THE INTEGRAL ON QUANTUM SUPER GROUPS OF TYPE $A_{r|s}$

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ABSTRACT. We compute the integral on matrix quantum (super) groups of type $A_{r|s}$ and derive from it the quantum analogue of (super) HCIZ integral.

INTRODUCTION

A formula for the integral on the group $U(n)$ was obtained by Itzykson and Zuber [3]. It can be given in the following form

$$\int_{U(n)} \mathrm{tr}(MUNU^{-1})^k[dU] = \sum_{\lambda\in\mathcal{P}_k^n} \frac{d_\lambda}{r_\lambda}\Phi_\lambda(M)\Phi_\lambda(N) \tag{1}$$

for any hermitian matrices M and N. Φ_λ is the irreducible character of $U(n)$, corresponding to λ. If $\xi_1,\xi_2,...,\xi_n$ are the eigenvalues of M then $\Phi_\lambda(M) = s_\lambda(\xi_1,\xi_2,...,\xi_n)$, s_λ are the Schur functions. d_λ is the dimension of the irreducible module of the symmetric function $\mathfrak{S}_n$, r_λ is the dimension of the irreducible representation of $U(n)$, corresponding to partition λ. Explicitly, $d_\lambda = n!\prod_{x\in[\lambda]} h(x)^{-1}$, $r_\lambda = \prod_{x\in[\lambda]} c(x)h(x)^{-1}$, where $c(x)$ is the content, $h(x)$ is the hook-length of the box x in the diagram $[\lambda]$. This formula turns out to be a special case of a formula obtained by Harish-Chandra [10]. The integral on the left-hand side of (1) is therefore referred as Harish-Chandra-Itzykson-Zuber (HCIZ) integral. A super analogue of this formula was obtained by Alfaro, Medina and Urrutia [1, 2], it reads

$$\int_{U(m|n)} \mathrm{str}(MUNU^{-1})^k[dU] = \sum_{\substack{\mu\in\mathcal{P}^m,\nu\in\mathcal{P}^n\\ |\mu|+|\nu|=k-mn}} \frac{k!}{|\mu|!|\nu|!}\frac{d_\mu d_\nu}{r_\mu r_\nu}\Phi_\lambda(M)\Phi_\lambda(N), \tag{2}$$

where $\lambda = (n^m) + \mu\cup\nu'$. The aim of this paper is to give a quantum analogue of the super HCIZ integral. In contrary to the method of Itzykson-Zuber and its super version developed by Alfaro *et. al.*, which is analytic, our method is purely algebraic. Thus, fist we want to give an algebraic definition of the integral.

It is well-known that the integral on a compact group is uniquely defined by its left (or right) invariance with respect to the group action. Thus, the integral can be considered as a linear functional on the function algebra of the group with an invariance property. Notice that the algebra of polynomial function on a compact (Lie) group is a commutative Hopf algebra. It turns out that for any Hopf algebra, one can give the notion of integral, although, such an integral does not always exist. Thus, by definition, a left integral on a Hopf algebra H is a linear form $\int$ on H such that $(\int \otimes \mathrm{id})\Delta = \int$. Analogously, one has the notion of right integral. Generally, left and right integrals may differ. For general properties of integral on Hopf algebras, the reader is referred to [16].

Originally, the integral is defined only on compact groups. The algebraically defined integral exists however on any cosemisimple Hopf algebras and on finite dimensional Hopf algebras. In particular, Hopf algebras of functions on linear reductive groups

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possess integrals. This is explained by the fact that each linear reductive group possesses a compact form and the corresponding integral coincides with the analytically defined one on this compact form. Having this relationship in mind, we shall find in this work the integral on the function algebras on quantum (super) groups of type $A_{r|s}$, such a quantum group is understood to be a generalization of the quantum general linear super group $GL_q(r+1|s+1)$. It is defined in terms of a Hecke symmetry of birank $(r+1, s+1)$.

In [7] we found the integral on quantum groups ot type $A_r = A_{r|0}$. In this work we extend the result to the case of $A_{r|s}$ for arbitrary r, s. Our main Theorem 2.2 computes the integral at every monomial in the entries of the generating matrix and its inverse. This is an advantage of our method to the analytical one. The original HCIZ formula computes the integral only at certain polynomials and it is hard to recover the value of the integral at each monomials, while it is easy to obtain the HCIZ formula from our formula.

The paper is organize as follows. In Section 1 we recall the definition of quantum (super) groups of type $A_{r|s}$. They are define in terms of Hecke symmetries. The reader, who is not interested in this general setting, can have in mind the standard quantum general linear super group $GL_q(r|s)$ [14]. The generating matrix is denoted by Z. We insert some signs in the definition of its tensor power $Z^{\otimes n}$ in order to provide that this matrix is still multiplicative. The inverse to Z is, again, upto some signs denoted by T. The signs provide T^t to be multiplicative. After recalling the general definition we give two main examples of Hecke symmetries, the Drinfeld-Jimbo's matrix of type A_r and it super version due to Manin.

In Section 2 we compute the integral on the entries of $Z^{\otimes n}T^{\otimes n}$. Our method bases on constructing a sequence of endomorphisms P_n of the tensor powers of the basic comodule. In Section 3 we introduce the notion of super characters. The reader who is not familiar with ribbon categories can skip the discussion preceding Lemma 3.1. Using the characters we establish the quantum supre HCIZ integral. In Section 4 we discuss the orthogonal relations for a class of H_R-comodules.

1. Quantum groups associated to Hecke symmetries

Let V be a super vector space over $\mathbb{K}$, a fixed field of characteristic zero. Fix a homogeneous basis $x_1, x_2, \ldots, x_d$ of V. We shall denote the parity of the basis element x_i by $\hat{i}$. An even operator R on $V \otimes V$ can be given by a matrix R_{ij}^{kl}: $R(x_i \otimes x_j) = x_k \otimes x_l R_{ij}^{kl}$. R is even means that the matrix elements R_{kl}^{ij} satisfy $\hat{i} + \hat{j} = \hat{k} + \hat{l}$. R is called *Hecke symmetry* if the following conditions are satisfied:

i) R satisfies the Yang-Baxter equation $R_1R_2R_1 = R_2R_1R_2$, $R_1 := R \otimes I$, $R_2 := I \otimes R$, I is the identity matrix of degree d.
ii) R satisfies the Hecke equation $(R-q)(R+1) = 0$ for some q which will be assumed not to be a root of unity.
iii) There exists a matrix P_{ij}^{kl} such that $P_{jn}^{im} R_{ml}^{nk} = \delta_l^i \delta_j^k$.

The matrix bialgebra E_R is define as follows. Let $\{z_j^i, t_j^i | 1 \le i, j \le d\}$ be a set of variables, $\hat{x_j^i} = \hat{t_j^i} = \hat{i} + \hat{j}$. Define

$$E_R := \mathbb{K} < \{z_j^i | 1 \le i, j \le d\} > \Big/ \Big((-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^s = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn} \Big)$$

$$H_R := \mathbb{K} < \{z_j^i, t_j^i | 1 \le i, j \le d\} > \Bigg/ \left(\begin{array}{c} (-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^s = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn}, \\ (-1)^{\hat{j}(\hat{j}+\hat{k})} z_j^i t_k^j = (-1)^{\hat{l}(\hat{l}+\hat{i})} t_l^i z_k^l = \delta_k^i \end{array} \right)$$

($\mathbb{K} < X >$ denotes the free algebra generated by elements of X). The coproduct on E_R and H_R is given by

$$\Delta(z_j^i) = z_k^i \otimes z_j^k, \quad \Delta(t_j^i) = t_j^k \otimes t_k^i.$$

The antipode on H_R is given by

$$S(z^i_j) = (-1)^{\hat{j}(\hat{i}+\hat{j})} t^i_j, \quad S(t^i_j) = (-1)^{\hat{i}(\hat{i}+\hat{j})} C^i_k z^k_l {C^{-1}}^l_j,$$

where $C^i_j := P^{il}_{jl}$. We also define $D^i_j := P^{li}_{lj}$. The matrices C and D play important roles in our work, they are call refection operator.

Since we are working in the category of super vector spaces, the rule of sign effects on the coproduct. More precisely, the compatibility of product and coproduct of a super bialgebra reads

$$\Delta(a \otimes b) = (-1)^{\hat{a}_2 \hat{b}_1} a_1 b_1 \otimes a_2 b_2.$$

Therefore we have

$$\Delta(z^{i_1}_{j_1} z^{i_2}_{j_2} \cdots z^{i_2}_{j_2}) = \mathrm{sign}(I, K)\mathrm{sign}(K, J)\mathrm{sign}(I, J) z^{i_1}_{k_1} z^{i_2}_{k_2} \cdots z^{i_n}_{k_n} \otimes z^{k_1}_{j_1} z^{k_2}_{j_2} \cdots z^{k_2}_{j_n},$$

where $I := (i_1, i_2, \ldots, i_n)$. $\mathrm{sign}(I, J)$ is given recurently by

$$\mathrm{sign}(i, j) = 1, \quad \mathrm{sign}(Ii, Ji) = (-1)^{\hat{i}(|\hat{I}|+|\hat{J}|)} \mathrm{sign}(I, J),$$

$\hat{I} := (\hat{i}_1, \hat{i}_2, ..., \hat{i}_n)$, $|\hat{I}|$ denotes the sum of its terms. Hence, for convenience, we denote

$$Z^I_J = {Z^{\otimes n}}^I_J := \mathrm{sign}(I, J) z^{i_1}_{j_1} z^{i_2}_{j_2} \cdots z^{i_2}_{j_2}.$$

Throughout this paper we shall always use this notation. Then we have $\Delta(Z^I_J) = Z^I_K \otimes Z^K_L$, $\Delta(T^I_J) = T^K_L \otimes T^I_K$, and $S(Z^I_J) = (-1)^{|\hat{J}|(|\hat{I}|+|\hat{J}|)} T^{I'}_{J'}$. Notice that

$$\mathrm{sign}(K', L') = (-1)^{|\hat{K_1}|(\hat{k}_n + \hat{j}_n)} \mathrm{sign}(K'_1, L'_1),$$

where for $K = (k_1, k_2, ..., k_n)$, $K_1 := (k_1, k_2, ..., k_{n-1})$ and $K' := (k_n, k_{n-1}, ..., k_1)$.

R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ (see Appendix) on the tensor powers $V^{\otimes n}$ of V, $\rho_n(T_i) = R_i := \mathrm{id}_V^{i-1} \otimes R \otimes \mathrm{id}_V^{n-i-1}$. We shall therefore use the notation $R_w := \rho(T_w)$. On the other hand, E_R coacts on V by $\delta(x_i) = x_j \otimes z^j_i$. Since E_R is a bialgebra, it acts on $V^{\otimes n}$ by means of the coproduct. These two actions are centralizers of each other in $\mathrm{End}_{\mathbb{K}}(V^{\otimes n})$ [6]. Hence, the algebra $\mathrm{End}^{E_R}(V^{\otimes n})$ is a factor algebra of $\mathcal{H}_n$ and $(E^n_R)^* \cong \mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})$. Let us denote $\mathrm{End}^{E_R}(V^{\otimes n})$ by $\overline{\mathcal{H}_n}$. Since $\mathcal{H}_n$ is semi-simple, provided q is not a root of unity, the algebras $(E^n_R)^*$ and $\overline{\mathcal{H}_n}$ are semi-simple, too.

The double centralizer theorem implies that a simple E_R-comodule is the image of the operator induced by a primitive idempotent of $\mathcal{H}_n$ and, conversely, each primitive idempotent of $\mathcal{H}_n$ induces an E_R comodule which is either zero or simple. Irreducible representations of $\mathcal{H}_n$ are parameterized by partitions of n. Thus, up to conjugation, primitive idempotents of $\mathcal{H}_n$ are parameterized by partitions of n, too.

The primitive idempotents $x_n := \sum_w T_w/[n]_q!$ and $y_n := \sum_w (-q)^{-l(w)} T_w/[n]_{1/q}!$ induce the symmetrizer and anti-symmetrizer operators on $V^{\otimes n}$. Let $S_n := \mathrm{Im}\rho_n(x_n)$ and $\Lambda_n := \mathrm{Im}\rho_n(y_n)$. Then one can show that $S := \bigoplus_{n=0}^{\infty} S_n$ and $\Lambda := \bigoplus_{n=0}^{\infty} \Lambda_n$ are algebras. They are called symmetric and exterior tensor algebras on the corresponding quantum space.

By definition, the Poincaré series $P_\Lambda(t)$ of Λ is $\sum_{n=0}^{\infty} t^n \dim_{\mathbb{K}}(\Lambda_n)$. It is proved that this series is a rational function having negative roots and positive poles. Let r be the number of its roots and s be the number of its poles. As a consequence, simple E_R-comodules are parameterized by hook-partitions $\{\lambda | \lambda \in \Gamma^{r,s}_n\}$, i.e., partitions λ, such that $\lambda_{r+1} \le s$. (r, s) is called the birank of R [9]. Consequently, in the algebra $\overline{\mathcal{H}_n}$ primitive idempotents are parameterized by hook-partitions from $\Gamma^{r,s}_n$, too.

Our main assumption on R is that

$$\mathrm{tr}(C) = -[s - r]_q, \tag{3}$$

C is the reflection operator introduced above. It can be proved that $\mathrm{tr}(C)$ should have the form $-[x]_q$ for some integer x in the interval $[-s, r]$. The above equation holds for any known Hecke symmetry. It is conjectured that it holds for all Hecke symmetry.

Simple H_R-comodules are much more complicated. The problem of classifying all its comodule is still open.

The Hopf algebra H_R (resp. the bialgebra E_R) is called the (function algebra on) quantum group (resp. quantum semigroup) of type $A_{r|s}$.

The following are two main examples of Hecke symmetries. The Drinfeld-Jimbo's R-matrix of type A_{r-1} [11], for $1 \leq i,j,k,l \leq r$, $p^2 = q$,

$$R_r{}_{ij}^{kl} := \begin{cases} q & \text{if} \quad i=j=k=l \\ q-1 & \text{if} \quad k=i>j=l \\ p & \text{if} \quad k=j\neq i=l \\ 0 & \text{otherwise.} \end{cases}$$

Assume that all parameters are even, the Hopf algebra associated to R_r is called the (function algebra on) standard quantum general linear group $\mathrm{GL}_q(r)$. R_r is a one-parameter deformation of the permuting operator. The super version of this operator was given by Manin [14]. Assume that the variable x_i, $i \leq r$ are even and the rest s variables are odd. Define, for $1 \leq i,j,k,l \leq r+s$, $p^2 = q$,

$$R_{r|s}{}_{ij}^{kl} := \begin{cases} q & \text{if} \quad i=j=k=l, \hat{i}=0 \\ -1 & \text{if} \quad i=j=k=l, \hat{i}=1 \\ q-1 & \text{if} \quad k=i<j=l \\ (-1)^{\hat{i}\hat{j}}p & \text{if} \quad k=j\neq i=l \\ 0 & \text{otherwise.} \end{cases}$$

$R_{r|s}$ is a deformation of the permuting operator in super symmetry. The associated Hopf algebra is called the (function algebra on) standard quantum general linear super group $\mathrm{GL}_q(r|s)$.

R_r has the birank $(r,0)$. Such an operator is called even Hecke symmetry. $R_{r|s}$ has the birank (r,s). Generally, if R is a Hecke symmetry of rank (r,s), then H_R is called the (function algebra on) quantum group of type $A_{r-1|s-1}$. The aim of this work is to compute the integral on such a quantum super group.

The integral on quantum groups of type $A_{r|0}$ was computed in [7]. Let us recall it here. First, remark that an arbitrary element of H_R can be represented as a linear combination of monomials of the form $Z_I^J T_K^L$. Second, remark that from definition the integral on $Z_I^J T_K^L$ is zero if $l(I) \neq l(K)$, $l(I)$ is the length of I. The integral on $Z_J^I T_L^K$ with $l(I) = l(K) = n$ is

$$\int (Z_I^J T_{K'}^{L'}) = \sum_{w \in \mathfrak{S}_n} q^{-l(w)} (P_n C^{\otimes n} R_w)_I^L R_w{}_K^J,$$

where for $K = (k_1, k_2, ..., k_n)$, $K' := (k_n, ..., k_1)$. P_n is given by $P_n := \prod_{k=1}^n (L_k - [-r]_q)^{-1}$. L_n are the Murphy operators, $L_1 := 0$, $L_n = \sum_{i=1}^{n-1} q^{-i} R_{(n-i,n)}$, $n \geq 2$. The proof bases on the fact that P_n are central in $\overline{\mathcal{H}_n}$ and satisfy the equation (in $\overline{\mathcal{H}_{n+1}}$)

$$P_n = P_{n+1}(L_{n+1} - [-r]).$$

2. The Integral on H_R

As we have seen in the previous section, the integral on H_R with R even Hecke symmetry is obtained by constructing a sequence of operators P_n with property $P_n = P_{n+1}(L_{n+1} - [-r])$. To find a integral on H_R with arbitrary R we construct an analogous sequence P_n with this property for r replaced by $r - s$, where (r,s) is the birank of R. Recall that we assume $\mathrm{tr}(C) = -[s-r]_q$. In the general case, the operator $L_n - [s-r]_q$ is not always invertible, so that we have to find a new construction.

A full set of mutual orthogonal primitive idempotents of $\mathcal{H}_n$ was found by by Dipper and James [4]:

$$E_{t_i(\lambda)} = \prod_{\substack{1\le m\le n\\ |k|\le m-1\\ k\neq c_{t_i(\lambda)}(m)}} \frac{L_m - [k]_q}{[c_{t_i(\lambda)}(m)]_q - [k]_q}, \quad 1\le i\le d_\lambda,\ \lambda\in\mathcal{P}_n.$$

$c_{t_i(\lambda)}(m)$ is the content of m in the standard tableau $t_i(\lambda)$ (see Appendix for notations). Notice that $\sum_{1\le i\le d_\lambda} E_{t_i(\lambda)} = F_\lambda$ – the minimal central idempotent corresponding to λ.

L_m satisfy the equation

$$\prod_{k=-m-1}^{m+1} (L_m - [k]_q) = 0.$$

Therefore, for $1\le m\le n$,

$$L_m E_{t_i(\lambda)} = E_{t_i(\lambda)} L_m = c_{t_i(\lambda)}(m) E_{t_i(\lambda)}. \tag{4}$$

Using this equation and the fact that P_n is central we obtain the follwing decompsition

$$P_n = \sum_{\lambda\vdash n} q_\lambda F_\lambda = \sum_{\substack{\lambda\vdash n\\ 1\le i\le d_\lambda}} q_\lambda E_{t_i(\lambda)}, \tag{5}$$

the coefficient q_λ is given by

$$q_\lambda = \prod_{x\in[\lambda]} \frac{q^r}{[c_\lambda(x)+r]_q}.$$

For an arbitrary R we shall find P_n in the form analogous to (5). The sum will be taken on the set $\{\lambda\in\Gamma_n^{r,s} | \lambda_r \ge s\} =: \Omega_n^{r,s}$. Geometrically, this is the set of λ's, the diagram $[\lambda]$ contains the diagram $[(s^r)]$. Each such λ decomposes into $\lambda = (s^r) + \mu \cup \nu'$, where $\mu\in\mathcal{P}^r$, $\nu\in\mathcal{P}^s$.

We define

$$q_\lambda := \prod_{x\in[\lambda]\setminus[(s^r)]} \frac{q^{r-s}}{[c_\lambda(x)+r-s]_q},$$

thus $q_\lambda = q^{r|\nu|-s|\mu|} p_\mu p^-{}_\nu$, where $p^o{}_\nu = \prod_{x\in\nu} \frac{q^{-s}}{[-c(x)-s]}$. Then, we define

$$P_n := \sum_{\substack{\lambda\in\Omega_n^{r,s}\\ 1\le i\le d_\lambda}} q_\lambda E_{t_i(\lambda)} = \sum_{\lambda\in\Omega_n^{r,s}} q_\lambda F_\lambda. \tag{6}$$

Recall that P_n are defined in the algebra $\overline{\mathcal{H}_n} \cong \mathrm{End}^{E_R}(V^{\otimes n})$, which is the factor algebra of $\mathcal{H}_n$ by the two-sided ideal generated by minimal central idempotents corresponding to partitions from $\mathcal{P}_n \setminus \Gamma_n^{r,s}$. Fixing an embedding $\overline{\mathcal{H}_n} \hookrightarrow \overline{\mathcal{H}_{n+1}}$, $\overline{\mathcal{H}_n} \ni W \longmapsto W\otimes \mathrm{id}_V \in \overline{\mathcal{H}_{n+1}}$, we identify $\overline{\mathcal{H}_n}$ with a subalgebra of $\overline{\mathcal{H}_{n+1}}$.

Lemma 2.1. *The operators P_n are central in $\overline{\mathcal{H}_n}$ and satisfy the equation (in $\overline{\mathcal{H}_{n+1}}$)*

$$P_{n+1}(L_{n+1} - [s-r]_q) = P_n.$$

Proof. The operator P_n^s is obviously central, for it is a sum of central elements.

We check the equation above. First, notice that, if $\lambda\in\Omega_{n+1}^{r,s}$ and $t_i(\lambda)$ is a standard λ-tableau, then the node of $[\lambda]$, containing $n+1$, is removable, i.e., having removed it we still have a standard tableau. The tableau $t_i(\lambda)$ is called essential if this node is not the node (r,s), otherwise, it is called non-essential. A tableau $t_i(\lambda)$ is essential iff the tableau, obtained from it by removing the node containing $n+1$ is again a γ-tableau with $\gamma\in\Omega_n^{r,s}$.

Observe, that if $t_i(\lambda)$ is non-essential, then $c_{t_i(\lambda)} = s - r$, hence

$$E_{t_i(\lambda)}(L_{n+1} - [s-r]_q) = 0,$$

by virtue of Equation (4). We reshuffle the terms of P_{n+1} in groups as follows

$$P_{n+1} = \sum_{\substack{\gamma\in\Omega_n^{r,s}\\ 1\le i\le d_\gamma}} \sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} q_\lambda E_{t(\lambda)} + \sum_{\substack{t(\lambda)\text{ is}\\ \text{not eessential}}} q_\lambda E_{t(\lambda)}.$$

That is, we pick up into group for each $t_i(\gamma)$, $\gamma \in \Omega_n^{r,s}$, those standard tableaux $t(\lambda)$, that contain $t_i(\gamma)$ as a subtableau. The preceding observation implies that the second sum in the right-hand side of the above equation multiplied by $L_{n+1} - [s-r]_q$ vanishes. Thus, it is sufficient to prove, for a fixed $t_i(\gamma)$, $\gamma \in \Omega_n^{r,s}$,

$$\sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} q_\lambda E_{t(\lambda)}(L_{n+1} - [s-r]_q) = q_\gamma E_{t_i(\gamma)}.$$

We have $(L_{n+1} - [s-r]_q)E_{t(\lambda)} = [c_{t(\lambda)}(n+1)]_q - [s-r]_q$ and $q_\lambda(c_\lambda(x) - [s-r]_q) = q_\gamma$ whenever $[\gamma]$ is obtained from $[\lambda]$ by removing the node x. Since, for any two standard tableaux $t(\gamma)$ and $t(\lambda)$ with $\gamma \subset \lambda$ as above, the number $n+1$ should lie in the node x, for which $[\lambda] \setminus [x] = [\gamma]$, we deduce that the equation to be proved is equivalent to

$$\sum_{\substack{\lambda\in\Omega_{n+1}^{r,s}\\ t(\lambda)\supset t_i(\gamma)}} E_{t(\lambda)} = E_{t_i(\gamma)}. \tag{7}$$

Since $\prod_{k=-n-1}^{n+1}(L_{n+1} - [k]_q) = 0$,

$$\sum_{m=-n-1}^{n+1} \prod_{\substack{k=-n-1,\\ k\neq m}}^{n+1} \frac{L_{n+1} - [k]_q}{[m]_q - [k]_q} = 1.$$

Therefore

$$E_{t_i(\gamma)} = \sum_{m=-n-1}^{n+1} E_{t_i(\gamma)} \prod_{\substack{k=-n-1,\\ k\neq m}}^{n+1} \frac{L_{n+1} - [k]_q}{[m]_q - [k]_q}.$$

Remember that we are working in the algebra $\overline{\mathcal{H}_{n+1}}$, in which $E_\lambda \neq 0$ iff $\lambda \in \Gamma_{n+1}^{r,s}$. Each term on the right-hand side of the above equation is either zero or a primitive idempotent of the form $E_{t(\lambda)}$ with $t(\lambda)$ containing $t_i(\gamma)$ as a subtableau. Since the left-hand side of (7) contains all primitive idempotents in $\overline{\mathcal{H}_{n+1}}$ that correspond to standard tableaux containing $t_i(\gamma)$ as a subtableau, the equation (7) follows. Lemma 2.1 is therefore proved.∎

We are now going to give the formula for an integral on the super Hopf algebra H_R. First, notice that as in the case of even Hecke symmetries, the integral on $Z_J^I T_L^K$ vanishes unless $l(I) = l(K)$. Further the integral should vanish at the identity if $rs > 0$, for H_R is then not cosemisimple. In fact, we shall set $\int(Z_I^J T_K^L) = 0$ if $l(I) = l(K) < rs$ and compute the integral only for $l(I) = l(K) \geq rs$. This is in fact not a restriction, for the integral if it exists then uniquely up to a constant [15].

Theorem 2.2. *The Hopf algebra H_R associated to a Hecke symmetry R satisfying the condition (3) possesses an integral, which is uniquely determined up to a scalar multiple. Let (r,s) be the birank of R. Then an integral can be given as follows: if $l(I) = l(K) < rs$, $\int(Z_I^J T_K^L) = 0$, if $l(I) = l(K) = n \geq rs$,*

$$\int(Z_I^J T_{K'}^{L'}) = (-1)^{|\hat K|(|\hat K|+|\hat L|)} \sum q^{-l(w)} (P_n C^{\otimes n} R_{w^{-1}})_I^L R_{w}{}_K^J,$$

($K' = (k_n, k_{n-1}, ..., k_1)$).

Notice that since the operators C, P, R_w are all even, the integral vanishes unless $|\hat{I}| = |\hat{L}|, |\hat{K}| = |\hat{J}|$.

Proof. (sketch) The proof of this theorem does not differ from the one for even Hecke symmetries, given in [7]. We give here only a sketch, for detail the reader is referred to [7].

It is to check that the above formula is invariant with respect to the defining relations

$$(-1)^{\hat{s}(\hat{\imath}+\hat{p})} R^{kl}_{ps} z^p_i z^s_j = (-1)^{\hat{l}(\hat{q}+\hat{k})} z^k_q z^l_n R^{qn}_{ij},$$
$$(-1)^{\hat{j}} z^i_j t^j_k = (-1)^{\hat{l}} t^i_l z^l_k = \delta^i_k.$$

Notice that the last relation, combined with the first one gives us

$$(-1)^{\hat{j}(\hat{\imath}+\hat{j})} z^l_k C^j_l t^i_j = C^i_k.$$

The invariance with respect to the first relation above follows from the fact that P_n and $\sum_{w\in\mathfrak{S}_n} q^{-l(w)} R_{w^{-1}} \otimes R_w$ are central. The invariance with respect to the other relations is equivalent to the following equations (all signs are cancelled):

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} (P_n R_{w^{-1}} C^{\otimes n})^{L_1 m}_{I_1 m} R_w{}^J_K = \sum_{w\in\mathfrak{S}_{n-1}} q^{-l(w)} (P_{n-1} R_{w^{-1}} C^{\otimes n-1})^{L_1}_{I_1} R_w{}^{J_1}_{K_1} \delta^{j_n}_{k_n},$$

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} (P_n R_{w^{-1}} C^{\otimes n-1})^{L_1 l_n}_{I_1 i_n} R_w{}^{J_1 n}_{K_1 p} C^p_n = \sum_{w\in\mathfrak{S}_{n-1}} q^{-l(w)} (P_{n-1} R_{w^{-1}} C^{\otimes n-1})^{L_1}_{I_1} R_w{}^{J_1}_{K_1} C^{l_n}_{i_n},$$

here, for a multi-index $K = (k_1, k_2, \ldots, k_n)$ we denote $K_1 := (k_1, k_2, \ldots, k_{n-1})$. By virtue of Lemma 2.1 this equations reduce to

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)} R_{w^{-1}} \otimes \mathrm{tr}^n(R_w) = \sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)} (L_n - [-r]_q) R_{w^{-1}} \otimes R_w,$$

Here the map $\mathrm{tr}^n : \overline{\mathcal{H}_n} \longrightarrow \overline{\mathcal{H}_{n-1}}$ is given by

$$\mathrm{tr}^n(R_w) = \left[\begin{array}{ll} R_{v_k \cdots v_{n-2} w_1} & \text{if } k \le n-1 \\ -[s-r]_q R_{w_1} & \text{if } k = n, \end{array} \right.$$

where $w = v_k v_{k+1} \cdots v_{n-1} w_1, w_1 \in \mathfrak{S}_{n-1}$ (cf. [7], Lemma 4.1.3). ∎

3. Characters of H_R and quantum analogue of super HCIZ integral

Following [8] we define characters of comodules over H_R. To do this we first need a co-ribbon structure on H_R. By definition, a coquasitriangular super Hopf algebra is a super Hopf algebra H, equipped with a $*$-invertible even bilinear form $R : H \otimes H \longrightarrow \mathbb{K}$, subject to the following conditions:

$$r(ab, c) = (-1)^{\hat{b}\hat{c}_1} r(a, c_1) r(b, c_2) \qquad r(a, bc) = r(a_2, b) r(a_1, c)$$
$$(-1)^{\hat{a}_2\hat{b}_1} r(a_1, b_1) a_2 b_2 = (-1)^{\hat{a}\hat{b}_1} b_1, a_1 r(a_2, b_2).$$

R is called a coquasitriangular (CQT) structure on H. A CQT structure on H_R is given by $r(z^i_j, z^k_l) = (-1)^{\hat{j}\hat{k}} R^{ki}_{jl}$, $r(t^i_j, t^k_l) = (-1)^{\hat{\imath}\hat{l}} R^{ki}_{jl}$, $r(t^i_j, z^k_l) = (-1)^{\hat{\imath}\hat{k}} R^{-1}{}^{ik}_{lj}$, $r(z^i_j, t^k_l) = (-1)^{\hat{\imath}\hat{l}} P^{ki}_{jl}$.

A co-ribbon structure on a CQT Hopf algebra H is a $*$-invertible linear form $t : H \longrightarrow \mathbb{K}$, subject to the following conditions:

$$t(a_{(1)}) a_{(2)} = a_{(1)} t(a_{(2)}), \quad t \circ S = t,$$
$$t(ab) = (-1)^{(\hat{a}_1+\hat{a}_1)(\hat{b}_1+\hat{b}_2)} t(a_{(1)}) t(b_{(1)}) r(b_{(2)} a_{(2)}) r(a_{(3)}, b_{(3)}).$$

The co-ribbon structure on H_R is given by $t(z^i_j) = t(t^i_j) = q^{(r-s+1)/2} \delta^i_j$.

It H is a coribbon Hopf algebra then H-comod is a ribbon category [12], the braiding and the twist are defined as follows:

$$\tau_{M,N}(m,n)=n_0\otimes m_0 r(m_1,n_1),\quad \theta_M(m):=m_0t(m_1),\quad m\in M,n\in N.$$

By definition, the character of an endomorphism on a comodule M over H_R is the element

$$\Phi(M):=(\mathrm{ev}_M\otimes \mathrm{id}_{H_R})(\mathrm{id}_{M^*}\otimes\delta_M)\tau_{M,M^*}(f\circ\theta_M\otimes \mathrm{id}_{M^*})\mathrm{db}(1_{\mathbb{K}}),$$

where $\mathrm{ev}_M,\mathrm{db}_M$ are the canonical morphism defining a structure of dual H_R-comodule on M^*: $\mathrm{ev}:M^*\otimes M\longrightarrow\mathbb{K}$, $\mathrm{db}:\mathbb{K}\longrightarrow M\otimes M^*$. The character $\Phi(M)$ of M is defined to be $\Phi(M):=\Phi(\mathrm{id}_M)$. The character map Φ is a trace map from $\mathrm{End}(M)$ in to H.

Let $x_1,x_2,\ldots,x_m$ be a basis of M and $a_i^j, 1\le i,j\le m$ be the corresponding coefficient matrix: $\delta(x_i)=x_j\otimes a_i^j$. Let ${T_M}_j^i:=t(a_j^i)$ and ${D_M}_j^i=r(a_l^i,S(a_j^l))$, $f(x_i):=x_jT_i^j$. Then the character of f and M is given by

$$\Phi(f)=\mathrm{tr}(FD_MT_MA),\quad \Phi(M)=\mathrm{tr}(D_MT_MA).$$

We have the following identity

$$\tau_{M^{\otimes n},M^{*\otimes n}}(\theta_{M^{\otimes n}}\otimes\mathrm{id}_{M^{*\otimes n}})=\mathrm{db}_M^n\circ\mathrm{db}_M^{n-1}\circ\cdots\circ\mathrm{db}_M:\mathbb{K}\longrightarrow M^{\otimes n}\otimes M^{*\otimes n},\quad (8)$$

where $\mathrm{db}^i_{M^{\otimes n}}:=\mathrm{id}_M^{i-1}\otimes\mathrm{db}_M\otimes\mathrm{id}_{M^*}^{i-1}$. Using it we can show that $\Phi(f\otimes g)=\Phi(f)\Phi(g)$. Moreover, let $f:M^{\otimes n}\longrightarrow M^{\otimes n}$. Then we have

$$\Phi(f)=\mathrm{tr}(D^{\otimes n}T^{\otimes n}FA^{\otimes n}),$$

where F is the matrix of f with respect to the basis $x_1,...,x_n$.

For each partition $\lambda\in\Gamma_n^{r,s}$, M_λ denotes the corresponding simple E_R-comodule. Since the map $E_R\longrightarrow H_R$ is injective (cf. [7]), M_λ is simple H_R-comodule, too. M_λ is isomorphic to $\mathrm{Im}\rho(E_\lambda)$ of a primitive idempotent E_λ. Therefore $\Phi(M_\lambda)=\Phi(E_\lambda)$. Thus, setting $D_j^i=r(z_l^i,S(z_j^l))$, we have

Lemma 3.1. *The characters of M_λ and M_λ^*, $\lambda\vdash n$ are given by*

$$S_\lambda:=\Phi(M_\lambda)=q^{n(r-s+1)/2}\mathrm{tr}(D^{\otimes n}E_\lambda Z^{\otimes n}),$$
$$S_{-\lambda}:=\Phi(M_\lambda^*)=q^{n(r-s+1)/2}\mathrm{tr}(C^{\otimes n}E_\lambda\overline{T^{\otimes n}}),$$

where $\overline{T^{\otimes n}}_J^I:=T_{J'}^{I'}$.

Notice that in the definition of $Z^{\otimes n}$, the signs are also inserted.

Since S_λ, $\lambda\in\Gamma_n^{r,s}$ belong to different simple subcoalgebras of E_R, they are linear independent in E_R, and hence in H_R, too.

Let K be an algebra. A K-point of a (super) bialgebra B is an algebra homomorphism $\mathcal{A}:B\longrightarrow K$. Let $\mathcal{A}$ be a K-point of the bialgebra E_R. Then the entries of A commute by the same rule as the entries of Z. We define $S_\lambda(A):=\mathcal{A}(S_\lambda)$. In other words, $S_\lambda(A)$ is S_λ computed at $Z=A$. We are now ready to formulate a formula to compute the quantum super HCIZ integral.

Theorem 3.2. *Let M be a K-point of E_R and N be a K-point of $E'_R=S(E_R)$ (i.e., the subbialgebra of H_R, generated by T). Assume that entries of M and N commute and that they commute with the entries of Z and T. Then*

$$\int q^{n(r-s)/2}\mathrm{tr}(C^{\otimes n}M^{\otimes n}Z^{\otimes n}\overline{N^{\otimes n}T^{\otimes n}})=\sum_{\lambda\in\Omega_n^{r,s}}\frac{d_\lambda p_\lambda^s}{k_\lambda}S_\lambda(M)S_{-\lambda}(N).$$

Here $S_{-\lambda}=S(S_\lambda)$, $S_{-\lambda}(N)$ is computed at $T=N$. $k_\lambda=q^{n(\lambda)}\prod_{x\in[\lambda]}[h(x)]_q^{-1}$, $n(\lambda)=\sum_i\lambda_i(i-1)$, $h(x)$ is the hook length of x in the diagram $[\lambda]$: $h_\lambda(x)=\lambda_i+\lambda'_j-i-j+1$, where (i,j) is the coordinate of x in the diagram $[\lambda]$.

Proof. The proof, again, does not differ from the one given in [8] for quantum HCIZ integral. The main observation is that

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)}R_{w^{-1}}R_w = \sum_{\substack{\lambda\vdash n\\ 1\le i,j\le d_\lambda}} \frac{1}{k_\lambda}E_\lambda^{ij},$$

and that $\Phi(E_\lambda^{ij}) = 0$ for $i \neq j$. Here E_λ^{ij} is a basis of the Hecke algebra $\mathcal{H}_n$ with the property $E_\lambda^{ij}E_\mu^{kl} = \delta_\lambda^\mu\delta_k^j E_\lambda^{il}$, thus, in particular, E_λ^{ii} are primitive idempotents. For the detailed proof the reader is referred to [8].∎

Example. Let us consider the case of standard quantum general linear super group $GL_q(r|s)$, determined in terms of the symmetry $R_{r|s}$ given Section 1. In this case, any diagonal matrix with commuting entries is a point of E_R. Thus, assume that M and N are diagonal matrix with entries commuting each other and with the entries of Z and T, $M = \mathrm{diag}(m_1, m_2, ..., m_{r+s})$, $N = \mathrm{diag}(n_1, n_2, ..., n_{r+s})$. Then, we have

$$S_{(n)} = \sum_{k=0}^{n} h_{n-k}(qm_1, ..., q^r, m_r)e_k(-q^r m_{r+1}, ..., -q^{r-s+1}m_{r+s}),$$

h_k, e_k are the k^{th} complete and elementar symmetric functions in r and s variables, respectively. Hence

$$S_\lambda(M) = s_\lambda(qm_1, q^2m_2, ..., q^r m_r / -q^r m_{r+1}, ..., -q^{r-s+1}m_{r+s}),$$

s_λ are the Hook-Schur functions in $r+s$ variables (cf. [13], Ex. I.3.23). Therefore, if $\lambda \in \Omega_n^{r,s}$, thus, $\lambda = (r^s) + \mu \cup \nu'$, $\mu \in \mathcal{P}^r, \nu \in \mathcal{P}^s$, we have [loc.cit.]

$$S_\lambda(M) = (-1)^{|\nu|}\prod_{i=1,j=1}^{r,s}(q^i m_i - q^{r-j+1}m_{r+j})s_\mu(\{q^i m_i\})s_\nu(\{q^{r+1-i}m_{r+i}\}),$$

where s_μ (resp. s_ν) are the Schur functions in r variables (resp. s variables). Analogously, we have

$$S_{-\lambda}(N) = (-1)^{|\nu|}\prod_{i=1,j=1}^{r,s}(q^{r-s-i+1}n_i - q^{j-s}n_{r+j})s_\mu(\{q^{r-s+1-i}n_i\})s_\nu(\{q^{i-s}n_{r+i}\}).$$

The quantum super HCIZ is then given by, $(n \ge rs)$,

$$\int \mathrm{tr}(C^{\otimes n}N^{\otimes n}Z^{\otimes n}M^{\otimes n}T^{\otimes n}) =$$

$$\sum_{\substack{\mu\in\mathcal{P}^r,\nu\in\mathcal{P}^s,\\ |\mu|+|\nu|=n-rs\\ \lambda=(s^r)+\mu\cap\nu'}} q^{r|\nu|-s|\mu|}p_\mu p_\nu^- \frac{d_\lambda}{k_\lambda}\prod_{\substack{i=1\\ j=1}}^{r,s}(q^i m_i - q^{r-j+1}m_{r+j})(q^{r-s-i+1}n_i - q^{j-s}n_{r+j})$$

$$\times s_\mu(\{q^i m_i\})s_\nu(\{q^{r+1-i}m_{r+i}\})s_\mu(\{q^{r-s+1-i}n_i\})s_\nu(\{q^{i-s}n_{r+i}\}).$$

4. The Orthogonal Relations

We are now interested in the orthogonal relations. Let M_λ, M_μ be two simple comodules corresponding to partitions λ and μ of n. Let $M_\lambda = \mathrm{Im}E_\lambda$, $M_\mu = \mathrm{Im}E_\mu$.

Then, using Theorem 2.2, we have

$$\begin{aligned}
(\Phi(M_\lambda), \Phi(M_\mu)) &= \int (\Phi(M_\lambda)\Phi(M_\mu^*)) \\
&= \sum q^{-l(w)} \mathrm{tr}(P_n C^{\otimes n} E_\lambda R_{w^{-1}} E_\mu R_w) \\
&= \delta_\lambda^\mu \delta_{\Omega^{r,s}}^\lambda \frac{q_\lambda}{k_\lambda} \mathrm{tr}(C^{\otimes n} E_\lambda) \\
&= 0.
\end{aligned}$$

Here $\delta_{\Omega^{r,s}}^\lambda$ indicates, whether λ belongs to $\Omega^{r,s}$, it is zero if $\lambda \notin \Omega^{r,s}$ and 1 otherwise. On the other hand, for λ form $\Omega^{r,s}$, $\mathrm{tr}(C^{\otimes n} E_\lambda) = 0$, for it is the quantum rank of M_λ.

We see that the scalar product above cannot be used to define the orthogonal relations. Following [1] we compute the integral $\int (Z_\lambda Z_{-\mu})$, where Z_λ is a coefficient matrix of the simple comodule M_λ, i.e., Z_λ is the multiplicative matrix corresponding to certain basis of M_λ.

Let us fix a primitive idempotent E_λ, $\lambda \vdash n$, and set $M_\lambda = \mathrm{Im}\rho(E_\lambda)$. To choose a basis of M_λ we proceed as follows. First notice that there exists a one-one correspondence between the set of bases of M_λ and the set of maximal operators P_b^a in $\mathrm{End}_{\mathbb{K}}(M_\lambda$ with property $P_b^a P_d^c = \delta_b^c P_d^a$. Further, since E_λ is an idempotent, $\mathrm{End}_{\mathbb{K}}(M_\lambda) = E_\lambda \mathrm{End}_{\mathbb{K}}(V^{\otimes n}) E_\lambda$. Therefore, instead of fixing a basis of M_λ we can equivalently choose a maximum set of operators $P_b^a = E_\lambda Q_b^a E_\lambda$, $P_b^a \in \mathrm{End}(V^{\otimes n})$, satisfying $P_b^a P_d^c = \delta_b^c P_d^a$.

Using standard argument we deduce that the corresponding multiplicative matrix is given by ${Z_\lambda}_n^a = \mathrm{tr}(P_b^a Z^{\otimes n})$. Analogously, ${Z_{-\lambda}}_n^a = \mathrm{tr}(P_b^a \overline{T^{\otimes n}})$. Now we can find the integral

$$\begin{aligned}
\int ({Z_\lambda}_b^a {Z_{-\lambda}}_d^c) &= \int \mathrm{tr}(P_b^a Z^{\otimes n}) \mathrm{tr}(P_d^c \overline{T^{\otimes n}}) \\
&= k_\lambda^{-1} q_\lambda \mathrm{tr}(C^{\otimes n} E_\lambda^{ij} P_b^a E_\lambda^{ji} P_d^c) \\
&= k_\lambda^{-1} q_\lambda \delta_b^c \mathrm{tr}(C^{\otimes n} P_d^a),
\end{aligned}$$

for we can choose E_λ^{ij} such that $E_\lambda = E_\lambda^{ii}$ for some i.

Appendix

A partition of a positive integer n is a sequence of non-increasing positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, whose sum $|\lambda|$ is equal to n. We shall write $\lambda \vdash n$ or $\lambda \in \mathcal{P}_n$. r is the length of λ. The set of partition of length r is denoted by $\mathcal{P}^r$. $\mathcal{P}_n^r := \mathcal{P}^r \cap \mathcal{P}_n$. To shorten notations, we shall, for example, use $(3^3, 2, 1^2)$ to denote the partition $(3, 3, 3, 2, 1, 1)$.

The diagram $[\lambda]$ of λ is like a matrix, its first row contains λ_1 nodes, second row contains λ_2 notes,... The conjugated to λ partition, denoted by λ' is the one, whose diagram is obtained from $[\lambda]$ by rotating along its diagonal $180°$. A standard λ-tableau it the diagram $[\lambda]$ filled with the numbers $1, 2, ..., |\lambda|$ in such a way that they are increasing along rows and columns. The content of the node x in the i^{th}-row and j^{th}-column is $j - i$, it is denoted by $c_\lambda(x)$. For a tableau $t(\lambda)$ we also speak of the content $c_{t(\lambda)}(m)$ of a number m which is the content of the node containing m. The hook-length of the node x is $h_\lambda(x) := \lambda_i + \lambda'_j - i - j + 1$. The number d_λ of all standard λ-tableau is equal to [5] $d_\lambda = n! \prod_{x \in [\lambda]} h_\lambda(x)^{-1}$.

A (r, s)-hook-partition is a partition, whose $(r+1)^{\text{th}}$-term is less than or equal to s. The set of all (r, s)-hook-partition is denoted by $\Gamma^{r,s}$, its subset of all partitions, of which the r^{th}-term is greater then or equal to s is denoted by $\Omega^{r,s}$. $\Gamma_n^{r,s} := \Gamma^{r,s} \cap \mathcal{P}_n$, $\Omega_n^{r,s} := \Omega^{r,s} \cap \mathcal{P}_n$.

The Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ is generated by the elements $T_i, i = 1, 2, ..., n-1$, subject to the relations: $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, and $T_i^2 = (q-1)T_i + q$. It is shown that $\mathcal{H}_n$ has a basis over $\mathbb{K}$ $\{T_w | w \in \mathfrak{S}_n\}$, $\mathfrak{S}_n$ is the symmetric group, for which $T_{(i,i+1)} = T_i$. If q is not a root of unity of order greater than 1, then $\mathcal{H}_n$ is semisimple. Thus it is decomposed into a direct sum of its minimal two-sided ideals, which are shown to be parameterized by partitions of n:

$$\mathcal{H}_n = \bigoplus_{\lambda \vdash n} \mathcal{A}_\lambda.$$

$\mathcal{A}_\lambda$ itself is a matrix ring over $\mathbb{K}$, its unit element is the minimal central idempotent F_λ. This element is uniquely determined by λ. Dipper and James showed in [4] that the set

$$E_{t_i(\lambda)} = \prod_{\substack{1 \leq m \leq n \\ 1-m \leq k \leq m-1 \\ k \neq c_{t_i(\lambda)}(m)}} \frac{L_m - [k]_q}{[c_{t_i(\lambda)}(m)]_q - [k]_q}, \quad 1 \leq i \leq d_\lambda,$$

is a full set of primitive idempotent of $\mathcal{A}_\lambda$. Their sum is F_λ.

References

[1] J. Alfaro, R. Medina, and L.F. Urrutia. The Itzykson-Zuber integral for U$(m|n)$. *J. Math. Phys.*, 36(6):3085–3093, 1995.

[2] J. Alfaro, R. Medina, and L.F. Urrutia. The orthogonality relations for the supergroup U$(m|n)$. *J. Phys. A*, 28(16):4581–4588, 1995.

[3] C. C. Itzykson and J.B. Zuber. The planar approximation. II. *J. Math. Phys*, 21:411–421, 1980.

[4] R. Dipper and G. James. Block and Idempotents of Hecke Algebras of General Linear Groups. *Proc. London Math. Soc.*, 54(3):57–82, 1987.

[5] J.A. Green. Polynomial Representations of GL$_n$. *Lecture Note in Math.*, 830, 1980.

[6] Phung Ho Hai. Koszul Property and Poincaré Series of Matrix Bialgebra of Type A_n. *Journal of Algebra*, 192(2):734–748, 1997.

[7] Phung Ho Hai. On Matrix Quantum Groups of Type A_n. *Preprint ICTP/97101,* `q-alg/9708007`, 1997.

[8] Phung Ho Hai. Characters for Quantum Groups of Type A_n. *Preprint ICTP,* `xxx.lanl.gov/dvi/q-alg/98-7045`, 1998.

[9] Phung Ho Hai. Poincaré Series of Quantum Spaces Associated to Hecke Operators. *Acta Math. Vietnam., to appear*, 1998. Available at `xxx.lanl.gov/dvi/q-alg/9711020`.

[10] Harish-Chandra. A formula for semisimple lie groups. *Amer. J. Math*, 79:733–760, 1957.

[11] M. Jimbo. A q-analogue of U$(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Lett. Math. Phys.*, 11:247–252, 1986.

[12] Ch Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics.* Springer-Verlag, 1995. 531p.

[13] I.G. Macdonald. *Symmetric functions and the Hall polynomials.* Oxford University Press, 1979 (Second edition 1995). New York.

[14] Yu.I. Manin. Multiparametric Quantum Deformation of the General Linear Supergroups. *Com. Math. Phys.*, 123:163–175, 1989.

[15] J.B. Sullivan. The Uniqueness of Integral for Hopf Algebras and Some Existence Theorems of Integrals for Commutative Hopf Algebras. *Journal of Algebra*, 19:426–440, 1971.

[16] M. Sweedler. *Hopf Algebras.* Benjamin, New York, 1969.

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