}', \mathbb{B}')R(\mathbb{A} \smallsetminus \mathbb{A}', \mathbb{B} \smallsetminus \mathbb{B}')}{R(\mathbb{A}', \mathbb{A} \smallsetminus \mathbb{A}')R(\mathbb{B}', \mathbb{B} \smallsetminus \mathbb{B}')}$$

2000 *Mathematics Subject Classification.* 05E05 , 13B25 , 14M15.

The authors were partially supported by grant No.5031 of French-Polish cooperation C.N.R.S.-P.A.N.

[1]James Joseph Sylvester (1814-97) English-American analyst, number theorist, and geometer who also trained as an actuary and a barrister, and published poetry. He held chairs at John Hopkins University and Oxford, and founded the American Journal of Mathematics. Perhaps his most famous result is Sylvester's law of inertia for rank and signature of a quadratic form over the field of real numbers. Another often used results are Sylvester's relations for minors and his determinant for the resultant.

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the sum being over all subsets $\mathbb{A}'$ of cardinality p and $\mathbb{B}'$ of cardinality q. Observe that $Sylv^{0,0} = R(\mathbb{A}, \mathbb{B})$ and if $p = 0$ or $q = 0$ then the Sylvester summation reduces to a single summation. More interesting Sylvester's summations are given in Example 3.6 and 3.8. Summations of this type and their generalizations (cf. [LS2]) play nowadays an important role e.g. in representation theory.

The main result connecting the remainders $\mathcal{R}_i$ and Sylvester's summations is the following theorem.

Theorem 0.1. *For $p+q = 0, 1, \dots, n-1$, specializing $\mathbb{A}$ and $\mathbb{B}$ in the roots of the polynomials P and Q respectively, one has*

$$Sylv^{p,q} = (-1)^{(m-p-q)(n-q)} \binom{p+q}{p} \mathcal{R}_{n-(p+q)} \, .$$

This result was stated without complete proof by Sylvester in [S] and we did not find any in the literature. The objective of the present paper is to give a proof of Theorem 0.1 with the help of techniques of multivariate polynomials, involving Schur multifunctions and divided differences.

The paper is organized as follows.

In the first section we recall some information on Schur multifunctions.

In the second section we discuss the consecutive remainders in the Euclidean process of division of two polynomials, and, in particular, two determinantal expressions for these remainders, one of them in the form of a Schur multifunction.

In the third section we provide a proof of Theorem 0.1. This proof goes by induction on $\deg(Q)$ and makes use of a formula which expresses the first remainder $\mathcal{R}_1$ with the help of divided differences.

This algebro-combinatorial computation can be interpreted in terms of Gysin maps for Grassmann bundles; we discuss this in the fourth section.

Acknowledgement. The second author thanks the MPIM for excellent work conditions during the preparation of this paper. He hopes not to be the last reader of Sylvester's Complete Works in the MPIM library.

1. Symmetric polynomials and Schur multifunctions.

Given a formal series in z, $\sum_0^\infty z^j S_j$, with $S_0 = 1$ and with coefficients from an arbitrary commutative ring, one associates with the series an infinite matrix of their coefficients:

$$\mathbb{S} = [S_{k-h}]_{h,k \geqslant 1} \tag{1.1}$$

(one has $S_j = 0$ if $j < 0$). For any natural number p and any $J = (j_1, \dots, j_p) \in \mathbb{Z}^p$ one associates with the series the *Schur function* S_J indexed by J: by definition S_J is the minor of $\mathbb{S}$ taken over the p first lines and over the columns $j_1+1, j_2+2, \dots, j_p+p$ (S_J is zero if one of these integers is negative).

More explicitly,

$$S_J = |S_{j_k+k-h}|_{1 \leqslant h,k \leqslant p} \, . \tag{1.2}$$

We note that $S_{0,\dots,0} = 1$

Given a (finite) *alphabet* $\mathbb{A}$, i.e. an ordered set of elements in a commutative ring (often indeterminates), we denote by $S_J(\mathbb{A})$ the Schur functions associated with the rational series

$$\prod_{a\in\mathbb{A}} (1-za)^{-1} .$$

This notion is extended to *virtual alphabets*[2] as follows. Given two alphabets $\mathbb{A}, \mathbb{B}$, we denote by $S_J(\mathbb{A}-\mathbb{B})$ the Schur functions associated with the rational series

$$\prod_{b\in\mathbb{B}} (1-zb) / \prod_{a\in\mathbb{A}} (1-za) .$$

The notation $\mathbb{A}-\mathbb{B}$ is compatible with the multiplication of series:

$$\sum z^i S_i(\mathbb{A}-\mathbb{B}) \cdot \sum z^j S_j(\mathbb{A}'-\mathbb{B}') = \sum z^i S_i\big((\mathbb{A}+\mathbb{A}')-(\mathbb{B}+\mathbb{B}')\big) , \tag{1.3}$$

the sum $\mathbb{A}+\mathbb{A}'$ denoting the union of two multisets $\mathbb{A}$ and $\mathbb{A}'$.

CONVENTION: We will often identify an alphabet $\mathbb{A} = \{a_1, \dots, a_m\}$ with the sum $a_1 + \cdots + a_m$ and perform usual algebraic operations on such elements. We will give priority to the algebraic notation over the set-theoretic one.

We have $(\mathbb{A}+\mathbb{C}) - (\mathbb{B}+\mathbb{C}) = \mathbb{A}+\mathbb{B}$, and this corresponds to simplification of the common factor for the rational series

$$\sum z^i S_i((\mathbb{A}+\mathbb{C}) - (\mathbb{B}+\mathbb{C})) = \sum z^i S_i(\mathbb{A}-\mathbb{B}) .$$

The cardinality of $\mathbb{B}$ being n, we can write the polynomial $\prod_{b\in\mathbb{B}}(x-b)$ in the compact form $S_n(x-\mathbb{B})$; after the previous definition for $\mathbb{A}=\{x\}$ one has

$$S_n(x-\mathbb{B}) = x^n + x^{n-1}S_1(-\mathbb{B}) + \cdots + x^0 S_n(-\mathbb{B}) . \tag{1.4}$$

Moreover, one has for $j>0$,

$$S_{n+j}(-\mathbb{B}) = 0 \quad \text{and} \quad S_{n+j}(x-\mathbb{B}) = x^j S_n(x-\mathbb{B}) . \tag{1.5}$$

The inverse of the rational series $\sum z^i S_i(\mathbb{A}-\mathbb{B})$ is the series $\sum z^i S_i(\mathbb{B}-\mathbb{A})$; the inversion of series correspond to the inversion of the matrices, and for their minors which are Schur functions, to the transposition of the partitions indexing them.

Given p virtual alphabets $\mathbb{A}^{(1)}, \dots, \mathbb{A}^{(p)}$, and a sequence $J \in \mathbb{Z}^p$, one associates with them the *Schur multifunction*

$$S_J\big(\mathbb{A}^{(1)}, \dots, \mathbb{A}^{(p)}\big) = |S_{j_k+k-h}(\mathbb{A}^{(k)})|_{1\leqslant h,k\leqslant p} . \tag{1.6}$$

[2]By a virtual alphabet, we understand a formal finite $\mathbb{Z}$-linear combination of alphabets. The precise sense of the operations "+" and "−" for alphabets, is explained in this section. Of course, using these operations, any virtual alphabet has the form $\mathbb{A}-\mathbb{B}$, for some alphabets $\mathbb{A}$ and $\mathbb{B}$.

Numerous applications of Schur multifunctions are discussed in [LS1].
NOTATION: We will group the parts of the indexing partition, corresponding to the same argument. So, if $i_p = i_{p+1} = \ldots = i_q = i$ are parts of I that are equal to i, and that correspond to the argument $\mathbb{A}$, where $\mathbb{A}$ is an alphabet, then we will write

$$S_{\ldots;i^{q-p};\ldots}(\ldots;\mathbb{A};\ldots)\,.$$

For example, the Schur multifunction $S_{1,1,1,1,2,2,4,4,4,4,5}$ associated with a sequence of virtual alphabets

$$\mathbb{A}^{(1)},\mathbb{A}^{(2)},\mathbb{A}^{(2)},\mathbb{A}^{(3)},\mathbb{A}^{(3)},\mathbb{A}^{(3)},\mathbb{A}^{(4)},\mathbb{A}^{(4)},\mathbb{A}^{(4)},\mathbb{A}^{(5)},\mathbb{A}^{(5)}$$

is denoted by

$$S_{1;1^2;1;2^2;4^3;4;5}(\mathbb{A}^{(1)};\mathbb{A}^{(2)};\mathbb{A}^{(3)};\mathbb{A}^{(4)};\mathbb{A}^{(5)};\mathbb{A}^{(5)})\,.$$

Using the relation

$$S_i(\mathbb{A}-x) = S_i(\mathbb{A}) - xS_{i-1}(\mathbb{A})\,,$$

where $\mathbb{A}$ is a virtual alphabet and x is a single variable, and performing elementary operations on the rows (or columns) of a determinant of the form (1.6), we arrive at the following transformation lemma:

Lemma 1.7. *Let $q \leqslant p$ be a positive integer and $\mathbb{X}$ an alphabet of cardinality $r \leqslant q$.*
(i) The replacement of $S_i(\mathbb{A}^{(k)})$ by $S_i(\mathbb{A}^{(k)}-\mathbb{X})$ in the first $q-r$ rows of a sequence of q consecutive rows of the determinant (1.6), does not change the value of the determinant.
(ii) Suppose that there exists k such that $k+q \leqslant p$, $j_{k+1} = j_{k+2} = \ldots = j_{k+q}$, and $\mathbb{A}^{(k+1)} = \ldots \mathbb{A}^{(k+q)} = \mathbb{A}$. Then the replacement $S_i(\mathbb{A})$ by $S_i(\mathbb{A}-\mathbb{X})$ in the $(k+q)$-th, $(k+q-1)$-th ,...,$(k-r+1)$-th column of the determinant (1.6), does not change the value of the determinant.

Example 1.8. *For virtual alphabets $\mathbb{A}$, $\mathbb{B}$, ... , and an alphabet $\mathbb{X}$ of cardinality 2, the following equalities hold:*

$$\begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}) & S_8(\mathbb{B}) & S_4(\mathbb{C}) & S_3(\mathbb{D}) & S_7(\mathbb{E}) \\ S_4(\mathbb{A}) & S_7(\mathbb{B}) & S_3(\mathbb{C}) & S_2(\mathbb{D}) & S_6(\mathbb{E}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix} = \begin{vmatrix} S_6(\mathbb{A}) & S_9(\mathbb{B}) & S_5(\mathbb{C}) & S_4(\mathbb{D}) & S_8(\mathbb{E}) \\ S_5(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{B}-\mathbb{X}) & S_4(\mathbb{C}-\mathbb{X}) & S_3(\mathbb{D}-\mathbb{X}) & S_7(\mathbb{E}-\mathbb{X}) \\ S_4(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{B}-\mathbb{X}) & S_3(\mathbb{C}-\mathbb{X}) & S_2(\mathbb{D}-\mathbb{X}) & S_6(\mathbb{E}-\mathbb{X}) \\ S_3(\mathbb{A}) & S_6(\mathbb{B}) & S_2(\mathbb{C}) & S_1(\mathbb{D}) & S_5(\mathbb{E}) \\ S_2(\mathbb{A}) & S_5(\mathbb{B}) & S_1(\mathbb{C}) & 1 & S_4(\mathbb{E}) \\ S_1(\mathbb{A}) & S_4(\mathbb{B}) & 1 & 0 & S_3(\mathbb{E}) \end{vmatrix}$$

and

$$\begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_9(\mathbb{A}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_2(\mathbb{C}) \end{vmatrix} = \begin{vmatrix} S_8(\mathbb{B}) & S_6(\mathbb{A}) & S_7(\mathbb{A}) & S_8(\mathbb{A}-\mathbb{X}) & S_9(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{C}) \\ S_7(\mathbb{B}) & S_5(\mathbb{A}) & S_6(\mathbb{A}) & S_7(\mathbb{A}-\mathbb{X}) & S_8(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{C}) \\ S_6(\mathbb{B}) & S_4(\mathbb{A}) & S_5(\mathbb{A}) & S_6(\mathbb{A}-\mathbb{X}) & S_7(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{C}) \\ S_5(\mathbb{B}) & S_3(\mathbb{A}) & S_4(\mathbb{A}) & S_5(\mathbb{A}-\mathbb{X}) & S_6(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{C}) \\ S_4(\mathbb{B}) & S_2(\mathbb{A}) & S_3(\mathbb{A}) & S_4(\mathbb{A}-\mathbb{X}) & S_5(\mathbb{A}-\mathbb{X}) & S_3(\mathbb{C}) \\ S_3(\mathbb{B}) & S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_3(\mathbb{A}-\mathbb{X}) & S_4(\mathbb{A}-\mathbb{X}) & S_2(\mathbb{C}) \end{vmatrix}.$$

Let now $\mathbb{A}$ and $\mathbb{B}$ be two alphabets. The element $R(\mathbb{A},\mathbb{B}) = \prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$ is the resultant of the polynomials with roots $\mathbb{A}$ and $\mathbb{B}$. Given an indeterminate x, the polynomial $S_m(x-\mathbb{A})$ can be denoted unambiguously $R(x,\mathbb{A})$. In general, we have the following expression:

$$R(\mathbb{A},\mathbb{B}) = S_{n^m}(\mathbb{A}-\mathbb{B})\,. \tag{1.9}$$

See [L1] for more about resultants.

In particular, we record the following expansion of $R(\mathbb{A},\mathbb{B})$ as a combination of products of Schur functions in $\mathbb{A}$ and $\mathbb{B}$:

$$R(\mathbb{A},\mathbb{B}) = \sum_{I\subset n^m} S_I(\mathbb{A})S_{I^*}(-\mathbb{B})\,, \tag{1.10}$$

where the sum is over partitions $I=(i_1,\dots,i_m)$ in n^m and I^* stands for the partition $(n-i_m,\dots,n-i_1)$.

Lemma 1.11. *For nonnegative integers d and r, and virtual alphabets $\mathbb{A}$ and $\mathbb{B}$, the following equality holds:*

$$S_{1^d;p^r}(\mathbb{A};\mathbb{B}) = \sum_{i=0}^{d}(-1)^i S_{1^{d-i}}(\mathbb{A})S_{p^{r-1},p+i}(\mathbb{B})\,.$$

Proof. The Laplace expansion of the determinant $S_{1^d;p^r}(\mathbb{A};\mathbb{B})$ along the first d columns gives a sum of products of Schur functions of $\mathbb{A}$ and $\mathbb{B}$ respectively. One checks that their indices are as stated. □

For example, consider the determinant of $S_{1,1;2,2,2}(\mathbb{A};\mathbb{B})$:

$$\begin{vmatrix} S_1(\mathbb{A}) & S_2(\mathbb{A}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) & S_6(\mathbb{B}) \\ 1 & S_1(\mathbb{A}) & S_3(\mathbb{B}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) \\ 0 & 1 & S_2(\mathbb{B}) & S_3(\mathbb{B}) & S_4(\mathbb{B}) \\ 0 & 0 & S_1(\mathbb{B}) & S_2(\mathbb{B}) & S_3(\mathbb{B}) \\ 0 & 0 & 1 & S_1(\mathbb{B}) & S_2(\mathbb{B}) \end{vmatrix}\,.$$

A term in the expansion is

$$-\begin{vmatrix} S_1(\mathbb{A}) & S_2(\mathbb{A}) \\ 0 & 1 \end{vmatrix}\;\begin{vmatrix} S_3(\mathbb{B}) & S_4(\mathbb{B}) & S_5(\mathbb{B}) \\ S_1(\mathbb{B}) & S_2(\mathbb{B}) & S_3(\mathbb{B}) \\ 1 & S_1(\mathbb{B}) & S_2(\mathbb{B}) \end{vmatrix}$$

in which one recognizes $-S_1(\mathbb{A})\,S_{223}(\mathbb{B})$, up to transpositions along the anti-diagonals.

2. Consecutive remainders.

Let $P = P(x) = S_m(x-\mathbb{A}) = R(x,\mathbb{A})$, $Q = Q(x) = S_n(x-\mathbb{B}) = R(x,\mathbb{B})$ be two polynomials in one variable (with respective roots $\mathbb{A}$ and $\mathbb{B}$). As usual we assume that $m \geqslant n$. Algebraic operations on these two polynomials preserve the symmetry in the roots of each of the two polynomials, and thus produce symmetric functions in $\mathbb{A}$ and $\mathbb{B}$ separately.

We want the remainder of degree zero to be the resultant,

$$\mathcal{R}^{(0)} = R(\mathbb{B},\mathbb{A}) = S_{m^n}(\mathbb{B}-\mathbb{A})\,;$$

thus we shall normalize the successive remainders by asking that $\mathcal{R}^{(d)}$ (the remainder of degree d) has the leading term

$$x^d S_{(m-d)^{n-d}}(\mathbb{B}-\mathbb{A})\,. \tag{2.1}$$

Theorem 0.1 holds true in this case trivially. More generally, suppose that the polynomials P and Q have a greatest common divisor ("GCD" for brevity). This is, for instance, the case when the ground ring is a unique factorization domain. The GCD of two polynomials $S_m(x-\mathbb{A})$ and $S_n(x-\mathbb{B})$ is of degree d iff the Schur functions

$$S_{m^n}(\mathbb{B}-\mathbb{A}),\quad \ldots\quad, S_{(m-d+1)^{n-d+1}}(\mathbb{B}-\mathbb{A})$$

vanish and $S_{(m-d)^{n-d}}(\mathbb{B}-\mathbb{A}) \neq 0$. (See [L1] and [P1] for more details.) In this case $GCD = \mathcal{R}^{(d)}$ is given by explicit determinants (2.4) and (2.7) below.

Euler found that the first remainder $\mathcal{R}_1$ can be characterized by the fact that it is the unique polynomial (up to normalization) of degree less than or equal to $n-1$ in the space

$$\mathrm{Lin}(P,Q,xQ,\ldots,x^{m-n}Q) = \mathrm{Lin}\big(S_m(x-\mathbb{A}), S_n(x-\mathbb{B}), S_{n+1}(x-\mathbb{B}),\ldots,S_m(x-\mathbb{B})\big)\,. \tag{2.2}$$

More generally, the remainder of order d is the unique polynomial (still up to normalization) of degree less than or equal to d in the space

$$\begin{aligned}&\mathrm{Lin}(P,xP,\ldots,x^{n-d-1}P,Q,xQ,\ldots,x^{m-d-1}Q)\\ &= \mathrm{Lin}\big(S_m(x-\mathbb{A}),\ldots,S_{m+n-d-1}(x-\mathbb{A}),S_n(x-\mathbb{B}),\ldots,S_{m+n-d-1}(x-\mathbb{B})\big)\,.\end{aligned} \tag{2.3}$$

Consider the determinant:

$$\mathcal{D}^{(d)} = \begin{vmatrix} S_0(-\mathbb{A}) & \ldots & S_{m+n-2d-2}(-\mathbb{A}) & S_{m+n-d-1}(x-\mathbb{A}) \\ \vdots & & \vdots & \vdots \\ S_{d-n+1}(-\mathbb{A}) & \ldots & S_{m-d-1}(-\mathbb{A}) & S_m(x-\mathbb{A}) \\ S_0(-\mathbb{B}) & \ldots & S_{m+n-2d-2}(-\mathbb{B}) & S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots & & \vdots & \vdots \\ S_{d-m+1}(-\mathbb{B}) & \ldots & S_{n-d-1}(-\mathbb{B}) & S_n(x-\mathbb{B}) \end{vmatrix}\,. \tag{2.4}$$

Euler's point of view gives us the following:

Proposition 2.5. *For $d = 0, 1, \dots, n$, we have $\mathcal{R}^{(d)} = (-1)^{(m-d)(n-d)} \mathcal{D}^{(d)}$.*

Proof. First observe that $\mathcal{D}^{(d)}$ is a linear combination of the elements in its last column, and therefore belongs to the space (2.3). But we can eliminate powers of x strictly greater than d, subtracting to the last column a linear combination of the preceding ones, because by (1.4), for all $0 \leqslant p \leqslant m+n-d-1$, we have

$$S_p(x-\mathbb{A}) = x^{m+n-d-1} S_{p-m-n+d+1}(-\mathbb{A}) + \cdots + x^{d+1} S_{p-d-1}(-\mathbb{A}) + x^d S_{p-d}(-\mathbb{A}) + \cdots + x^0 S_p(-\mathbb{A})\,.$$

We conclude by checking the leading term of $\mathcal{D}^{(d)}$, which is

$$x^d S_{(n-d)^{m-d}}(-\mathbb{B}+\mathbb{A}) = (-1)^{(m-d)(n-d)} x^d S_{(m-d)^{n-d}}(\mathbb{B}-\mathbb{A})\,. \quad \square$$

We record now the following alternative formula for the remainder of order d, expressing it as a Schur multifunction:

Proposition 2.6. *One has*

$$\mathcal{R}^{(d)} = (-1)^d S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x; \mathbb{B}-\mathbb{A})\,.$$

We give a proof by specialization, using some simple transformations of determinants (cf. [L2], where this formula was originally stated, for a connection with the work of Wroński[3].)

Proof. It suffices to show that the n specializations $x = b \in \mathbb{B}$ of $\mathcal{R}^{(d)}$ and $\mathcal{D}^{(d)}$ coincide because these two polynomials have degree $< n$.

We recall that $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x; \mathbb{B}-\mathbb{A})$ is the determinant:

$$\text{(2.7)}\qquad \begin{vmatrix} S_1(\mathbb{B}-x) & \dots & S_d(\mathbb{B}-x) & S_m(\mathbb{B}-\mathbb{A}) & \dots & S_{m+n-d-1}(\mathbb{B}-\mathbb{A}) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{2-n}(\mathbb{B}-x) & \dots & S_{d-n+1}(\mathbb{B}-x) & S_{m-n+1}(\mathbb{B}-\mathbb{A}) & \dots & S_{m-d}(\mathbb{B}-\mathbb{A}) \end{vmatrix}.$$

We specialize the determinant (2.7). First, using the transformation Lemma 1.7(i), we subtract $\mathbb{B}-b$ in the top row. We perform operations on alphabets only, and therefore will write only them. So, starting from

$$\begin{vmatrix} \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix},$$

[3]Józef Maria Hoene-Wroński (1778-1853) Polish-French analyst, combinatorialist, physicist, and universal philosopher. His most famous contribution to mathematics is the Wronskian; Wroński considered even more general functional determinants, at a time when the theory of determinants was still in infancy, and pioneered what is called today "algorithmic mathematics". For a glimpse of some of his many-sided achievements, cf. [L3].

(where $\mathbb{B}-b$ appears in the first d columns), we see that the previous determinant is equal to the following one:

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & \dots & b-\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \dots & \mathbb{B}-\mathbb{A} \end{vmatrix},$$

(where the zeros appear in the first d columns). Now, using transformation Lemma 1.7(ii), we subtract b in the last $m-d+1$ columns; we see that the previous determinant becomes

$$\begin{vmatrix} 0 & \dots & 0 & b-\mathbb{A} & -\mathbb{A} & \dots & -\mathbb{A} \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbb{B}-b & \dots & \mathbb{B}-b & \mathbb{B}-\mathbb{A} & \mathbb{B}-b-\mathbb{A} & \dots & \mathbb{B}-b-\mathbb{A} \end{vmatrix}.$$

This last determinant clearly factorizes, because in the first row, the elements:

$$S_{m+1}(-\mathbb{A}), \dots, S_{m+r-1}(-\mathbb{A})$$

are null. It is equal to

$$(-1)^d R(b,\mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

Similarly, we specialize the determinant (2.4)

$$\mathcal{D}^{(d)} = \begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & x-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & x-\mathbb{B} \end{vmatrix},$$

in $x = b \in \mathbb{B}$. We obtain

$$\begin{vmatrix} -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

The bottom part of the last column:

$$\begin{matrix} S_{m+n-d-1}(x-\mathbb{B}) \\ \vdots \\ S_n(x-\mathbb{B}) \end{matrix}$$

is null, and one can write 0 instead of $b-\mathbb{B}$.

Now, use the transformation Lemma 1.7(i) and subtract b in the first $n-d-2$ rows; we see that the previous determinant is equal to the following one:

$$\begin{vmatrix} -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ \vdots & & \vdots & \vdots \\ -\mathbb{A}-b & \dots & -\mathbb{A}-b & -\mathbb{A} \\ -\mathbb{A} & \dots & -\mathbb{A} & b-\mathbb{A} \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \\ \vdots & & \vdots & \vdots \\ -\mathbb{B} & \dots & -\mathbb{B} & 0 \end{vmatrix}.$$

Taking finally the Laplace expansion of the last determinant w.r.t. its last column, we see that this determinant is equal to

$$(-1)^{(m-d)(n-d)} R(b,\mathbb{A}) S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A}).$$

(Here we also use formula (1.10) giving the expansion of $S_{(m-d)^{n-d-1}}(\mathbb{B}-b-\mathbb{A})$ as a combination of the products of Schur functions in $-\mathbb{B}$ and $-\mathbb{A}-b$.) $\square$

3. Proof of the main theorem.

Let $\mathbb{B}=\{b_1,\dots,b_n\}$ be a set of indeterminates. Divided differences, due to Newton [N], are linear operators ∂_i defined by

$$\partial_i : \mathbb{Z}[\mathbb{B}] \to \mathbb{Z}[\mathbb{B}] \qquad \text{(of degree } -1)$$

defined by

$$\partial_i(f) = (f-\tau_i f)/(b_i - b_{i+1}) \qquad i=1,\dots,n-1, \tag{3.1}$$

where τ_i denotes the i-th simple transposition exchanging i with $i+1$. More generally, to every reduced decomposition $\sigma=\tau_{i_1}\cdots\tau_{i_k}$ one associates the composition $\partial_{i_1}\cdots\partial_{i_k}$ which is independent of the choice of the reduced decomposition of σ (because the ∂_i's satisfy the braid relations), and thus can be denoted ∂_σ.

NOTATION: Given an alphabet $\mathbb{B}=\{b_1,b_2,\dots\}$, we write $\mathbb{B}_i := \{b_1,\dots,b_i\}$.

Lemma 3.2. *For every $p \in \mathbb{Z}$, and a virtual alphabet $\mathbb{A}$ which is independent*[4] *of $\mathbb{B}$, one has*

$$\partial_i S_p(\mathbb{B}_i - \mathbb{A}) = S_{p-1}(\mathbb{B}_{i+1} - \mathbb{A}) .$$

Proof. Introducing an extra variable z, and writing $\mathbb{B}_i - \mathbb{A} = -b_{i+1} + (\mathbb{B}_{i+1} - \mathbb{A})$, one factorizes $\sum_p z^p S_p(\mathbb{B}_i - \mathbb{A})$ into fg, with

$$f = 1 - zb_{i+1} \quad \textit{and} \quad g = \sum_p z^p S_p(\mathbb{B}_{i+1} - \mathbb{A}) \ .$$

The second series is symmetrical in b_i, b_{i+1}, therefore commutes with ∂_i. The image of $1 - zb_{i+1}$ under ∂_i being z, the assertion follows. $\square$

The first remainder $\mathcal{R}_1$ can be conveniently presented through divided differences (the same operator expresses the remainder in Lagrange interpolation of a function in one variable).

Lemma 3.3. *The first remainder $\mathcal{R}_1$ of the division of $P(x) = S_m(x - \mathbb{A})$ by $Q(x) = S_n(x - \mathbb{B})$ equals*

$$\mathcal{R}_1 = \partial_{n-1} \cdots \partial_1 \big(P(b_1) R(x, \mathbb{B} - b_1) \big) .$$

Proof. We have

$$R(x, \mathbb{B} - b_1) = (x - b_2) \cdots (x - b_n) = S_{n-1}(x - (\mathbb{B} - b_1)) = (-1)^{n-1} S_{1^{n-1}}((\mathbb{B} - x) - b_1) ,$$

and by the transformation Lemma 1.7(i), the product $P(b_1)R(x, \mathbb{B} - b_1)$ equals

$$(-1)^{n-1} S_{1^{n-1};m}(\mathbb{B} - x; b_1 - \mathbb{A}) .$$

Thus, using Lemma 3.2, we see that the image of $P(b_1)R(x, \mathbb{B} - b_1)$ under $\partial_{n-1} \cdots \partial_1$ is

$$(-1)^{n-1} S_{1^{n-1};m-n+1}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}) = \mathcal{R}_1 . \quad \square$$

CONVENTION: In the following, by the "top coefficient" of a polynomial in x, we shall mean the coefficient of the highest power of x.

The following lemma will be used in the proof several times:

Lemma 3.4. *(i) The specialization $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; a)$, $a \in \mathbb{A}$, is equal to*

$$(-1)^p R(a, \mathbb{B}) \ c ,$$

where c is the top coefficient of $Sylv^{p,q}(\mathbb{A} - a, \mathbb{B}; x)$.

(ii) The specialization $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; b)$, $b \in \mathbb{B}$, is equal to

$$(-1)^{m-p-q} R(b, \mathbb{A}) \ c ,$$

[4]It suffices that τ_i preserves $\mathbb{A}$.

where c is the top coefficient of $Sylv^{p,q}(\mathbb{A}, \mathbb{B}-b; x)$.

Proof. We consider, for example, (ii). We have

$$
\begin{aligned}
Sylv^{p,q}(b) &= \sum_{\mathbb{A}'\subset\mathbb{A}, \mathbb{B}'\subset\mathbb{B}-b} R(b,\mathbb{A}')R(b,\mathbb{B}')\frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}-\mathbb{A}',\mathbb{B}-\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')R(\mathbb{B}',\mathbb{B}-\mathbb{B}')} \\
&= (-1)^{m-p-q}R(b,\mathbb{A}) \sum_{\mathbb{A}'\subset\mathbb{A}, \mathbb{B}'\subset\mathbb{B}-b} \frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}-\mathbb{A}',\mathbb{B}-b-\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')R(\mathbb{B}',\mathbb{B}-b-\mathbb{B}')}\,,
\end{aligned}
$$

as claimed. □

We need also the following proposition concerning the limit case when $p+q=n$.

Proposition 3.5. *(i) Suppose that $p+q=n<m$. Then we have*

$$Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x) = (-1)^{(m-n)p}\binom{p+q}{p}R(x,\mathbb{B})\,.$$

(ii) Suppose that $p+q=n=m$. Then we have

$$Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x) = \binom{p+q-1}{q}R(x,\mathbb{A}) + \binom{p+q-1}{p}R(x,\mathbb{B})\,.$$

Proof. Both assertions depend on the four-tuple $\big(m=\mathrm{card}(\mathbb{A}), n=\mathrm{card}(\mathbb{B}), p, q\big)$. We prove (i) and (ii) simultaneously using induction on $m+n$.

i) Let $m>n$. By Lemma 3.4, the specialization of x in a, $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; a)$, is equal, modulo sign, to $R(a,\mathbb{B})$ times the top coefficient of $Sylv^{p,q}(\mathbb{A}-a, \mathbb{B}; x)$.

If $m-1>n$, then by (i) for $(m-1,n,p,q)$, it is equal to $\binom{p+q}{p}R(a,\mathbb{B})$ and these m specializations determine the polynomial.

If $m-1=n$, then according to ii) for (n,n,p,q), the top coefficient of $Sylv^{p,q}(\mathbb{A}-a, \mathbb{B}; x)$ is equal to $\binom{n-1}{p}+\binom{n-1}{q}=\binom{n}{p}$, and therefore $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x)$ coincides with $\binom{n}{p}R(x,\mathbb{B})$.

ii) Let $m=n=p+q$. By Lemma 3.4, the specialization of x in a, $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; a)$, is equal, modulo sign, to $R(a,\mathbb{B})$ times the top coefficient of $Sylv^{p,q}(\mathbb{A}-a, \mathbb{B}; x)$. Note that this last coefficient is equal to the top coefficient of $Sylv^{q-1,p}(\mathbb{B}, \mathbb{A}-a; x)$. Then by (ii) for $(n,n-1,q-1,p)$, the considered specialization is $\binom{n-1}{p}R(a,\mathbb{B})$.

Similarly, the specialization of x in b, $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; b)$, is equal, modulo sign, to $R(b,\mathbb{A})$ times the top coefficient of $Sylv^{p,q}(\mathbb{A}, \mathbb{B}-b; x)$. Note that this last coefficient is equal to the top coefficient of $Sylv^{q,p-1}(\mathbb{A}, \mathbb{B}-b; x)$. Then by (ii) for $(n,n-1,q,p-1)$ the considered specialization is $\binom{n-1}{q}R(b,\mathbb{A})$.

The polynomial $Sylv^{p,q}(\mathbb{A}, \mathbb{B}; x)$ is determined by these specializations. □

Example 3.6. *(i) For $m=3$, $n=2$, and $p=q=1$,*

$$\frac{(a_1-b_1)(a_2-b_2)(a_3-b_2)}{(a_1-a_2)(a_1-a_3)(b_1-b_2)}(x-a_1)(x-b_1)+\frac{(a_1-b_2)(a_2-b_1)(a_3-b_1)}{(a_1-a_2)(a_1-a_3)(b_2-b_1)}(x-a_1)(x-b_2)$$

$$+\frac{(a_2-b_1)(a_1-b_2)(a_3-b_2)}{(a_2-a_1)(a_2-a_3)(b_1-b_2)}(x-a_2)(x-b_1)+\frac{(a_2-b_2)(a_1-b_1)(a_3-b_1)}{(a_2-a_1)(a_2-a_3)(b_2-b_1)}(x-a_2)(x-b_2)$$

$$+\frac{(a_3-b_1)(a_1-b_2)(a_2-b_2)}{(a_3-a_1)(a_3-a_2)(b_1-b_2)}(x-a_3)(x-b_1)+\frac{(a_3-b_2)(a_1-b_1)(a_2-b_1)}{(a_3-a_1)(a_3-a_2)(b_2-b_1)}(x-a_3)(x-b_2)$$

is equal to $-2(x-b_1)(x-b_2)$.

(ii) For $m=n=3$, $p=2$, and $q=1$,

$$\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_2)(x-b_1)$$

$$+\frac{(a_1-b_1)(a_3-b_1)(a_2-b_2)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_1-b_2)(b_1-b_3)}(x-a_1)(x-a_3)(x-b_1)$$

$$+\frac{(a_2-b_1)(a_3-b_1)(a_1-b_2)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_1-b_2)(b_1-b_3)}(x-a_2)(x-a_3)(x-b_1)$$

$$+\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_3-b_3)}{(a_1-a_3)(a_2-a_3)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_2)(x-b_2)$$

$$+\frac{(a_1-b_2)(a_3-b_2)(a_2-b_1)(a_2-b_3)}{(a_1-a_2)(a_3-a_2)(b_2-b_1)(b_2-b_3)}(x-a_1)(x-a_3)(x-b_2)$$

$$+\frac{(a_2-b_2)(a_3-b_2)(a_1-b_1)(a_1-b_3)}{(a_2-a_1)(a_3-a_1)(b_2-b_1)(b_2-b_3)}(x-a_2)(x-a_3)(x-b_2)$$

$$+\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_3-b_2)}{(a_1-a_3)(a_2-a_3)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_2)(x-b_3)$$

$$+\frac{(a_1-b_3)(a_3-b_3)(a_2-b_1)(a_2-b_2)}{(a_1-a_2)(a_3-a_2)(b_3-b_1)(b_3-b_2)}(x-a_1)(x-a_3)(x-b_3)$$

$$+\frac{(a_2-b_3)(a_3-b_3)(a_1-b_1)(a_1-b_2)}{(a_2-a_1)(a_3-a_1)(b_3-b_1)(b_3-b_2)}(x-a_2)(x-a_3)(x-b_3)$$

is equal to $2(x-a_1)(x-a_2)(x-a_3)+(x-b_1)(x-b_2)(x-b_3)$.

We now can pass to the proof of Theorem 0.1. Write $d:=p+q$. By Proposition 2.5, the assertion of the theorem is equivalent to the identity:

$$Sylv^{p,q}(\mathbb{A},\mathbb{B};x)=(-1)^{(m-d)(n-q)+d}\binom{d}{p}S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A}) \tag{3.7}$$

if $d<n$.

Observe that $Sylv^{p,q}$, as a polynomial in x, is of degree $d<n$. Hence it coincides with its first remainder modulo $R(x,\mathbb{B})$, and one can use Lemma 3.3 to obtain it starting from $Sylv^{p,q}(b_1)$:

$$Sylv^{p,q}(x)=\partial_{n-1}\cdots\partial_1\big(Sylv^{p,q}(b_1)R(x,\mathbb{B}-b_1)\big)\,.$$

By Lemma 3.4, the specialization of $Sylv^{p,q}(\mathbb{A},\mathbb{B};x)$ in $x=b_1$ is equal to $(-1)^{m-d}R(b_1,\mathbb{A})$ times the top coefficient in $Sylv^{p,q}(\mathbb{A},\mathbb{B}-b_1;x)$. If $d<n-1$ then we can apply the induction assumption, and we see that this coefficient is equal to

$$(-1)^{(m-d)(n-1-q)}\binom{d}{p}S_{\square}(\mathbb{B}-b_1-\mathbb{A})\,,$$

with $\square=(m-d)^{n-d-1}$. If $d=n-1$ then the coefficient is given by the same formula by Proposition 3.5.

We are left with computation

$$\partial_{n-1}\cdots\partial_1\big(R(x,\mathbb{B}-b_1)R(b_1,\mathbb{A})S_{\square}(\mathbb{B}-b_1-\mathbb{A})\big)\,.$$

We rewrite

$$\begin{aligned}&R(x,\mathbb{B}-b_1)=S_{n-1}(x-(\mathbb{B}-b_1))\\&=(-1)^{n-1}S_{1^{n-1}}((\mathbb{B}-x)-b_1)=(-1)^{n-1}\big(\sum_i(-1)^iS_{1^{n-i-1}}(\mathbb{B}-x)b_1^i\big)\,.\end{aligned}$$

We have by (1.5)

$$b_1^iR(b_1,\mathbb{A})=S_{m+i}(b_1-\mathbb{A})\,,$$

and by the transformation Lemma 1.7(i),

$$S_{m+i}(b_1-\mathbb{A})S_{\square}(\mathbb{B}-b_1-\mathbb{A})=S_{\square;m+i}(\mathbb{B}-\mathbb{A};b_1-\mathbb{A})\,.$$

Finally, we get with $\epsilon=(-1)^{(m-d)(n-q)+n-1}$,

$$Sylv^{p,q}(b_1)R(x,\mathbb{B}-b_1)=\epsilon\binom{d}{p}\sum_{i=0}^{n-1}(-1)^iS_{1^{n-1-i}}(\mathbb{B}-x)S_{\square;m+i}(\mathbb{B}-\mathbb{A};b_1-\mathbb{A})\,.$$

Its image under $\partial_{n-1}\ldots\partial_1$ is equal to

$$\epsilon\binom{d}{p}\sum_{i=0}^{n-1}(-1)^iS_{1^{n-1-i}}(\mathbb{B}-x)S_{\square;m+i-(n-1)}(\mathbb{B}-\mathbb{A})\,.$$

In this last summation, the terms for $i=0,\ldots,n-d-2$ disapear because they correspond to determinants having two identical columns. In the remaining sum for $i=n-d-1,\ldots,n-1$:

$$\sum_{j=0}^{d}(-1)^jS_{1^{d-j}}(\mathbb{B}-x)S_{\square;m+j}(\mathbb{B}-\mathbb{A})\,,$$

one recognizes the Laplace expansion of the determinant $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$ along the first d columns (cf. Lemma (1.11)).

This ends the proof of the theorem. $\square$

We illustrate the theorem by the following example.

Example 3.8. *Let $m=5, n=4$. Then the Sylvester sum $Sylv^{2,1}(\mathbb{A},\mathbb{B};x)$:*

$$
\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_1-b_2)(b_1-b_3)(b_1-b_4)}\\
&\times(x-a_1)(x-a_2)(x-b_1)\\
+&\frac{(a_1-b_2)(a_2-b_2)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_3)(a_4-b_3)(a_5-b_3)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_2-b_1)(b_2-b_3)(b_2-b_4)}\\
&\times(x-a_1)(x-a_2)(x-b_2)\\
+&\frac{(a_1-b_3)(a_2-b_3)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_3-b_1)(b_3-b_2)(b_3-b_4)}\\
&\times(x-a_1)(x-a_2)(x-b_3)\\
+&\frac{(a_1-b_4)(a_2-b_4)(a_3-b_1)(a_4-b_1)(a_5-b_1)(a_3-b_2)(a_4-b_2)(a_5-b_2)(a_3-b_4)(a_4-b_4)(a_5-b_4)}{(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_2-a_3)(a_2-a_4)(a_2-a_5)(b_4-b_1)(b_4-b_2)(b_4-b_3)}\\
&\times(x-a_1)(x-a_2)(x-b_4)\\
+&\,9\times 4\ \textit{similar terms obtained by replacing}\ \{a_1,a_2\}\ \textit{by}\ \{a_i,a_j\}
\end{aligned}
$$

is equal to $3\mathcal{R}^{(3)}$ for the division of $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ by $(x-b_1)(x-b_2)(x-b_3)(x-b_4)$.

We will now sketch another possible method for getting the equality (3.7), using only divided differences, and not specializations. It is based upon the following two lemmas:

Lemma 3.9. *Let $i \leqslant n-1$ be a nonnegative integer and let $\mathbb{C}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}) \leqslant i$. *Then for a nonnegative integer j and a virtual alphabet $\mathbb{D}$,*

$$\partial_{n-1}\cdots\partial_1\big(S_i(b_1-\mathbb{C})S_j(b_1+\mathbb{D})\big) = S_{i+j-n+1}(\mathbb{B}-\mathbb{C}+\mathbb{D})\,,$$

provided $\mathbb{C}$ and $\mathbb{D}$ are independent of $\mathbb{B}$.

Proof. We have

$$S_i(b_1-\mathbb{C})S_j(b_1+\mathbb{D}) = \sum_{k,l} b_1^{k+l} S_{i-k}(-\mathbb{C})S_{j-l}(\mathbb{D})\,.$$

This element is sent via $\partial_{n-1}\cdots\partial_1$ to

$$\sum_k S_{i-k}(-\mathbb{C})S_{j+k-n+1}(\mathbb{B}+\mathbb{D})$$

because $k \leqslant n-1$. Since $S_k(-\mathbb{C})=0$ for $k \geqslant i$, this last sum is the expression of $S_{i+j-n+1}(\mathbb{B}-\mathbb{C}+\mathbb{D})$ and the assertion of the lemma follows. □

NOTATION: Given an alphabet $\mathbb{B}=\{b_1,b_2,\dots\}$, we write $\mathbb{B}^i := \{b_i, b_{i+1},\dots\}\,.$

Lemma 3.10. *Let i_0 be a nonnegative integer and let $\mathbb{C}^{(0)}$ be an alphabet such that* $\mathrm{card}(\mathbb{C}^{(0)}) \leqslant i_0$. *Then for integers $i_1, \dots, i_{n-1}$, and virtual alphabets $\mathbb{C}^{(1)}, \dots, \mathbb{C}^{(n-1)}$,*

$$\begin{aligned}\partial_{n-1}\cdots\partial_1\big(S_{i_0}(b_1-\mathbb{C}^{(0)})S_{i_1,\dots,i_{n-1}}(\mathbb{B}^2-\mathbb{C}^{(1)},\dots,\mathbb{B}^2-\mathbb{C}^{(n-1)})\big)\\ =(-1)^{n-1}S_{i_0,i_1-1,\dots,i_{n-1}-1}(\mathbb{B}-\mathbb{C}^{(0)},\mathbb{B}-\mathbb{C}^{(1)},\dots,\mathbb{B}-\mathbb{C}^{(n-1)}),\end{aligned}$$

provided $\mathbb{C}^{(0)}, \dots, \mathbb{C}^{(n-1)}$ are independent of $\mathbb{B}$.

Proof. Using the transformation Lemma 1.7(i), we have the factorization

$$\begin{aligned}&S_{i_0}(b_1-\mathbb{C}^{(0)})S_{i_1,\dots,i_{n-1}}(\mathbb{B}^2-\mathbb{C}^{(1)},\dots,\mathbb{B}^2-\mathbb{C}^{(n-1)})\\ &=S_{i_1,\dots,i_{n-1},i_0}(\mathbb{B}-\mathbb{C}^{(1)},\dots,\mathbb{B}-\mathbb{C}^{(n-1)},b_1-\mathbb{C}^{(0)}).\end{aligned}$$

The image of this element via via $\partial_{n-1}\cdots\partial_1$ is

$$\begin{aligned}&S_{i_1,\dots,i_{n-1},i_0-n+1}(\mathbb{B}-\mathbb{C}^{(1)},\dots,\mathbb{B}-\mathbb{C}^{(n-1)},\mathbb{B}-\mathbb{C}^{(0)})\\ &=(-1)^{n-1}S_{i_0,i_1-1,\dots,i_{n-1}-1}(\mathbb{B}-\mathbb{C}^{(0)},\mathbb{B}-\mathbb{C}^{(1)},\dots,\mathbb{B}-\mathbb{C}^{(n-1)}).\end{aligned}$$

The lemma is proved. □

NOTATION: We denote $\partial_{\mathbb{B}} := \partial_{[n,n-1,\dots,2,1]}$, and for two complementary subsequences $\mathbb{B}'$ and $\mathbb{B}$" of $\mathbb{B}$, we write $\partial_{\mathbb{B}"\mathbb{B}'} := \partial_\sigma$ where the permutation σ reorders the sequence $\mathbb{B}"\mathbb{B}'$ to $\mathbb{B}$. For example, $\partial_{n-1}\cdots\partial_1$ is denoted by $\partial_{\mathbb{B}^2 b_1}$.

In this computation, we fix $\mathbb{A}' := \mathbb{A}_p$, $\mathbb{A}" := \mathbb{A}^{p+1}$, $\mathbb{B}' := \mathbb{B}_q$, and $\mathbb{B}" := \mathbb{B}^{q+1}$. (Here, $a_1, \dots, a_m$ are variables.)

Recall (cf., e.g., [P2, §3]) that the divided difference operator $\partial_{\mathbb{A}"\mathbb{A}'}$ has the following interpretation as a symmetrizing operator occuring in Sylvester's summations. For a polynomial $P(a_1, \dots, a_m)$,

$$\partial_{\mathbb{A}"\mathbb{A}'}(P)=\frac{1}{p!(m-p)!}\sum_\sigma\Big(\frac{P}{R(\mathbb{A}_p,\mathbb{A}^{p+1})}\Big)^\sigma,$$

the sum over permutations σ of $\{1, \dots, m\}$. Hence the equality (3.7) is equivalent to

$$\begin{aligned}&\partial_{\mathbb{A}"\mathbb{A}'}\partial_{\mathbb{B}"\mathbb{B}'}[R(x,\mathbb{A}'+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")]\\ &=(-1)^{(m-p-q)(n-q)+p+q}\binom{p+q}{p}S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A}).\end{aligned}\tag{3.11}$$

NOTATION: For an alphabet $\mathbb{A} = \{a_1, \dots, a_m\}$ we write $\Delta(\mathbb{A}) := \prod_{i<j\leqslant m}(a_i - a_j)$, and for an auxillary alphabet $\mathbb{B}$, we set $\Delta(\mathbb{A}+\mathbb{B}) := \Delta(\mathbb{A})\Delta(\mathbb{B})R(\mathbb{A},\mathbb{B})$.

Since

$$\partial_{\mathbb{A}'}\partial_{\mathbb{B}'}\big((1/p!)(1/q!)\Delta(x+\mathbb{A}')(x+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\big)=R(x,\mathbb{A}'+\mathbb{B}')R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}",\mathbb{B}")$$

the equation (3.11) is equivalent to
(3.12)
$$\nabla\big(\Delta(x+\mathbb{A}'+\mathbb{B}')R(\mathbb{A}",\mathbb{B}")\big) = (-1)^{(m-p-q)(n-q)+p+q}(p+q)!S_{1^{p+q};(m-p-q)^{n-p-q}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,,$$

where

$$\nabla := \partial_{\mathbb{B}^2\,b_1}\cdots\partial_{\mathbb{B}^{q-p+1}\,b_{q-p}}\partial_{\mathbb{A}^2\,a_1}\partial_{\mathbb{B}^{q-p+2}\,b_{q-p+1}}\cdots\partial_{\mathbb{A}^p\,a_{p-1}}\partial_{\mathbb{B}^q\,b_{q-1}}\partial_{A"\,a_p}\partial_{B"\,b_q}\,.$$

(Without loss of generality we can assume that $p \leqslant q$.)

We will now show (3.12) for $m = \mathrm{card}(\mathbb{A}) = 11, n = \mathrm{card}(\mathbb{B}) = 9$, $p = 2$, and $q = 3$. We will disregard the signs, for simplicity. We want to show

$$\partial_{\mathbb{B}^2\,b_1}\partial_{\mathbb{A}^2\,a_1}\partial_{\mathbb{B}^3\,b_2}\partial_{\mathbb{A}^3\,a_2}\partial_{\mathbb{B}^4\,b_3}\big(\Delta(x+\mathbb{A}_2+\mathbb{B}_3)R(\mathbb{A}^3,\mathbb{B}^4)\big) = -5!S_{1^5;6^4}(\mathbb{B}-x,\mathbb{B}-\mathbb{A})\,.$$

We have

$$\begin{aligned}&\Delta(\mathbb{B}_3+\mathbb{A}_2+x)R(\mathbb{B}^4,\mathbb{A}^3) = \Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_5(b_3-(\mathbb{B}_2+\mathbb{A}_2+x))S_{9^6}(\mathbb{B}^4-\mathbb{A}^3)\\ &= \Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_{9^6;5}(\mathbb{B}^3-\mathbb{A}^3;b_3-(\mathbb{B}_2+\mathbb{A}_2+x))\end{aligned}$$

(the last equality follows from the transformation Lemma 1.7(i)). By Lemma 3.10, this element goes via $\partial_{\mathbb{B}^4\,b_3}$ to

$$\pm\Delta(\mathbb{B}_2+\mathbb{A}_2+x)S_{5;8^6}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3)\,.$$

We explain the effect of applying $\partial_{A^3\,a_2}$ to this last element. By the transformation Lemma 1.7(ii), we have

$$S_{5;8^6}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3) = S_{5;8;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_2+x);\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2)\,,$$

and by a well-known linearity formula this last element is

$$S_{5;8;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x)-a_2;\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2) = S_{5;8;8^5}-a_2S_{4;8;8^5}$$

the Schur functions in $(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^3;\mathbb{B}^3-\mathbb{A}^2)$. Factorizing Δ and using (1.5), to compute is the image under $\partial_{\mathbb{A}^3\,a_2}$ of

$$\Delta(\mathbb{B}_2+\mathbb{A}_1+x)[S_4(a_2-(\mathbb{B}_2+\mathbb{A}_1+x))S_{5;8^6}-S_5(a_2-(\mathbb{B}_2+\mathbb{A}_1+x))S_{4;8^6}]\,.$$

the Schur functions in $(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2+a_2;\mathbb{B}^3-\mathbb{A}^2)$. Using Lemma 3.9, the first summand in the bracket [ ] is sent to

$$S_{5,3;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2) = -S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)\,.$$

The second summand in the bracket [ ] is sent to $-S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)$. In sum, the initial element goes via $\partial_{\mathbb{A}^3\,a_2}\partial_{\mathbb{B}^4\,b_3}$ to

$$\pm1\cdot2\Delta(\mathbb{B}_2+\mathbb{A}_1+x)S_{4^2;8^5}(\mathbb{B}^3-(\mathbb{B}_2+\mathbb{A}_1+x);\mathbb{B}^3-\mathbb{A}^2)\,.$$

We explain now the effect of applying $\partial_{\mathbb{B}^3 b_2}$ to the last element. Using consecutively the transformation Lemma 1.7(ii), the linearity formula, and (1.5), we rewrite this element as

$$\Delta(\mathbb{B}_1 + \mathbb{A}_1 + x)[S_3\ S_{4,4;8;8^4} - S_4\ S_{3,4;8;8^4} + S_5\ S_{3,3;8;8^4}]\,,$$

the complete functions S_i in $(b_2 - (\mathbb{B}_1 + \mathbb{A}_1 + x))$ and the remaining Schur functions in $(\mathbb{B}^3 - (\mathbb{B}_1 + \mathbb{A}_1 + x); \mathbb{B}^3 - \mathbb{A}^2; \mathbb{B}^2 - \mathbb{A}^2)$. Now, using 3 times Lemma 3.9, we see that this last element goes via $\partial_{B^3 b_2}$ to

$$\pm 3\Delta(\mathbb{B}_1 + \mathbb{A}_1 + x) S_{3^3;7^5}(\mathbb{B}^2 - (\mathbb{B}^1 + \mathbb{A}^1 + x); \mathbb{B}^2 - \mathbb{A}^2)\,.$$

Similarly, we have

$$\partial_{\mathbb{A}^2 a_1}[\Delta(\mathbb{B}_1 + \mathbb{A}_1 + x) S_{3^3;7^5}(\mathbb{B}^2 - (\mathbb{B}^1 + \mathbb{A}^1 + x); \mathbb{B}^2 - \mathbb{A}^2)] = \pm 4\Delta(\mathbb{B}_1 + x) S_{2^4;7^5}(\mathbb{B}^2 - (\mathbb{B}^1 + x); \mathbb{B}^2 - \mathbb{A}^1)$$

and

$$\partial_{B^2 b_1}(\Delta(\mathbb{B}_1 + x) S_{2^4;7^5}(\mathbb{B}^2 - (\mathbb{B}^1 + x); \mathbb{B}^2 - \mathbb{A}^1) = \pm 5 S_{1^4;5^4}(\mathbb{B}^1 - x; \mathbb{B}^1 - \mathbb{A}^1)\,.$$

In sum, we get

$$\partial_{\mathbb{B}_2 b_1}\partial_{\mathbb{A}_2 a_1}\partial_{\mathbb{B}_3 b_2}\partial_{\mathbb{A}_3 a_2}\partial_{\mathbb{B}_4 b_3}\Delta(\mathbb{B}_3 + \mathbb{A}_2 + x) R(\mathbb{B}^4, \mathbb{A}^3) = -(2+3)!\, S_{1^5;6^4}(\mathbb{B} - x; \mathbb{B} - \mathbb{A})\,,$$

as wanted in (3.12).

4. Gysin maps for Grassmann bundles.

Let X be a smooth variety, and let E and F be vector bundles on X of ranks m and n respectively. For integers $0 \leqslant p \leqslant m$ and $0 \leqslant q \leqslant n$, let

$$\pi_E : Gr^p(E) \to X \quad \text{and} \quad \pi_F : Gr^q(F) \to X$$

be two Grassmann bundles on X parametrizing rank p quotients of E and rank q quotients of F. These Grassmannians are equipped with the two tautological exact sequences:

$$0 \to R_E \to E \to Q_E \to 0 \ \text{ and } \ 0 \to R_F \to F \to Q_F \to 0\,,$$

where $\mathrm{rank}(Q_E) = p$ and $\mathrm{rank}(Q_F) = q$. (We use the same notation for vector bundles and their pullbacks.) We form a product over X:

$$\pi := \pi_E \times_X \pi_F : Gr^p(E) \times_X Gr^q(F) \to X\,.$$

Suppose now that $E^{(1)}, \ldots, E^{(p)}$ are p virtual bundles.[5] Given a sequence $J \in \mathbb{Z}^p$, one defines the Schur multifunction

$$S_J\big(E^{(1)}, \ldots, E^{(p)}\big) := |S_{j_k+k-h}(\mathbb{A}^{(k)})|_{1 \leqslant h,k \leqslant p}\,.$$

[5] As is common, by a virtual bundle we understand a finite $\mathbb{Z}$-linear combination of bundle classes in the Grothendieck group.

where $\mathbb{A}^{(1)}, \dots, \mathbb{A}^{(p)}$ are the virtual alphabets of Chern roots associated with $E^{(1)}, \dots, E^{(p)}$.
NOTATION: For a vector bundle E, we write $c_{\mathrm{top}}E$ for the Chern class $c_m(E)$, where $m = \mathrm{rank}(E)$.

The next result will give a cohomological interpretation of the equality (3.11). Recall (cf., e.g., [P2, §3]) that the operator $\partial_{\mathbb{A}''\mathbb{A}'}$ appearing in (3.11) induces the Gysin map $(\pi_E)_*$. More precisely, for a polynomial $P(a_1, \dots, a_m)$ symmetric in the first p variables and in the last $m-p$ variabes separately, one has

$$(\pi_E)_*(P(x_1, \dots, x_m) = \partial_{\mathbb{A}''\mathbb{A}'}\big(P(a_1, \dots, a_m)\big)|_{a_1=x_1, \dots, a_m=x_m} ,$$

where $x_1, \dots, x_p$ are the Chern roots of Q_E, and $x_{p+1}, \dots, x_m$ are the ones of R_E. Using this and the splitting principle, the equality (3.11) is translated into the following formula about the Gysin map $\pi_* : H(Gr^p(E) \times_X Gr^q(F)) \to H(X)$:

Proposition 4.1. *Let L be a line bundle on X. Then, in the above notation,*

$$\pi_*\Big(c_{\mathrm{top}}\,\mathrm{Hom}(Q_E, L)\ c_{\mathrm{top}}\,\mathrm{Hom}(Q_F, L)\ c_{\mathrm{top}}\,\mathrm{Hom}(Q_F, Q_E)\ c_{\mathrm{top}}\,\mathrm{Hom}(R_F, R_E)\Big)$$
$$= (-1)^{(m-p-q)(n-q)+p+q}\binom{p+q}{p} S_{1^{p+q};(m-p-q)^{n-p-q}}(F-L; F-E)\,.$$

In particular, we get the following expression for a Schur polynomial of the form $S_{(m-r)^{n-r}}(F-E)$:

Corollary 4.2. *In the above notation,*

$$\pi_*\Big(c_{\mathrm{top}}\,\mathrm{Hom}(Q_F, Q_E)\ c_{\mathrm{top}}\,\mathrm{Hom}(R_F, R_E)\Big) = (-1)^{(m-p-q)(n-q)}\binom{p+q}{p}S_{(m-p-q)^{n-p-q}}(F-E)\,.$$

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