A NOTE ON THE SYMPLECTIC STRUCTURE ON THE SPACE OF G-MONOPOLES

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1. Introduction

1.1. Let $\mathbf{G}$ be a semisimple complex Lie group with the Cartan datum $(I, \cdot)$ and the root datum $(Y, X, \dots)$. Let $\mathbf{H} \subset \mathbf{B} = \mathbf{B}_+, \mathbf{B}_- \subset \mathbf{G}$ be a Cartan subgroup and a pair of opposite Borel subgroups respectively. Let $\mathbf{X} = \mathbf{G}/\mathbf{B}$ be the flag manifold of $\mathbf{G}$. Let $C = \mathbb{P}^1 \ni \infty$ be the projective line. Let $\alpha = \sum_{i \in I} a_i i \in \mathbb{N}[I] = H_2(\mathbf{X}, \mathbb{Z})$.

The moduli space of $\mathbf{G}$-*monopoles of topological charge* α (see e.g. [4]) is naturally identified with the space $\mathcal{M}_b(\mathbf{X}, \alpha)$ of based maps from (C, ∞) to $(\mathbf{X}, \mathbf{B}_+)$. The moduli space of $\mathbf{G}$-monopoles carries a natural hyperkähler structure, and hence a holomorphic symplectic structure. We propose a simple explicit formula for the symplectic structure on $\mathcal{M}_b(\mathbf{X}, \alpha)$. It generalizes the well known formula for $\mathbf{G} = SL_2$ [1].

1.2. Recall that for $\mathbf{G} = SL_2$ we have $(\mathbf{X}, \mathbf{B}_+) = (\mathbb{P}^1, \infty)$. Recall the natural local coordinates on $\mathcal{M}_b(\mathbb{P}^1, a)$ (see [1]). We fix a coordinate z on C such that $z(\infty) = \infty$. Then a based map $\phi : (C, \infty) \to (\mathbb{P}^1, \infty)$ of degree a is a rational function $\dfrac{p(z)}{q(z)}$ where $p(z)$ is a degree a polynomial with the leading coefficient 1, and $q(z)$ is a degree $< a$ polynomial. Let U be the open subset of based maps such that the roots $x^1, \dots, x^a$ of $p(z)$ are multiplicity free. Let y^k be the value of $q(z)$ at x^k. Then $x^1, \dots, x^a, y^1, \dots, y^a$ form an étale coordinate system on U. The symplectic form Ω on $\mathcal{M}_b(\mathbb{P}^1, a)$ equals $\sum_{k=1}^a \dfrac{dy^k \wedge dx^k}{y^k}$. In other words, the Poisson brackets of these coordinates are as follows: $\{x^k, x^m\} = 0 = \{y^k, y^m\}$; $\{x^k, y^m\} = \delta_{km} y^m$.

For an arbitrary $\mathbf{G}$ and $i \in I$ let $\mathbf{X}_i \subset \mathbf{X}$ be the corresponding codimension 1 $\mathbf{B}_-$-orbit (Schubert cell), and let $\overline{\mathbf{X}}_i \supset \mathbf{X}_i$ be its closure (Schubert variety). For $\phi \in \mathcal{M}_b(\mathbf{X}, \alpha)$ we define $x_i^1, \dots, x_i^{a_i} \in \mathbb{A}^1$ as the points of intersection of $\phi(\mathbb{P}^1)$ with $\overline{\mathbf{X}}_i$. This way we obtain the projection $\pi^\alpha : \mathcal{M}_b(\mathbf{X}, \alpha) \to \mathbb{A}^\alpha$ (the configuration space of I-colored divizors of degree α on $\mathbb{A}^1$). Let $U \subset \mathcal{M}_b(\mathbf{X}, \alpha)$ be the open subset of based maps such that $\phi(\mathbb{P}^1) \cap \overline{\mathbf{X}}_i \subset \mathbf{X}_i$ for any $i \in I$, and $x_i^k \neq x_j^l$ for any $i, j \in I$, $1 \le k \le a_i$, $1 \le l \le a_j$. Locally in $\mathbf{X}$ the cell $\mathbf{X}_i$ is the zero divizor of a function φ_i (globally, φ_i is a section of the line bundle L_{ω_i} corresponding to the fundamental weight $\omega_i \in X$). The rational function

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$\varphi_i \circ \phi$ on C is of the form $\dfrac{p_i(z)}{q_i(z)}$ where $p_i(z)$ is a degree a_i polynomial with the leading coefficient 1, and $q_i(z)$ is a degree $< a_i$ polynomial. Let y_i^k be the value of $q_i(z)$ at x_i^k. Then x_i^k, y_i^k, $i \in I$, $1 \leq k \leq a_i$, form an étale coordinate system on U. The Poisson brackets of these coordinates are as follows:

$$\{x_i^k, x_j^l\} = 0 = \{y_i^k, y_i^l\};\ \{x_i^k, y_j^l\} = \delta_{ij}\delta_{kl}y_j^l;\ \{y_i^k, y_j^l\} = i \cdot j \frac{y_i^k y_j^l}{x_i^k - x_j^l} \text{ for } i \neq j.$$

1.3. It follows that the symmetric functions of the x-coordinates (well defined on the whole $\mathcal{M}_b(\mathbf{X}, \alpha)$) are in involution. In other words, the projection $\pi^\alpha:\ \mathcal{M}_b(\mathbf{X}, \alpha) \to \mathbb{A}^\alpha$ is an integrable system on $\mathcal{M}_b(\mathbf{X}, \alpha)$. The fibers of $\pi^\alpha:\ U \to \mathbb{A}^\alpha$ are Lagrangian submanifolds of U. It is known that all the fibers of $\pi^\alpha:\ \mathcal{M}_b(\mathbf{X}, \alpha) \to \mathbb{A}^\alpha$ are equidimensional of the same dimension $|\alpha|$ (see [3]), hence π^α is flat, hence all the fibers are Lagrangian.

1.4. Let $\mathbf{P} \supset \mathbf{B}$ be a parabolic subgroup. The construction of the Poisson structure on $\mathcal{M}_b(\mathbf{X}, \alpha)$ generalizes *verbatim* to the space of based maps $\mathcal{M} = \mathcal{M}_b(\mathbf{G}/\mathbf{P}, \beta)$. In most cases the corresponding map $T\mathcal{M} \to T^*\mathcal{M}$ is not an isomorphism, i.e. $\mathcal{M}$ splits into nontrivial symplectic leaves.

1.5. The above Poisson structure is a baby (rational) version of the Poisson structure on the moduli space of $\mathbf{B}$-bundles over an elliptic curve [2]. We learnt of its definition (as a differential in the hypercohomology spectral sequence, see §2) from N.Markaryan. Thus, our modest contribution reduces just to a proof of Jacobi identity. Note that the Poisson structure of [2] arises as a quasiclassical limit of *elliptic algebras*. On the other hand, $\mathcal{M}_b(\mathbf{X}, \alpha)$ is an open subset in the moduli space $\widehat{\mathcal{M}_b}(\mathbf{X}, \alpha)$ of $\mathbf{B}$-bundles on C trivialized at ∞, such that the induced $\mathbf{H}$-bundle has degree α. One can see easily that $\widehat{\mathcal{M}_b}(\mathbf{X}, \alpha)$ is isomorphic to an affine space $\mathbb{A}^{2|\alpha|}$, and the symplectic structure on $\mathcal{M}_b(\mathbf{X}, \alpha)$ extends to the Poisson structure on $\widehat{\mathcal{M}_b}(\mathbf{X}, \alpha)$. The latter one can be quantized along the lines of [2].

It is clear from the above discussion that the present note owes its existense to the generous explanations of B.Feigin and N.Markaryan.

1.6. **Notations.** For a subset $J \subset I$ we denote by $\mathbf{P}_J \supset \mathbf{B}$ the corresponding parabolic subgroup. Thus, $\mathbf{P}_\emptyset = \mathbf{B}$. Denote by $\mathbf{X}_J = \mathbf{G}/\mathbf{P}_J$ the corresponding parabolic flag variety; thus, $\mathbf{X}_\emptyset = \mathbf{X}$. We denote by $\varpi:\ \mathbf{X} \to \mathbf{X}_J$ the natural projection. We denote by $x \in \mathbf{X}_J$ the marked point $\varpi(\mathbf{B}_+)$.

Let $\mathcal{M} = \mathcal{M}_b(\mathbf{X}_J, \alpha)$ denote the space of based algebraic maps $\phi : (C, \infty) \to (\mathbf{X}_J, x)$ of degree $\alpha \in H_2(\mathbf{X}_J, \mathbb{Z})$.

Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$. Let $\mathfrak{g}_{\mathbf{X}_J}$ denote the trivial vector bundle with the fiber $\mathfrak{g}$ over $\mathbf{X}_J$ and let $\mathfrak{p}_{\mathbf{X}_J} \subset \mathfrak{g}_{\mathbf{X}_J}$ (resp. $\mathfrak{r}_{\mathbf{X}_J} \subset \mathfrak{p}_{\mathbf{X}_J} \subset \mathfrak{g}_{\mathbf{X}_J}$) be its subbundle with the fiber over a point P equal to the corresponding Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ (resp. its nilpotent radical $\mathfrak{r} \subset \mathfrak{p} \subset \mathfrak{g}$). In case $J = \emptyset$ we will also denote $\mathfrak{r}_{\mathbf{X}_\emptyset}$ by $\mathfrak{n}_{\mathbf{X}}$, and $\mathfrak{p}_{\mathbf{X}_\emptyset}$ by $\mathfrak{b}_{\mathbf{X}}$. Note that the quotient bundle $\mathfrak{h}_{\mathbf{X}} := \mathfrak{b}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}}$ is trivial (abstract Cartan algebra).

Recall that the tangent bundle $T\mathbf{X}_J$ of $\mathbf{X}_J$ (resp. cotangent bundle $T^*\mathbf{X}_J$) is canonicaly isomorphic to the bundle $\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J}$ (resp. $\mathfrak{r}_{\mathbf{X}_J}$).

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2. The Poisson structure

2.1. The fibers of the tangent and cotangent bundles of the space $\mathcal{M}$ at the point ϕ are computed as follows:

$$T_\phi\mathcal{M} = H^0(C, (\phi^*T\mathbf{X}_J)\otimes\mathcal{O}_C(-1)) = H^0(C, (\phi^*\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1)),$$
$$T^*_\phi\mathcal{M} = H^1(C, (\phi^*T^*\mathbf{X}_J)\otimes\mathcal{O}_C(-1)) = H^1(C, (\phi^*\mathfrak{r}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1)).$$

The second identification follows from the first by the Serre duality.

We have a tautological complex of vector bundles on $\mathbf{X}_J$:

$$\mathfrak{r}_{\mathbf{X}_J}\to\mathfrak{g}_{\mathbf{X}_J}\to\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J} \tag{1}$$

The pull-back via ϕ of this complex twisted by $\mathcal{O}_C(-1)$ gives the following complex of vector bundles on C

$$(\phi^*\mathfrak{r}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1)\to(\phi^*\mathfrak{g}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1)\to(\phi^*\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1) \tag{2}$$

Consider the hypercohomology spectral sequence of the complex (2). Since $\mathfrak{g}_{\mathbf{X}_J}$ is the trivial vector bundle we have $H^\bullet(C, (\phi^*\mathfrak{g}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1)) = 0$, hence the second differential of the spectral sequence induces a map

$$d_2 : H^1(C, (\phi^*\mathfrak{r}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1))\to H^0(C, (\phi^*\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J})\otimes\mathcal{O}_C(-1))$$

that is a map $P^{\mathbf{X}_J}_\phi : T^*_\phi\mathcal{M}\to T_\phi\mathcal{M}$. This construction easily globalizes to give a morphism $P^{\mathbf{X}_J} :\ T^*\mathcal{M}\to T\mathcal{M}$.

Theorem 1. *P defines a Poisson structure on $\mathcal{M}$.*

Here we will reduce the Theorem to the case $\mathbf{X}_J = \mathbf{X}$. This case will be treated in the next section.

2.2. Let $\varpi_* : H_2(\mathbf{X}, \mathbb{Z})\to H_2(\mathbf{X}_J, \mathbb{Z})$ be the push-forward map. The map ϖ induces a map $\Pi : \mathcal{M}_b(\mathbf{X}, \alpha)\to\mathcal{M}_b(\mathbf{X}_J, \varpi_*\alpha)$.

Proposition 1. *The map Π respects P, that is the following square is commutative*

$$\begin{array}{ccc}
T^*\mathcal{M}_b(\mathbf{X},\alpha) & \xrightarrow{P^{\mathbf{X}}} & T\mathcal{M}_b(\mathbf{X},\alpha) \\
{\scriptstyle\Pi^*}\uparrow & & \downarrow{\scriptstyle\Pi_*} \\
\Pi^*T^*\mathcal{M}_b(\mathbf{X}_J,\varpi_*\alpha) & \xrightarrow{P^{\mathbf{X}_J}} & \Pi^*T\mathcal{M}_b(\mathbf{X}_J,\varpi_*\alpha)
\end{array}$$

Proof. We have the following commutative square on $\mathbf{X}$

$$\begin{array}{ccccc}
\mathfrak{n}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}}/\mathfrak{b}_{\mathbf{X}} \\
\uparrow & & \| & & \| \\
\varpi^*\mathfrak{r}_{\mathbf{X}_J} & \longrightarrow & \mathfrak{g}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}}/\mathfrak{b}_{\mathbf{X}} \\
\| & & \| & & \downarrow \\
\varpi^*\mathfrak{r}_{\mathbf{X}_J} & \longrightarrow & \varpi^*\mathfrak{g}_{\mathbf{X}_J} & \longrightarrow & \varpi^*(\mathfrak{g}_{\mathbf{X}_J}/\mathfrak{p}_{\mathbf{X}_J})
\end{array}$$

Consider its pull-back via $\phi \in \mathcal{M}_b(\mathbf{X},\alpha)$ twisted by $\mathcal{O}_C(-1)$. Let d_2 denote the second differential of the hypercohomology spectral sequence of the middle row. Then we evidently have

$$P^{\mathbf{X}} \cdot \Pi^* = d_2, \qquad \Pi_* \cdot d_2 = P^{\mathbf{X}_J}$$

and the Proposition follows. □

Now, assume that we have proved that $P^{\mathbf{X}}$ defines a Poisson structure. For any $\beta \in H_2(\mathbf{X}_J,\mathbb{Z})$ we can choose $\alpha \in H_2(\mathbf{X},\mathbb{Z})$ such that $\varpi_*\alpha = \beta$ and the map Π is open. Then the algebra of functions on $\mathcal{M}_b(\mathbf{X}_J,\beta)$ is embedded into the algebra of functions on $\mathcal{M}_b(\mathbf{X},\alpha)$ and the Proposition 1 shows that the former bracket is induced by the latter one. Hence it is also a Poisson bracket.

3. The case of $\mathbf{X}$

In this section we will denote $\mathcal{M}_b(\mathbf{X},\alpha)$ simply by $\mathcal{M}$.

3.1. Since $\mathfrak{h}$ is a trivial vector bundle on $\mathbf{X}$ the exact sequences

$$0 \to \mathfrak{n}_{\mathbf{X}} \to \mathfrak{b}_{\mathbf{X}} \to \mathfrak{h}_{\mathbf{X}} \to 0, \qquad 0 \to \mathfrak{h}_{\mathbf{X}} \to \mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}} \to \mathfrak{g}_{\mathbf{X}}/\mathfrak{b}_{\mathbf{X}} \to 0$$

induce the isomorphisms

$$\begin{aligned}
T_\phi\mathcal{M} &= H^0(C,(\phi^*\mathfrak{g}_{\mathbf{X}}/\mathfrak{b}_{\mathbf{X}})\otimes\mathcal{O}_C(-1)) = H^0(C,(\phi^*\mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}})\otimes\mathcal{O}_C(-1)),\\
T^*_\phi\mathcal{M} &= H^1(C,(\phi^*\mathfrak{n}_{\mathbf{X}})\otimes\mathcal{O}_C(-1)) \qquad = H^1(C,(\phi^*\mathfrak{b}_{\mathbf{X}})\otimes\mathcal{O}_C(-1))
\end{aligned}$$

Applying the construction of 2.1 to the following tautological complex of vector bundles on $\mathbf{X}$

$$\mathfrak{b}_{\mathbf{X}} \to \mathfrak{g}_{\mathbf{X}} \oplus \mathfrak{h}_{\mathbf{X}} \to \mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}} \tag{3}$$

and taking into account the above isomorphisms we get a map $\widetilde{P}^{\mathbf{X}}_\phi : T^*_\phi\mathcal{M} \to T_\phi\mathcal{M}$.

Lemma 1. *We have $\widetilde{P}^{\mathbf{X}}_\phi = P^{\mathbf{X}}_\phi$.*

Proof. The same reasons as in the proof of the Proposition 1 work if we consider the following commutative diagram

$$\begin{array}{ccccc}
\mathfrak{n}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}}/\mathfrak{b}_{\mathbf{X}} \\
\| & & \| & & \uparrow \\
\mathfrak{n}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}} \\
\downarrow & & \downarrow & & \| \\
\mathfrak{b}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}} \oplus \mathfrak{h}_{\mathbf{X}} & \longrightarrow & \mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}}
\end{array}$$

□

It will be convenient for us to use the complex (3) for the definition of the map $P^{\mathbf{X}}_{\phi}$ instead of (1).

3.2. Here we will describe the Plücker embedding of the space $\mathcal{M}$.

Let $X \supset \mho \cong I$ be the set of fundamental weights: $\langle i, \omega_j \rangle = \delta_{ij}$. We denote by $(,)$ the scalar product on X such that $(i', j') = i \cdot j$ for the simple roots i', j'. For a dominant weight $\lambda \in X$ we denote by V_λ the irreducible G-module with highest weight λ.

Recall that $\mathbf{X}$ is canonically embedded into the product of projective spaces

$$\mathbf{X} \subset \prod_{\omega \in \mho} \mathbb{P}(V_\omega)$$

This induces the embedding

$$\mathcal{M} \subset \prod_{\omega \in \mho} \mathcal{M}_b(\mathbb{P}(V_\omega), \langle \alpha, \omega \rangle)$$

Note that the marked point of the space $\mathbb{P}(V_\omega)$ is just the highest weight vector v_ω with respect to the Borel subgroup $\mathbf{B}$.

A degree d based map $\phi_\omega : (C, \infty) \to (\mathbb{P}(V_\omega), v_\omega)$ can be represented by a V_ω-valued degree d polynomial in z, taking the value v_ω at infinity. Let us denote the affine space of such polynomials by $R_d(V_\omega)$.

The *Plücker embedding* of the space $\mathcal{M}$ is the embedding into the product of affine spaces

$$\mathcal{M} \subset \prod_{\omega \in \mho} R_{\langle \alpha, \omega \rangle}(V_\omega).$$

A map $\phi \in \mathcal{M}$ will be represented by a collection of polynomials $(\phi_\omega \in R_{\langle \alpha, \omega \rangle}(V_\omega))_{\omega \in \mho}$.

3.3. **The coordinates.** The dual representation V^*_ω decomposes into the sum of weight subspaces

$$V^*_\omega = \oplus_{\lambda \in X} {V^*}^{\lambda}_{\omega}.$$

We choose a weight base (f^λ_ω) of V^*_ω, such that $f^{-\omega}_\omega(v_\omega) = 1$. Suppose $\langle i, \omega \rangle = 1$. Then $\dim {V^*}^{-\omega}_{\omega} = \dim {V^*}^{-\omega+i'}_{\omega} = 1$, and $\dim {V^*}^{-\omega+i'+j'}_{\omega} = 0$ if $i \cdot j = 0$, and $\dim {V^*}^{-\omega+i'+j'}_{\omega} = 1$ if $i \cdot j \neq 0$. Hence, in the latter case, the vectors $f^{-\omega}_\omega$, $f^{i'-\omega}_\omega$ and $f^{i'+j'-\omega}_\omega$ are defined uniquely up to multiplication by a constant. Let E_i, F_i, H_i be the standard generators of $\mathfrak{g}$. Then we will take $f_\omega^{i'-\omega} := E_i f_\omega^{-\omega}$, $f_\omega^{i'+j'-\omega} := E_j E_i f_\omega^{-\omega}$.

We consider the polynomials $\phi_\omega^\lambda := f_\omega^\lambda(\phi_\omega)$: the λ weight components of ϕ_ω. In particular, $\phi_\omega^{-\omega}$ is the degree $\langle\alpha,\omega\rangle$ unitary polynomial and $\phi_\omega^{i'-\omega}$ is the degree $< \langle\alpha,\omega\rangle$ polynomial.

Let $x_\omega^1, \dots, x_\omega^{\langle\alpha,\omega\rangle}$ be the roots of $\phi_\omega^{-\omega}$ and $y_\omega^1, \dots, y_\omega^{\langle\alpha,\omega\rangle}$ be the values of $\phi_\omega^{i'-\omega}$ at the points $x_\omega^1, \dots, x_\omega^{\langle\alpha,\omega\rangle}$ respectively. Consider the open subset $U \subset \mathcal{M}$ formed by all the maps ϕ such that all x_ω^k are distinct and all y_ω^k are non-zero. On this open set we have

$$\phi_\omega^{-\omega}(z) = \prod_{k=1}^{\langle\alpha,\omega\rangle}(z - x_\omega^k), \qquad \phi_\omega^{i'-\omega}(z) = \sum_{k=1}^{\langle\alpha,\omega\rangle} \frac{y_\omega^k \phi_\omega^{-\omega}(z)}{(\phi_\omega^{-\omega})'(x_\omega^k)(z - x_\omega^k)}.$$

The collection of $2|\alpha|$ functions

$$(x_\omega^k, y_\omega^k), \qquad (\omega \in \mho,\ 1 \le k \le \langle\alpha,\omega\rangle) \tag{4}$$

is an étale coordinate system in U. One can either check this straightforwardly, or just note that the matrix of $P^{\mathbf{X}}$ in these coordinates has a maximal rank, see the Remark 2 below.

So let us compute the map $P^{\mathbf{X}}$ in these coordinates.

3.4. The action of $\mathfrak{g}$ on V_ω induces an embedding

$$\mathfrak{g}_{\mathbf{X}}/\mathfrak{n}_{\mathbf{X}} \subset \bigoplus_{\omega\in\mho} V_\omega \otimes L_\omega$$

of vector bundles over $\mathbf{X}$ and the dual surjection

$$\bigoplus_{\omega\in\mho} V_\omega^* \otimes L_\omega^* \to \mathfrak{b}_{\mathbf{X}},$$

where L_ω stands for the line bundle, corresponding to the weight ω. Hence we have the following complex

$$\bigoplus_{\omega\in\mho} V_\omega^* \otimes L_\omega^* \to \mathfrak{g}_{\mathbf{X}} \oplus \mathfrak{h}_{\mathbf{X}} \to \bigoplus_{\omega\in\mho} V_\omega \otimes L_\omega \tag{5}$$

Remark 1. The differentials in the above complex in the fiber over a point $\mathbf{B}' \in \mathbf{X}$ are computed as follows:

$$\varphi \in V_\omega^* \mapsto \left(\sum \varphi(\xi^k v')\xi_k\right) \oplus \left(\sum \omega(h^i)\varphi(v')h_i\right) \in \mathfrak{g} \oplus \mathfrak{h},$$

$$\xi \oplus h \in \mathfrak{g} \oplus \mathfrak{h} \mapsto \xi v' - \omega(h)v' \in V_\omega.$$

Here v' is a highest weight vector of V_ω with respect to $\mathbf{B}'$; (ξ_k), (ξ^k) are dual (with respect to the standard scalar product) bases of $\mathfrak{g}$; and (h_i), (h^i) are dual bases of $\mathfrak{h}$.

3.5. In order to compute the brackets of the coordinates (4) at a point $\phi \in \mathcal{M}$ we need to take the pull-back of the complex (5) via ϕ, twist it by $\mathcal{O}_C(-1)$ and compute the second differential of the hypercohomology spectral sequence. The following Lemma describes this differential in general situation.

Lemma 2. *Consider a complex* $K^\bullet = (\mathcal{F} \xrightarrow{f} A \otimes \mathcal{O}_C \xrightarrow{g} \mathcal{G})$ *on* C, *where* A *is a vector space and*

$$\begin{array}{lcl} f \in Hom(\mathcal{F}, A \otimes \mathcal{O}_C) & = & A \otimes H^0(C, \mathcal{F}^*), \\ g \in Hom(A \otimes \mathcal{O}_C, \mathcal{G}) & = & A^* \otimes H^0(C, \mathcal{G}). \end{array}$$

Consider

$$D = \mathrm{tr}(f \otimes g) \in H^0(C, \mathcal{F}^*) \otimes H^0(C, \mathcal{G}) = H^0(C \times C, \mathcal{F}^* \boxtimes \mathcal{G}),$$

where $\mathrm{tr} : A \otimes A^* \to \mathbb{C}$ *is the trace homomorphism. Then*

1) *The restriction of* D *to the diagonal* $\Delta \subset C \times C$ *vanishes, hence* $D = \widetilde{D}\Delta$ *for some*

$$\widetilde{D} \in H^0(C \times C, (\mathcal{F}^* \boxtimes \mathcal{G})(-\Delta)) = H^0(C, \mathcal{F}^*(-1)) \otimes H^0(C, \mathcal{G}(-1)) = \\ = H^1(C, \mathcal{F}(-1))^* \otimes H^0(C, \mathcal{G}(-1)).$$

2) *The second differential* $d_2 : H^1(C, \mathcal{F}(-1)) \to H^0(C, \mathcal{G}(-1))$ *of the hypercohomology spectral sequence of* $K^\bullet \otimes \mathcal{O}_C(-1)$ *is induced by the section* $\widetilde{D}$.

Proof. The first statement is evident. To prove the second statement consider the following commutative diagram on $C \times C$

$$\begin{array}{ccccc} \mathcal{F}(-1) \boxtimes \mathcal{O}_C & \xrightarrow{f(-1)\boxtimes 1} & A \otimes \mathcal{O}_{C\times C}(-1,0) & \xrightarrow{(1\boxtimes g)|_\Delta} & (\mathcal{O}_C(-1) \boxtimes \mathcal{G})|_\Delta \\ {\scriptstyle \widetilde{D}}\big\downarrow & & {\scriptstyle 1\boxtimes g}\big\downarrow & & \big\| \\ \mathcal{O}_C(-2) \boxtimes \mathcal{G}(-1) & \xrightarrow{\Delta} & \mathcal{O}_C(-1) \boxtimes \mathcal{G} & \xrightarrow{|_\Delta} & (\mathcal{O}_C(-1) \boxtimes \mathcal{G})|_\Delta \end{array}$$

Both rows are complexes with acyclic middle term, hence the second differentials of the hypercohomology spectral sequences commute with the maps induced on cohomology by the vertical arrows:

$$\begin{array}{ccc} H^1(C \times C, \mathcal{F}(-1) \boxtimes \mathcal{O}_C) & \xrightarrow{d_2} & H^0(C \times C, (\mathcal{O}_C(-1) \boxtimes \mathcal{G})|_\Delta) \\ {\scriptstyle \widetilde{D}}\big\downarrow & & \big\| \\ H^1(C \times C, \mathcal{O}_C(-2) \boxtimes \mathcal{G}(-1)) & \xrightarrow{d_2} & H^0(C \times C, (\mathcal{O}_C(-1) \boxtimes \mathcal{G})|_\Delta) \end{array}$$

Now it remains to note that

$$\begin{array}{lcl} H^1(C \times C, \mathcal{F}(-1) \boxtimes \mathcal{O}_C) & = & H^1(C, \mathcal{F}(-1)), \\ H^1(C \times C, \mathcal{O}_C(-2) \boxtimes \mathcal{G}(-1)) & = & H^0(C, \mathcal{G}(-1)), \\ H^0(C \times C, (\mathcal{O}_C(-1) \boxtimes \mathcal{G})|_\Delta) & = & H^0(C, \mathcal{G}(-1)), \end{array}$$

and that the map $H^0(C, \mathcal{G}(-1)) \to H^0(C, \mathcal{G}(-1))$ induced by the map d_2 in the second row of the above diagram is identity. □

3.6. Consider the pullback of (5) via $\phi \in \mathcal{M}$, and twist it by $\mathcal{O}_C(-1)$. We want to apply Lemma 2 to compute the (ω_i, ω_j)-component of the second differential of the hypercohomology spectral sequence.

In notations of the Lemma we have

$$D_{\omega_i,\omega_j}(z,w) = \sum \xi^k \phi_{\omega_i}(z) \otimes \xi_k \phi_{\omega_j}(w) - \sum \omega_i(h^i)\phi_{\omega_i}(z) \otimes \omega_j(h_i)\phi_{\omega_j}(w) =$$
$$= \sum \xi^k \phi_{\omega_i}(z) \otimes \xi_k \phi_{\omega_j}(w) - (\omega_i,\omega_j)\phi_{\omega_i}(z) \otimes \phi_{\omega_j}(w) \in V_{\omega_i} \otimes V_{\omega_j}(z,w).$$

Lemma 3. *The operator $\sum \xi^k \otimes \xi_k - (\omega_i, \omega_j)$ acts as a scalar multiplication on every irreducible summand $V_\lambda \subset V_{\omega_i} \otimes V_{\omega_j}$.*

On $V_{\omega_i+\omega_j}$ it acts as a multiplication by 0.

If $\omega_i = \omega_j$, then on $V_{2\omega_i - i'} \subset V_{\omega_i} \otimes V_{\omega_i}$ it acts as a multiplication by (-2).

If $i \neq j$, $i \cdot j \neq 0$ then on $V_{\omega_i+\omega_j-i'-j'} \subset V_{\omega_i} \otimes V_{\omega_j}$ it acts as a multiplication by $(i \cdot j - 2)$.

Proof. It is easy to check that $\sum \xi^k \otimes \xi_k$ commutes with the natural action of $\mathfrak{g}$ on $V_{\omega_i} \otimes V_{\omega_j}$. The first part of the Lemma follows. The rest of the Lemma can be checked by the straightforward computation of the action of $\sum \xi^k \otimes \xi_k$ on the highest vectors of the corresponding subrepresentations. □

3.7. If we want to compute the brackets of the coordinates (4) we are interested in the components of $D_{\omega_i,\omega_j}(z,w)$ in the weights

$$\omega_i + \omega_j, \quad \omega_i + \omega_j - i', \quad \omega_i + \omega_j - j', \quad \omega_i + \omega_j - i' - j'. \tag{6}$$

The following Lemma describes the corresponding weight components of the tensor product $V_{\omega_i} \otimes V_{\omega_j}$.

Lemma 4. *The embedding $V_{\omega_i+\omega_j} \subset V_{\omega_i} \otimes V_{\omega_j}$ induces an isomorphism in the weights* (6) *with the following two exceptions:*
(1) $(V_{\omega_i} \otimes V_{\omega_i})^{2\omega_i - i'} = V_{2\omega_i}^{2\omega_i - i'} \oplus V_{2\omega_i - i'}^{2\omega_i - i'}$*; the G-projection to the second summand is given by the formula*

$$a(v_{\omega_i} \otimes F_i v_{\omega_i}) + b(F_i v_{\omega_i} \otimes v_{\omega_i}) \mapsto \frac{a-b}{2}(v_{\omega_i} \otimes F_i v_{\omega_i} - F_i v_{\omega_i} \otimes v_{\omega_i}).$$

(2) $(V_{\omega_i} \otimes V_{\omega_j})^{\omega_i+\omega_j-i'-j'} = V_{\omega_i+\omega_j}^{\omega_i+\omega_j-i'-j'} \oplus V_{\omega_i+\omega_j-i'-j'}^{\omega_i+\omega_j-i'-j'}$ *if $i \neq j$ and $i \cdot j \neq 0$; the G-projection to the second summand is given by the formula*

$$a(v_{\omega_i} \otimes F_i F_j v_{\omega_j}) + b(F_i v_{\omega_i} \otimes F_j v_{\omega_j}) + c(F_j F_i v_{\omega_i} \otimes v_{\omega_j}) \mapsto$$
$$\mapsto \frac{b-a-c}{i \cdot j - 2}(v_{\omega_i} \otimes F_i F_j v_{\omega_j} + (i \cdot j) F_i v_{\omega_i} \otimes F_j v_{\omega_j} + F_j F_i v_{\omega_i} \otimes v_{\omega_j}).$$

Proof. Straightforward. □

3.8. Hence (see Lemma 3, Lemma 4) when λ is one of the weights (6) the λ-component $\widetilde{D}^{\lambda}_{\omega_i,\omega_j}(z,w)$ of the polynomial $\widetilde{D}_{\omega_i,\omega_j}(z,w)=\dfrac{D_{\omega_i,\omega_j}(z,w)}{z-w}$ is zero with the following two exceptions

$$\widetilde{D}^{2\omega_i-i'}_{\omega_i,\omega_i}=\frac{\phi^{\omega_i}_{\omega_i}(z)\phi^{\omega_i-i'}_{\omega_i}(w)-\phi^{\omega_i-i'}_{\omega_i}(z)\phi^{\omega_i}_{\omega_i}(w)}{z-w}(F_iv_{\omega_i}\otimes v_{\omega_i}-v_{\omega_i}\otimes F_iv_{\omega_i}) \tag{7}$$

$$\begin{aligned}\widetilde{D}^{\omega_i+\omega_j-i'-j'}_{\omega_i,\omega_j}=&\frac{\phi^{\omega_i-i'}_{\omega_i}(z)\phi^{\omega_j-j'}_{\omega_j}(w)-\phi^{\omega_i}_{\omega_i}(z)\phi^{\omega_j-i'-j'}_{\omega_j}(w)-\phi^{\omega_i-i'-j'}_{\omega_i}(z)\phi^{\omega_j}_{\omega_j}(w)}{z-w}\cdot\\&\cdot(v_{\omega_i}\otimes F_iF_jv_{\omega_j}+(i\cdot j)F_iv_{\omega_i}\otimes F_jv_{\omega_j}+F_jF_iv_{\omega_i}\otimes v_{\omega_j})\end{aligned} \tag{8}$$

Note that the scalar multiplicators of the Lemma 3 canceled with the denominators of the Lemma 4.

3.9. Now we can compute the brackets.

Proposition 2. *We have*

$$\begin{aligned}\{x^k_{\omega_i},x^l_{\omega_j}\}&=0;\\ \{x^k_{\omega_i},y^l_{\omega_j}\}&=\delta_{kl}\delta_{ij}y^l_{\omega_j};\\ \{y^k_{\omega_i},x^l_{\omega_j}\}&=-\delta_{kl}\delta_{ij}y^k_{\omega_i};\\ \{y^k_{\omega_i},y^l_{\omega_j}\}&=i\cdot j\frac{y^k_{\omega_i}y^l_{\omega_j}}{x^k_{\omega_i}-x^l_{\omega_j}},\qquad \textit{if } i\neq j;\\ \{y^k_{\omega_i},y^l_{\omega_i}\}&=0.\end{aligned} \tag{9}$$

Proof. Note that if $p\in V_{\omega_i}(z)$ then

$$dy^k_{\omega_i}(p)=\left\langle f^{i'-\omega_i}_{\omega_i},p(x^k_{\omega_i})\right\rangle,\qquad dx^k_{\omega_i}(p)=\left\langle f^{-\omega_i}_{\omega_i},\frac{p(x^k_{\omega_i})}{(\phi^{\omega_i}_{\omega_i})'(x^k_{\omega_i})}\right\rangle,$$

where $\langle\bullet,\bullet\rangle$ stands for the natural pairing. Note also that

$$\phi^{\omega_i}_{\omega_i}(x^k_{\omega_i})=0,\qquad \phi^{\omega_i-i'}_{\omega_i}(x^k_{\omega_i})=y^k_{\omega_i}$$

by definition and

$$\langle f^{i'-\omega_i}_{\omega_i},F_iv_{\omega_i}\rangle=\langle E_if^{-\omega_i}_{\omega_i},F_iv_{\omega_i}\rangle=-\langle f^{-\omega_i}_{\omega_i},E_iF_iv_{\omega_i}\rangle=-1.$$

Now the Proposition follows from the Lemma 2 and from the formulas of 3.8. $\square$

Remark 2. The matrix of the bivector field $P^{\mathbf{X}}$ in the coordinates (x^k_ω, y^k_ω) looks as follows

$$\left(\begin{array}{c|c} 0 & \operatorname{diag}(y^k_\omega) \\ \hline -\operatorname{diag}(y^k_\omega) & * \end{array}\right)$$

Since on the open set U this matrix is evidently nondegenerate it follows that the functions (x^k_ω, y^k_ω) indeed form an étale coordinate system.

3.10. Now we can prove Theorem 1.

Proof of the Theorem 1. The reduction to the case $J = \emptyset$ has been done in 2.2. The latter case is straightforward by the virtue of Proposition 2. □

Corollary 1. *The map $P^{\mathbf{X}}$ provides the space $\mathcal{M}_b(\mathbf{X}, \alpha)$ with a holomorphic symplectic structure.*

Proof. Since $P^{\mathbf{X}}$ gives a Poisson structure it suffices to check that $P^{\mathbf{X}}$ is nondegenrate at any point. To this end recall that the hypercohomology spectral sequence of a complex $K^\bullet$ converges to $H^\bullet(\mathbf{X}, K^\bullet)$. Since the only nontrivial cohomology of the complex (1) is $\mathfrak{h}_{\mathbf{X}}$ in degree zero, the complex (2) is quasiisomorphic to $(\phi^*\mathfrak{h}) \otimes \mathcal{O}_C(-1)$ in degree zero, hence the hypercohomology sequence of the complex (2) converges to zero, hence $P^{\mathbf{X}}_\phi$ is an isomorphism. □

Remark 3. One can easily write down the corresponding symplectic form in the coordinates (4):

$$\sum \frac{dy^k_\omega \wedge dx^k_\omega}{y^k_\omega} + \frac{1}{2}\sum_{i \neq j} i \cdot j \frac{dx^k_{\omega_i} \wedge dx^l_{\omega_j}}{x^k_{\omega_i} - x^l_{\omega_j}}$$

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