Roots of 3-manifolds and cobordisms

C. Hog-Angeloni[1], S. Matveev[2]

1 Introduction

Given a set of *simplifying moves* on 3-manifolds, we apply them to a given 3-manifold M as long as possible. What we get is a *root* of M. For us, it makes sense to consider three types of moves: compressions along 2-spheres, proper discs and proper annuli having boundary circles in different components of ∂M. Our main result is that for the above moves the root of any 3-manifold exists and is unique. The same result remains true if instead of manifolds we apply the moves to 3-cobordisms of the type $(M, \partial_- M, \partial_+ M)$. The only difference between moves on manifolds and moves on cobordisms is that one boundary circle of every annulus participating in a compression of a cobordism must lie in $\partial_- M$ while the other in $\partial_+ M$. We can also restrict ourselves to considering compressions along only spheres or only spheres and discs. The existence and uniqueness in the first case is well-known and essentially comprise the content of the Milnor theorem on unique decomposition of a 3-manifold into a connected sum. For the second case our result is close to theorems about *characteristic compression bodies* and about *cores* of irreducible manifolds, presented by F. Bonahon [1] and S. Matveev [4], respectively.

We use Kneser existence [3], but perhaps the proof of the uniqueness part is easier with the method we are developing.

We point out that considering roots of cobordisms was motivated by the paper [2] of R. Gadgil, which is interesting although the proof of his main theorem seems to be incomplete.

We thank C. Gordon, W. Metzler, E. Pervova, C. Petronio, and S. Zentner for helpful discussions.

The final version of the paper has been written by the second author during his stay at MPIM Bonn. He thanks the institute for hospitality and financial support.

[1]Partially supported by the INTAS Project "CalcoMet-GT" 03-51-3662

[2]Partially supported by the INTAS Project "CalcoMet-GT" 03-51-3662 and the RFBR grant 05-01-0293-a

2 Moves, roots, and complexity

We introduce several moves on 3-manifolds. In this paper all 3-manifolds are assumed to be orientable.

1. *S-move (compression along a 2-sphere).* Let S be a 2-sphere in a 3-manifold M. Then we cut M along S and fill the two spheres arising in this way with two 3-balls.
2. *D-move (compression along a disc).* Let D be a proper disc in M. Then we cut M along D.
3. *A-move (compression along an annulus).* Let A be an annulus in M such that its boundary circles lie in different components of ∂M. Then we cut M along A and attach two plates $D_1^2 \times I, D_2^2 \times I$ by identifying their base annuli $\partial D_1^2 \times I, \partial D_2^2 \times I$ with the two copies of A, which appear under cutting.

Definition 1. *Let F be a proper surface in a 3-manifold M such that F is either a sphere, or a disc, or an annulus. Then F is called* essential*, if one of the following holds:*

1. *F is a sphere not bounding a ball;*
2. *F is a disc such that the circle ∂D is nontrivial in ∂M;*
3. *F is an incompressible annulus having boundary circles in different components of ∂M.*

If F is essential, then the corresponding F-move (i.e. the compression of M along F) is also called essential.

Remark 1. *Later on under essential surface we will understand either an essential sphere, or an essential disc, or an essential annulus. The condition that the boundary circles of any essential annulus A must lie in different components of ∂M guarantees us that A is boundary incompressible.*

2.1 Roots and complexity

Definition 2. *Let M be a 3-manifold. Then a 3-manifold N is called a* root *of M, if*

1. *N can be obtained from M by essential compressions along spheres, discs, and annuli.*

2. N admits no further essential compressions.

Theorem 1. *For any compact 3-manifold M the root of M exists and is unique up to homeomorphism and removing disjoint 3-spheres and balls.*

We postpone the proof of the theorem to Section 3. Note that the condition on boundary circles of compression annuli to lie in different components of ∂M is essential. Below we present an example of a 3-manifold M with two incompressible boundary incompressible annuli $A, B \subset M$ such that ∂M is connected and compressions of M along A and along B lead us to two different 3-manifolds admitting no further essential compression, i.e. to two different "roots".

Example. *Let Q be the complement space of the figure eight knot. We assume that the torus ∂Q is equipped with a coordinate system such that the slope of the meridian is (1,0). Choose two pairs (p,q), (m,n) of coprime integers such that $|q|, |n| \geq 2$ and $|p| \neq |m|$. Let a and b be corresponding curves in ∂Q. Then the manifolds $Q_{p,q}$ and $Q_{m,n}$ obtained by Dehn filling of Q are not homeomorphic. By [Th], they are hyperbolic.*

Consider the thick torus $X = S^1 \times S^1 \times I$ and locate its exterior meridian $\mu = S^1 \times \{*\} \times \{1\}$ and interior longitude $\lambda = \{*\} \times S^1 \times \{0\}$. Then we attach to X two copies Q', Q'' of Q as follows. The first copy Q' is attached to X by identifying an annular regular neighborhood $N(a)$ of a in ∂Q with an annular regular neighborhood $N(\mu)$ of μ in ∂X. The second copy Q'' is attached by identifying $N(b)$ with $N(\lambda)$. Denote by M the resulting manifold $Q' \cup X \cup Q''$.

Since Q is hyperbolic, M contains only two incompressible boundary incompressible annuli A and B, where A is the common image of $N(a)$ and $N(\mu)$, and B is the common image of $N(b)$ and $N(\lambda)$. It is easy to see that compression of M along A gives us a disjoint union of a punctured $Q'_{p,q}$ and a punctured Q'' while the compression along B leaves us with a punctured Q' and a punctured $Q''_{m,n}$. After filling the punctures (by compressions along spheres surrounding them), we get two different manifolds, homeomorphic to $Q_{p,q} \cup Q$ and $Q_{m,n} \cup Q$. Since their connected components (i.e. $Q_{p,q}, Q_{m,n}, Q$) are hyperbolic, they are irreducible, boundary irreducible and contain no essential annuli. Hence $Q_{p,q} \cup Q$ and $Q_{m,n} \cup Q$ are different roots of M.

Let M be a compact 3-manifold. Let us apply to it essential

S-moves as long as possible. It follows from Kneser finiteness theorem [3] that the number of possible moves is bounded by a constant depending on M only. Denote by $s(M)$ the maximum of these numbers taken over all chains of essential S-moves.

The following notion will be the main inductive parameter in our proofs.

Definition 3. *Let M be a 3-manifold. Then the complexity $\mathbf{c}(M)$ of M is the pair $(g^{(2)}(\partial M), s(M))$, where $g^{(2)}(M) = \sum_i g^2(F_i)$, $g(F_i)$ is the genus of a component $F_i \subset \partial M$, and the sum is taken over all components of ∂M. The pairs are considered in lexicographical order.*

The use of complexity as an inductive parameter is justified by the following fact.

Lemma 1. *Each essential S, D, A-move strictly decreases $\mathbf{c}(M)$.*

Proof. If an essential D-move cuts ∂M along a nonseparating curve ℓ on some component F of ∂M, then it strictly decreases $g(F)$ and hence $c(M)$. If the move turns F into two components F', F'', then $g(F) = g(F') + g(F'')$ and, since ℓ is nontrivial and thus $g(F')$, $g(F'') \neq 0$, we have $g^2(F) > g^2(F') + g^2(F'')$. This implies that $c(M)$ is decreased again. The case of the A-move is similar.

As follows from the definition of $s(M)$, each essential S-move strictly decreases $s(M)$. The boundary of M remains the same, hence so does $g^{(2)}(M)$. ∎

Remark 2. *It is easy to show that inessential S- and D-moves preserve the complexity. However, an inessential A-move can increase it, but only at the expense of $s(M)$ ($g^{(2)}(M)$ cannot increase). For example, if an annulus A cuts off a $D^2 \times I$ from M, then the corresponding move results in the appearing of an additional component of the type $S^2 \times I$.*

2.2 Equivalence of essential surfaces

Throughout this section, surface means sphere or disc or annulus.

Definition 4. *Let M be a 3-manifold and F, G be two essential surfaces in M. Then F, G are* equivalent *(we write $F \sim G$) if there exists a finite sequence of essential surfaces $X_1, X_2, \ldots, X_n$ such that the following holds:*

1. $F = X_1$ *and* $X_n = G$*;*

2. For each $i, 1 \leq i < n$*, the surfaces* X_i *and* X_{i+1} *are disjoint.*

Lemma 2. *Let* M *be a 3-manifold not homeomorphic to* $S^1 \times S^1 \times I$*. Then any two essential surfaces in* M *are equivalent.*

Proof. Let F, G be two essential surfaces in M in general position. Then the number of curves (circles and arcs) in the intersection of F and G will be denoted by $\#(F \cap G)$. Arguing by induction, we may assume that any two essential surfaces F, G with $\#(F \cap G) < n$ are equivalent. The base of the induction is evident: if $\#(F \cap G) = 0$, then $F \sim G$ by definition. Let F, G be two essential surfaces such that $\#(F \cap G) = n$.

Case 1. Suppose that $F \cap G$ contains a circle s which is trivial in F. By an innermost circle argument we may assume that s bounds a disc $D \subset F$ such that $D \cap G = s$. Compressing G along D, we get a two-component surface G' such that one component is a sphere, the other is homeomorphic to G, and $\#(F \cap G') = n - 1$. Since G is an interior connected sum of the components of G', at least one of them (denote it by X) is essential and thus $F \sim X$ by the inductive assumption. On the other hand, X can be shifted away from G by a small isotopy. It follows that $X \sim G$ and thus $F \sim G$.

Case 2. Suppose that $F \cap G$ does not contain trivial circles, but contains an arc a which is trivial in F. By an outermost arc argument we may assume that a cuts off a disc $D \subset F$ from F such that $D \cap G = a$. Compressing G along D, we get a two-component surfaces G' such that one component is a proper disc, the other is homeomorphic to G, and $\#(F \cap G') = n - 1$. Since G is an interior boundary connected sum of the components of G', at least one of them (denote it by X) is essential and thus equivalent to F by the inductive assumption. On the other hand, X can be shifted away from G by a small isotopy. It follows that $G \sim X$ and thus $F \sim G$.

Case 3. Suppose that F and G are annuli such that $F \cap G$ consists of circles parallel to the core circles of F and G. Then one can find two different components A, B of ∂M such that a circle of ∂F is in A and a circle of ∂G is in B. Denote by s the first circle of $F \cap G$ we meet at our radial way along F from the circle $\partial F \cap A$ to the other boundary circle of F. Let F' be the subannulus of F bounded by $\partial F \cap A$ and s, and G' the subannulus of G bounded by s and $\partial G \cap B$. Then the annulus $F' \cup G'$ is essential and is isotopic to an

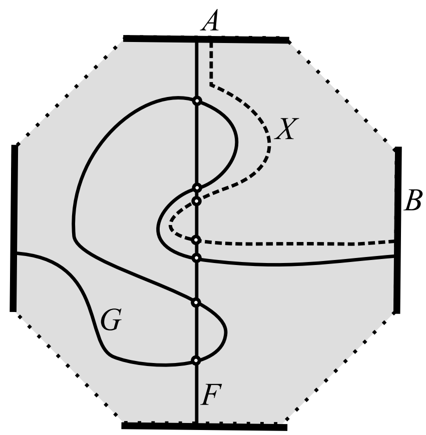

Figure 1:

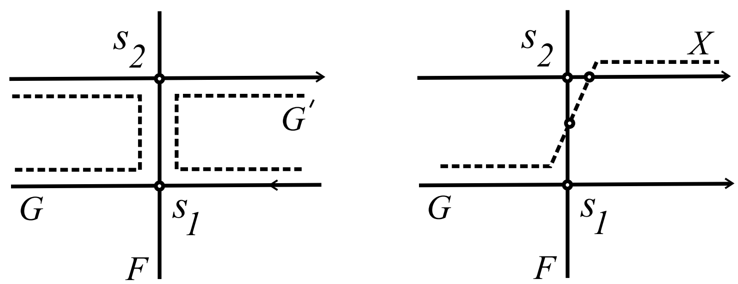

Figure 2:

annulus X such that $\#(X \cap F) < n$ and $\#(X \cap G) = 0$, see Fig. 1 (to get a real picture, multiply by S^1). It follows that $F \sim G$.

Case 4. Let F and G be annuli such that $F \cap G$ consists of more than one radial segments, each having endpoints in different components of ∂F and different components of ∂G.

Case 4.1. Suppose that there are two neighboring segments $s_1, s_2 \subset F \cap G \subset F$ such that G crosses F at s_1, s_2 in opposite directions. Denote by D the quadrilateral part of F between them. Then we cut G along s_1, s_2 and attach to it two parallel copies of D lying on different sides of F. We get a new surface G' consisting of two disjoint annuli, at least one of which (denote it by X) is essential, see Fig. 2 to the left. Since $\#(X \cap F) = n - 2$ and, after a small isotopy of X, $\#(X \cap G) = \emptyset$, we get $F \sim X \sim G$.

Case 4.2. Suppose that at all segments G crosses F in the same direction (say, from the left to the right). Let s_1, s_2 be two neighboring segments spanned by a quadrilateral part $D \subset F$ between them. Then s_1, s_2 decompose G into two strips L_1, L_2 such that L_1 approaches s_1 from the left side of F and s_2 from the right side. Then the annulus $L_1 \cup D$ is isotopic to an annulus X such that $\#(X \cap F) \leq n-1$ and $\#(X \cap G) = 1$, see Fig. 2 to the right. Since X crosses F one or more times in the same direction, it is essential. Therefore, $F \sim X \sim G$.

Case 5. This is the last logical possibility. Suppose that F and G are annuli such that $F \cap G$ consists of one radial segment. Denote by G' the relative boundary $\partial_{rel}(N) = \mathrm{Cl}(N \cap \mathrm{Int}\ M)$ of a regular neighborhood N of $F \cup G$ in M. Then G' is an annulus having boundary circles in different components of ∂M.

Case 5.1. If G' is incompressible, then we put $X = G'$.

Case 5.2. If G' admits a compressing disc D, then the relative boundary of a regular neighborhood N of $G' \cup D$ consists of a parallel copy of G' and two proper discs D_1, D_2. If at least one of these discs (say, D_1) is essential, then we put $X = D_1$.

Case 5.3. Suppose that discs D_1, D_2 are not essential. Then the circles $\partial D_1, \partial D_2$ bound discs D_1', D_2' contained in the corresponding components of ∂M. We claim that at least one of the spheres $S_1 = D_1 \cup D_1', S_2 = D_2 \cup D_2'$ (denote it by X) must be essential. Indeed, if both bound balls, then M is homeomorphic to $S^1 \times S^1 \times I$, contrary to our assumption.

In all three cases 5.1-5.3 X is disjoint to F as well as to G. Therefore, $F \sim X \sim G$. ∎

3 Proof of the main theorem

Let F be a sphere, a disc or an annulus in a 3-manifold M. It is convenient to denote by $C_F(M)$ the result of the F-move, i.e. the manifold obtained by compressing M along F.

Lemma 3. *If F is a sphere or a disc or an essential annulus, then any root of $M_F = C_F(M)$ is a root of M. If F is an inessential annulus, then M_F and M have at least one common root.*

Proof. It is convenient to decompose the proof into four steps.

(1) If F is essential, then any root of M_F is a root of M by definition of the root.

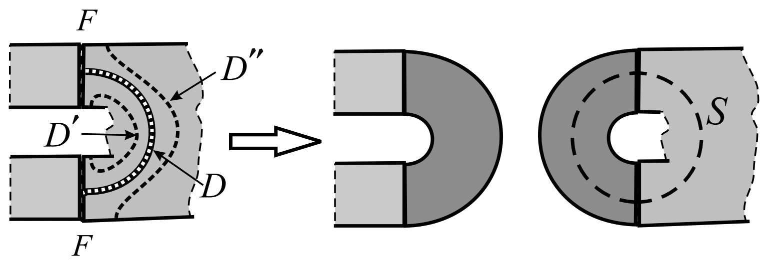

Figure 3:

(2) If F is an inessential sphere, then M_F is a union of M and a disjoint 3-sphere. Therefore, all roots of M and M_F are the same.

(3) Let F be an inessential disc. Then its boundary circle bounds a disc $D \subset \partial M$. Choose a 2-sphere S inside M which is parallel to the 2-sphere $F \cup D$. Then the manifold M_F is obtained from the manifold $M_S = C_S(M)$ by puncturing (cutting off a ball $V \subset M_S$). We claim that any root R of $M_F = M_S \setminus \text{Int } V \subset M_S$, which can be obtained from M_F by successive compressions along essential subsurfaces, is a root of M_S. Indeed, we simply compress M_S along the same surfaces and get either R (if one of those subsurfaces is a sphere surrounding V) or a punctured R (if the puncture survives all compressions). One more compression along a sphere surrounding the puncture is sufficient to convert the punctured R to R (modulo disjoint 3-spheres and balls, which are irrelevant). It follows from (1), (2), and (3) that any root of M_F is a root of M.

(4) Let F be an inessential (i.e. compressible) annulus and D a compressing disc for F such that $D \cap F = \partial D$ is a core circle of F. Denote by N a regular neighborhood of $F \cup D$ in M. Then the relative boundary $\partial_{rel} N = \text{Cl}(\partial N \cap \text{Int } M)$ consists of a parallel copy of F and two proper discs D', D''. Denote by S a 2-sphere in $C_F(M)$ composed from a copy of D and a core disc of one of the attached plates, see Fig. 3. Then the manifolds $C_{D''}(C_{D'}(M))$ and $C_S(C_F(M))$ are homeomorphic. Applying (1) - (3), we conclude that any root of the manifold $C_{D''}(C_{D'}(M)) = C_S(C_F(M))$ is a root of both M and $C_F(M)$.

∎

Proof of Theorem 1. Existence. Let us apply to M essential S, D, A-moves in arbitrary order as long as possible. By Lemma 1, each move strictly decreases the complexity. Since every set of pairs of nonnegative integers has a minimal pair, the process stops and we get a root.

Uniqueness. Assume the converse: suppose that there is a 3-manifold having two different roots. Among all such manifolds we choose a manifold M having minimal complexity. Then there exist two sequences of essential moves producing two different roots. Denote by C_F and C_G the first moves of the sequences, where F, G are essential surfaces in M. By Lemma 2, there are essential surfaces $X_1, X_2, \ldots, X_n$ such that $F = X_1$, $X_n = G$, and that the surfaces X_i and X_{i+1} are disjoint for all $i, 1 \leq i < n$. We may begin the construction of a root starting with the compression along any of them. Evidently, for at least two neighboring surfaces X_k, X_{k+1} the roots thus obtained are different. For convenience, we rename X_k, X_{k+1} by F, G thus getting two disjoint surfaces such that $C_F(M)$ and $C_G(M)$ have different roots. Then F is a subsurface of M and of $M_G = C_G(M)$ while G is a subsurface of M and of $C_F(M)$. Denote by N the manifold, obtained from M by compressions along both surfaces F, G. Of course, it coincides with $C_G(C_F(M))$ and $C_F(C_G(M))$.

We claim that the complexity of N is strictly less than the one of M. Indeed, if F is either a sphere or a disc, then $c(N) \leq c(M_G)$ (since compression along a sphere or a disc does not increase complexity) while $c(M_G) < c(M)$ by Lemma 1. Suppose that F is an annulus. Then $g^{(2)}(\partial N)$ is no greater than $g^{(2)}(\partial M_F)$, since no compression move increases the genus of the boundary. On the other hand, since F is essential, then $g^{(2)}(\partial M_F) < g^{(2)}(\partial M)$, which implies $c(N) < c(M)$.

Using the inductive assumption, we may conclude that N has a unique root. The same is true for M_F and M_G, since by Lemma 1 their complexities are also smaller than $c(M)$. Now we have:

1. M_F and N have the same root (since they have a common root by Lemma 3).

2. M_G and N have the same root (same reason);

3. Hence M_F and M_G have the same root, which is a contradiction.

4 Other roots

Roots of cobordisms. Recall that a *3-cobordism* is a triple $(M, \partial_- M, \partial_+ M)$, where M is a compact 3-manifold and $\partial_- M$, $\partial_+ M$ are unions of connected components of ∂M such that $\partial_- M \cap \partial_+ M = \emptyset$ and $\partial_- M \cup \partial_+ M = \partial M$. One can define S- and D-moves on cobordisms just in the same way as for manifolds. The A-move on cobordisms differs from the one for manifolds only in that one boundary circle of A must lie in $\partial_- M$ while the other in $\partial_+ M$.

Theorem 2. *For any compact 3-cobordism $(M, \partial_- M, \partial_+ M)$ its root exists and is unique up to homeomorphisms of cobordisms and removing disjoint 3-spheres and balls.*

The proof of this theorem is the same as the proof of Theorem 1.

S-Roots of manifolds. We define an S-root of M as a manifold which can be obtained from M by essential S-moves and does not admit any further essential S-moves.

Theorem 3. *For any compact 3-manifold M, its S-root exists and is unique up to homeomorphism and removing disjoint 3-spheres.*

This theorem is actually equivalent to the theorem on the unique decomposition into a connected sum of prime factors. Indeed, the S-root of M coincides with the union of the irreducible prime factors of M.

(S, D)-Roots of manifolds. An (S, D)-root of M is a manifold which can be obtained from M by essential S- and D- moves and does not admit any further essential S-moves and D-moves.

Theorem 4. *For any compact 3-manifold M its (S, D)-root exists and is unique up to homeomorphism and removing disjoint 3-spheres and balls.*

For irreducible manifolds this theorem can be deduced from the theorem of F. Bonahon [1] on characteristic compression bodies as well as from [4], where D-roots of irreducible manifolds had been considered under the name *cores*.

Our way for proving Theorem 1 works also for S- and (S, D)-roots. All we need is to forget about discs and annuli in the first case and about annuli in the second. This makes the proof significantly shorter.

References

[1] F. Bonahon, Cobordism of automorphisms of surfaces, Ann. Éc. Norm. Sup. 83 (1983), 237-270.

[2] S. Gadgil, On the Andrews-Curtis conjecture and algorithms from topology, math.GT/0108116.

[3] H. Kneser, Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. Jahresbericht der Deut. Math. Verein, 28:248-260, 1929.

[4] S. Matveev Algorithmic topology and classification of 3-manifolds, Springer ACM-monographs, V. 9 (2003), 480 pp.