A remark on the global indices of $\mathbb{Q}$-Calabi-Yau 3-folds

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A REMARK ON THE GLOBAL INDICES OF $\mathbb{Q}$-CALABI-YAU 3-FOLDS

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Introduction.

It is well known that so called Beauville number $B := 2^5.3^3.5^2.7.11.13.17.19$ is a universal bound of the global indices of $\mathbb{Q}$-Calabi-Yau 3-folds, but it has been unknown whether this number is best possible or not.

In this short note, very much inspired by a recent paper of S. Kondo "Automorphisms of algebraic K3 surfaces which act trivially on Picard groups", we shall show the best possibility of this number:

Main Theorem. *Beauville number $B := 2^5.3^3.5^2.7.11.13.17.19$ is best possible as a universal bound of the global indices of $\mathbb{Q}$-Calabi-Yau 3-folds. More precisely, for each i with $1 \le i \le 8$, there exists a (necessarily smooth) $\mathbb{Q}$-Calabi-Yau 3-fold X_i whose global index $I(X)$ is p_i, where $p_1 = 2^5$, $p_2 = 3^3$, $p_3 = 5^2$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, and $p_8 = 19$.*

We should explain some terms in the main theorem and related known results. By a $\mathbb{Q}$-Calabi-Yau 3-fold ($\mathbb{Q}$-C.Y. 3-fold, for short), we mean a complex projective 3-fold with only terminal singularities and with numerically trivial canonical (Weil) divisor. For a $\mathbb{Q}$-C.Y. 3-fold X, it is shown by Kawamata [Ka1] that there is a positive integer m_X such that $\mathcal{O}_X(m_X K_X) \simeq \mathcal{O}_X$ and the global index $I(X)$ of X is defined as $I(X) := min\{m \in \mathbb{Z}_{>0} | \mathcal{O}_X(mK_X) \simeq \mathcal{O}_X\}$. Note that $I(X)|m$ if and only if $\mathcal{O}_X(mK_X) \simeq \mathcal{O}_X$. By a universal bound of the global indices of $\mathbb{Q}$-C.Y. 3-folds, we mean a positive integer I such that $I(X)|I$ for all $\mathbb{Q}$-C.Y. 3-folds. The existence of a universal bound was first shown by Kawamata [K2]. On the other hand, in [B, Proposition 8, Problem 1 in page 612], Beauville found that Beauville number is a universal bound of the global indices of smooth $\mathbb{Q}$-C.Y. 3-folds, and after these results, Morrison [Mo] proved that we can take the number $120 = 2^3.3.5$ as a universal bound of the global indices of $\mathbb{Q}$-C.Y. 3-folds with at least one singular point and consequently that, apart from its best possibility, Beauville number is a universal bound of the global indices of all $\mathbb{Q}$-C.Y. 3-folds.

We shall prove our main theorem by constructing a K3 surface S_i with a finite automorphism group whose representation on $H^{2,0}(S_i) = \mathbb{C}\omega_{S_i}$ is the p_i-th cyclic group $\{z \in \mathbb{C} | z^{p_i} = 1\} \simeq \mathbb{Z}_{p_i}$ for each $1 \le i \le 8$, where p_i are the integers defined in our main theorem (cf. Proposition 2). For $i \ge 2$, such a K3 surface is already constructed in [Ko, §7]. But, for $i = 1$, or equivalently, for $p_1 = 2^5$, previously

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there seems to be no known examples of such K3 surfaces and our example seems to be new (cf. [Ko], [Ni], [Mu]). In fact, Kondo classified in [Ko] all the finite automorphism groups of K3 surfaces which act trivially on Picard groups, but the 2^5-th cyclic group never has such actions ([Ko, Lemma 6.3]).

Anyway, proof of our main theorem is extremally easy and short. But, our main theorem is still worth mentioning because this establishes a 3-dimensional analogue of the following well known theorem on surfaces in a completely effective way:

Theorem. *The number 12 is the best possible universal bound of the global indices of minimal algebraic surfaces with numerical trivial canonical divisor.*

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Proof of the Main Theorem.

In what follows, we use the following notation:
$p_1 = 2^5$, $p_2 = 3^3$, $p_3 = 5^2$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, and $p_8 = 19$;
e_m = a primitive m-th root of unity in $\mathbb{C}$.

We describe an elliptic surface $\varphi : S \longrightarrow \mathbb{P}^1$ with a section by its affine equation $y^2 = x^3 + a(t)x + b(t)$. For a K3 surface S, we denote by ω_S a non-zero holomorphic 2-form on S.

Lemma 1. *Let S be an algebraic K3 surface on which an automorphism group $< g >\simeq \mathbb{Z}_m$ acts as $g^*\omega_S = e_m\omega_S$. Let E be an elliptic curve and t_m a translation of order m on E. Then the quotient 3-fold $X := S \times E/ < g \times t_m >$ is a (smooth) $\mathbb{Q}$-C.Y. 3-fold whose global index is m.*

Proof. Since $< g \times t_m >\simeq \mathbb{Z}_m$ acts on $S \times E$ freely and since $(g \times t_m)^*\omega_{S\times E} = e_m\omega_{S\times E}$ by definition, the natural etale quotient map $S \times E \longrightarrow X$ of degree m is nothing but the global canonical cover of X.

Now, in order to complete the proof, it is enough to show the following proposition.

Proposition 2. *For each $1 \leq i \leq 8$, there exists a K3 surface S_i with an automorphism group $< g_i >\simeq \mathbb{Z}_{p_i}$ such that $g_i^*\omega_{S_i} = e_{p_i}^{a_i}\omega_{S_i}$, where $(a_i, p_i) = 1$. More concretely, the following pairs (p_i, S_i, g_i) satisfy this requirement:*

(1) $p_1 = 2^5$,
$S_1 : y^2 = x^3 + t^2x + t^{11}$,
$g_1 : (x, y, t) \mapsto (e_{32}^{18}x, e_{32}^{11}y, e_{32}^{2}t)$

(2) $p_2 = 3^3$,
$S_2 : y^2 = x^3 + t(t^9 - 1)$,
$g_2 : (x, y, t) \mapsto (e_{27}^{2}x, e_{27}^{3}y, e_{27}^{6}t)$

(3) $p_3 = 5^2$,
$S_3 : \{z^2 = x_0^6 + x_0x_1^5 + x_1x_2^5\} \subset \mathbb{P}(1,1,1,3)$ *(the finite double covering of $\mathbb{P}^2$ ramified along the non-singular sixtic $\{x_0^6 + x_0x_1^5 + x_1x_2^5 = 0\} \subset \mathbb{P}^2$),*

$g_3 : [x_0 : x_1 : x_2 : z] \mapsto [x_0 : e_{25}^5 x_1 : e_{25}^4 x_2 : z]$

(4) $p_4 = 7$,
$S_4 : y^2 = x^3 + t^3 x + t^8$,
$g_1 : (x, y, t) \mapsto (e_7^3 x, e_7 y, e_7^2 t)$

(5) $p_5 = 11$,
$S_5 : y^2 = x^3 + t^5 x + t^2$,
$g_5 : (x, y, t) \mapsto (e_{11}^5 x, e_{11}^2 y, e_{11}^2 t)$

(6) $p_6 = 13$,
$S_6 : y^2 = x^3 + t^5 x + t$,
$g_6 : (x, y, t) \mapsto (e_{13}^5 x, e_{13} y, e_{13}^2 t)$

(7) $p_7 = 17$,
$S_7 : y^2 = x^3 + t^7 x + t^2$,
$g_7 : (x, y, t) \mapsto (e_{17}^7 x, e_{17}^2 y, e_{17}^2 t)$

(8) $p_8 = 19$,
$S_7 : y^2 = x^3 + t^7 x + t$,
$g_7 : (x, y, t) \mapsto (e_{19}^7 x, e_{19} y, e_{19}^2 t)$

Remark. *As was mentioned in the introduction, examples (2)-(8) already appeared in [Ko, §7] while an example (1) is new. In example (1), g_1 acts on $Pic\, S_1$ as an involution, while in (2)-(8) g_i acts on $Pic\, S_i$ as the identity. Moreover, as was remarked in [Ko, 7.12], there does not exist an elliptic K3 surface with an automorphism group of order 5^2 which acts faithfully on the space of holomorphic 2-forms.*

Proof. We shall prove that the pair (p_1, S_1, g_1) in (1) satisfies our requirement. One argues similarly for the remaining cases (2)-(8) and we leave details of (2)-(8) to the reader. Since the discriminant (resp. the j-invariant) of the elliptic surface $\varphi : S_1 \longrightarrow \mathbb{P}^1$ is $t^6(4+27t^{16})$ (resp. $\frac{4}{4+27t^{16}}$), by [Ne, page 124-125], we know that φ has 16 singular fibers of type I_1 over $4+27t^{16} = 0$, one singular fiber of type I_0^* over $t = 0$, and one singular fiber of type II over $t = \infty$. Thus, $c_2(S_1) = 16 + 6 + 2 = 24$ and S_1 is a K3 surface. It is clear that g_1 acts on S_1 and $< g_1 > \simeq \mathbb{Z}_{32}$ as an automorphism group of S_1. Moreover, since we can take $\frac{dx \wedge dt}{y}$ as ω_{S_1} and since $g_1^*(\frac{dx \wedge dt}{y}) = e_{32}^9 \frac{dx \wedge dt}{y}$ by definition of g_1, the pair (p_1, S_1, g_1) in (1) actually satisfies our desired requirement.

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